## Hidden Convexity in the $\ell_{0}$ Pseudonorm

> Algorithms in Generalized Convexity and Application to Sparse Optimization

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with the contributions of
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Here are the level sets
of the (highly nonconvex) $\ell_{0}$ pseudonorm in $\mathbb{R}^{2}$


## The $\ell_{0}$ pseudonorm is not a norm

Let $n \in \mathbb{N}^{*}$ be a fixed natural number

- For any vector $x \in \mathbb{R}^{n}$, we define its $\ell_{0}$ pseudonorm $(x)$ by

$$
\ell_{0}(x)=\text { number of nonzero components of } x=\sum_{i=1}^{n} \mathbf{1}_{\left\{x_{i} \neq 0\right\}}
$$

- The function $\ell_{0}$ pseudonorm : $\mathbb{R}^{n} \rightarrow \llbracket 0, n \rrbracket=\{0,1, \ldots, n\}$ satisfies 3 out of 4 axioms of a norm
- we have $\ell_{0}(x) \geq 0$
- we have $\left(\ell_{0}(x)=0 \Longleftrightarrow x=0\right)$
- we have $\ell_{0}\left(x+x^{\prime}\right) \leq \ell_{0}(x)+\ell_{0}\left(x^{\prime}\right)$
- But... instead of 1-homogeneity, it is 0 -homogeneity that holds true

$$
\ell_{0}(\rho x)=\ell_{0}(x), \quad \forall \rho \neq 0
$$

## WHY STUDY A FUNCTION <br> THAT IS <br> ALMOST SURELY CONSTANT?

The $\ell_{0}$ pseudonorm is used in typical sparse optimization problems

- Spark of a matrix $A$

$$
\operatorname{spark}(A)=\min \left\{\ell_{0}(x) \mid A x=0, x \neq 0\right\}
$$

- Compressed sensing: recovery of a sparse signal $x \in \mathbb{R}^{n}$ from a measurement $b=A x$

$$
\min _{\substack{x \in \mathbb{R}^{n} \\ A x=b}} \ell_{0}(x)
$$

- Least squares sparse regression (best subset selection):

$$
\text { for } k \in \llbracket 1, n \rrbracket \quad \min _{\substack{x \in \mathbb{R}^{n} \\ \ell_{0}(x) \leq k}}\|A x-b\|^{2}
$$

"explaining" the output $b$ by at most $k$ components of $x$

Fenchel conjugacy ( $\star$ ) versus E-Capra conjugacy ( $\grave{( })$ for the $\ell_{0}$ pseudonorm

- Fenchel conjugacy ( $\star$ )

$$
\ell_{0}^{\star^{\prime}}=0
$$

- E-Capra conjugacy (¢)

$$
\ell_{0}^{C C^{\prime}}=\ell_{0}
$$

[Chancelier and De Lara, 2021]

The $\ell_{0}$ pseudonorm coincides, on the unit sphere, with the proper convex lower semicontinuous $\ell_{0}$-cup function $\mathcal{L}_{0}=\ell_{0}^{\dot{C}^{\prime} \star^{\prime}}$



## Towards algorithms?

- As motivation, we consider the sparse optimization problem, where $C$ is a nonempty closed convex subset of $\mathbb{R}^{n}$,
where $¢$ is the so-called E-Capra coupling
- Can we design algorithms using the above property that the pseudonorm $\ell_{0}$ is E-Capra convex?


## END OF THE TEASER

## Talk outline

Crash course on generalized convexity [5 min]
Capra conjugacies [15 min]
Euclidean Capra conjugacy
Capra conjugacies
Towards Capra-algorithms in sparse optimization? [15 min]
Good and bad news about the Fermat rule (with Adrien Le Franc and Seta Rakotomandimby)
Capra-cuts method
(with Seta Rakotomandimby)
The geometry of sparsity-inducing unit balls (with Antoine Deza and Lionel Pournin)

Conclusion [1 min]
Additional material

## Outline of the presentation

Crash course on generalized convexity [5 min]

Capra conjugacies [15 min]

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Conclusion [1 min]

Additional material

Motivation: Legendre transform and
Fenchel conjugacy in convex analysis

## Definition

Two vector spaces $\mathcal{X}$ and $\mathcal{Y}$, paired by a bilinear form $\langle$,$\rangle ,$ give rise to the classic Fenchel conjugacy

$$
f \in \overline{\mathbb{R}}^{\mathcal{X}} \mapsto f^{\star} \in \overline{\mathbb{R}}^{\mathcal{Y}}
$$

given by the Legendre transform

$$
f^{\star}(y)=\sup _{x \in \mathcal{X}}(\langle x, y\rangle+(-f(x))), \quad \forall y \in \mathcal{Y}
$$

## Coupling functions

## Coupling function between sets

- Let be given two sets $\mathcal{X}$ ("primal") and $\mathcal{Y}$ ("dual") not necessarily paired vector spaces (nodes and arcs, etc.)
- We consider a coupling function

$$
c: \mathcal{X} \times \mathcal{Y} \rightarrow \overline{\mathbb{R}}
$$

We also use the notation $\mathcal{X} \stackrel{C}{\leftrightarrow} \mathcal{Y}$ for a coupling
[Moreau, 1966-1967, 1970]

In duality in convex analysis, one uses the bilinear coupling

$$
c(x, y)=\langle x, y\rangle
$$

and, on a Hilbert space, the scalar product

$$
c(x, y)=\langle x \mid y\rangle
$$

## Euclidean Constant Along Primal RAys (Capra) coupling

- On the Euclidean space $\mathbb{R}^{n}$, the Euclidean-Capra coupling (E-Capra) $\mathbb{R}^{n} \stackrel{\text { ¢ }}{\longleftrightarrow} \mathbb{R}^{n}$ is given by

$$
\forall y \in \mathbb{R}^{n},\left\{\begin{array}{l}
\dot{c}(x, y)=\frac{\langle x \mid y\rangle}{\|x\|_{2}}=\frac{\langle x \mid y\rangle}{\sqrt{\langle x \mid x\rangle}}, \forall x \in \mathbb{R}^{n} \backslash\{0\} \\
\dot{(0, y)}=0
\end{array}\right.
$$

- The coupling E-Capra has the property of being Constant Along Primal RAys (Capra)

Fenchel-Moreau conjugacies

## Fenchel-Moreau conjugate of a function

$$
f \in \overline{\mathbb{R}}^{\mathcal{X}} \mapsto f^{c} \in \overline{\mathbb{R}}^{\mathcal{Y}}
$$

## Definition

The $c$-Fenchel-Moreau conjugate $f^{c}: \mathcal{Y} \rightarrow \overline{\mathbb{R}}$ of a function $f: \mathcal{X} \rightarrow \overline{\mathbb{R}}$ is defined by

$$
f^{c}(y)=\sup _{x \in \mathcal{X}}(c(x, y)+(-f(x))), \quad \forall y \in \mathcal{Y}
$$

We use the Moreau lower and upper additions on $\overline{\mathbb{R}}$ that extend the usual addition with

$$
\begin{aligned}
(+\infty)+(-\infty) & =(-\infty)+(+\infty)=-\infty \\
(+\infty)+(-\infty) & =(-\infty)+(+\infty)=+\infty
\end{aligned}
$$

## E-Capra-conjugate of the $\ell_{0}$ pseudonorm

$$
\begin{aligned}
\ell_{0}^{c}(y) & =\sup _{x \in \mathbb{R}^{n}}\left\{¢(x, y)+\left(-\ell_{0}(x)\right)\right\} \\
& =\sup \left\{0, \sup _{x \neq 0}\left\{\frac{\langle x \mid y\rangle}{\|x\|_{2}}-\ell_{0}(x)\right\}\right\} \\
& =\sup \left\{0, \sup _{s \in S_{2}}\left\{\langle s \mid y\rangle-\ell_{0}(s)\right\}\right\}
\end{aligned}
$$

where $S_{2} \subset \mathbb{R}^{n}$ is the Euclidean unit sphere

$$
\begin{aligned}
& =\sup \{0, \sup _{j \in \llbracket 1, d \rrbracket}\{\underbrace{\substack{\begin{subarray}{c}{s \in \mathcal{S}_{2} \\
\ell(s)=j} }}}_{\substack{\text { top- } \left.(2, j) \text { norm } \\
\|y\|_{2, j}=\sqrt{\sum_{l=1}^{k}=1 y_{\nu(1)}}\right)^{2}}}\langle s \mid y\rangle-j\}\} \\
& =\sup _{j \in[1, d]}\left[\|y\|_{2, j}^{\top}-j\right]_{+}
\end{aligned}
$$

Biconjugates and duality

Motivation: duality in convex analysis


## Reverse coupling and Fenchel-Moreau biconjugate

With the coupling $c$, we associate the reverse coupling $c^{\prime}$

$$
\begin{gathered}
c^{\prime}: \mathcal{Y} \times \mathcal{X} \rightarrow \overline{\mathbb{R}}, \quad c^{\prime}(y, x)=c(x, y), \quad \forall(y, x) \in \mathcal{Y} \times \mathcal{X} \\
f \in \overline{\mathbb{R}}^{\mathcal{X}} \mapsto f^{c} \in \overline{\mathbb{R}}^{\mathcal{Y}} \\
g \in \overline{\mathbb{R}}^{\mathcal{Y}} \mapsto g^{c^{\prime}} \in \overline{\mathbb{R}}^{\mathcal{X}}
\end{gathered}
$$

## Reverse coupling and Fenchel-Moreau biconjugate

With the coupling $c$, we associate the reverse coupling $c^{\prime}$

$$
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c^{\prime}: \mathcal{Y} \times \mathcal{X} \rightarrow \overline{\mathbb{R}}, \quad c^{\prime}(y, x)=c(x, y), \quad \forall(y, x) \in \mathcal{Y} \times \mathcal{X} \\
f \in \overline{\mathbb{R}}^{\mathcal{X}} \mapsto f^{c} \in \overline{\mathbb{R}}^{\mathcal{Y}} \\
g \in \overline{\mathbb{R}}^{\mathcal{Y}} \mapsto g^{c^{\prime}} \in \overline{\mathbb{R}}^{\mathcal{X}} \\
g^{c^{\prime}}(x)=\sup _{y \in \mathcal{Y}}(c(x, y)+(-g(y))), \quad \forall x \in \mathcal{X} \\
f^{c c^{\prime}}(x)=\left(f^{c}\right)^{c^{\prime}}(x)=\sup _{y \in \mathcal{Y}}\left(c(x, y)+\left(-f^{c}(y)\right)\right), \quad \forall x \in \mathcal{X}
\end{gathered}
$$

In generalized convexity, one defines so-called c-convex functions

$$
f \in \overline{\mathbb{R}}^{\mathcal{X}} \mapsto f^{c} \in \overline{\mathbb{R}}^{\mathcal{Y}} \mapsto f^{c c^{\prime}} \in \overline{\mathbb{R}}^{\mathcal{X}}
$$

For any function $f: \mathcal{X} \rightarrow \overline{\mathbb{R}}$, one has that

$$
f^{c c^{\prime}} \leq f
$$

## Definition

The function $f: \mathcal{X} \rightarrow \overline{\mathbb{R}}$ is said to be $c$-convex if

$$
f^{c c^{\prime}}=f
$$

## c-convex functions have dual representations as suprema of elementary functions (abstract convexity)

If the function $f: \mathcal{X} \rightarrow \overline{\mathbb{R}}$ is $c$-convex, we have that

$$
f(x)=\sup _{y \in \mathcal{Y}} \underbrace{\left(c(x, y)+\left(-f^{c}(y)\right)\right)}_{\text {elementary function of } x}, \forall x \in \mathcal{X}
$$

Example: *-convex functions $=$ closed convex functions
$=$ proper convex Isc or $\equiv-\infty$ or $\equiv+\infty$
$=$ suprema of affine functions

# Subdifferential 

Motivation: subgradients in convex analysis


## (Upper) subdifferential $\partial^{c} f: \mathcal{X} \rightrightarrows \mathcal{Y}$ of a conjugacy

For any function $f: \mathcal{X} \rightarrow \overline{\mathbb{R}}$ and $x \in \mathcal{X}, y \in \mathcal{Y}$

## Definition

Upper subdifferential (following [Martinez-Legaz and Singer, 1995])

$$
y \in \partial^{c} f(x) \Longleftrightarrow f(x)=c(x, y)+\left(-f^{c}(y)\right)
$$

The upper subdifferential $\partial^{c} f$ has the property that

$$
\begin{aligned}
\partial^{c} f(x) \neq \emptyset & \Longrightarrow f(x)=\max _{y \in \partial^{c} f(x)}\left(c(x, y)+\left(-f^{c}(y)\right)\right) \\
& \Longrightarrow \underbrace{f(x)=f^{c c^{\prime}}(x)}_{\text {the function } f \text { is } c \text {-convex at } x}
\end{aligned}
$$

## Wrap-up on generalized/abstract convexity

- Generalized convexity
- coupling function between two sets

$$
c: \mathcal{X} \times \mathcal{Y} \rightarrow \overline{\mathbb{R}}
$$

- conjugacy and biconjugacy $f \in \overline{\mathbb{R}}^{\mathcal{X}} \mapsto f^{c} \in \overline{\mathbb{R}}^{\mathcal{Y}} \mapsto f^{c c^{\prime}} \in \overline{\mathbb{R}}^{\mathcal{X}}$
- generalized convex functions
$f=f^{c c^{\prime}}$
- subdifferential
$\partial^{c} f(x) \subset \mathcal{Y}$
- Abstract convexity
- set of elementary functions
- abstract convex envelope:
supremum of lower elementary functions
- abstract convex function:
equal to its abstract convex envelope
- subdifferential:
tight lower elementary functions


## Outline of the presentation

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Capra conjugacies [15 min]

Towards Capra-algorithms in sparse optimization? [15 min]

Conclusion [1 min]

Additional material

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Conclusion [1 min]
Additional material

We introduce the coupling E-Capra between $\mathbb{R}^{n}$ and itself

## Definition

The Euclidean-Capra coupling (E-Capra) $\mathbb{R}^{n} \stackrel{\S}{\longleftrightarrow} \mathbb{R}^{n}$ is given by

$$
\forall y \in \mathbb{R}^{n},\left\{\begin{array}{l}
\dot{c}(x, y)=\frac{\langle x \mid y\rangle}{\|x\|_{2}}=\frac{\langle x \mid y\rangle}{\sqrt{\langle x \mid x\rangle}}, \forall x \in \mathbb{R}^{n} \backslash\{0\} \\
\dot{c}(0, y)=0=\frac{0}{0}
\end{array}\right.
$$

The coupling E-Capra has the property of being
Constant Along Primal RAys (Capra)

## E-Capra $=$ Fenchel coupling after primal normalization

- We define the primal radial projection $\varrho$ as

$$
\varrho: \mathbb{R}^{n} \rightarrow S_{2} \cup\{0\}, \varrho(x)= \begin{cases}\frac{x}{\|x\|_{2}} & \text { if } x \neq 0 \\ \frac{0}{0}=0 & \text { if } x=0\end{cases}
$$

- so that the coupling E-Capra

$$
\dot{c}(x, y)=\langle\varrho(x) \mid y\rangle, \forall x \in \mathbb{R}^{n}, \quad \forall y \in \mathbb{R}^{n}
$$

appears as the Fenchel coupling after primal normalization (and the coupling E-Capra is one-sided linear)

The E-Capra conjugacy shares properties with the Fenchel conjugacy

## Proposition

- For any function $f: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$, the $\phi$-Fenchel-Moreau conjugate is given by

$$
\begin{gathered}
f^{\dot{C}}=(\inf [f \mid \varrho])^{\star} \\
\inf [f \mid \varrho](x)= \begin{cases}\inf _{p>0} f(\rho x) & \text { if } x \in S_{2} \cup\{0\} \\
+\infty & \text { if } x \notin S_{2} \cup\{0\}\end{cases}
\end{gathered}
$$

- For any function $g: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$, the $\oint^{\prime}$-Fenchel-Moreau conjugate is given by

$$
g^{\dot{\zeta}^{\prime}}=g^{\star} \circ \varrho
$$

The E-Capra-convex functions are 0-homogeneous and coincide, on the unit sphere, with a closed convex function

## Proposition

¢-convexity of the function $h: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$
$\Longleftrightarrow h=h^{\text {¢ } C^{\prime}}$
$\Longleftrightarrow h=\underbrace{\left(h^{\dot{C}}\right)^{\star^{\prime}}} \circ \varrho$
convex lsc function
$\Longleftrightarrow$ hidden convexity in the function $h: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ there exists a closed convex function $f: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ such that $h=f \circ \varrho$, that is, $h(x)=f\left(\frac{x}{\|x\|_{2}}\right)$

The $\ell_{0}$ pseudonorm is E-Capra-convex

## Notation

- The Euclidean top- $(2, k)$ norm is also known as the (2, k)-symmetric gauge norm, or Ky Fan vector norm

$$
\|y\|_{2, k}^{\top}=\sqrt{\sum_{l=1}^{k}\left|y_{\nu(I)}\right|^{2}},\left|y_{\nu(1)}\right| \geq\left|y_{\nu(2)}\right| \geq \cdots \geq\left|y_{\nu(n)}\right|
$$

- We denote the level sets of the $\ell_{0}$ pseudonorm by

$$
\ell_{0}^{\leq k}=\left\{x \in \mathbb{R}^{n} \mid \ell_{0}(x) \leq k\right\}, \quad \forall k \in \llbracket 0, n \rrbracket
$$

and its elements are call $k$-sparse vectors

- For any subset $W \subset \mathbb{R}^{n}$, its indicator function $\iota_{W}$ is

$$
\iota W(w)= \begin{cases}0 & \text { if } w \in W \\ +\infty & \text { if } w \notin W\end{cases}
$$

## The $\ell_{0}$ pseudonorm and the E-Capra-coupling

## Theorem

The $\ell_{0}$ pseudonorm, the indicator functions $\iota_{\ell_{0}^{\leq k}}$ of its level sets and the Euclidean top- $(2, k)$ norms $\|\cdot\|_{2, k}^{\top}$ are related by

$$
\begin{aligned}
& \iota_{\ell_{0}^{\leq k}}^{\dot{C}}=\|\cdot\|_{2, k}^{\top}, \quad k \in \llbracket 0, n \rrbracket \\
& \ell_{0}^{\dot{C}}=\sup _{j \in \llbracket 0, n \rrbracket}\left[\|\cdot\|_{2, j}^{\top}-j\right] \\
& \ell_{0}^{\dot{C}{C^{\prime}}^{\prime}}=\ell_{0}
\end{aligned}
$$

The $\ell_{0}$ pseudonorm displays hidden convexity

## The $\ell_{0}$ pseudonorm displays a convex factorization property

## Theorem

As the $\ell_{0}$ pseudonorm is E-Capra-convex, we get that

$$
\ell_{0}=\ell_{0}^{\dot{c} \dot{\varphi}^{\prime}}=\ell_{0}^{\dot{\star^{\prime}}} \circ \varrho=\underbrace{\left(\ell_{0}^{c}\right)^{\star^{\prime}}}_{\text {convex lsc function } \mathcal{L}_{0}}
$$

As a consequence, the $\ell_{0}$ pseudonorm coincides, on the Euclidean unit sphere $S_{2}$, with a proper convex Isc function, the Euclidean $\ell_{0}$-cup function $\mathcal{L}_{0}=\ell_{0}^{\dot{c} \star^{\prime}}$

$$
\ell_{0}(x)=\mathcal{L}_{0}(x), \quad \forall x \in S_{2}
$$

## Graph of the Euclidean $\ell_{0}$-cup function $\mathcal{L}_{0}=\ell_{0}^{\dagger \star^{\prime}}$

Best proper convex Isc lower approximation of the $\ell_{0}$ pseudonorm on the Euclidean unit ball

## Theorem

The Euclidean $\ell_{0}$-cup function $\mathcal{L}_{0}=\ell_{0}^{\zeta \star^{\prime}}$ is the best convex Isc lower approximation of the $\ell_{0}$ pseudonorm on the Euclidean unit ball $B_{2}$
best convex lsc function $\quad \mathcal{L}_{0}(x) \leq \ell_{0}(x), \forall x \in B_{2}$
and, as seen above, coincides with the $\ell_{0}$ pseudonorm
on the Euclidean unit sphere $S_{2}$

$$
\ell_{0}(x)=\mathcal{L}_{0}(x), \quad \forall x \in S_{2}
$$

E-Capra subdifferential of the $\ell_{0}$ pseudonorm (with Adrien Le Franc)

## Capra-subdifferential of the $\ell_{0}$ pseudonorm on $\mathbb{R}^{2}$

Illustration at three points (black dots)


$$
\partial_{\dot{C}} \ell_{0}(0,0), \partial_{C} \ell_{0}(1,0), \partial_{\dot{C}} \ell_{0}\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)
$$

## Capra-subdifferential of the $\ell_{0}$ pseudonorm on $\mathbb{R}^{2}$



$$
\partial_{c} \ell_{0}(0) \cup\left\{\underset{\ell_{0}(x)=1}{\cup} \partial_{c} \ell_{0}(x)\right\} \cup\left\{\underset{\ell_{0}(x)=2}{\cup} \partial_{\dot{c}} \ell_{0}(x)\right\}
$$

Lower approximation of the $\ell_{0}$ pseudonorm by a finite number of elementary E-Capra-functions



Variational formulas

## We recall the Euclidean $(2, k)$-support norms $\|\cdot\|_{2, k}^{T_{\star}}$

- The dual norm of the top- $(2, k)$ norm $\|\cdot\|_{2, k}^{\top}$

$$
\|\cdot\|_{2, k}^{T \star}=\left(\|\cdot\|_{2, k}^{\top}\right)_{\star}
$$

is called the (Euclidean) $(2, k)$-support norm
[Argyriou, Foygel, and Srebro, 2012]

- We have the following inclusions between unit balls

$$
B_{(1)}^{T_{\star}} \subset \cdots \subset B_{(\ell-1)}^{T_{\star}} \subset B_{(\ell)}^{T_{\star}} \subset \cdots \subset B_{(n)}^{T_{\star}}=B
$$

## Proposition

The proper convex Isc function $\mathcal{L}_{0}$ is the convex envelope of the following piecewise constant function

$$
L_{0}(x)= \begin{cases}0 & \text { if } x=0 \\ \ell & \text { if } x \in B_{(\ell)}^{T_{\star}} \backslash B_{(\ell-1)}^{T_{\star}}, \quad \ell \in \llbracket 1, n \rrbracket \\ +\infty & \text { if } x \notin B_{(n)}^{T_{\star}}=B\end{cases}
$$



## Variational formulas for the $\ell_{0}$ pseudonorm

## Proposition

$$
\begin{gathered}
\ell_{0}(x)=\frac{1}{\|x\|_{2}} \min _{\substack{x^{(1)} \in \mathbb{R}^{n}, \ldots, x^{(n)} \in \mathbb{R}^{n} \\
\sum_{\ell=1}^{n}\left\|x^{(\ell)}\right\|_{2, \ell}^{T \star} \leq\|x\|_{2}}} \sum_{\ell=1}^{n} \ell\left\|x^{(\ell)}\right\|_{2, \ell}^{T_{\star}}, \quad \forall x \in \mathbb{R}^{n} \\
\sum_{\ell=1}^{n} x^{(\ell)}=x
\end{gathered}
$$

$$
\ell_{0}(x)=\sup _{y \in \mathbb{R}^{n}} \inf _{\ell \in \llbracket 1, n \rrbracket}\left(\frac{\langle x \mid y\rangle}{\|x\|_{2}}-\left[\|y\|_{2, \ell}^{\top}-\ell\right]_{+}\right), \forall x \in \mathbb{R}^{n} \backslash\{0\}
$$

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Conclusion [1 min]
Additional material

## Work has gone on along two paths

|  | Norm Euclidean | Norm orthant-strictly monotonic | Norm any | 1-homogeneous nonnegative function |
| :---: | :---: | :---: | :---: | :---: |
| $\ell_{0}$ pseudonorm |  | difference of norms <br> [Chancelier and De Lara, 2022b] |  |  |
| $\begin{gathered} \varphi \circ \ell_{0} \\ \varphi: \mathbb{N} \rightarrow \overline{\mathbb{R}} \\ \text { nondecreasing } \end{gathered}$ |  | $\begin{gathered} \text { ¢-convex }\left(\left(\varphi \circ \ell_{0}\right)^{\phi c^{\prime}}=\varphi \circ \ell_{0}\right) \\ \text { hidden convexity } \\ \text { variational formula } \\ \text { subdifferential } \\ \text { [Chancelier and De Lara, 2022c] } \\ \hline \end{gathered}$ |  |  |
| $\begin{aligned} & \varphi \circ \ell_{0} \\ & \varphi: \bar{N} \rightarrow \bar{R} \\ & \text { any } \end{aligned}$ |  |  | $\left(\varphi \circ \ell_{0}\right)^{\text {¢ }^{\prime}}$ variational inequality subdifferential [Chancelier and De Lara, 2022a] |  |
| Fo support $\begin{gathered} F: 2_{\text {any }}^{\llbracket 1, d \rrbracket} \end{gathered} \rightarrow \overline{\mathbb{R}}$ |  |  | $\begin{aligned} & (F \circ \text { support })^{c c^{\prime}} \\ & \text { variational inequality } \\ & \text { subdifferential } \\ & \text { [preprint] } \\ & \hline \end{aligned}$ |  |
| 0-homogeneous function |  |  |  | best lower approximation [preprint] |

## We introduce the coupling Capra

- Let be given $\mathcal{X}$ and $\mathcal{Y}$, two vector spaces paired by a bilinear form $\langle\cdot, \cdot\rangle$
- Suppose that $\mathcal{X}$ is equipped with a (source) norm $\|\cdot\|$


## Definition

[Chancelier and De Lara, 2022a]
The coupling Capra $\mathcal{X} \stackrel{¢}{\longleftrightarrow} \mathcal{Y}$ is given by

$$
\forall y \in \mathcal{Y},\left\{\begin{array}{l}
\dot{c}(x, y)=\frac{\langle x, y\rangle}{\|x\|}, \forall x \in \mathcal{X} \backslash\{0\} \\
\dot{c}(0, y)=0
\end{array}\right.
$$

In what follows, $\mathcal{X}=\mathcal{Y}=\mathbb{R}^{n}$
with norm $\|\cdot\|$ having unit ball $B$ and unit sphere $S$

Orthant-monotonic and orthant-strictly monotonic norms

## Orthant-monotonic norms

For any $x \in \mathbb{R}^{n}$, we denote by $|x|$ the vector of $\mathbb{R}^{n}$ with components $\left|x_{i}\right|, i \in \llbracket 1, n \rrbracket$

## Definition

A norm $\|\cdot\|$ on the space $\mathbb{R}^{n}$ is called orthant-monotonic [Gries, 1967] if, for all $x, x^{\prime}$ in $\mathbb{R}^{n}$, we have

$$
|x| \leq\left|x^{\prime}\right| \text { and } x \circ x^{\prime} \geq 0 \Longrightarrow\|x\| \leq\left\|x^{\prime}\right\|
$$

where $x \circ x^{\prime}=\left(x_{1} x_{1}^{\prime}, \ldots, x_{n} x_{n}^{\prime}\right)$
is the Hadamard (entrywise) product

$$
\left.\begin{array}{l}
\left|x_{1}\right| \leq\left|x_{1}^{\prime}\right|, \ldots, \quad\left|x_{n}\right| \leq\left|x_{n}^{\prime}\right| \\
x_{1} x_{1}^{\prime} \geq 0, \ldots, x_{n} x_{n}^{\prime} \geq 0
\end{array}\right\} \Longrightarrow\|x\| \leq\left\|x^{\prime}\right\|
$$

## Example of unit sphere of a non orthant-monotonic norm



## Orthant-strictly monotonic norms

[Chancelier and De Lara, 2022b]

## Definition

A norm $\|\cdot\|$ on the space $\mathbb{R}^{n}$ is called orthant-strictly monotonic if, for all $x, x^{\prime}$ in $\mathbb{R}^{n}$, we have

$$
|x|<\left|x^{\prime}\right| \text { and } x \circ x^{\prime} \geq 0 \Longrightarrow\|x\|<\left\|x^{\prime}\right\|
$$

where $|x|<\left|x^{\prime}\right|$ means that there exists $j \in \llbracket 1, n \rrbracket$ such that $\left|x_{j}\right|<\left|x_{j}^{\prime}\right|$

Intuition: $\epsilon \neq 0 \Longrightarrow\|(0, *, 0, *, *, 0)\|<\|(0, *, \epsilon, *, *, 0)\|$

## Examples of orthant-strictly monotonic norms

$$
\|x\|_{\infty}=\sup _{i \in \llbracket 1, n \rrbracket}\left|x_{i}\right| \text { and }\|x\|_{p}=\left(\sum_{i=1}^{n}\left|x_{i}\right|^{p}\right)^{1 / p} \text { for } p \in[1, \infty[
$$

with unit ball $B_{p}$ and unit sphere $S_{p}$

- All the $\ell_{p}$-norms $\|\cdot\|_{p}$ on the space $\mathbb{R}^{n}$, for $p \in[1, \infty]$, are monotonic, hence orthant-monotonic

$$
\ell_{1}, \ell_{2}, \ell_{\infty}
$$

- All the $\ell_{p}$-norms $\|\cdot\|_{p}$ on the space $\mathbb{R}^{n}$, for $p \in[1, \infty[$, are orthant-strictly monotonic

$$
\begin{gathered}
\ell_{1}, \ell_{2}, \ell 6 \\
|\epsilon|<1 \Longrightarrow\|(1,0)\|_{\infty}=1=\|(1, \epsilon)\|_{\infty}
\end{gathered}
$$

Orthant-strictly monotonic norms and Capra-convexity

## Capra-subdifferentiability properties of the $\ell_{0}$ pseudonorm

[Chancelier and De Lara, 2022c]

## Proposition

If both the norm $\|\cdot\|$ and the dual norm $\|\cdot\|_{\star}$ are orthant-strictly monotonic, we have that

$$
\partial_{C} \ell_{0}(x) \neq \emptyset, \quad \forall x \in \mathbb{R}^{n},
$$

that is, the pseudonorm $\ell_{0}$ is Capra-subdifferentiable on $\mathbb{R}^{n}$ and, as a consequence,

$$
\ell_{0}^{c C^{\prime}}=\ell_{0}
$$

Best convex lower approximation of the $\ell_{0}$ pseudonorm on the $\ell_{p}$-unit balls, $p \in[1, \infty]$

## Theorem

The best convex Isc lower approximation $\mathcal{L}_{0}$ of $\ell_{0}$
best convex lsc function $\quad \mathcal{L}_{0}(x) \leq \ell_{0}(x), \forall x \in B_{p}$
on the unit ball $B_{p}$ is $\ell_{0}^{\dot{c} \star^{\prime}}$, and coincides with the $\ell_{0}$ pseudonorm

$$
\ell_{0}(x)=\mathcal{L}_{0}(x), \quad \forall x \in S_{p}
$$

on the unit sphere $S_{p}$

Tightest closed convex function below the $\ell_{0}$ pseudonorm on the $\ell_{p}$-unit balls on $\mathbb{R}^{2}$ for $p \in\{1.1,2,4,300\}$


Capra-subdifferential of the $\ell_{0}$ pseudonorm

## Exposed faces and normal cones

For any nonempty closed convex subset $\subset \subset \mathcal{X}$, where $\mathcal{X}=\mathcal{Y}=\mathbb{R}^{n}$,

- the exposed face $F_{\perp}(C, y)$ of $C$ by any dual vector $y \in \mathcal{Y}$ is

$$
F_{\perp}(C, y)=\underset{x \in C}{\arg \max }\langle x \mid y\rangle
$$

- the normal cone $N(C, x)$ of $C$ at any primal vector $x \in C$ is defined by the conjugacy relation

$$
x \in C \text { and } y \in N(C, x) \Longleftrightarrow x \in F_{\perp}(C, y)
$$

The family of all normal cones is the normal fan $\mathcal{N}(C)$

## Capra-subdifferential of the $\ell_{0}$ pseudonorm

- $\left\{\|\cdot\|_{(j)}^{\mathcal{R}}\right\}_{j \in \llbracket 1, n \rrbracket}$ and $\left\{\|\cdot\|_{(j), \star}^{\mathcal{R}}\right\}_{j \in \llbracket 1, n \rrbracket}$, associated coordinate-k and dual coordinate-k norms
- $\left\{B_{(j)}^{\mathcal{R}}\right\}_{j \in \llbracket 1, n \rrbracket}$ and $\left\{B_{(j), \star}^{\mathcal{R}}\right\}_{j \in \llbracket 1, n \rrbracket}$, corresponding unit balls


## Proposition

[Chancelier and De Lara, 2022a]
The Capra-subdifferential of the $\ell_{0}$ pseudonorm is given by

$$
\begin{aligned}
\text { if } x & =0, \quad \partial_{\dot{C}} \ell_{0}(0)
\end{aligned}=\bigcap_{j \in \llbracket 1, n \rrbracket} j B_{(j), \star}^{\mathcal{R}}, \quad \begin{aligned}
& \text { if } x \neq 0 \text { and } \ell_{0}(x)=\ell, \quad \partial_{C} \ell_{0}(x)=N\left(B_{(\ell)}^{\mathcal{R}}, \frac{x}{\|x\|_{(\ell)}^{\mathcal{R}}}\right) \cap Y_{\ell}
\end{aligned}
$$

where $\quad Y_{\ell}=\left\{y \in \mathcal{Y} \mid \ell \in \underset{j \in \llbracket 0, n \rrbracket}{\arg \max }\left(\|y\|_{(j), \star}^{\mathcal{R}}-j\right)\right\}, \quad \forall \ell \in \llbracket 0, n \rrbracket$

Coordinate- $k$ norms and their dual norms

## Courtesy of Basile and Lionel Pournin



Figure: Unit ball $\overline{\operatorname{co}}\left(\ell_{0}^{\leq 2} \cap S_{1}\right)$ when $n=3$

## Extreme points of the coordinate- $k$ norm unit ball

 are $k$-sparseFor any source norm $\|\cdot\|$ on $\mathbb{R}^{n}$, and for $k \in \llbracket 1, d \rrbracket$,

- the coordinate- $k$ norm $\|\cdot\|_{(k)}^{\mathcal{R}}$ has unit ball

$$
B_{(k)}^{\mathcal{R}}=\underbrace{\overline{\operatorname{co}}\left(\ell_{0}^{\leq k} \cap S\right)}_{\text {closed convex hull }}=\underbrace{\operatorname{co}\left(\ell_{0}^{\leq k} \cap S\right)}_{\text {convex hull }}
$$

- hence the extreme points of $B_{(k)}^{\mathcal{R}}$ belong to $\ell_{0}^{\leq k} \cap S \subset \ell_{0}^{\leq k}$, hence are $k$-sparse vectors


## Extreme points of the coordinate- $k$ norm unit ball

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$$

- hence the extreme points of $B_{(k)}^{\mathcal{R}}$ belong to $\ell_{0}^{\leq k} \cap S \subset \ell_{0}^{\leq k}$, hence are $k$-sparse vectors
This is how we define
- a sequence $\left\{\|\cdot\|_{(k)}^{\mathcal{R}}\right\}_{k \in \llbracket 1, n \rrbracket}$ of coordinate- $k$ norms
- a sequence $\left\{\|\cdot\|_{(k), \star}^{\mathcal{R}}\right\}_{k \in \llbracket 1, n \rrbracket}$ of dual coordinate- $k$ norms


## Courtesy of Basile and Lionel Pournin


(a) Unit ball $\overline{\mathrm{co}}\left(\ell_{0}^{\leq 2} \cap S_{1}\right)$ when $n=3$
(b) Unit ball $\overline{\mathrm{co}}\left(\ell_{0}^{\leq 2} \cap S_{2}\right)$ when $n=3$

## Coordinate and dual coordinate norms

## induced by the $\ell_{p}$-norms $\|\cdot\|_{p}$

For $y \in \mathbb{R}^{n}, \nu$ is a permutation of $\llbracket 1, n \rrbracket$ such that $\left|y_{\nu(1)}\right| \geq\left|y_{\nu(2)}\right| \geq \cdots \geq\left|y_{\nu(n)}\right|$

| $\\|\cdot\\|$ | $\\|\cdot\\|_{(k)}^{\mathcal{R}}$ | $\\|\cdot\\|_{(k), \star}^{\mathcal{R}}$ |
| :---: | :---: | :---: |
| $\\|\cdot\\|_{p}$ | top- $(p, k)$ norm | $(q, k)$-support norm |
|  | $\\|x\\|_{p, k}^{\top}$ | $\\|y\\|_{q, k}^{\top+}$ |
|  | $=\left(\sum_{j=1}^{k}\left\|x_{\nu(j)}\right\|^{p}\right)^{1 / p}$ | $1 / p+1 / q=1$ |
| $\\|\cdot\\|_{1}$ | top- $(1, k)$ norm | $\\|x\\|_{1, k}^{\top}=\sum_{l=1}^{k}\left\|x_{\nu(I)}\right\|$ |

Why do top- $k$ and $k$-support norms pop up?

Generalized top and support norms

## We reformulate sparsity in terms of coordinate subspaces

- For any $K \subset \llbracket 1, n \rrbracket$, we introduce the (coordinate) subspace

$$
\mathcal{R}_{K}=\left\{y \in \mathbb{R}^{n} \mid y_{j}=0, \forall j \notin K\right\} \subset \mathbb{R}^{n}
$$

- The connection with the level sets of the $\ell_{0}$ pseudonorm is

$$
\ell_{0}^{\leq k}=\bigcup_{|K| \leq k} \mathcal{R}_{K}, \quad \forall k \in \llbracket 0, n \rrbracket
$$

- We denote by $\pi_{K}: \mathbb{R}^{n} \rightarrow \mathcal{R}_{K}$ the orthogonal projection
- For any vector $y \in \mathbb{R}^{n}, \pi_{K}(y) \in \mathbb{R}^{n}$ is the vector whose components coincide with those of $y$, except for those outside of $K$ that vanish

$$
y=(*, *, *, *, *, *) \rightarrow \pi_{\{2,4,5\}}(y)=(0, *, 0, *, *, 0)
$$

## We define generalized top- $k$ and $k$-support dual norms

## Definition

For any source norm $\|\cdot\|$ on $\mathbb{R}^{n}$, for any $k \in \llbracket 1, n \rrbracket$, we call

- generalized top- $k$ dual norm the norm

$$
\|y\|_{\star,(k)}^{\top}=\underbrace{\sup _{|K| \leq k} \overbrace{\pi_{K}(y)}^{\begin{array}{c}
k \text {-sparse } \\
\text { projection } \\
\text { on } \mathcal{R}_{K}
\end{array}} \|_{\star}}_{\text {exploring all } \text { k-sparse projections }}, \forall y \in \mathbb{R}^{n}
$$

- generalized $k$-support dual norm the dual norm

$$
\|\cdot\|_{\star,(k)}^{\top \star}=\left(\|\cdot\|_{\star,(k)}^{\top}\right)_{\star}
$$

## Coordinate norms and dual norms versus

 generalized top- $k$ and $k$-support dual norms
## Proposition

If the source norm $\|\cdot\|$ is orthant monotonic, for all $k \in \llbracket 1, n \rrbracket$,

so that, if $S$ is the unit sphere of the source norm $\|\cdot\|$,

$$
B_{(k)}^{\mathcal{R}}=\operatorname{co}\left(\ell_{0}^{\leq k} \cap S\right)=B_{\star,(k)}^{T_{\star} \star}
$$

## Where do we stand?

- We have Capra couplings $\dot{\text { c }}$ for which the pseudonorm $\ell_{0}$
- has nonempty Capra-subdifferential

$$
\partial_{\dot{C}} \ell_{0} \neq \emptyset
$$

- hence is Capra-convex (equal to its Capra-biconjugate)

$$
\ell_{0}^{\mathrm{C} \zeta^{\prime}}=\ell_{0}
$$

- This looks promising to study sparse optimization problems

But. . .

## Outline of the presentation

> Crash course on generalized convexity [5 min]

> Capra conjugacies [15 min]

Towards Capra-algorithms in sparse optimization? [15 min]

Conclusion [1 min]

Additional material

## Archetypal sparse optimization problems

- For $X \subset \mathbb{R}^{n}$ a nonempty set,

$$
\min _{x \in X} \ell_{0}(x)
$$

is an optimization problem for which any point in $X$
is a local minimizer! Jean-Baptiste Hiriart-Urruty and Hai Le. A variational approach of the
rank function. TOP: An Official Journal of the Spanish Society of Statistics and Operations Research, 21
(2):207-240, 2013.

- For $k \in \llbracket 1, n \rrbracket$ and a function $f: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$,

$$
\min _{\ell_{0}(x) \leq k} f(x)
$$

- For $\gamma>0$ and a function $f: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$,

$$
\min _{x \in \mathbb{R}^{n}}\left(f(x)+\gamma \ell_{0}(x)\right)
$$

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# Good news :-) <br> the Fermat rule holds true for the Capra coupling 

$$
x^{*} \in \arg \min f \Longleftrightarrow 0 \in \partial_{\dot{C}} f\left(x^{*}\right)
$$

# Good news :-) <br> the Fermat rule holds true for the Capra coupling 

$$
x^{*} \in \arg \min f \Longleftrightarrow 0 \in \partial_{\dot{C}} f\left(x^{*}\right)
$$

As an application, we get that

$$
x^{*} \in \underset{x \in X}{\arg \min } \ell_{0}(x) \Longleftrightarrow 0 \in \partial_{\dot{C}}\left(\ell_{0}+\iota_{X}\right)\left(x^{*}\right)
$$

But...

Bad news :-( when zero is in the subdifferential of the sum...


$$
x^{*} \in \underset{X}{\arg \min } \ell_{0} \Longleftrightarrow 0 \in \underbrace{\partial_{\varphi}\left(\ell_{0}+\iota x\right)\left(x^{*}\right)}_{\text {subdifferential of the sum }}
$$

... but zero is not in the sum of the subdifferentials

$$
\underbrace{\partial_{C^{e}} \ell_{0}\left(x^{*}\right)+\partial_{C^{\iota}} \iota_{X}\left(x^{*}\right)}_{0 \notin} \subsetneq \underbrace{\partial_{¢}\left(\ell_{0}+\iota_{X}\right)\left(x^{*}\right)}_{0 \in}
$$




Who is to blame? Capra or $\ell_{0}$ ? (with Seta Rakotomandimby)

## Primal-dual pair

in the Capra-subdifferential of an absolute function

## Proposition

Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be an absolute function and $\|\cdot\|: \mathbb{R}^{n} \rightarrow \mathbb{R}_{+}$be an absolute norm, meaning that

$$
\begin{aligned}
f(x) & =f(|x|), \quad \forall x \in \mathbb{R}^{n} \\
\|x\| & =\||x|\|, \quad \forall x \in \mathbb{R}^{n}
\end{aligned}
$$

Then, we have that

$$
y \in \partial_{C} f(x) \Longrightarrow x \circ y \geq 0
$$

where $x \circ y=\left(x_{1} y_{1}, \ldots, x_{n} y_{n}\right)$
NB: this property also holds true with the classic Rockafellar-Moreau subdifferential in convex analysis

## Illustration of $x \circ y \geq 0$



## Capra-subdifferential of an indicator function

## Proposition

Let $X \subset \mathbb{R}^{n}$ be a nonempty set. Then, for any $x \in \mathbb{R}^{n}$

$$
\partial_{\mathrm{C}^{\iota}} \iota(x)= \begin{cases}\overbrace{N(\overline{\operatorname{co}}(\varrho(X)), \varrho(x))}^{\text {normal cone }} & \text { if } x \in X \\ \emptyset & \text { if } x \notin X\end{cases}
$$



- The Capra-subdifferential of $\iota_{X}$ at $x^{*}$ is the normal cone of the convex subset $\overline{\mathrm{co}}(\varrho(X)) \subset B$ at $\varrho\left(x^{*}\right) \in S$, hence points outward
- The Rockafellar-Moreau subdifferential of $\iota_{X}$ at $x^{*}$ is the normal cone of $X$ at $x^{*}$


## $0 \in \partial_{\mathcal{C}^{\prime}} f(x)+\partial_{\mathcal{C}^{\iota}} \iota(x)$ is much too strong a condition

Under the previous assumptions, we get that

$$
\begin{aligned}
0 \in \partial_{C^{\prime}} f(x)+\partial_{\dot{C}^{\prime}} \iota x(x) & \Longrightarrow 0=\overbrace{\underbrace{y^{\prime}}_{x \circ y^{\prime} \geq 0}}^{\partial_{C^{f(x)}}^{f(x)}}+\overbrace{y^{\prime \prime}}^{\partial_{\dot{C}^{\iota x}(x)}} \\
& \Longrightarrow \underbrace{y^{\prime \prime} \in N(\overline{\operatorname{co}}(\varrho(X)), \varrho(x))}_{y^{\prime \prime} \text { is outward }} \text { and } \underbrace{x \circ y^{\prime \prime} \leq 0}_{y^{\prime \prime} \text { is inward }}
\end{aligned}
$$

- In general, this will give $y^{\prime \prime}=0$, that is, $0 \in \partial_{\dot{C}} f(x)$
- Thus, necessarily, $x \in X$ must be a global minimum of $f$ over all $\mathbb{R}^{n}$, which is much too strong. . .


## Where do we stand?

- We had good hope to handle sparse optimization problems with the Capra coupling that makes the pseudonorm $\ell_{0}$ Capra convex
- But, in a simple sparse optimization problem, it is not true that the subdifferential of the sum
is equal to the sum of the subdifferentials
- And not having practical qualification conditions is an obstacle to many numerical methods


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Additional material

## Minimization problems from compressed sensing

- Goal: recovery of a sparse signal $x \in \mathbb{R}^{n}$ from a measurement $b \in \mathbb{R}^{m} \backslash\{0\}$, where $m<n$
- Measurements are modeled by $A \in \mathbb{R}^{m \times n}$ such that

$$
A x=b
$$

- Minimization approach for the recovery

$$
\min _{\substack{x \in \mathbb{R}^{n} \\ A x=b}} \ell_{0}(x)
$$

## Using a Capra-polyhedral approximation for $\ell_{0}$

- For a suitable (infinite) subset $Y \subset \bigcup_{x^{\prime}} \partial_{C^{c}} \ell_{0}\left(x^{\prime}\right)$ of Capra-subgradients of $\ell_{0}$, we have that

$$
\ell_{0}(x)=\sup _{y \in Y}\langle\varrho(x), y\rangle-\ell_{0}^{C}(y), \quad \forall x \in \mathbb{R}^{n}
$$

- Idea: using a Capra-" polyhedral" approximation $f$ of $\ell_{0}$ in the minimization problem

$$
f(x)=\max _{y \in \tilde{Y}}\langle\varrho(x), y\rangle-\ell_{0}^{\dot{C}}(y)
$$

where $\tilde{Y} \subset Y$ and $\tilde{Y}$ finite $\leadsto$ cutting plane-like method

## Illustration of a Capra-polyhedral approximation for $\ell_{0}$



## Abstract cutting plane method

[Rubinov, 2000, §9.2.3]

## Definition

Let $\mathcal{W}$ be a set, $H \subset \overline{\mathbb{R}}^{\mathcal{W}}$ be a set of elementary functions, and $f: \mathcal{W} \rightarrow \overline{\mathbb{R}}$ be a $H$-convex function

1. Set $k:=0$. Choose an arbitrary initial point $w_{0} \in \mathcal{W}$
2. Find an abstract subgradient $h_{k} \in \partial^{H} f\left(w_{k}\right)$

Let $f_{-1}=-\infty$ and set

$$
f_{k}=\max \{f_{k-1}, \underbrace{h_{k}}_{\begin{array}{c}
\text { new cut } \\
\text { in } \partial^{H} f\left(w_{k}\right)
\end{array}}\}
$$

3. Find an optimal solution $\widehat{w} \in \arg \min _{w \in \mathcal{W}} f_{k}(w)$
4. Set $k:=k+1, w_{k}=\widehat{w}$ Repeat from Step 2 until a stop condition is satisfied

## Still problems with $\ell_{0}$

- The pseudonorm $\ell_{0}$ is abstract Capra-convex
- ...but $\ell_{0}$ is not continuous and its abstract

Capra-subgradients

$$
\left\{x \mapsto\langle\varrho(x), y\rangle-\ell_{0}^{\oint}(y)\right\}_{y \in \cup_{x^{\prime}} \partial_{C^{\prime}} \ell_{0}\left(x^{\prime}\right)}
$$

are not uniformly continuous

- So the pseudonorm $\ell_{0}$ does not satisfy any assumptions of established theoretical convergence results [Pallaschke and Rolewicz, 1997, Theorem 9.1.1]
- Also, numerically, we observe no convergence for simple examples in dimension $n=3$


## However for $\ell_{1} / \ell_{2}$ !

- $\ell_{1} / \ell_{2}$ is a surrogate function for $\ell_{0}$ in compressed sensing
- $\ell_{1} / \ell_{2}$ is Capra-convex
(and an absolute function so Fermat rule is no help)
- and $\ell_{1} / \ell_{2}$ is continuous
and the following Capra-abstract subgradients

$$
\left\{x \mapsto\langle\varrho(x), y\rangle-\ell_{0}^{c}(y)\right\}_{y \in\{-1,0,1\}^{n}}
$$

are uniformly continuous

- Most assumptions of theoretical convergence results [Pallaschke and Rolewicz, 1997, Theorem 9.1.1] are satisfied

Solving time for the ratio of two norms


## Work needs to be done for theoretical guarantees

- Convergence results [Pallaschke and Rolewicz, 1997, Theorem 9.1.1] [Rubinov, 2000, Proposition 9.2]
- But the assumptions do not fit our case: need to be adapted


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Conclusion [1 min]

## The intuition behind lasso



$$
\min _{x \in \mathbb{R}^{n}}\left(f(x)+\gamma\|x\|_{2}\right)
$$

Comments of
[Tibshirani, 1996, Figure 2]
"The lasso solution is the first place that the contours touch the square, and this will sometimes occur at a corner, corresponding to a zero coefficient. The picture for ridge regression is shown in Fig. 2(b): there are no corners for the contours to hit and hence zero solutions will rarely result."

## Geometric (alignment) expression of optimality condition

- We consider an optimal solution $x^{*}$ of

$$
\min _{x \in \mathbb{R}^{n}}(f(x)+\gamma\|x\|)
$$

where $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is a smooth convex function, $\gamma>0$ and $\|\cdot\|$ is a norm with unit ball $B$

- By the Fermat rule, when $x^{*} \neq 0$,

$$
0 \in \nabla f\left(x^{*}\right)+\gamma \partial\|\cdot\|\left(x^{*}\right) \Longleftrightarrow \frac{x^{*}}{\left\|x^{*}\right\|} \in \underbrace{F_{\perp}\left(B,-\nabla f\left(x^{*}\right)\right)}_{\begin{array}{l}
\text { face of the unit ball. } B \\
\text { exposed by }-\nabla f\left(x^{*}\right)
\end{array}}
$$

- The norm ||•\| may be qualified as sparsity-inducing if information about the support of $x^{*}$ and the exposed faces of the unit ball $B$ can be recovered from one another [Fan, Jeong, Sun, and Friedlander, 2020]

Design of sparsity inducing norms/balls

## Courtesy of Basile and Lionel Pournin



Figure: Unit ball $\overline{\operatorname{co}}\left(\ell_{0}^{\leq 2} \cap S_{1}\right)$ when $n=3$

## How to design a sparsity inducing unit ball?

For $k \in \llbracket 1, d \rrbracket$

- consider the $k$-sparse vectors in $\ell_{0}^{\leq k}$
- as they do not form a compact set, intersect $\ell_{0}^{\leq k}$ with a unit sphere $S$ (or a unit ball $B$ )
- form the convex hull and obtain a new

$$
\text { unit ball } \quad B_{(k)}^{\mathcal{R}}=\operatorname{co}\left(\ell_{0}^{\leq k} \cap S\right)
$$

whose extreme points belong to $\ell_{0}^{\leq k} \cap S \subset \ell_{0}^{\leq k}$, hence are $k$-sparse vectors

Does this procedure induces sparsity? If yes, in what sense?

Support identification of a $k$-sparse vector in the exposed face of a generalized $k$-support dual norm (1/2)

## Theorem

Let $k \in \llbracket 1, n \rrbracket$. If the source norm $\|\cdot\|$ is orthant-monotonic, then

$$
B_{(k)}^{\mathcal{R}}=\operatorname{co}\left(\ell_{0}^{\leq k} \cap S\right)=B_{\star,(k)}^{T_{\star} \star}
$$

and, for any nonzero dual vector $y \in \mathcal{Y} \backslash\{0\}$, the two following statements are equivalent
(i) $x \in \ell_{0}^{\leq k} \cap F_{\perp}\left(B_{\star,(k)}^{T_{\star}}, y\right)$
(ii) there exists $K^{*} \in \arg \max _{|K| \leq k}\left\|\pi_{K}(y)\right\|_{\star}$ such that $x \in \pi_{K^{*}}\left(B \cap F_{\perp}\left(B, \pi_{K^{*}}(y)\right)\right) \subset \mathcal{R}_{K^{*}}$

As a consequence, we get that

$$
\operatorname{supp}(x) \subset K^{*}
$$

Support identification of a $k$-sparse vector in the exposed face of a generalized $k$-support dual norm (2/2)


1. From $x \in \ell_{0}^{\leq k}$, we only know that there exists $K \subset \llbracket 1, n \rrbracket$ with $|K| \leq k$ such that

$$
\operatorname{supp}(x) \subset K
$$

2. From $x \in F_{\perp}\left(B_{\star,(k)}^{\top_{\star}}, y\right)$, we add information and obtain that
there exists $K^{*} \in \arg \max \left\|\pi_{K}(y)\right\|_{\star}$ such that

$$
|K| \leq k
$$

$$
\operatorname{supp}(x) \subset K^{*}
$$

## Support identification

## Corollary

Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a smooth convex function, $\gamma>0$ and $\|\cdot\|$ be a norm
Then, an optimal solution $x^{*}$ of

$$
\min _{x \in \mathbb{R}^{n}}\left(f(x)+\gamma\|x\|_{\star,(k)}^{T_{\star}}\right)
$$

has support

$$
\operatorname{supp}\left(x^{*}\right) \subset \bigcup_{\substack{K^{*} \in \arg \max _{|K|} \leq K \\\left\|\pi_{K}\left(-\nabla f\left(x^{*}\right)\right)\right\|_{\star}}} K^{*}
$$

Especially interesting when the arg $\max _{|K| \leq k}$ is unique, because then the optimal solution $x^{*}$ is $k$-sparse

Geometry of sparsity inducing balls

(a) Unit ball $B_{\infty, 2}^{\top_{\star}}$ when $n=3$

(c) Unit ball $B_{1,2}^{\top}$ when $n=3$

(b) Unit ball $B_{2,2}^{\top,}$ when $n=3$

(d) Unit ball $B_{2,2}^{\top}$ when $n=3$

Figure: Four top ( 6 c and 6 d ) and support ( 7 a and 7 b ) unit balls, either obtained from the $\ell_{1}$ source norm ( 7 a and 6 c ) or from the $\ell_{2}$ source norm (7b and 6d)

## Additional geometric properties

## Proposition

For any $k \in \llbracket 1, n \rrbracket$, all the proper faces of $B_{2, k}^{T_{\star}}$ are hypersimplices, and the normal fan of $B_{2, k}^{\top \star}$ refines the normal fan of $B_{\infty, k}^{\top \star}$

(a) Unit ball $B_{\infty, 2}^{\top \star}$ when $n=3$

(b) Unit ball $B_{2,2}^{\top \star}$ when $n=3$

Figure: Two support norm unit balls, either obtained from the $\ell_{1}$ source norm (7a) or from the $\ell_{2}$ source norm (7b)

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```

Additional material

- So-called generalized convexity and Fenchel-Moreau conjugacy are extensions of duality beyond convex analysis
- The Capra-coupling $\&$ and induced Capra-conjugacy seem promising to handle sparsity in optimization as the pseudonorm $\ell_{0}$ satisfies

$$
\partial_{C^{\prime}} \ell_{0} \neq \emptyset \text { hence } \ell_{0}^{c c^{\prime}}=\ell_{0}
$$

but we have problems handling sums like $\ell_{0}+\iota_{X}$ :-(

- So, our working program is now to study
- the $\ell_{0}$-cup function $\mathcal{L}_{0}=\ell_{0}^{\zeta \star^{\prime}}$
- the geometry of unit balls of norms related to the Capra-coupling $\&$ and to the pseudonorm $\ell_{0}$
- lower bound convex programs

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Thank you :-)


## Outline of the presentation

```
Crash course on generalized convexity [5 min
Capra conjugacies [15 min]
Towards Capra-algorithms in sparse optimization? [15 min]
Conclusion [1 min]
```

Additional material

The $\ell_{0}$ pseudonorm is (almost)
a convex-composite function

- [Chancelier and De Lara, 2021]

$$
\ell_{0}(x)=\underbrace{\mathcal{L}_{0}}_{\text {proper convex lsc }}\left(\frac{x}{\|x\|}\right), \forall x \in \mathbb{R}^{n} \backslash\{0\}
$$

- As a consequence, if $C \subset \mathbb{R}^{n}$ is a closed convex set with $0 \notin C$,

$$
\min _{x \in C} \ell_{0}(x)=\min _{x \in \mathbb{R}^{n}}\left\{\mathcal{L}_{0}\left(\frac{x}{\|x\|}\right)+\iota_{C}(x)\right\}
$$

or if $f: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ is a proper convex Isc function,

$$
\min _{x \in \mathbb{R}^{n}, \ell_{0}(x) \leq k} f(x)=\min _{x \in \mathbb{R}^{n}}\{f(x)+\underbrace{\iota_{B_{(k)}^{\top}}^{\iota_{k}}}_{\substack{(2, k) \text {-support norm } \\ \text { unit ball }}}\left(\frac{x}{\|x\|}\right)\}
$$

Graded sequence of norms

## We define graded sequence of norms

A graded sequence of norms detects the number of nonzero components of a vector in $\mathbb{R}^{n}$
when the sequence becomes stationary

## Definition

We say that a sequence $\left\{\|\cdot\|_{k}\right\}_{k \in \llbracket 1, n \rrbracket}$ of norms is (increasingly) graded with respect to the $\ell_{0}$ pseudonorm if, for any $y \in \mathbb{R}^{n}$ and $I \in \llbracket 1, n \rrbracket$, we have

$$
\ell_{0}(y)=\ell \Longleftrightarrow\|y\|_{1} \leq \cdots \leq\|y\|_{\ell-1}<\|y\|_{\ell}=\cdots=\|y\|_{n}
$$

or, equivalently, $k \in \llbracket 1, n \rrbracket \mapsto\|y\|_{k}$ is nondecreasing and

$$
\ell_{0}(y) \leq \ell \Longleftrightarrow\|y\|_{\ell}=\|y\|_{n}
$$

Graded sequences are suitable for so-called "difference of convex" (DC) optimization methods to tackle sparse $\ell_{0}(y) \leq I$ constraints

## Orthant-strictly monotonic dual norms produce graded sequences of norms

## Proposition

If the dual norm $\|\cdot\|_{\star}$ of the source norm $\|\cdot\|$ is orthant-strictly monotonic, then the sequence

is graded with respect to the $\ell_{0}$ pseudonorm

Thus, we can produce families of graded sequences of norms suitable for "difference of convex" (DC) optimization methods to tackle sparse constraints

## Fenchel versus Capra conjugacies for $\ell_{0}$

[Chancelier and De Lara, 2022a], [Chancelier and De Lara, 2022c] If both the source norm and its dual are orthant-strictly monotonic

| Fenchel conjugacy | Capra conjugacy |
| :---: | :---: |
| $\iota_{\ell_{0}^{\leq} \leq k}^{\star}=\iota_{\{0\}}, k \neq 0$ | $\iota_{\ell_{0}^{\leq k}}^{C}=\\|\cdot\\|_{(k), \star}^{\mathcal{R}}=\\|\cdot\\|_{\star,(k)}^{\top}$ |
| $\ell_{0}^{\star}=\iota_{\{0\}}$ | $\begin{aligned} \ell_{0}^{C} & =\sup _{\ell \in \llbracket 0, n \rrbracket}\left[\\|\cdot\\|_{(\ell), \star}^{\mathcal{R}}-\ell\right] \\ & =\sup _{\ell \in \llbracket 0, n \rrbracket}\left[\\|\cdot\\|_{\star,(\ell)}^{\top+}-\ell\right] \end{aligned}$ |
|  |  |
| $\ell_{0}^{\star \star^{\prime}}=0$ | $\ell_{0}^{\text {¢ } ¢^{\prime}}=\ell_{0}$ |

Lower bounds for the pseudonorm $\ell_{0}$

## Best ratio of norms [Chancelier and De Lara, 2022a]

- For any $\varphi: \llbracket 0, d \rrbracket \rightarrow[0,+\infty[$, such that $\varphi(j)>\varphi(0)=0$ for all $j \in \llbracket 1, d \rrbracket$, there exists a norm $\|\cdot\|_{(\varphi)}^{\mathcal{R}}$ such that

$$
\frac{\|x\|_{(\varphi)}^{\mathcal{R}}}{\|x\|} \leq \varphi\left(\ell_{0}(x)\right), \forall x \in \mathbb{R}^{n} \backslash\{0\}
$$

where $\|\cdot\|_{(\varphi)}^{\mathcal{R}}$ is characterized by its dual norm

$$
\|y\|_{(\varphi), \star}^{\mathcal{R}}=\sup _{j \in \llbracket 1, d \rrbracket} \frac{\|y\|_{(j), \star}^{\mathcal{R}}}{\varphi(j)}, \quad \forall y \in \mathbb{R}^{n}
$$

- For $\|\cdot\|=\|\cdot\|_{p}$ with $p>1$, and $\varphi_{\alpha}(j)=j^{1 / \alpha}$ for $\alpha>0$,

$$
\begin{aligned}
& \left(\frac{\left(\|x\|_{p}\right)_{\left(\varphi_{\alpha}\right)}^{\mathcal{R}}}{\|x\|_{p}}\right)^{\alpha} \leq \ell_{0}(x), \quad \forall x \in \mathbb{R}^{n} \backslash\{0\} \\
& \left(\frac{\|x\|_{1}}{\|x\|_{p}}\right)^{p} \leq \ell_{0}(x), \quad \forall x \in \mathbb{R}^{n} \backslash\{0\}
\end{aligned}
$$

Lower bound convex programs for exact sparse optimization

## Concave dual problem for exact sparse optimization

From $\sup _{y \in \mathcal{Y}}\left(\left(-f^{\mathcal{C}}(y)\right)+\left(-\iota_{X}^{-\dot{\phi}}(y)\right)\right) \leq \inf _{x \in \mathcal{X}}\left(f(x)+\iota_{X}(x)\right)$
we deduce that

$$
\sup _{y \in \mathbb{R}^{n}}(-(\inf [f \mid \varrho])^{\star}(y)+(-\underbrace{\left.-\iota_{\ell-k}^{-\delta}(y)\right)}_{\|y\|_{2, k}^{T}}) \leq \inf _{\ell_{0}(x) \leq k}^{\iota_{0}^{-c}} f(x)
$$

## Proposition

For any function $f: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$, we have the following lower bound

$$
\begin{aligned}
\sup _{y \in \mathbb{R}^{n}} \overbrace{\left(-(\inf [f \mid \varrho])^{\star}(y)-\|y\|_{2, k}^{\top}\right)}^{\text {concave usc function }} & \leq \inf _{\ell_{0}(x) \leq k} f(x) \\
& =\inf _{\ell_{0}(x) \leq k} \inf [f \mid \varrho](x)
\end{aligned}
$$

## Convex primal problem for exact sparse optimization

## Proposition

Under a mild technical assumption ("à la" Fenchel-Rockafellar), namely if $(\inf [f \mid \varrho])^{\star}$ is a proper function, we have the following lower bound

$$
\min _{\|x\|_{2, k}^{\Pi_{x} \leq \leq 1}}(\inf [f \mid \varrho])^{\star \star^{\prime}}(x) \leq \inf _{\ell_{0}(x) \leq k} f(x)=\inf _{\ell_{0}(x) \leq k} \inf [f \mid \varrho](x)
$$

The primal problem is the minimization of a closed convex function on the unit ball of the $(2, k)$-support norm $\|\cdot\|_{2, k}^{T_{*}}$ (introduced in [Argyriou, Foygel, and Srebro, 2012])

## Duality

## Perturbation scheme

- Functions $k: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}, \varphi: \llbracket 0, n \rrbracket \rightarrow \overline{\mathbb{R}}$ nondecreasing (ex: identity, $\iota_{\llbracket 0, k \rrbracket}$ ) and original minimization problem

$$
\inf _{w \in \mathbb{R}^{n}}\left\{k(w) \dot{+} \varphi\left(\ell_{0}(w)\right)\right\}=\inf _{w \in \mathbb{R}^{n}}\left\{k(w) \dot{+}\left(\varphi \circ \ell_{0}\right)^{\dot{c} \star^{\prime}}(\varrho(w))\right\}
$$

because $\varphi \circ \ell_{0}=\left(\varphi \circ \ell_{0}\right)^{\dot{c}{c^{\prime}}^{\prime}}=\left(\varphi \circ \ell_{0}\right)^{c \star^{\prime}} \circ \varrho$
[Chancelier and De Lara, 2022c]

- Rockafellian (perturbation scheme) $R: \mathbb{R}^{n} \times \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$

$$
R(w, x)=k(w) \dot{+}\left(\varphi \circ \ell_{0}\right)^{\dot{c} *^{\prime}}(\varrho(w)+x), \quad \forall(w, x) \in \mathbb{R}^{n} \times \mathbb{R}^{n}
$$

- Value function

$$
\varphi(x)=\inf _{w \in \mathbb{R}^{n}}\left\{k(w)+\left(\varphi \circ \ell_{0}\right)^{\dot{c} *^{\prime}}(\varrho(w)+x)\right\}, \forall x \in \mathbb{R}^{n}
$$

## Lagrangian and dual problem

- Fenchel coupling $\mathbb{R}^{n} \stackrel{\langle | \cdot| \rangle}{\leftrightarrow} \mathbb{R}^{n}$, and Lagrangian
$\mathcal{L}: \mathbb{R}^{n} \times \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ given, for any $(w, y) \in \mathbb{R}^{n} \times \mathbb{R}^{n}$, by

$$
\begin{aligned}
\mathcal{L}(w, y) & =\inf _{x \in \mathbb{R}^{n}}\left\{k(w)+\left(\varphi \circ \ell_{0}\right)^{\dot{c} \star^{\prime}}(\varrho(w)+x)-\langle x, y\rangle\right\} \\
& =k(w)+\left(\langle\varrho(w), y\rangle-\left(\varphi \circ \ell_{0}\right)^{c}(y)\right)
\end{aligned}
$$

- Dual maximization problem

$$
\varphi^{\star \star^{\prime}}(0)=\sup _{y \in \mathbb{R}^{n}} \inf _{w \in \mathbb{R}^{n}} \mathcal{L}(w, y)=\sup _{y \in \mathbb{R}^{n}}\left\{\left(-k^{-غ}(y)\right)+\left(-\left(\varphi \circ \ell_{0}\right)^{¢}(y)\right)\right\}
$$

- Original minimization problem (case " $\dot{+}=+$ " when $k$ proper)

$$
\varphi(0)=\inf _{w \in \mathbb{R}^{n}} \sup _{y \in \mathbb{R}^{n}} \mathcal{L}(w, y)=\inf _{w \in \mathbb{R}^{n}}\left\{k(w) \dot{+} \varphi\left(\ell_{0}(w)\right)\right\}
$$

Numerics

## A toy example

$$
\begin{aligned}
& \min _{w \in \mathbb{R}^{2}} \overbrace{\left(\left(w_{1}-b_{1}\right)^{2}+\left(w_{2}-b_{2}\right)^{2}\right)}^{k(w)}+\ell_{0}(w) \\
& \text { with } \quad b=(0.8,1.1)
\end{aligned}
$$

We have that $\left\{\left(0, b_{2}\right)\right\}=\{(0,1.1)\}=\underset{w \in \mathbb{R}^{2}}{\arg \min }\left\{k(w)+\ell_{0}(w)\right\}$


## The toy example as a min-max problem

As $\ell_{0}(w)=\max _{y \in \mathbb{R}^{2}}\left\{\dot{c}(w, y)-\ell_{0}^{¢}(y)\right\}$, we obtain that

$$
\min _{w \in \mathbb{R}^{2}}\left\{k(w)+\ell_{0}(w)\right\}=\min _{w \in \mathbb{R}^{2}} \max _{y \in \mathbb{R}^{2}}\left\{k(w)+c(w, y)-\ell_{0}^{c}(y)\right\}
$$

with

$$
\ell_{0}^{C}(y)=\sup _{k \in \llbracket 1, n \rrbracket}\left[\|y\|_{2, k}^{\top}-k\right]_{+}
$$

## Generalized primal-dual proximal splitting

GPDPS Algorithm Christian Clason, Stanislav Mazurenko, and Tuomo Valkonen. Primal-dual proximal splitting and generalized conjugation in non-smooth non-convex optimization. Applied Mathematics and Optimization, 84(2):1239-1284, apr 2020.

Given a starting point ( $w_{0}, y_{0}$ ) and step lengths $\tau_{i}, \omega_{i}, \sigma_{i}>0$, iterate

$$
\begin{aligned}
w^{(i+1)} & :=\operatorname{prox}_{\tau_{i} k}\left(w^{(i)}-\oint_{w}\left(w^{(i)}, y^{(i)}\right)\right) \\
\bar{w}^{(i+1)} & :=w^{(i+1)}+\omega_{i}\left(w^{(i+1)}-w^{(i)}\right) \\
y^{(i+1)} & :=\operatorname{prox}_{\sigma_{i} \ell_{0}^{¢}}\left(y^{(i)}+\sigma_{i} غ_{y}\left(\bar{w}^{(i+1)}, y^{(i)}\right)\right)
\end{aligned}
$$

The prox of $k$ is analytically computed (quadratic function), whereas the prox of $\ell_{0}^{\ell}$ is numerically computed with the optimization algorithm newuoa by M.J.D. Powell

## GPDPS convergence, varying the starting point



