Game Theory with Information: Witsenhausen Intrinsic Model

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Information in Game Theory

- Game theory is concerned with strategic interactions:
  my best choice depends on the other players
- Strategic interactions originate from two sources
  - Payoffs
    - a player's payoff may (possibly) depend
      on other players decisions (and on Nature moves)
    - with iconic examples *prisonners dilemma*, *hawk and dove*
  - Information
    - a player's decision may (possibly) depend
      on what he knows of other players decisions
      (and on Nature moves)
    - with iconic examples Akerlof market for lemons,
      Spence job market signaling

Information is the fuel of strategies

To speak about information, one must distinguish

- Decisions/actions : elements of a decision set
  ("taking an umbrella or not")
- Strategies: mappings from a set SET towards decision sets
  ("if it is raining, I take an umbrella",
  "if not, I do not take an umbrella")

What is the set SET? What is information?

Our roadmap

1. Present existing models with information: SET=tree
   - the celebrated finite tree extensive form of Kuhn
   - the "infinite continuous" tree form of Alòs-Ferrer and Ritzberger

   information is defined with reference to predecessors in the tree:

   i) tree $\rightsquigarrow$ ii) information

2. Introduce Witsenhausen model: SET$\neq$tree
   and information makes no reference to predecessors
   (may possibly be induced by proper information structures)

   i) set $\rightsquigarrow$ ii) information ($\rightsquigarrow$ iii) possible tree)

3. Display connections between them

Outline of the presentation

Three models of games with information: K, AFR, W

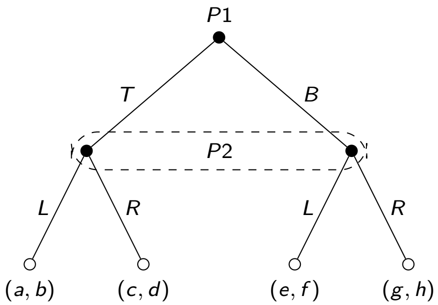
Witsenhausen intrinsic model (W-model)

From W to AFR: W-model + causality $\subseteq$ AFR-model

Conclusion

Outline of the presentation

K-model (Kuhn 1953): general setting



- Players
- Tree:
  - vertices: locii of decision
  - edges: decisions
- Information sets
- Strategies

Comparative table

- K=Kuhn (1953)
- AFR=Alòs-Ferrer and Ritzberger (2005)
- W=Witsenhausen (1975)

Table: Basics of three models

<table>
<tr><th>K-model</th><th>AFR-model</th><th>W-model</th></tr>
<tr><td>Players</td><td>Players</td><td>Agents</td></tr>
<tr><td rowspan="2">Tree: (finite)
• root
• vertices
• edges</td><td>Tree (infinite, continuous)</td><td>No ~~tree~~ structure</td></tr>
<tr><td rowspan="2">Choices/
Information partition</td><td>Actions</td></tr>
<tr><td>Information partition</td><td>Information subfield</td></tr>
<tr><td>Strategies</td><td>Strategies</td><td>Strategies</td></tr>
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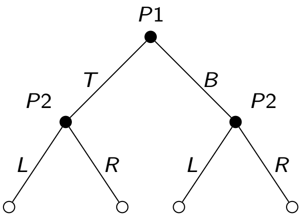
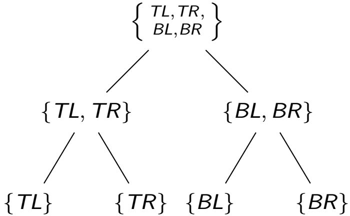
Outline of the presentation

Trees as posets where every upset is a chain

Consider a simple 2 ∗ 2 game where

- First player chooses between actions called Top and Bottom
- Second player chooses between actions called Left and Right





This game tree can be also represented as a poset,
where each vertex corresponds to the set of plays that pass by it
and where every upset is a chain (totally ordered by inclusion)

AFR-model (Alós-Ferrer, Ritzberger)

Definition of an abstract tree

- Plays
  - $\mathbb{W}$ is the set of plays
- Vertices
  - Set $V \subset 2^{\mathbb{W}}$ of vertices is called a $\mathbb{W}$-poset $(V, \supseteq)$ when it is equipped with set inclusion
    Partially ordered set because the relation $\supseteq$ is $\left\{ \begin{array}{c} \text{reflexive} \\ \text{transitive} \\ \text{antisymmetric} \end{array} \right.$
  - Given a $\mathbb{W}$-poset $(V, \supseteq)$ and a vertex $v \in V$, define its up-set by

$$\uparrow v = \{v' \in V | v' \supseteq v\}$$

  - A nonempty subset $c \in 2^V$ is a chain if for any $v, v' \in c$: $\left[\begin{array}{c} v \subseteq v' \\ \text{or} \\ v' \subseteq v \end{array} \right.$
    (a chain is a totally ordered set)
- Tree
  - Definition: a tree is a $\mathbb{W}$-poset $(V, \supseteq)$ such that $\uparrow v$ is a chain for all vertex $v \in V$

Information in K and AFR "tree" models

Information of a player = partition of the player vertices of the tree

- K-model
  - each player has an exogenous partition of his vertices
    an element of this partition is called a player's information set
- AFR-model
  - each player has an exogenous partition of his vertices
    an element of this partition is called a player's choice
  - player's information partition is the image of his choice partition
    under immediate predecessor mapping
  - Player's partition of choices $\xrightarrow{\text{immediate predecessor mapping}}$ Player's information partition

The tree comes first, information comes second

What comes next

- Can we define information
  without reference to predecessors and tree?
- Yes. Witsenhausen intrinsic model
- This is especially useful when players are scattered on a network (electric grids)

Outline of the presentation

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Witsenhausen intrinsic model (W-model)

From W to AFR: W-model + causality $\subseteq$ AFR-model

Conclusion

Outline of the presentation

W-model: agents

- An individual who makes a first, followed by a second decision, is represented by two agents (two decision makers)
- An individual who makes a sequence of decisions — one for each period $t = 0, 1, 2, \dots, T - 1$ — is represented by T agents, labelled $t = 0, 1, 2, \dots, T - 1$

Agents, decisions and decision space

- Let $\mathbb{A}$ be a finite set, whose elements are called agents (or decision-makers)
- Each agent $a \in \mathbb{A}$ is supposed to make one decision $u_a \in \mathbb{U}_a$ where
  - the set $\mathbb{U}_a$ is the set of decisions for agent a
  - and is equipped with a σ-field $\mathcal{U}_a$

Examples

- $\mathbb{A} = \{0, 1, 2, \ldots, T-1\}$ (T sequential decisions)
- $\mathbb{A} = \{\mathtt{Pr}, \mathtt{Ag}\}$ (principal-agent models)

States of Nature and configuration space

- A state of Nature (or uncertainty, or scenario) is $\omega \in \Omega$ where
  - the set Ω is a measurable set, the sample space,
  - equipped with a σ-field $\mathcal{F}$
    (at this stage of the presentation, we do not need probability distribution, as we focus only on information)
- The configuration space is the product space

$$\mathbb{H} = \prod_{a \in \mathbb{A}} \mathbb{U}_a \times \Omega$$

equipped with the product configuration field

$$\mathcal{H} = \bigotimes_{a \in \mathbb{A}} \mathcal{U}_a \otimes \mathcal{F}$$

W-model: Information fields

- The information field of agent $a \in \mathbb{A}$ is a σ-field

$$\mathcal{I}_a \subset \mathcal{H} = \bigotimes_{a \in \mathbb{A}} \mathcal{U}_a \otimes \mathcal{F}$$

- In this representation, $\mathcal{I}_a$ is a subfield of the configuration field $\mathcal{H}$ which represents the information available to agent a when he makes a decision
- Therefore, the information of agent a may depend
  - on the states of Nature
  - and on other agents' decisions

Examples 1/3: "Alice and Bob"

Example

- no Nature
- two agents a (Alice) and b (Bob)
- two possible actions each $\mathbb{U}_a = \{u_a^+, u_a^-\}$, $\mathbb{U}_b = \{u_b^+, u_b^-\}$
- configuration space (4 elements)

$$\mathbb{H} = \{u_a^+, u_a^-\} \times \{u_b^+, u_b^-\}$$

- information structure
  - $\mathcal{I}_b = \{\emptyset, \{u_a^+, u_a^-\}\} \otimes \{\emptyset, \{u_b^+, u_b^-\}\}$
    Bob knows nothing
  - $\mathcal{I}_a = \{\emptyset, \{u_a^+, u_a^-\}\} \otimes \{\emptyset, \{u_b^+\}, \{u_b^-\}, \{u_b^+, u_b^-\}\}$
    Alice knows what Bob does
    (as she can distinguish between Bob's actions $\{u_b^+\}$ and $\{u_b^-\}$)

Examples 2/3: "Alice and Bob are tossing a coin"

Example

- two states of Nature $\Omega = \{\omega^+, \omega^-\}$ (heads/tails)
- two agents a and b
- two possible actions each: $\mathbb{U}_a = \{u_a^+, u_a^-\}$, $\mathbb{U}_b = \{u_b^+, u_b^-\}$
- configuration space (8 elements)

$$\mathbb{H} = \{\omega^+, \omega^-\} \times \{u_a^+, u_a^-\} \times \{u_b^+, u_b^-\}$$

- information structure

$$\mathcal{I}_b = \overbrace{\{\emptyset, \{\omega^+\}, \{\omega^-\}, \{\omega^+, \omega^-\}\}}^{\text{Bob knows Nature's move}} \otimes \overbrace{\{\emptyset, \{u_a^+, u_a^-\}\}}^{\text{Bob does not know what Alice does}} \otimes \{\emptyset, \mathbb{U}_b\}$$

$$\mathcal{I}_a = \underbrace{\{\emptyset, \{\omega^+\}, \{\omega^-\}, \{\omega^+, \omega^-\}\}}_{\text{Alice knows Nature's move}} \otimes \{\emptyset, \mathbb{U}_a\} \otimes \underbrace{\{\emptyset, \{u_b^+\}, \{u_b^-\}, \{u_b^+, u_b^-\}\}}_{\text{Alice knows what Bob does}}$$

Examples 3/3: Principal-agent models with two players

Example

- A branch of Economics studies so-called principal-agent models

$$\mathbb{H} = \mathbb{U}_{\mathtt{Pr}} \times \mathbb{U}_{\mathtt{Ag}} \times \Omega$$
$$\mathcal{H} = \mathcal{U}_{\mathtt{Pr}} \otimes \mathcal{U}_{\mathtt{Ag}} \otimes \mathcal{F}$$

- There are two decision-makers
  - the principal $\mathtt{Pr}$ (leader), makes decisions $u_{\mathtt{Pr}} \in \mathbb{U}_{\mathtt{Pr}}$, where the set of decisions is equipped with a σ-field $\mathcal{U}_{\mathtt{Pr}}$
  - the agent $\mathtt{Ag}$ (follower) makes decisions $u_{\mathtt{Ag}} \in \mathbb{U}_{\mathtt{Ag}}$, where the set of decisions is equipped with a σ-field $\mathcal{U}_{\mathtt{Ag}}$
- and Nature, corresponding to private information (or type) of the agent $\mathtt{Ag}$
  - Nature selects $\omega \in \Omega$, where Ω is equipped with a σ-field $\mathcal{F}$

Classical information patterns in game theory

Now, we will make the information structure more specific

- Stackelberg leadership model
- Hidden action (moral hazard)
- Hidden type (adverse selection, market for lemons)
- Signaling a private type through action display (peacock's tail, diplomas on the job market)

Stackelberg leadership model

- In the Stackelberg leadership model of game theory,
- the leader $\mathtt{Pr}$ observes at most the state of Nature

$$\mathcal{I}_{\mathtt{Pr}} \subset \underbrace{\{\emptyset, \mathbb{U}_{\mathtt{Ag}}\}}_{\mathtt{Pr}\text{ does not know }\mathtt{Ag}\text{ action}} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \mathcal{F}$$

- whereas the follower $\mathtt{Ag}$ may partly observe the action of the leader $\mathtt{Pr}$

$$\mathcal{I}_{\mathtt{Ag}} \subset \{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \mathcal{U}_{\mathtt{Pr}} \otimes \mathcal{F}$$

- As a consequence, the system is sequential
  - with the principal $\mathtt{Pr}$ as first player (leader)
  - and the agent $\mathtt{Ag}$ as second player (follower)

Hidden action (moral hazard)

- An insurance company (the principal $\texttt{Pr}$) cannot observe the efforts of the insured (the agent $\texttt{Ag}$) to avoid risky behavior
- The firm faces the hazard that insured persons behave "immorally" (playing with matches at home)
- Moral hazard or hidden action occurs when the decisions of the agent $\texttt{Ag}$ are hidden to the principal $\texttt{Pr}$

$$\mathcal{I}_{\texttt{Pr}} \subset \underbrace{\{\emptyset, \mathbb{U}_{\texttt{Ag}}\}}_{\text{hidden action}} \otimes \{\emptyset, \mathbb{U}_{\texttt{Pr}}\} \otimes \mathcal{F}$$

- In case of moral hazard, the system is sequential with the principal as first player, (which does not preclude to choose the agent as first player in some special cases, as in a static team situation)

Hidden type (adverse selection, market for lemons)

- In the absence of observable information on potential customers (the agent $\mathtt{Ag}$), an insurance company (the principal $\mathtt{Pr}$) offers a unique price for a contract hence screens and selects the "bad" ones
- Adverse selection occurs when
  - the agent $\mathtt{Ag}$ knows the state of nature (his type, or private information)

$$\{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \underbrace{\mathcal{F}}_{\text{known inner type}} \subset \mathcal{I}_{\mathtt{Ag}}$$

    (the agent $\mathtt{Ag}$ can possibly observe the principal $\mathtt{Pr}$ action)
  - but the principal $\mathtt{Pr}$ does not know the agent type

$$\mathcal{I}_{\mathtt{Pr}} \subset \mathcal{U}_{\mathtt{Ag}} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \underbrace{\{\emptyset, \Omega\}}_{\text{unknown } \mathtt{Ag} \text{ type}}$$

    (the principal $\mathtt{Pr}$ can possibly observe the agent $\mathtt{Ag}$ action)

Signaling (peacock's tail,diplomas)

- In economics, a worker signals his working ability (productivity) by his educational level (diplomas)
- There is room for signaling
  - when the agent Ag knows the state of nature (his private type)

$$\{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \underbrace{\mathcal{F}}_{\text{known inner "quality"}} \subset \mathfrak{I}_{\mathtt{Ag}}$$

    (the agent Ag can possibly observe the principal Pr action)
  - whereas the principal Pr does not know the state of nature, but the principal Pr observes the agent Ag action

$$\mathfrak{I}_{\mathtt{Pr}} = \underbrace{\mathcal{U}_{\mathtt{Ag}}}_{\mathtt{Ag}\text{ effort}} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \{\emptyset, \Omega\}$$

as the agent Ag may reveal his type
by his decision which is observable by the principal Pr

What comes next

- We have just seen the great flexibility of Witsenhausen intrinsic model to express influence relations between agents without reference to a tree structure
- However, is it possible to build a tree in Witsenhausen intrinsic model?
- Not always
- But yes when the information structure displays causality

Outline of the presentation

We lay out mathematical ingredients to speak of actions order

- Let Σ denote the set of total orderings of agents in $\mathbb{A}$, that is, injective mappings from $\{1, \dots, |\mathbb{A}|\}$ to $\mathbb{A}$, where $|\mathbb{A}| = \mathrm{card}(\mathbb{A})$

$$\Sigma \ni \sigma : \{1, \dots, |\mathbb{A}|\} \to \mathbb{A}$$

- Configuration-ordering is a mapping from configurations towards orderings

$$\varphi : \mathbb{H} \to \Sigma$$

Along each configuration $h \in \mathbb{H}$, the agents are ordered by $\varphi(h) \in \Sigma$

Configuration orderings: "Alice and Bob"

Example

- no Nature
- two agents a (Alice) and b (Bob)
- two possible actions each $\mathbb{U}_a = \{u_a^+, u_a^-\}$, $\mathbb{U}_b = \{u_b^+, u_b^-\}$
- configuration space $\mathbb{H} = \{u_a^+, u_a^-\} \times \{u_b^+, u_b^-\}$ (4 elements)
- set of total orderings (2 elements: a plays first or b plays first)
  $$\Sigma = \left\{ (ab) = \begin{pmatrix} \sigma:\{1,2\}\to\{a,b\} \\ \sigma(1)=a \\ \sigma(2)=b \end{pmatrix}, (ba) = \begin{pmatrix} \sigma:\{1,2\}\to\{a,b\} \\ \sigma(1)=b \\ \sigma(2)=a \end{pmatrix} \right\}$$
- There are $2^4 = 16$ possible configuration orderings $\mathbb{H} \to \Sigma$

Configuration orderings: "Alice and Bob are tossing a coin"

Example

- two agents a (Alice) and b (Bob)
- two possible actions each $\mathbb{U}_a = \{u_a^+, u_a^-\}$, $\mathbb{U}_b = \{u_b^+, u_b^-\}$
- configuration space $\mathbb{H} = \{\omega^+, \omega^-\} \times \{u_a^+, u_a^-\} \times \{u_b^+, u_b^-\}$ (8 elements)
- set of total orderings (2 elements: a plays first or b plays first)
$$\Sigma = \left\{ (ab) = \begin{pmatrix} \sigma:\{1,2\}\to\{a,b\} \\ \sigma(1)=a \\ \sigma(2)=b \end{pmatrix}, (ba) = \begin{pmatrix} \sigma:\{1,2\}\to\{a,b\} \\ \sigma(1)=b \\ \sigma(2)=a \end{pmatrix} \right\}$$
- There are $2^8 = 256$ possible configuration orderings $\mathbb{H} \to \Sigma$
- Here is an example of non-constant configuration ordering

$$\varphi(h) = \begin{cases} (ab), \text{for } h \in \{\omega^+\} \times \mathbb{U}_a \times \mathbb{U}_b \\ (ba), \text{for } h \in \{\omega^-\} \times \mathbb{U}_a \times \mathbb{U}_b \end{cases}$$

Alice plays first when head shows up, whereas
Bob plays first when tail shows up

Causality intuition

Illustration: "Alice and Bob"

- Consider the following information structure:
  - $\mathcal{I}_b = \{\emptyset, \{u_a^+, u_a^-\}\} \otimes \{\emptyset, \{u_b^+, u_b^-\}\}$
    Bob knows nothing
  - $\mathcal{I}_a = \{\emptyset, \{u_a^+, u_a^-\}\} \otimes \{\emptyset, \{u_b^+\}, \{u_b^-\}, \{u_b^+, u_b^-\}\}$
    Alice knows what Bob does
- As Alice can distinguish between Bob's actions,
  we have the intuition that Alice cannot play before Bob;
  indeed, if Alice played first, she would know the future
  (the actions decided by Bob who plays after)
- By contrast, as Bob knows nothing, Bob can play first;
  then, Alice plays second and observes Bob's "past" actions
- We say that the constant ordering
  - $\varphi(h) = (ab)$, for all $h \in \mathbb{H}$ (a plays first) is non causal
  - $\varphi(h) = (ba)$, for all $h \in \mathbb{H}$ (b plays first) is causal

Here is how Witsenhausen defines causality

Causality

A collection $\{\mathcal{I}_a\}_{a\in\mathbb{A}}$ of information subfields is causal
if there exists (at least one) configuration-ordering φ from $\mathbb{H}$ towards Σ,
with the property that for any $k \in \{1, \ldots, |\mathbb{A}|\}$ and $\kappa \in \Sigma_k$,
the set $\mathbb{H}^{\varphi}_{k,\kappa}$ satisfies

$$\mathbb{H}^{\varphi}_{k,\kappa} \cap G \in \mathcal{F} \otimes \mathcal{U}_{\{\kappa(1),\ldots,\kappa(k-1)\}} \ , \quad \forall G \in \mathcal{I}_{\kappa(k)}$$

- In other words, when the first k agents are known
  and ordered by $(\kappa(1), \ldots, \kappa(k))$,
  the information $\mathcal{I}_{\kappa(k)}$ of the agent $\kappa(k)$ with rank k
  depends at most on the decisions of agents with rank $< k$,
  that is, $\kappa(1)$, ..., $\kappa(k-1)$

Information comes first, tree (possibly) comes second

What comes next

- K-model and AFR-model define information with reference to predecessors and tree
- W-model defines information without reference to predecessors and tree
- When, in the W-model, the information structure displays causality, we will see that we can build a tree and that

$$\text{W-model} + \text{causality} \subseteq \text{AFR-model}$$

Outline of the presentation

Three models of games with information: K, AFR, W

Witsenhausen intrinsic model (W-model)

From W to AFR: W-model + causality $\subseteq$ AFR-model

Conclusion

Roadmap: causal W-model $\subseteq$ AFR-model

Construct

- WtoAFR-tree
- WtoAFR-choices and information
- WtoAFR-strategies

Outline of the presentation

"Alice and Bob": tree structure

$$\mathbb{W} = \left\{ \begin{array}{cc} u_a^+ u_b^+(ba) \ , & u_a^+ u_b^-(ba) \ , \\ u_a^- u_b^+(ba) \ , & u_a^- u_b^-(ba) \end{array} \right\}$$

$$\left\{ u_a^+ u_b^+(ba) \ , \ \ u_a^- u_b^+(ba) \right\} \qquad \left\{ u_a^+ u_b^-(ba) \ , \ \ u_a^- u_b^-(ba) \right\}$$

$$\{ u_a^+ u_b^+(ba) \} \qquad \left\{ u_a^- u_b^+(ba) \right\} \qquad \left\{ u_a^+ u_b^-(ba) \right\} \qquad \left\{ u_a^- u_b^-(ba) \right\}$$

Construction of the WtoAFR-tree

$$\underbrace{\mathbb{W}}_{\text{plays}} = \mathbb{H} \times \varphi(\mathbb{H}) \subset \underbrace{\mathbb{H}}_{\text{configurations}} \times \underbrace{\Sigma}_{\text{orderings}}$$

Claim

For any configuration ordering φ there exist an increasing sequence $\{\mathfrak{V}_k^\varphi\}_{k \in \{0,\dots,|\mathbb{A}|\}}$ of equivalence relations, where each $\mathfrak{V}_k^\varphi$ is called vertex relation of level k, such that $\mathbb{W}$-poset $(V, \subset)$ is a tree, where

$$V = \mathbb{W} \cup \bigcup_{k \in \{0,\dots,|\mathbb{A}|\}} \mathbb{W} \big/ \mathfrak{V}_k^\varphi$$

Outline of the presentation

Roadmap: WtoAFR-choices

Information relation

For a fixed W-agent $a \in \mathbb{A}$, we suppose that the information subfield $\mathfrak{I}_a$ is generated by a partition ${}^{\mathbb{W}}\!/_{\mathfrak{I}_a}$ of $\mathbb{W}$

- Information $\mathfrak{I}_a$ partitions the set $\mathbb{W}$ of plays into the information partition ${}^{\mathbb{W}}\!/_{\mathfrak{I}_a}$

Action relation

For a fixed W-agent $a \in \mathbb{A}$, *a's action equivalence relation* $\mathfrak{U}_a$ is defined on the set of plays $\mathbb{W}$ in the following way:

$$(h, \sigma)\mathfrak{U}_a(h', \sigma') \text{ iff } h_a = h'_a \text{ iff } u_a = u'_a$$

- The action partition ${}^{\mathbb{W}}\!/_{\mathfrak{U}_a} \equiv \mathbb{U}_a$ defines a's possible actions

Construction of the WtoAFR-choices

Claim

- *The atoms of the following partition*

$$ {}^{\mathbb{W}}\!/_{\mathfrak{I}_a} \bigvee {}^{\mathbb{W}}\!/_{\mathfrak{U}_a} $$

  are intersections of atoms of the information partition with atoms of the action partition
- *They are called WtoAFR-choices*
- *If the history ordering φ is causal, the WtoAFR-choices of W-agent a satisfy the AFR-axioms for choices*

AFR-choice = (what I know, what I do)

Connection between WtoAFR-information and -choices

Definition

Immediate predecessor family P is a family of mappings

$$P = (P_k)_{k\in\{0,\ldots,|\mathbb{A}|\}}\,,\ \text{where}$$

$$\mathbb{W}\Big/\mathfrak{V}_k^{\varphi} \xrightarrow{P_k} \mathbb{W}\Big/\mathfrak{V}_{k-1}^{\varphi},\ \text{for } k \neq 0 \qquad \mathbb{W}\Big/\mathfrak{V}_0^{\varphi} \xrightarrow{P_0} \mathbb{W},\ \text{for } k = 0$$

and for any $v \in \mathbb{W}\Big/\mathfrak{V}_k^{\varphi}$ holds the parent relation: $v \subset P_k(v)$

How does it work?

$$P\Big(\underbrace{\mathbb{W}\Big/\mathfrak{U}_a \bigvee \mathbb{W}\Big/\mathfrak{I}_a}_{a\text{'s choices}}\Big) = \underbrace{\mathbb{W}\Big/\mathfrak{I}_a}_{a\text{'s information}}$$

Each agent's choice is mapped to the information atom where it was made

Outline of the presentation

Construction of the WtoAFR-strategies

W-strategy

Adapted strategy

An adapted strategy for agent a is a mapping

$$\lambda_a : (\mathbb{H}, \mathcal{H}) \to (\mathbb{U}_a, \mathcal{U}_a)$$

which is measurable w.r.t. the information field $\mathcal{I}_a$ of agent a, that is,

$$\lambda_a^{-1}(\mathcal{U}_a) \subset \mathcal{I}_a$$

Characterization of an adapted strategy

λ_a is a W-strategy iff there exists $\widetilde{s}_a$ such that $\lambda_a = \widetilde{s}_a \circ \pi_{\mathfrak{I}_a}$

$$\begin{tikzcd}
\mathbb{W} \arrow[r, "\lambda_a"] \arrow[d, "\pi_{\mathfrak{I}_a}"] & \mathbb{U}_a \equiv \mathbb{W} / \mathfrak{U}_a \\
\mathbb{W} / \mathfrak{I}_a \arrow[ur, "\widetilde{s}_a"'] &
\end{tikzcd}$$

Construction of the WtoAFR-strategies

Claim

To any W-strategy λ_a we can associate a WtoAFR-strategy s_a

$$s_a : \underbrace{\mathbb{W}/\mathfrak{U}_a}_{\text{agent } a\text{'s information}} \longrightarrow \underbrace{\mathbb{W}/\mathfrak{U}_a \bigvee \mathbb{W}/\mathfrak{I}_a}_{\text{agent } a\text{'s choices}}$$

The mapping s_a satisfies the definition of AFR-strategy

Sketch of construction

$$\begin{tikzcd}
\mathbb{W} \arrow[r, "\lambda_a"] \arrow[d, "\pi_{\mathfrak{I}_a}"'] & \mathbb{U}_a \equiv \mathbb{W}/\mathfrak{U}_a \\
\mathbb{W}/\mathfrak{I}_a \arrow[ur, "\widetilde{s}_a"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{W}/\mathfrak{U}_a \times \mathbb{W}/\mathfrak{I}_a \arrow[d, "\cap"] \\
\mathbb{W}/\mathfrak{I}_a \arrow[ur, "(\widetilde{s}_a, \mathrm{id})"] \arrow[r, "s_a"] & \mathbb{W}/\mathfrak{U}_a \bigvee \mathbb{W}/\mathfrak{I}_a
\end{tikzcd}$$

Roadmap completed: causal W-model $\subseteq$ AFR-model

We have constructed

- WtoAFR-tree
- WtoAFR-choices and information
- WtoAFR-strategies

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Conclusion

We have presented a language adapted to handle information

Research program

- Continuous games in W-framework
(in the sense of continuous locii of decisions/agents)
- Embedding Bayesian games in W-framework
- Definition of Nash equilibrium
- Definition of subgames, of subgame perfect equilibrium
and of backward induction in W-framework
thanks to the notion of subsystem

$$\text{subsystem } B \subset \mathbb{A} \iff \bigvee_{b \in B} \mathcal{I}_b \subset \bigotimes_{b \in B} \mathcal{U}_b \otimes \bigotimes_{c \notin B} \{\emptyset, \mathbb{U}_c\} \otimes \mathcal{F}$$