Witsenhausen intrinsic model
Games in Product Form
Kuhn's Equivalence Theorem

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Charles Darwin and the peacock's tail

- In a letter to botanist Asa Gray — dated 3 April 1860, one year after the publication of *The Origin of Species* — Charles Darwin writes

  > *The sight of a feather in a peacock's tail,*
  > *whenever I gaze at it, makes me sick!*

  Indeed, this embarrasing cumbersome tail is a handicap for survival (like escaping predators)
- In 1871, Charles Darwin published *The Descent of Man, and Selection in Relation to Sex* and proposed that the peacock's tail had evolved because females preferred to mate with males with more elaborate ones (sexual selection)

Informational asymetry in mating

- In 1975, biologist Amotz Zahavi published
  Mate Selection-A Selection for a Handicap

  > *These handicaps are of use to the selecting sex since they test the quality of the mate. [...] The understanding that a handicap, which tests for quality, can evolve as a consequence of its advantage to the individual, may provide an explanation for many puzzling evolutionary problems.*

- In 2013, mathematicians Pierre Bernhard and Frédéric Hamelin published
  Simple signaling games of sexual selection (Grafen's revisited)

Illustrations of informational asymetries

- In biology, a peacock signals its “good genes” (genotype) by its lavish tail (phenotype)
- In economics, a worker signals her/his working ability (productivity) by her/his educational level (hard-to-get diplomas)
- An insurance company cannot observe whether insured persons play with matches at home or not
- A restaurant offering ”all you can eat” at a fixed price will attract customers with a larger than average appetite, resulting in a loss for the restaurant
- George Akerlof’s market for used cars with hidden flaws (”lemons”)

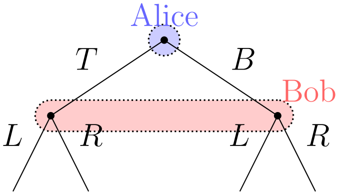
Information in game theory

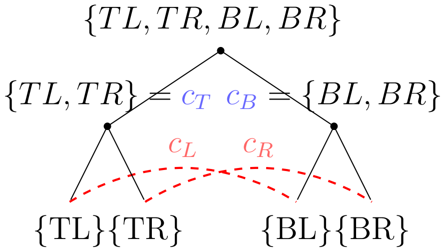
Game theory is concerned with strategic interactions:
my best choice depends on the other players

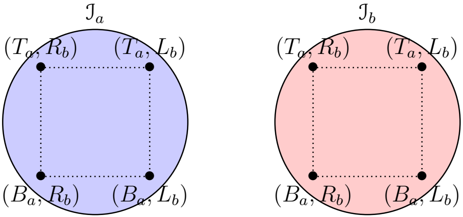
Strategic interactions originate from two sources

- Payoffs and beliefs
  - My payoff depends on the other players actions
  - I have beliefs about Nature (like other players types)
- Information
  - Information — who knows what and when — plays a crucial role in competitive contexts
  - Concealing, cheating, lying, deceiving are effective strategies

Three game forms (for two players Alice and Bob): Kuhn, Alós-Ferrer and Ritzberger, Witsenhausen







Kuhn's Equivalence Theorem

When a player satisfies perfect recall, for any mixed strategy, there is an equivalent behavioral strategy (and the converse)

- Tree extensive form (finite action sets) [Kuhn, 1953]
  Harold W. Kuhn
  Extensive games and the problem of information, 1953
- Extensive form (infinite action sets) [Aumann, 1964]
  Robert Aumann
  Mixed and behavior strategies in infinite extensive games, 1964
- Product form (infinite action sets)
  [Heymann, De Lara, and Chancelier, 2022]
  Benjamin Heymann, Michel De Lara, Jean-Philippe Chancelier.
  Kuhn's Equivalence Theorem for Games in Product Form, 2022

Roadmap

1. Introduce the Witsenhausen intrinsic model (W-model),
   and illustrate its potential to handle informational interactions,
   especially for games in product form (W-games)
2. State a Kuhn Theorem — equivalence between perfect recall
   and restriction to behavioral strategies —
   for games in product form
3. Provide a very general mathematical language for game theory,
   especially suited for the analysis of noncooperative decision settings
   without common clock,
   and for their resolution by agent decomposition

Outline of the presentation

Outline of the presentation

Witsenhausen intrinsic model

Games in product form

Classification of information structures

Algebras, σ-algebras/fields, partition fields

Let $\mathcal{Z}$ be a set

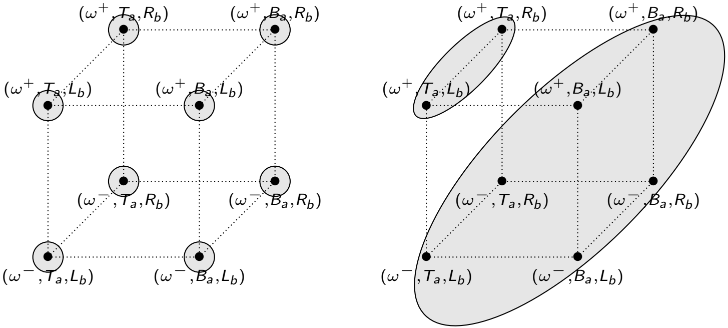
- An algebra (or field) on $\mathcal{Z}$ is a nonempty collection $\mathfrak{Z}$ of subsets of $\mathcal{Z}$ (identified with a subset $\mathfrak{Z} \subset 2^{\mathcal{Z}}$) which is stable under complementation and finite union (hence, under finite intersection)
- A σ-algebra (or σ-field) on $\mathcal{Z}$ is a nonempty collection $\mathfrak{Z}$ of subsets of $\mathcal{Z}$ (identified with a subset $\mathfrak{Z} \subset 2^{\mathcal{Z}}$) which is stable under complementation and countable union (hence, under countable intersection)
- A partition field (or π-field) on $\mathcal{Z}$ is a nonempty collection $\mathfrak{Z}$ of subsets of $\mathcal{Z}$ (identified with a subset $\mathfrak{Z} \subset 2^{\mathcal{Z}}$) which is stable under complementation and unlimited union (hence, under unlimited intersection)

The couple $(\mathcal{Z}, \mathfrak{Z})$ is called a measurable space

Examples of σ-fields and partition fields

Let $\mathcal{Z}$ be a set

- $\mathfrak{Z} = \{\emptyset, \mathcal{Z}\}$ is the trivial σ-field (or trivial π-field)
- $\mathfrak{Z} = 2^{\mathcal{Z}}$ is the complete σ-field (or complete π-field)
- The atoms of a partition field are
  the minimal elements for the inclusion $\subset$ relation,
  and they form a partition of $\mathcal{Z}$ into undistinguishable elements



Operations on σ-fields

Let $\mathcal{Z}$ be a set and $\{\mathfrak{Z}_i\}_{i\in I}$ be a family of σ-fields

- $\bigwedge_{i\in I}\mathfrak{Z}_i$ is the largest σ-field included in all the $\mathfrak{Z}_i$, for $i \in I$ (it coincides with $\bigcap_{i\in I}\mathfrak{Z}_i$)
- $\bigvee_{i\in I}\mathfrak{Z}_i$ is the smallest σ-field that contains all the $\mathfrak{Z}_i$, for $i \in I$

Let $\{(\mathcal{Z}_i, \mathfrak{Z}_i)\}_{i\in I}$ be a family of measurable spaces

- $\bigotimes_{i\in I}\mathfrak{Z}_i$ is a (product) σ-field on the (product) set $\prod_{i\in I}\mathcal{Z}_i$ ($\bigotimes_{i\in I}\mathfrak{Z}_i$ is the smallest σ-field that contains all the cylinders)

Outline of the presentation

Agents, actions, Nature, configuration space

We distinguish an individual from an agent

- An individual who makes a first, followed by a second action, is represented by two agents (two decision makers)
- An individual who makes a sequence of actions — one for each period $t = 0, 1, 2, \ldots, T-1$ — is represented by T agents, labelled $t = 0, 1, 2, \ldots, T-1$
- N individuals — each i of whom makes a sequence of actions, one for each period $t = 0, 1, 2, \ldots, T_i - 1$ — is represented by $\prod_{i=1}^{N} T_i$ agents, labelled by

$$(i, t) \in \bigcup_{j=1}^{N} \{j\} \times \{0, 1, 2, \ldots, T_j - 1\}$$

Agents, actions and action spaces

- Let A be a (finite or infinite) set, whose elements are called agents (or decision-makers)
- With each agent $a \in A$ is associated a measurable space

$$(\mathcal{U}_a, \mathfrak{U}_a)$$

where

  - the set $\mathcal{U}_a$ is the set of actions for agent a, where he makes one action $u_a \in \mathcal{U}_a$
  - the set $\mathfrak{U}_a \subset 2^{\mathcal{U}_a}$ is a σ-field (σ-algebra)

Examples

- $A = \{0, 1, 2, \ldots, T-1\}$ (T sequential actions), $(\mathcal{U}_a, \mathfrak{U}_a) = (\mathbb{R}^d, \mathfrak{B}^{\mathrm{o}}_{\mathbb{R}^d})$
- $A = \{\text{Principal, Agent}\}$ (principal-agent models)

Nature space

With Nature is associated a measurable space

$$(\Omega, \mathfrak{F})$$

where

- the set Ω is the set of states of Nature (uncertainties, scenarios, etc.) $\omega \in \Omega$
- the set $\mathfrak{F} \subset 2^{\Omega}$ is a σ-field (σ-algebra)
  (at this stage of the presentation, we do not need to equip $(\Omega, \mathfrak{F})$ with a probability distribution, as we only focus on information)

Examples

States of Nature Ω can include
types of players, randomness, stochastic processes, etc.

The configuration space is a product space

Configuration space

The configuration space is the product space

$$\mathcal{H} = \Omega \times \mathcal{U}_A = \Omega \times \prod_{a \in A} \mathcal{U}_a$$

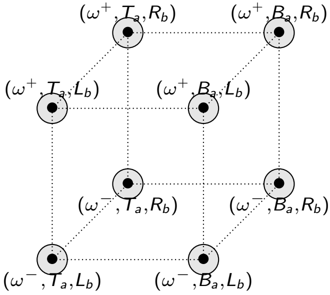
equipped with the product σ-field, called configuration field

$$\mathfrak{H} = \mathfrak{F} \otimes \mathfrak{U}_A = \mathfrak{F} \otimes \bigotimes_{a \in A} \mathfrak{U}_a$$

so that $(\mathcal{H}, \mathfrak{H})$ is a measurable space

Example of configuration space

$(\mathcal{H}, \mathfrak{H})$



- product configuration space

$$\mathcal{H} = \Omega \times \prod_{a \in A} \mathcal{U}_a$$

- product configuration field

$$\mathfrak{H} = \mathfrak{F} \otimes \bigotimes_{a \in A} \mathfrak{U}_a$$

Remark: a finite σ-field is represented by the partition of its atoms (minimal elements for inclusion)
Here, $\mathfrak{H} = 2^{\mathcal{H}}$ is represented by the partition of singletons

Information fields

Information fields express dependencies

Information field of an agent

The information field of agent $a \in A$ is a σ-field

$$\mathfrak{I}_a \subset \mathfrak{H} = \mathfrak{F} \otimes \bigotimes_{a \in A} \mathfrak{U}_a$$

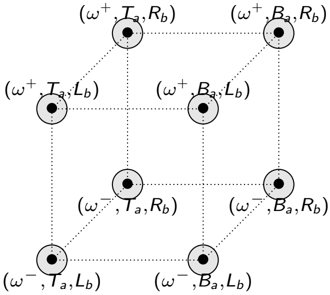
which is a subfield of the product configuration field

- The subfield $\mathfrak{I}_a$ of the configuration field $\mathfrak{H}$ represents the information available to agent a when the agent chooses an action
- Therefore, the information of agent a may depend
  - on the states of Nature
  - and on other agents' actions

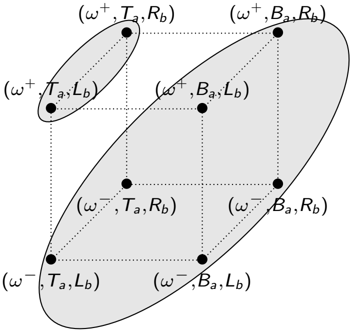
In the finite case, information fields are represented by the partition of its atoms

The information field of agent $a \in A$ is a subfield $\mathfrak{I}_a \subset \mathfrak{H} = \mathfrak{F} \otimes \bigotimes_{a \in A} \mathfrak{U}_a$ which can, in the finite case, be represented by the partition of its atoms

$(\mathcal{H}, \mathfrak{H})$



$\mathfrak{I}_a \subset \mathfrak{H}$



Definition of the W-model (2 basic objects, 1 axiom)

W-model

A W-model $\left(A, (\Omega, \mathfrak{F}), (\mathcal{U}_a, \mathfrak{U}_a)_{a \in A}, (\mathfrak{I}_a)_{a \in A}\right)$
consists of 2 basic objects

(W-BO1a) the sample space $(\Omega, \mathfrak{F})$
equipped with a σ-field

(W-BO1b) the collection $(\mathcal{U}_a, \mathfrak{U}_a)_{a \in A}$
of agents' actions equipped with σ-fields

(W-BO2) the collection $(\mathfrak{I}_a)_{a \in A}$
of agents' information subfields of $\mathfrak{H} = \mathfrak{F} \otimes \bigotimes_{a \in A} \mathfrak{U}_a$

and 1 axiom imposed on them

(W-Axiom1) for all agent $a \in A$, absence of self-information holds

$$\mathfrak{I}_a \subset \mathfrak{F} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \bigotimes_{b \in A \setminus \{a\}} \mathfrak{U}_b$$

We consider W-models that display absence of self-information

Absence of self-information

A W-model displays absence of self-information when

$$\mathfrak{I}_a \subset \mathfrak{F} \otimes \mathfrak{U}_{A\setminus\{a\}} = \mathfrak{F} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \bigotimes_{b \in A\setminus\{a\}} \mathfrak{U}_b$$

for any agent $a \in A$

- Absence of self-information means that the information of agent a may depend on the states of Nature and on all the other agents' actions, but not on his own (yet to take) action
- Absence of self-information makes sense as we have distinguished an individual from an agent (else, it would lead to paradoxes)

In absence of self-information, information fields are cylindrical

For any agent $a \in A$

$$\mathfrak{I}_a \subset \mathfrak{F} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \bigotimes_{b \in A \setminus \{a\}} \mathfrak{U}_b$$

$$\Longrightarrow$$

$$\mathfrak{I}_a = \underbrace{\{\emptyset, \mathcal{U}_a\} \otimes \hat{\mathfrak{I}}_a}_{\text{cylindrical } \sigma\text{-field (w.r.t. } \mathcal{U}_a)}$$

where $\hat{\mathfrak{I}}_a \subset \mathfrak{F} \otimes \bigotimes_{b \in A \setminus \{a\}} \mathfrak{U}_b$

Outline of the presentation

Alice and Bob

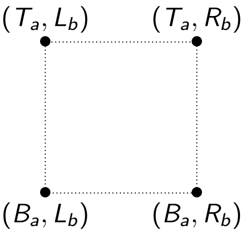
"Alice and Bob" configuration space

Alice and Bob are playing simultaneously

Example

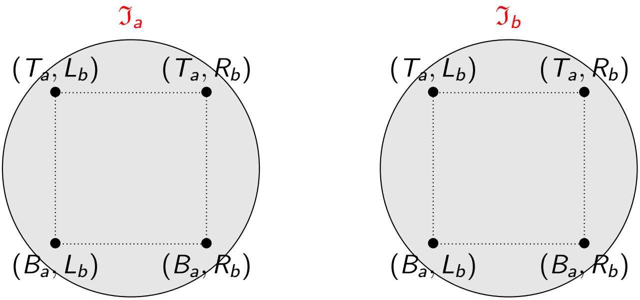
- no Nature
- two agents a (Alice) and b (Bob)
- two possible actions each $\mathcal{U}_a = \{T_a, B_a\}$, $\mathcal{U}_b = \{R_b, L_b\}$
- product configuration space (4 elements)

$$\mathcal{H} = \{T_a, B_a\} \times \{R_b, L_b\}$$



"Alice and Bob" information partitions

Alice and Bob are playing simultaneously

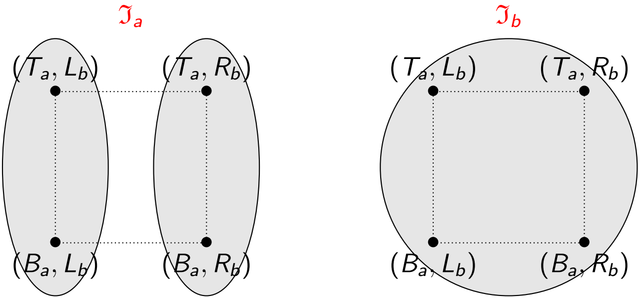


- $\mathfrak{I}_a = \{\emptyset, \{T_a, B_a\}\} \otimes \{\emptyset, \{R_b, L_b\}\}$ (trivial σ-field)
  Alice knows nothing
- $\mathfrak{I}_b = \{\emptyset, \{T_a, B_a\}\} \otimes \{\emptyset, \{R_b, L_b\}\}$ (trivial σ-field)
  Bob knows nothing

Alice knows Bob's action

"Alice and Bob" information partitions

Alice knows Bob's action



- $\mathfrak{I}_b = \{\emptyset, \{T_a, B_a\}\} \otimes \{\emptyset, \{R_b, L_b\}\}$ (trivial σ-field)
  Bob knows nothing
- $\mathfrak{I}_a = \{\emptyset, \{T_a, B_a\}\} \otimes \{\emptyset, \{R_b\}, \{L_b\}, \{R_b, L_b\}\}$
  (cylindrical σ-field by absence of self-information)
  Alice knows what Bob does
  (as she can distinguish between Bob's actions $\{R_b\}$ and $\{L_b\}$)

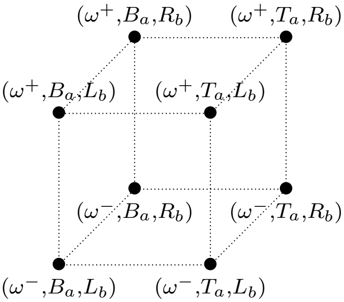
Alice, Bob and a coin tossing

"Alice, Bob and a coin tossing" configuration space

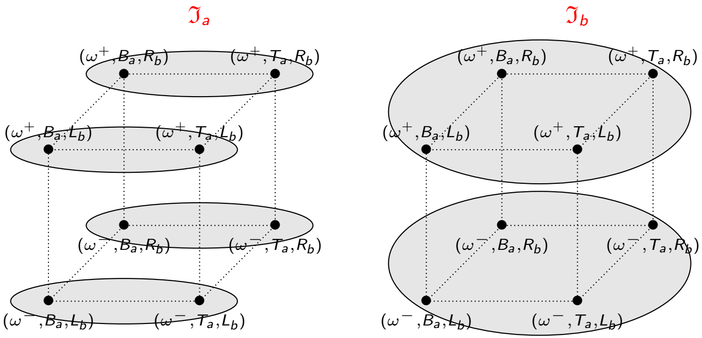
Example

- two states of Nature $\Omega = \{\omega^+, \omega^-\}$ (heads/tails)
- two agents a and b
- two possible actions each: $\mathcal{U}_a = \{T_a, B_a\}$, $\mathcal{U}_b = \{R_b, L_b\}$
- product configuration space (8 elements)

$$\mathcal{H} = \{\omega^+, \omega^-\} \times \{T_a, B_a\} \times \{R_b, L_b\}$$



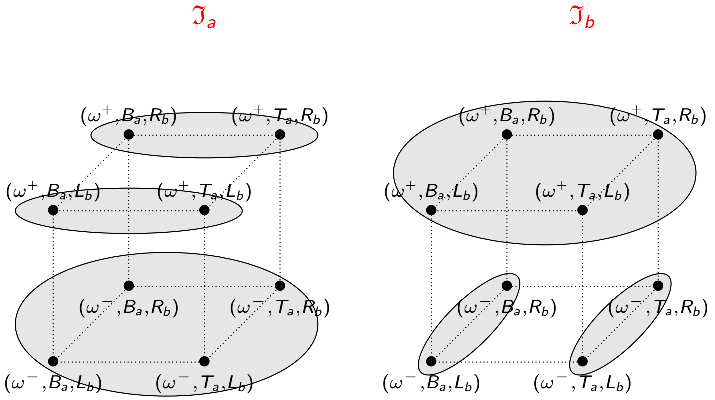
"Alice, Bob and a coin tossing" information partitions



$$\mathfrak{I}_b = \overbrace{\{\emptyset, \{\omega^+\}, \{\omega^-\}, \{\omega^+, \omega^-\}\}}^{\text{Bob knows Nature's move}} \otimes \overbrace{\{\emptyset, \{T_a, B_a\}\}}^{\text{Bob does not know what Alice does}} \otimes \{\emptyset, \mathcal{U}_b\}$$

$$\mathfrak{I}_a = \underbrace{\{\emptyset, \{\omega^+\}, \{\omega^-\}, \{\omega^+, \omega^-\}\}}_{\text{Alice knows Nature's move}} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \underbrace{\{\emptyset, \{R_b\}, \{L_b\}, \{R_b, L_b\}\}}_{\text{Alice knows what Bob does}}$$

"Alice, Bob and a coin tossing" information partitions



Outline of the presentation

Stochastic control

Stochastic control

- Infinite (nonatomic) agents $A = [0, +\infty[$
- Decision of agent t taken in a set $\mathcal{U}_t$
- Filtration $\{\mathfrak{F}_t\}_{t \geq 0}$ of the sample space $(\Omega, \mathfrak{F})$

$$s \leq t \implies \mathfrak{F}_s \subset \mathfrak{F}_t \subset \mathfrak{F}$$

- Information of (nonanticipative) agent t is either modeled as

$$\mathfrak{I}_t \subset \underbrace{\mathfrak{F}_t}_{\text{partial observation of nature}} \otimes \underbrace{\bigotimes_{s \geq 0} \{\emptyset, \mathcal{U}_s\}}_{\text{no observation of actions}}$$

or as

$$\mathfrak{I}_t \subset \mathfrak{F}_t \otimes \underbrace{\bigotimes_{r < t} \mathfrak{U}_r}_{\text{memory of past actions}} \otimes \underbrace{\bigotimes_{s \geq t} \{\emptyset, \mathcal{U}_s\}}_{\text{no observation of future actions}}$$

Mean-field/dynamic game

Mean-field/dynamic game: data for information structures

- Infinite (mean-field) number of players $p \in P$ (finite for dynamical games)
- Time either discrete, $t \in \mathbb{N}$, or continuous, $t \in [0, +\infty[$
- Agents are couples $a = (p, t)$ making decisions in measurable sets $(\mathcal{U}_t^p, \mathfrak{U}_t^p)$
- Filtration $\{\mathfrak{F}_t\}_{t \geq 0}$ of the sample space $(\Omega, \mathfrak{F})$

$$s \leq t \implies \mathfrak{F}_s \subset \mathfrak{F}_t \subset \mathfrak{F}$$

Mean-field/dynamic game: information structures

Information $\mathfrak{I}_{(p,t)}$ of (nonanticipative) agent (p, t)

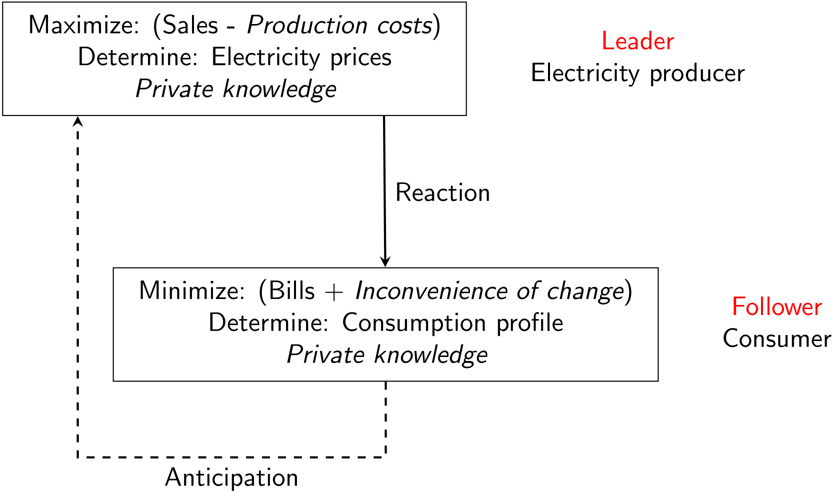
$$\mathfrak{I}_{(p,t)} \subset \overbrace{\mathfrak{F}_t}^{\text{partial observation of nature}} \otimes \overbrace{\bigotimes_{p \in P} \bigotimes_{s \geq 0} \{\emptyset, \mathcal{U}_s^p\}}^{\text{no observation of actions}}$$

$$\mathfrak{I}_{(p,t)} \subset \mathfrak{F}_t \otimes \Big(\underbrace{\bigotimes_{r<t} \mathfrak{U}_r^p}_{\text{memory of one's past actions}} \otimes \underbrace{\bigotimes_{s \geq t} \{\emptyset, \mathcal{U}_s^p\}}_{\text{no observation of one's future actions}} \Big) \otimes \underbrace{\bigotimes_{q \in P \setminus \{p\}} \bigotimes_{s \geq 0} \{\emptyset, \mathcal{U}_s^q\}}_{\text{no observation of other players actions}}$$

$$\mathfrak{I}_{(p,t)} \subset \mathfrak{F}_t \otimes \bigotimes_{q \in P} \Big(\underbrace{\bigotimes_{r<t} \mathfrak{U}_r^q}_{\text{memory of any player's past actions}} \otimes \underbrace{\bigotimes_{s \geq t} \{\emptyset, \mathcal{U}_s^q\}}_{\text{no observation of any player's future actions}} \Big)$$

Principal-agent models
or
Leader-follower models

Demand response: a leader-follower problem



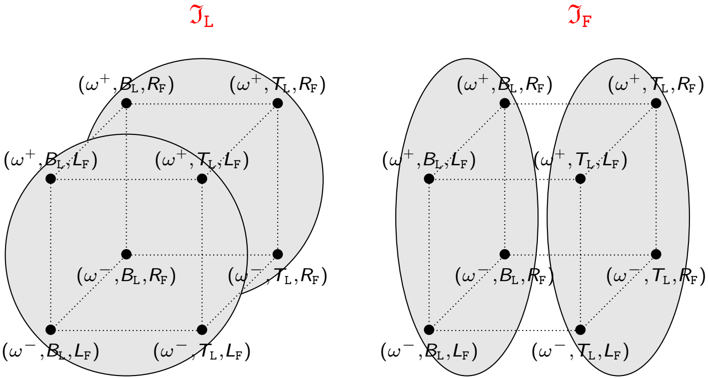
Principal-agent or leader-follower models with two decision-makers

A branch of Economics studies so-called principal-agent models, which can easily be expressed with Witsenhausen intrinsic model (to avoid confusion, we will shift to the vocable leader-follower models

- The model exhibits two decision-makers
  - the leader $\mathtt{L}$ makes actions $u_{\mathtt{L}}$ in $(\mathcal{U}_{\mathtt{L}}, \mathfrak{U}_{\mathtt{L}})$
  - the follower $\mathtt{F}$ makes actions $u_{\mathtt{F}}$ in $(\mathcal{U}_{\mathtt{F}}, \mathfrak{U}_{\mathtt{F}})$
- and Nature, corresponding to private information (or type) of the follower $\mathtt{F}$
  - Nature selects $\omega \in (\Omega, \mathfrak{F})$

Example of a (binary) leader-follower W-model

Information fields are cylinders, by absence of self-information



Here is the most general information structure of leader-follower models

$$\mathfrak{I}_{\mathtt{L}} \subset \mathfrak{F} \otimes \mathfrak{U}_{\mathtt{F}} \otimes \{\emptyset, \mathcal{U}_{\mathtt{L}}\}$$

$$\mathfrak{I}_{\mathtt{F}} \subset \mathfrak{F} \otimes \{\emptyset, \mathcal{U}_{\mathtt{F}}\} \otimes \mathfrak{U}_{\mathtt{L}}$$

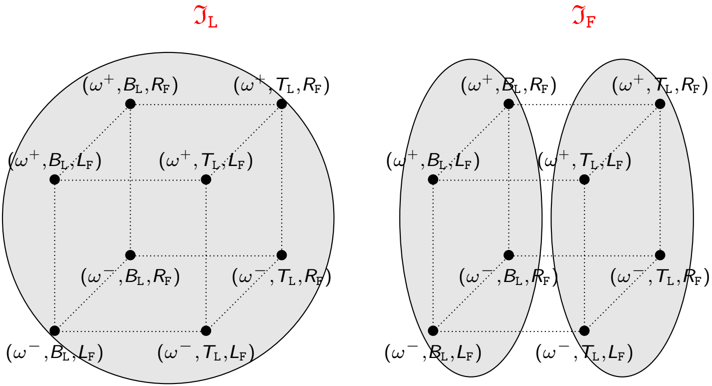
- By these expressions of the information fields
  - $\mathfrak{I}_{\mathtt{L}}$ of the leader $\mathtt{L}$
  - $\mathfrak{I}_{\mathtt{F}}$ of the follower $\mathtt{F}$
- we have excluded self-information

Classical information patterns in game theory

Now, we will make the information structure more specific

- Stackelberg leadership model
- Moral hazard (hidden action)
- Adverse selection (hidden type)
- Signaling

Example of a (binary) Stackelberg leadership W-model



Stackelberg leadership model

- The follower $\mathtt{F}$ may partly observe the action of the leader $\mathtt{L}$

$$\mathfrak{I}_{\mathtt{F}} \subset \mathfrak{F} \otimes \{\emptyset, \mathcal{U}_{\mathtt{F}}\} \otimes \mathfrak{U}_{\mathtt{L}}$$

- whereas the leader $\mathtt{L}$ observes at most the state of Nature

$$\mathfrak{I}_{\mathtt{L}} \subset \mathfrak{F} \otimes \{\emptyset, \mathcal{U}_{\mathtt{F}}\} \otimes \{\emptyset, \mathcal{U}_{\mathtt{L}}\}$$

- As a consequence, the system is sequential
  - with the leader $\mathtt{L}$ as first decision-maker
  - and the follower $\mathtt{F}$ as second decision-maker

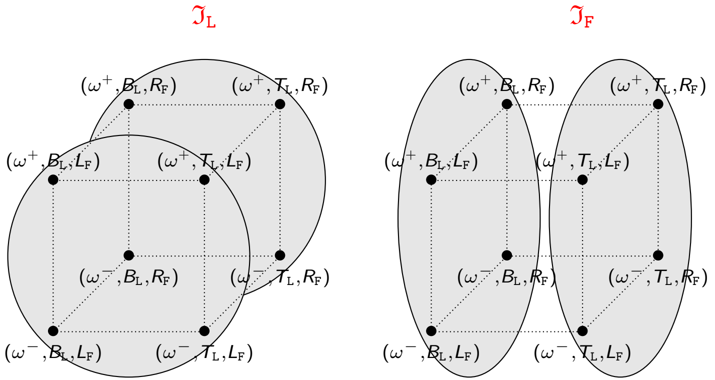
Moral hazard (hidden action)

- An insurance company (the leader $\mathtt{L}$) cannot observe the efforts of the insured (the follower $\mathtt{F}$) to avoid risky behavior, whereas the firm faces the hazard that insured persons behave "immorally" (playing with matches at home)
- Moral hazard (hidden action) occurs when the actions of the follower $\mathtt{F}$ are hidden to the leader $\mathtt{L}$

$$\mathfrak{I}_{\mathtt{L}} \subset \mathfrak{F} \otimes \{\emptyset, \mathcal{U}_{\mathtt{F}}\} \otimes \{\emptyset, \mathcal{U}_{\mathtt{L}}\}$$

- In case of moral hazard, the system is sequential with the principal as first decision-maker, (which does not preclude to choose the follower as first decision-maker in some special cases, as in a static team situation)

Example of a (binary) adverse selection W-model



Adverse selection

- In the absence of observable information on potential customers (the follower $\mathtt{F}$), an insurance company (the leader $\mathtt{L}$) offers a unique price for a contract, hence screens and selects the "bad" ones
- Adverse selection occurs when
  - the follower $\mathtt{F}$ knows the state of nature (her/his own type, or private information)

$$\mathfrak{F} \otimes \{\emptyset, \mathcal{U}_{\mathtt{F}}\} \otimes \{\emptyset, \mathcal{U}_{\mathtt{L}}\} \subset \mathfrak{I}_{\mathtt{F}}$$

    (the follower $\mathtt{F}$ can possibly observe the leader $\mathtt{L}$ action)
  - but the leader $\mathtt{L}$ does not know the state of nature, that is, the agent $\mathtt{F}$ type

$$\mathfrak{I}_{\mathtt{L}} \subset \{\emptyset, \Omega\} \otimes \mathfrak{U}_{\mathtt{F}} \otimes \{\emptyset, \mathcal{U}_{\mathtt{L}}\}$$

    (the leader $\mathtt{L}$ can possibly observe the follower $\mathtt{F}$ action)
- In case of adverse selection, the system may or may not be sequential

Signaling

- In biology, a peacock signals its "good genes" (genotype) by its lavish tail (phenotype)
- In economics, a worker signals her/his working ability (productivity) by her/his educational level (diplomas)
- There is room for signaling
  - when the follower $\mathtt{F}$ knows the state of nature (her/his own type)

$$\mathfrak{F} \otimes \{\emptyset, \mathcal{U}_{\mathrm{F}}\} \otimes \{\emptyset, \mathcal{U}_{\mathrm{L}}\} \subset \mathfrak{I}_{\mathrm{F}}$$

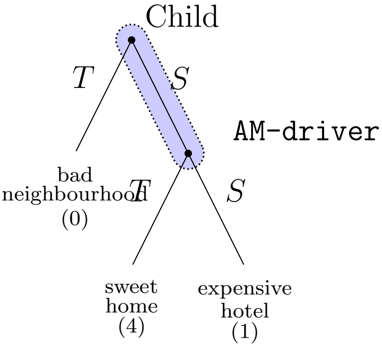
    (the follower $\mathtt{F}$ can possibly observe the leader $\mathtt{L}$ action)
  - whereas the leader $\mathtt{L}$ does not know the state of nature, that is, the follower $\mathtt{F}$ type, but the leader $\mathtt{L}$ observes the follower $\mathtt{F}$ action

$$\mathfrak{I}_{\mathrm{L}} = \{\emptyset, \Omega\} \otimes \mathfrak{U}_{\mathrm{F}} \otimes \{\emptyset, \mathcal{U}_{\mathrm{L}}\}$$

as the follower $\mathtt{F}$ may reveal her/his type by her/his action which is observable by the leader $\mathtt{L}$

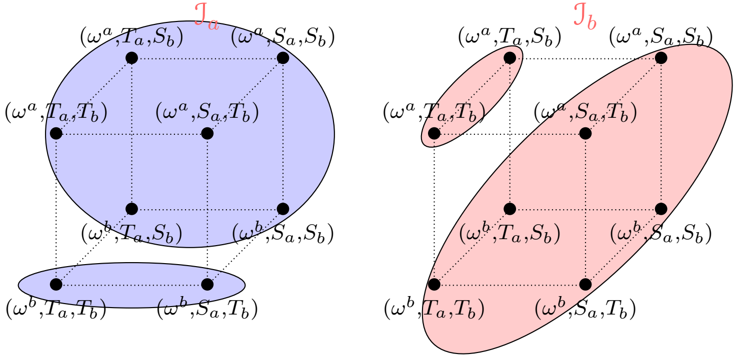
Absent-minded driver

Absent-minded driver



- S=Stay, T=Turn
- "paradox" that raised a problem in game theory
- the player looses public time, as plays "SS" "ST" cross the information set twice
- cannot be modelled *per se* in tree models (violates "no-AM" axiom)

A W-model for the absent-minded driver



$$\mathfrak{I}_a = \{\emptyset, \overbrace{\underbrace{\{\omega_a\} \times \mathcal{U}_a \times \mathcal{U}_b}_{\substack{\text{agent } a \text{ is whether} \\ \text{the first one to act}}} \cup \underbrace{\{\omega_b\} \times \{S_b\} \times \mathcal{U}_a}_{\substack{\text{or he acts second after} \\ \text{agent } b \text{ has chosen } S}}}^{\text{agent } a \text{ makes a move}}, \overbrace{\underbrace{\{\omega_b\} \times \{T_b\} \times \mathcal{U}_a}_{\substack{\text{agent } b \text{ chose } T \\ \text{and finished the game}}}}^{\text{agent } a \text{ doesn't make a move}}, \mathcal{H}\}$$

$$\mathfrak{I}_b = \{\emptyset, \{\omega_b\} \times \mathcal{U}_a \times \mathcal{U}_b \cup \{\omega_a\} \times \{S_a\} \times \mathcal{U}_b, \{\omega_a\} \times \{T_a\} \times \mathcal{U}_b, \mathcal{H}\}$$

What land have we covered?
What comes next?

- The stage is in place; so are the actors
  - agents
  - Nature
  - information
- How can actors play?
  - strategies
  - playability

Outline of the presentation

Strategies

Information is the fuel of W-strategies

W-strategy of an agent

A (pure) W-strategy of agent a is a mapping

$$\lambda_a : (\mathcal{H}, \mathfrak{H}) \to (\mathcal{U}_a, \mathfrak{U}_a)$$

which is measurable w.r.t. the information field $\mathfrak{I}_a$, that is,

$$\lambda_a^{-1}(\mathfrak{U}_a) \subset \mathfrak{I}_a$$

This condition expresses the property that
a W-strategy for agent a
may only depend upon the information $\mathfrak{I}_a$ available to the agent

Set of W-strategies

Set of W-strategies of an agent

We denote the set of (pure) W-strategies of agent a by

$$\Lambda_a = \left\{ \lambda_a : (\mathcal{H}, \mathfrak{H}) \to (\mathcal{U}_a, \mathfrak{U}_a) \,\middle|\, \lambda_a^{-1}(\mathfrak{U}_a) \subset \mathfrak{I}_a \right\}$$

and the set of W-strategies of all agents is

$$\Lambda = \Lambda_A = \prod_{a \in A} \Lambda_a$$

Examples of W-strategies

Consider a W-model with two agents a and b,
and suppose that σ-fields $\mathfrak{U}_a$, $\mathfrak{U}_b$ and $\mathfrak{F}$ contain the singletons

- Absence of self-information

$$\mathfrak{I}_a \subset \mathfrak{F} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \mathfrak{U}_b \ , \ \ \mathfrak{I}_b \subset \mathfrak{F} \otimes \mathfrak{U}_a \otimes \{\emptyset, \mathcal{U}_b\}$$

  Then, W-strategies λ_a and λ_b have the form

$$\lambda_a(\omega, \cancel{u_a}, u_b) = \widetilde{\lambda_a}(\omega, u_b) \ , \ \ \lambda_b(\omega, u_a, \cancel{u_b}) = \widetilde{\lambda_b}(\omega, u_a)$$

- Sequential W-model

$$\mathfrak{I}_a = \mathfrak{F} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \mathfrak{U}_b \ , \ \ \mathfrak{I}_b = \mathfrak{F} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \{\emptyset, \mathcal{U}_b\}$$

  Then, W-strategies λ_a and λ_b have the form

$$\lambda_a(\omega, u_b, \cancel{u_a}) = \widetilde{\lambda}_a(\omega, u_b) \ , \ \ \lambda_b(\omega, \cancel{u_b}, \cancel{u_a}) = \widetilde{\lambda}_b(\omega)$$

Playability

Playability

- In the Witsenhausen's intrinsic model,
  agents make actions in an order which is not fixed in advance
- Briefly speaking, playability ("solvability" in Witsenhausen's terms)
  is the property that, for each state of Nature,
  the agents' actions are uniquely determined by their W-strategies

Playability problem

The playability (solvability) problem consists in finding

- for any collection $\lambda = \{\lambda_a\}_{a \in A} \in \Lambda_A$ of W-strategies
- for any state of Nature $\omega \in \Omega$

actions $u \in \mathcal{U}_A$ satisfying
the implicit ("closed loop") equation

$$u = \lambda(\omega, u)$$

or, equivalently, the family of "closed loop" equations

$$u_a = \lambda_a(\omega, \{u_b\}_{b \in A}) \ , \ \ \forall a \in A$$

Playability property

Playability property

A W-model displays the playability property when the "closed loop" equation $u = \lambda(\omega, u)$ has a unique solution for any collection $\lambda = \{\lambda_a\}_{a \in A} \in \Lambda_A$ of W-strategies and for any state of Nature $\omega \in \Omega$, that is,

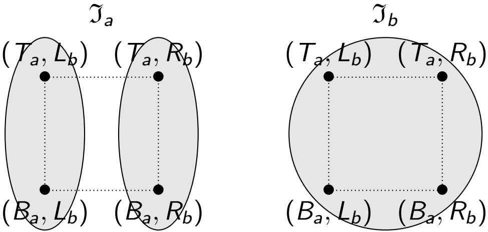
$$\forall \lambda = (\lambda_a)_{a \in A} \in \Lambda_A \; , \;\; \forall \omega \in \Omega \; , \;\; \exists ! u \in \mathcal{U}_A \; , \;\; u = \lambda(\omega, u)$$

or, equivalently, when

$$\begin{aligned} \forall \lambda = (\lambda_a)_{a \in A} \in \Lambda_A \; , \;\; \forall \omega \in \Omega \; , \;\; \exists ! u \in \mathcal{U}_A \; , \\ u_a = \lambda_a(\omega, \{u_b\}_{b \in A}) \; , \;\; \forall a \in A \end{aligned}$$

Playability is a property of the information structure

Sequentiality



Sequential W-model

$$\mathfrak{I}_a = \mathfrak{F} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \mathfrak{U}_b \ , \ \ \mathfrak{I}_b = \mathfrak{F} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \{\emptyset, \mathcal{U}_b\}$$

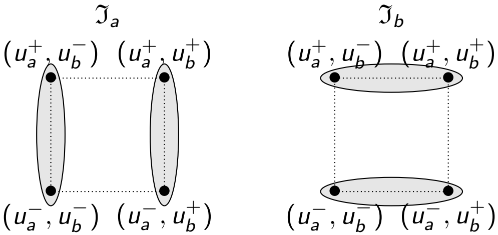
The closed-loop equations

$$u_a = \lambda_a(\omega, u_b, \cancel{u_a}) = \widetilde{\lambda}_a(\omega, u_b) \ , \ \ u_b = \lambda_b(\omega, \cancel{u_b}, \cancel{u_a}) = \widetilde{\lambda}_b(\omega)$$

always displays a unique solution (u_a, u_b),
whatever $\omega \in \Omega$ and W-strategies λ_a and λ_b

Playability is a property of the information structure

Deadlock



W-model with deadlock

$$\mathfrak{I}_a = \{\emptyset, \Omega\} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \mathfrak{U}_b \ , \ \ \mathfrak{I}_b = \{\emptyset, \Omega\} \otimes \mathfrak{U}_a \otimes \{\emptyset, \mathcal{U}_b\}$$

The closed-loop equations

$$u_a = \lambda_a(\cancel{u_a}, u_b) = \tilde{\lambda}_a(u_b) \ , \ \ u_b = \lambda_b(u_a, \cancel{u_b}) = \tilde{\lambda}_b(u_a)$$

may display zero solutions, one solution or multiple solutions, depending on the W-strategies λ_a and λ_b

Playability makes it possible to define a solution map from states of Nature towards configurations

Suppose that the playability property holds true

Solution map

We define the solution map

$$S_\lambda : \Omega \to \mathcal{H} = \Omega \times \mathcal{U}_A = \Omega \times \prod_{a \in A} \mathcal{U}_a$$

that maps states of Nature towards configurations, by

$$(\omega, u) = S_\lambda(\omega) \iff u = \lambda(\omega, u) \ , \ \forall (\omega, u) \in \Omega \times \mathcal{U}_A$$

We include the state of Nature ω in the image of $S_\lambda(\omega)$, so that we map the set Ω towards the configuration space $\mathcal{H}$, making it possible to interpret $S_\lambda(\omega)$ as a configuration driven by the W-strategy λ (in classical control theory, a state trajectory is produced by a policy)

In the sequential case, the solution map is given by iterated composition

- In the sequential case

$$\mathfrak{I}_b = \mathfrak{F} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \{\emptyset, \mathcal{U}_b\} \ , \ \ \mathfrak{I}_a = \mathfrak{F} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \mathfrak{U}_b$$

- W-strategies λ_b and λ_a have the form

$$\lambda_b(\omega, \cancel{u_b}, \cancel{u_a}) = \widetilde{\lambda}_b(\omega) \ , \ \ \lambda_a(\omega, \cancel{u_a}, u_b) = \widetilde{\lambda}_a(\omega, u_b)$$

- so that the solution map is

$$S_\lambda(\omega) = \Big(\omega, \widetilde{\lambda}_a\Big(\omega, \widetilde{\lambda}_b(\omega)\Big), \widetilde{\lambda}_b(\omega)\Big)$$

- because the system of equations $u = \lambda(\omega, u)$ here writes

$$u_b = \lambda_b(\omega, \cancel{u_a}, \cancel{u_b}) = \widetilde{\lambda}_b(\omega) \ , \ \ u_a = \lambda_a(\omega, \cancel{u_a}, u_b) = \widetilde{\lambda}_a(\omega, u_b)$$

With playability, hence with a solution map, one obtains a game form

Game form

A playable W-model induces a game form by means of the outcome mapping

$$\begin{aligned} S(\cdot,\cdot) : \Omega \times \Lambda &\to \mathcal{H} \\ (\omega, \lambda) &\mapsto S_\lambda(\omega) \end{aligned}$$

If the W-model is not playable, we get a set-valued mapping (correspondence)

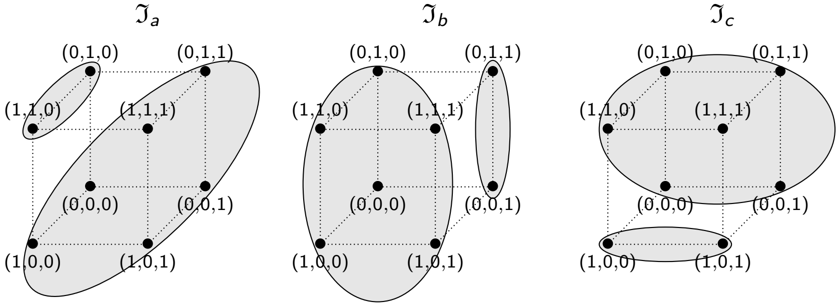
$$\begin{aligned} &\Omega \times \Lambda \rightrightarrows \mathcal{H} \\ &(\omega, \lambda) \mapsto \left\{ h \in \mathcal{H} \,\middle|\, h = (\omega, u) \;, \;\; u = \lambda(\omega, u) \right\} \end{aligned}$$

A game that can be played but that cannot start: the clapping hand game

- [Three players:] Alice, Bob and Carol
  are sitting around a circular table, with their eyes closed
- [Two decisions:] Each of them has to decide either to extend her/his left hand to the left or to extend her/his right hand to the right
- [Information:] when two hands touch, the remaining player is informed (say, a clap is directly conveyed to her/his ears);
  when two hands do not touch, the remaining player is not informed
- [Strategies:] for each player, a strategy is a mapping
  $\{\text{clap}, \text{no clap}\} \to \{\text{left}, \text{right}\}$
- [Playability:] for each triplet of strategies — one for each of Alice, Bob and Carol — there is a unique outcome of extended hands:
  the game is playable
- [No tree:] however, the game cannot start,
  hence this playable game cannot be written on a tree

Playable noncausal example [Witsenhausen, 1971]

- No Nature, $A = \{a, b, c\}$, $\mathcal{U}_a = \mathcal{U}_b = \mathcal{U}_c = \{0, 1\}$
- Set of configurations $\mathcal{H} = \{0, 1\}^3$, and information fields $\mathfrak{I}_a = \sigma(u_b(1 - u_c))$, $\mathfrak{I}_b = \sigma(u_c(1 - u_a))$, $\mathfrak{I}_c = \sigma(u_a(1 - u_b))$
- The "game" can be played but... cannot be started (no first agent)



What land have we covered?
What comes next?

- The stage is in place; so are the actors
  - agents
  - Nature
  - information
- Actors know how they can play
  - W-strategies
  - playability
- In a noncooperative context,
  we will now define players as "team leaders of agents"
  - playing mixed strategies
  - (possibly endowed with objectives and beliefs)

What comes next?

- Players and W-games
- Mixed and behavioral strategies
- Perfect recall
- Kuhn's equivalence Theorem

Outline of the presentation

Outline of the presentation

Players

A player holds a team of executive agents

- The set of players is denoted by P (finite or infinite set)
- Every player $p \in P$ has
  a team of executive agents

  $$A^p \subset A$$

  where $(A^p)_{p \in P}$ forms a partition of the set A of agents

  $$A = \underbrace{\bigcup_{p \in P} A^p}_{\text{partition}}$$

- A player is a team leader

Example: Don Juan wants to get married

Don Juan wants to get married

- Player Don Juan p is considering giving a phone call
  to his ex-lovers q, r (players),
  asking them if they want to marry him
- Don Juan selects one of his ex-lovers in the set $\{q, r\}$ and phones her
- If the answer to the first phone call is "yes",
  Don Juan marries the first called ex-lover
  (and decides not to give a second phone call)
- If the answer to the first phone call is "no",
  Don Juan makes a second phone call to the remaining ex-lover
- In that case, the remaining ex-lover answers "yes" or "no"

Agents, decisions, players

- Four agents partitioned in three players

$$A = \{ \overbrace{p_1, p_2}^{\text{Don Juan } p} , \overbrace{q}^{\text{ex-lover } q} , \overbrace{r}^{\text{ex-lover } r} \}$$

  because player Don Juan p makes decisions
  at possibly two occasions, hence has two executive agents p_1, p_2
- No Nature, but finite decisions sets

$$\mathcal{U}_{p_1} = \{q, r\} \; , \; \mathcal{U}_{p_2} = \{q, r, \partial\} \; , \; \mathcal{U}_q = \{Y, N\} \; , \; \mathcal{U}_r = \{Y, N\}$$

  - Agent p_1 selects an ex-lover in the set $\mathcal{U}_{p_1} = \{q, r\}$ and phones her
  - Agent p_2 either stops (decision ∂) or selects an ex-lover in $\{q, r\}$
  - Agents q, r either say "yes" or "no", hence select a decision in the set $\{Y, N\}$
- The finite decisions sets $\mathcal{U}_{p_1}, \mathcal{U}_{p_2}, \mathcal{U}_q, \mathcal{U}_r$ are equipped with the complete finite σ-fields $\mathfrak{U}_{p_1} = 2^{\mathcal{U}_{p_1}}, \mathfrak{U}_{p_2} = 2^{\mathcal{U}_{p_2}}, \mathfrak{U}_q = 2^{\mathcal{U}_q}, \mathfrak{U}_r = 2^{\mathcal{U}_r}$

Information structure: Don Juan

$$\mathcal{H} = \mathcal{U}_{p_1} \times \mathcal{U}_{p_2} \times \mathcal{U}_q \times \mathcal{U}_r$$

- When agent Don Juan p_1 makes the first phone call, he knows nothing, represented by his trivial information field

$$\mathfrak{I}_{p_1} = \{\emptyset, \mathcal{U}_{p_1}\} \otimes \{\emptyset, \mathcal{U}_{p_2}\} \otimes \{\emptyset, \mathcal{U}_q\} \otimes \{\emptyset, \mathcal{U}_r\}$$

- The agent Don Juan p_2 remembers who Don Juan p_1 called first, and knows the answer, which is represented by his information field

$$\begin{aligned}\mathfrak{I}_{p_2} = \{\emptyset, &\mathcal{U}_{p_1} \times \mathcal{U}_{p_2} \times \mathcal{U}_q \times \mathcal{U}_r, \\ &\underbrace{\{q\}}_{\text{remembering}} \times \overbrace{\{\emptyset, \mathcal{U}_{p_2}\}}^{\text{absence of self-information}} \times \underbrace{\mathfrak{U}_q}_{\text{knowing the answer}} \times \{\emptyset, \mathcal{U}_r\}, \\ &\{r\} \times \{\emptyset, \mathcal{U}_{p_2}\} \times \{\emptyset, \mathcal{U}_q\} \times \mathfrak{U}_r\}\end{aligned}$$

Information structure: ex-lovers

- If ex-lover q receives a phone call from Don Juan, she does not know if she was called first or second, hence she cannot distinguish the elements in the set

$$\{\underbrace{(q,q),(q,r),(q,\partial)}_{\text{called first}}, \quad \underbrace{(r,q)}_{\text{called second}}\}$$

so that her information field is

$$\mathfrak{I}_q = \{\emptyset, \underbrace{\{(q,q),(q,r),(q,\partial),(r,q)\}}_{\text{called}}, \underbrace{\{(r,r),(r,\partial)\}}_{\text{not called}}, \mathcal{U}_{p_1} \times \mathcal{U}_{p_2}\}$$
$$\otimes \{\emptyset, \mathcal{U}_q\} \otimes \{\emptyset, \mathcal{U}_r\}$$

- Conversely, ex-lover r is equipped with the σ-field

$$\mathfrak{I}_r = \{\emptyset, \{(r,r),(r,q),(r,\partial),(q,r)\}, \{(q,q),(q,\partial)\}, \mathcal{U}_{p_1} \times \mathcal{U}_{p_2}\} \otimes \{\emptyset, \mathcal{U}_q\} \otimes \{\emptyset, \mathcal{U}_r\}$$

A causal but nonsequential system

If Don Juan p_1 calls ex-lover q first, the agents play in the following order

$$p_1 \to q \to p_2 \to r$$

and conversely

- Configuration space

$$\mathcal{H} = \mathcal{U}_{p_1} \times \mathcal{U}_{p_2} \times \mathcal{U}_q \times \mathcal{U}_r$$

- Configuration space partition

$$\mathcal{H}_q = \{q\} \times \mathcal{U}_{p_2} \times \mathcal{U}_q \times \mathcal{U}_r \ , \ \ \mathcal{H}_r = \{r\} \times \mathcal{U}_{p_2} \times \mathcal{U}_q \times \mathcal{U}_r$$

- A non constant history-ordering mapping is

$$\varphi : \mathcal{H} \to \{(p_1, q, p_2, r), (p_1, r, p_2, q)\}$$

  such that

$$\varphi_{|\mathcal{H}_q} \equiv (p_1, q, p_2, r) \ , \ \ \varphi_{|\mathcal{H}_r} \equiv (p_1, r, p_2, q)$$

Outline of the presentation

Pure W-strategies profiles

- A pure W-strategy for player p is an element of

$$\Lambda_{A^p} = \prod_{a \in A^p} \Lambda_a$$

- The set of pure W-strategies for all players is

$$\prod_{p \in P} \Lambda_{A^p} = \prod_{p \in P} \prod_{a \in A^p} \Lambda_a = \prod_{a \in A} \Lambda_a = \Lambda_A$$

- A W-strategy profile is

$$\lambda = (\lambda^p)_{p \in P} \in \prod_{p \in P} \Lambda_{A^p}$$

- When we focus on player p, we write

$$\lambda = (\lambda^p, \lambda^{-p}) \in \Lambda_{A^p} \times \underbrace{\prod_{p' \neq p} \Lambda_{A^{p'}}}_{\Lambda_{A^{-p}}}$$

Mixed and behavioral strategies "à la Aumann"

For any player $p \in P$ and agent $a \in A^p$, we denote by

- $(\mathcal{W}_a, \mathfrak{W}_a)$ a copy of the Borel space $([0,1], \mathfrak{B}^{o}_{[0,1]})$
- ℓ_a a copy of the Lebesgue measure on $(\mathcal{W}_a, \mathfrak{W}_a) = ([0,1], \mathfrak{B}^{o}_{[0,1]})$

and we define a probability space (random generator) $(\mathcal{W}^p, \mathfrak{W}^p, \ell^p)$ attached to player p by

$$\mathcal{W}^p = \prod_{a \in A^p} \mathcal{W}_a \ , \ \ \mathfrak{W}^p = \bigotimes_{a \in A^p} \mathfrak{W}_a \ , \ \ \ell^p = \bigotimes_{a \in A^p} \ell_a$$

and we also set

$$\mathcal{W} = \prod_{p \in P} \mathcal{W}^p \ , \ \ \mathfrak{W} = \bigotimes_{p \in P} \mathfrak{W}^p \ , \ \ \ell = \bigotimes_{p \in P} \ell^p$$

Mixed, behavioral and pure strategies “à la Aumann”: definition

For the player $p \in P$,

- an A-mixed strategy is a family $m^p = \{m_a\}_{a \in A^p}$ of measurable mappings

$$m_a : \Big(\prod_{b \in A^p} \mathcal{W}_b \times \mathcal{H}, \bigotimes_{b \in A^p} \mathfrak{W}_b \otimes \mathfrak{I}_a \Big) \to (\mathcal{U}_a, \mathfrak{U}_a) \; , \quad \forall a \in A^p$$

- an A-behavioral strategy is an A-mixed strategy $m^p = \{m_a\}_{a \in A^p}$ with the property that

$$m_a^{-1}(\mathfrak{U}_a) \subset \Big(\mathfrak{W}_a \otimes \bigotimes_{b \in A^p \setminus \{a\}} \{\emptyset, \mathcal{W}_b\} \otimes \mathfrak{I}_a \Big) \; , \quad \forall a \in A^p$$

- an A-pure strategy is an A-mixed strategy $m^p = \{m_a\}_{a \in A^p}$ with the property that

$$m_a^{-1}(\mathfrak{U}_a) \subset \bigotimes_{b \in A^p} \{\emptyset, \mathcal{W}_b\} \otimes \mathfrak{I}_a \; , \quad \forall a \in A^p$$

Mixed, behavioral and pure strategies "à la Aumann": interpretation

For the player $p \in P$,

- an A-mixed strategy is a family $m^p = \{m_a\}_{a \in A^p}$ such that, for any configuration $h \in \mathcal{H}$,

$$m_a(\cdot, h) : \Big(\prod_{b \in A^p} \mathcal{W}_b, \bigotimes_{b \in A^p} \mathfrak{W}_b, \bigotimes_{a \in A^p} \ell_a \Big) \to (\mathcal{U}_a, \mathfrak{U}_a) , \quad \forall a \in A^p$$

  is a random variable
- an A-behavioral strategy is an A-mixed strategy $m^p = \{m_a\}_{a \in A^p}$ with the property that, for any configuration $h \in \mathcal{H}$, the random variables $\{m_a(\cdot, h)\}_{a \in A^p}$ are independent
- an A-pure strategy is an A-mixed strategy $m^p = \{m_a\}_{a \in A^p}$ with the property that, for any configuration $h \in \mathcal{H}$, the random variables $\{m_a(\cdot, h)\}_{a \in A^p}$ are constant

A-pure strategies and pure W-strategies

If $m^p = \{m_a\}_{a \in A^p}$ is an A-mixed strategy, every mapping

$$m_a^{w^p} = m_a(w^p, \cdot) : (\mathcal{H}, \mathfrak{I}_a) \to (\mathcal{U}_a, \mathfrak{U}_a)$$

belongs to Λ_a — that is, is a pure W-strategy — for $a \in A^p$, and thus

$$\left\{m_a^{w^p}\right\}_{a \in A^p} = \{m_a(w^p, \cdot)\}_{a \in A^p} \in \Lambda^p = \prod_{a \in A^p} \Lambda_a$$

Outline of the presentation

Partial orderings

Partial orderings

We denote $[\![1,k]\!] = \{1,\ldots,k\}$ for $k \in \mathbb{N}^*$
We consider a focus player $p \in P$ and we suppose that
the set A^p of her executive agents is finite with cardinality $|A^p|$

Partial orderings

The sets of k-orderings of player p is

$$\Sigma_k^p = \left\{\kappa : [\![1,k]\!] \to A^p \,\middle|\, \kappa \text{ is an injection} \right\}, \quad \forall k \in [\![1,|A^p|]\!]$$

The set of orderings of player p, shortly set of p-orderings is

$$\Sigma^p = \bigcup_{k=1}^{|A^p|} \Sigma_k^p$$

Range, cardinality, last element, first elements

For any partial ordering $\kappa \in \Sigma^p$, we define
the range $\|\kappa\| \subset A^p$ of the ordering κ as the subset of agents

$$\|\kappa\| = \{\kappa(1), \ldots, \kappa(k)\} \subset A^p \ , \ \ \forall \kappa \in \Sigma^p_k$$

the cardinality $|\kappa| \in \mathbb{N}^*$ of the ordering κ as the integer

$$|\kappa| = k \in [\![1, |A^p|]\!] \ , \ \ \forall \kappa \in \Sigma^p_k$$

the last element $\kappa_\star \in A^p$ of the ordering κ as the agent

$$\kappa_\star = \kappa(k) \in A^p \ , \ \ \forall \kappa \in \Sigma^p_k$$

the first elements $\kappa_- \in \Sigma^p$ of the ordering κ to the first $k-1$ elements

$$\kappa_- = \kappa_{|\{1,\ldots,k-1\}} \in \Sigma^p_{k-1} \ , \ \ \forall \kappa \in \Sigma^p_k$$

Player p-configuration-orderings

The set of total orderings of player p, shortly total p-orderings, is

$$\Sigma^{p}_{|A^p|} = \left\{ \kappa : [\![1, |A^p|]\!] \to A^p \;\middle|\; \kappa \text{ is a bijection} \right\}$$

Player p-configuration-ordering

A (player) p-configuration-ordering is a mapping

$$\varphi : \underbrace{\mathcal{H}}_{\text{configurations}} \to \underbrace{\Sigma^{p}_{|A^p|}}_{\text{total } p\text{-orderings}}$$

Thus, with each configuration $h \in \mathcal{H}$,
one associates a total ordering $\varphi(h) \in \Sigma^{p}_{|A^p|}$
of the executive agents of player p

Configurations compatible with a partial ordering

- For any $k \in [\![1, |A^p|]\!]$, there is a canonical mapping ψ_k

$$\psi_k : \Sigma^p_{|A^p|} \to \Sigma^p_k \ , \quad \kappa \mapsto \kappa_{|[\![1,k]\!]}$$

  which is the restriction of any (total) p-ordering of A^p to $[\![1, k]\!]$
- The configurations that are compatible with a partial ordering $\kappa \in \Sigma^p_k$ belong to the set

$$\mathcal{H}^\varphi_\kappa = \left\{ h \in \mathcal{H} \,\middle|\, \underbrace{\psi_{|\kappa|}\big(\varphi(h)\big) = \kappa}_{\substack{\text{configuration } h \\ \text{is ordered by } \kappa}} \right\}$$

Perfect recall

Perfect recall (without mathematics)

A player satisfies perfect recall if each of her agents,
when called upon to move last at a given ordering,
remembers everything that his predecessors
— according to the ordering, and who belong to the player —
knew and did

Perfect recall

Perfect recall for a player

We say that a player $p \in P$ in a W-model satisfies perfect recall if there exists a p-configuration-ordering $\varphi : \mathcal{H} \to \Sigma^{|A^p|}$ such that

$$\forall \kappa \in \Sigma_k^p\,,\ \forall H \in \bigvee_{a \in \|\kappa_-\|} \mathfrak{U}_a \vee \mathfrak{I}_a \quad \text{then} \quad \mathcal{H}_\kappa^\varphi \cap H \in \mathfrak{I}_{\kappa_\star}$$

where $\mathfrak{U}_a$ has to be understood as $\{\emptyset, \Omega\} \otimes \bigotimes_{b \in A \setminus \{a\}} \{\emptyset, \mathcal{U}_b\} \otimes \mathfrak{U}_a$

- $\kappa_\star$ is the last agent of κ
- $\|\kappa\|$ is the range of agents of the player p in κ
- $\mathcal{H}_\kappa^\varphi \subset \mathcal{H}$ contains the configurations compatible with the partial ordering κ
- κ_- are the previous agents of κ
- $\|\kappa_-\|$ is the range of agents of the player p in κ_-

Perfect recall (with and without mathematics)

$$\overbrace{\mathfrak{I}_{\underbrace{\kappa_\star}_{\text{last}}}}^{\substack{\text{what agent } \kappa_\star \\ \text{knows}}} \ni \overbrace{\mathcal{H}^{\varphi}_{\kappa}}^{\substack{\text{when called upon} \\ \text{to move last} \\ \text{at a given ordering}}} \cap \overbrace{\bigvee_{\underbrace{a \in \|\kappa_-\|}_{\substack{\text{predecessor} \\ \text{who belongs} \\ \text{to the player}}}} \underbrace{\mathfrak{U}_a}_{\text{did}} \vee \underbrace{\mathfrak{I}_a}_{\text{knew}}}^{\substack{\text{is everything that} \\ \text{his predecessors —} \\ \text{according to the ordering,} \\ \text{and who belong to the player —} \\ \text{did and knew}}}$$

where $\mathfrak{U}_a$ has to be understood as $\{\emptyset, \Omega\} \otimes \bigotimes_{b \in A \setminus \{a\}} \{\emptyset, \mathcal{U}_b\} \otimes \mathfrak{U}_a$

Outline of the presentation

Kuhn's Equivalence Theorem

When a player satisfies perfect recall, for any mixed strategy, there is an equivalent behavioral strategy (and the converse)

- Tree extensive form (finite action sets) [Kuhn, 1953]
  Harold W. Kuhn.
  Extensive games and the problem of information, 1953
- Extensive form (infinite action sets) [Aumann, 1964]
  Robert Aumann.
  Mixed and behavior strategies in infinite extensive games, 1964
- Product form (infinite action sets)
  [Heymann, De Lara, and Chancelier, 2022]
  Benjamin Heymann, Michel De Lara, Jean-Philippe Chancelier.
  Kuhn's Equivalence Theorem for Games in Product Form, 2022

Kuhn's Equivalence Theorem

Theorem (Heymann-De Lara-Chancelier)

We consider a playable W-model, a focus player $p \in P$ and additional technical assumptions
Then, the two following assertions are equivalent

1. *The player $p \in P$ satisfies perfect recall*
2. *For any A-mixed strategy $\overline{m}^{-p} = \{\overline{m}_a\}_{a \in A^{-p}}$ of the other players and for any A-mixed strategy $m^p = \{m_a\}_{a \in A^p}$, of the player p, there exists an A-behavioral strategy $m'^p = \{m'_a\}_{a \in A^p}$ such that*

$$\mathbb{Q}^{\omega}_{(\overline{m}^{-p}, m^p)} = \mathbb{Q}^{\omega}_{(\overline{m}^{-p}, m'^p)} \ , \quad \forall \omega \in \Omega$$

where $\mathbb{Q}^{\omega}_{(m^{-p}, m^p)}$ is the probability on the space $\big(\prod_{b \in A} \mathcal{U}_b, \bigotimes_{b \in A} \mathfrak{U}_b\big)$ defined as follows

Pushforward probability

$$\mathbb{Q}^{\omega}_{(m^{-p},m^{p})} = \Big(\bigotimes_{p\in P} \ell^{p}\Big) \circ \Big(M(\omega, m^{\cdot})\Big)^{-1} \in \Delta\Big(\prod_{b\in A} \mathcal{U}_b\Big)$$

is the pushforward probability, on the space $\big(\prod_{b\in A} \mathcal{U}_b, \bigotimes_{b\in A} \mathfrak{U}_b\big)$

of the product probability distribution $\bigotimes_{p\in P} \ell^{p}$

on $\big(\prod_{p\in P} \mathcal{W}^{p}, \bigotimes_{p\in P} \mathfrak{W}^{p}\big)$

by the composition of mappings

$$\prod_{p\in P} \mathcal{W}^{p} \to \Lambda \to \prod_{b\in A} \mathcal{U}_b$$

$$w \mapsto m^{w} \mapsto M_{m^{w}}(\omega)$$

where $S_{\lambda}(\omega) = \big(\omega, M_{\lambda}(\omega)\big)$

Outline of the presentation

Players can be endowed with objective functions and beliefs

Every player $p \in P$ has

- a team of executive agents

$$A^p \subset A$$

where $(A^p)_{p \in P}$ forms a partition of the set A of agents

- a criterion (objective function)

$$j^p : \mathcal{H} \to \mathbb{R} \quad (\text{or } \overline{\mathbb{R}})$$

a $\mathfrak{H}$-measurable function over the configuration space $\mathcal{H}$

- a belief

$$\beta^p : \mathfrak{F} \to [0,1]$$

a probability distribution over the states of Nature $(\Omega, \mathfrak{F})$

Game in product form

Game in product form

A game in product form is a W-model

- with a partition of the set of agents,
  whose atoms are the players
- where each player is endowed with
  - a preference relation on outcomes
    (configurations, probability distributions on configurations, etc.)
  - a belief on Nature
    (or, more generally, a risk measure)

Potential of W-models and W-games

W-models and W-games cover

- deterministic games (with finite or measurable action sets)
- deterministic dynamic games (countable time span)
- Bayesian games
- stochastic dynamic games (countable time span)
- games in Kuhn extensive form (countable time span)

For games with continuous time span,
the W-model has to be adapted (configuration-orderings)

Research questions

- Define a Nash equilibrium (doable from the normal form)
- How do we define a W-subgame?
  What is the relation with subsystems?
- How does the notion of subgame perfect equilibrium translate within this framework?
- When do we have a generalized backward induction mechanism?
- Target applications in nonsequential games, games on networks,
  distributed games in computer science,
  decentralized (energy) systems

What comes next?

- Classification of information structures
- Causality
  - as an ingredient for playability
  - as a bridge with tree models
    (H. Kuhn [Kuhn, 1953], C. Alós-Ferrer and K. Ritzberger [Alós-Ferrer and Ritzberger, 2016])
- Backward induction

Outline of the presentation

Handling subgroups of agents by means of cylindric extensions

Cylindric extension of a subgroup of agents

For any subset $B \subset A$ of agents, we define

$$\mathfrak{H}_B = \mathfrak{F} \otimes \bigotimes_{b \in B} \mathfrak{U}_b \otimes \bigotimes_{a \notin B} \{\emptyset, \mathcal{U}_a\}$$

$$\mathfrak{U}_B = \bigotimes_{b \in B} \mathfrak{U}_b \otimes \bigotimes_{a \notin B} \{\emptyset, \mathcal{U}_a\} \subset \bigotimes_{a \in A} \mathfrak{U}_a$$

$$\mathfrak{H}_B = \mathfrak{F} \otimes \mathfrak{U}_B = \mathfrak{F} \otimes \bigotimes_{b \in B} \mathfrak{U}_b \otimes \bigotimes_{a \notin B} \{\emptyset, \mathcal{U}_a\} \subset \mathfrak{H}$$

$$(\text{when } B \neq \emptyset) \qquad h_B = \{h_b\}_{b \in B} \in \prod_{b \in B} \mathcal{U}_b \ , \quad \forall h \in \mathcal{H}$$

$$(\text{when } B \neq \emptyset) \qquad \lambda_B = \{\lambda_b\}_{b \in B} \in \prod_{b \in B} \Lambda_b \ , \quad \forall \lambda \in \Lambda$$

Typology of W-models

- Static team
- Station
- Sequential W-model
- Partially nested W-model
- Quasiclassical W-model
- Classical W-model
- Hierarchical W-model
- Parallel coordinated W-model
- Causal W-model

Outline of the presentation

Precedence relation P

What are the agents whose actions might affect the information of a focal agent?

- The precedence binary relation identifies the agents whose actions affect the observations of a given agent
- For a given agent $a \in A$, we consider the set $\mathcal{P}_a \subset 2^A$ of subsets $C \subset A$ of agents such that

$$\mathfrak{I}_a \subset \mathfrak{F} \otimes \mathfrak{U}_C = \mathfrak{F} \otimes \bigotimes_{c \in C} \mathfrak{U}_c \otimes \bigotimes_{b \notin C} \{\emptyset, \mathcal{U}_b\}$$

- Any subset $C \in \mathcal{P}_a$ contains agents whose actions affect the information $\mathfrak{I}_a$ available to the focal agent a
- As the set $\mathcal{P}_a$ is stable under intersection, the following definition makes sense

The precedence relation P

Precedence relation P

1. For any agent $a \in A$, we define the subset $\mathrm{P}a \subset A$ of agents as the intersection of subsets $C \subset A$ of agents such that

$$\mathfrak{I}_a \subset \mathfrak{F} \otimes \mathfrak{U}_C$$

2. We define a precedence binary relation P on A by

$$b\,\mathrm{P}\,a \iff b \in \mathrm{P}a$$

and we say that b is a predecessor of a (or a precedent of a)

In other words, the actions of any predecessor of an agent affect the information of this agent: any agent is influenced by its predecessors (when they exist, because $\mathrm{P}a$ might be empty)

Characterization of the predecessors of a focal agent

- For any agent $a \in A$, the subset $\mathrm{P}a$ of agents is the smallest subset $C \subset A$ such that

$$\mathfrak{I}_a \subset \mathfrak{F} \otimes \mathfrak{U}_C$$

- In other words, $\mathrm{P}a$ is characterized by

$$\mathfrak{I}_a \subset \mathfrak{F} \otimes \mathfrak{U}_{\mathrm{P}a} \text{ and } (\mathfrak{I}_a \subset \mathfrak{F} \otimes \mathfrak{U}_C \Rightarrow \mathrm{P}a \subset C)$$

Potential for signaling

- Whenever $\mathrm{P}a \neq \emptyset$, there is a potential for signaling, that is, for information transmission
- Indeed, any agent b in $\mathrm{P}a$ influences the information $\mathfrak{I}_a$ upon which agent a bases its actions
- Therefore, whenever agent b is a predecessor of agent a, the former can, by means of its actions, send a signal to the latter
- In case $\mathrm{P}a = \emptyset$, the actions of agent a depend, at most, on the state of Nature, and there is no room for signaling

Iterated predecessors

- Let $C \subset A$ be a subset of agents
- We introduce the following subsets of agents

$$\mathrm{P}C = \bigcup_{b \in C} \mathrm{P}b \,, \;\; \mathrm{P}^0 C = C \;\text{ and }\; \mathrm{P}^{n+1} C = \mathrm{P}\mathrm{P}^n C \,, \;\; \forall n \in \mathbb{N}$$

  that correspond to the iterated predecessors of the agents in C
- When C is a singleton $\{a\}$, we denote $\mathrm{P}^n a$ for $\mathrm{P}^n\{a\}$

Successor relation P^{-1}

Successor relation P^{-1}

The converse of the precedence relation P
is the successor relation P^{-1} characterized by

$$b \, \mathrm{P}^{-1} \, a \iff a \, \mathrm{P} \, b$$

Quite naturally, b is a successor of a iff a is a predecessor of b

Subsystem relation S

A subsystem is a subset of agents closed w.r.t. information

We define the information $\mathfrak{I}_C \subset \mathfrak{H}$ of the subset $C \subset A$ of agents by

$$\mathfrak{I}_C = \bigvee_{b \in C} \mathfrak{I}_b$$

that is, the smallest σ-fields that contains all the σ-fields $\mathfrak{I}_b$, for $b \in C$

Subsystem

A nonempty subset C of agents in A is a subsystem if the information field $\mathfrak{I}_C$ at most depends on the actions of the agents in C, that is,

$$\mathfrak{I}_C \subset \mathfrak{F} \otimes \mathfrak{U}_C$$

Thus, the information received by agents in C depends upon states of Nature and actions of members of C only

Generated subsystem

- The subsystem $\overline{C}$ generated by a nonempty subset C of agents in A is the intersection of all subsystems that contain C, that is, the smallest subsystem that contain C
- A subset $C \subset A$ is a subsystem iff it coincides with the generated subsystem, that is,

$$C \text{ is a subsystem} \quad \Longleftrightarrow \quad C = \overline{C}$$

The subsystem relation S

Subsystem relation S

We define the subsystem relation S on A by

$$b \,\mathrm{S}\, a \iff \overline{\{b\}} \subset \overline{\{a\}} \,, \quad \forall (a,b) \in A^2$$

Therefore, $b \,\mathrm{S}\, a$ means that

- agent b belongs to the subsystem generated by agent a
- or, equivalently, that the subsystem generated by agent a contains the one generated by agent b

The subsystem relation S is a preorder

Proposition ([Witsenhausen, 1975a])

The subsystem relation S is a preorder,
namely it is reflexive and transitive

Proposition

1. *A subset $C \subset A$ is a subsystem iff $\mathrm{P}C \subset C$, that is, iff the predecessors of agents in C belong to C:*

$$C \text{ is a subsystem} \iff \overline{C} = C \iff \mathrm{P}C \subset C$$

2. *For any agent $a \in A$, the subsystem generated by agent a is the union of $\{a\}$ and of all its iterated predecessors, that is,*

$$\overline{\{a\}} = \bigcup_{n \in \mathbb{N}} \mathrm{P}^n a$$

Information-memory relation M

The information-memory relation M

Information-memory relation M

1. With any agent $a \in A$, we associate
   the subset $\mathrm{M}a$ of agents who pass on their information to a, that is,
   $$\mathrm{M}a = \left\{ b \in A \,\middle|\, \mathfrak{I}_b \subset \mathfrak{I}_a \right\}$$
2. We define an information memory binary relation M on A by
   $$b \,\mathrm{M}\, a \iff b \in \mathrm{M}a \iff \mathfrak{I}_b \subset \mathfrak{I}_a \,, \quad \forall (a, b) \in A^2$$

- When $b \,\mathrm{M}\, a$, we say that
  agent b information is remembered by or passed on to agent a,
  or that agent b is an informer of agent a, or that
  the information of agent b is embedded in the information of agent a
- When agent b belongs to $\mathrm{M}a$,
  the information available to b is also available to agent a

The information memory relation M is a preorder

Proposition

The information memory relation M *is a preorder,*
namely M *is reflexive and transitive*

Action-memory relation D

The action-memory relation D

We recall that the action subfield $\mathfrak{D}_b$ is

$$\mathfrak{D}_b = \{\emptyset, \Omega\} \otimes \mathfrak{U}_b \otimes \bigotimes_{c \neq b} \{\emptyset, \mathcal{U}_c\}$$

Action-memory relation [Carpentier, Chancelier, Cohen, and De Lara, 2015]

1. With any agent $a \in A$, we associate

$$\mathrm{D}a = \left\{ b \in A \,\middle|\, \mathfrak{D}_b \subset \mathfrak{I}_a \right\}$$

   the subset of agents b whose action is passed on to a

2. We define a action-memory binary relation D on A by

$$b \,\mathrm{D}\, a \iff b \in \mathrm{D}a \iff \mathfrak{D}_b \subset \mathfrak{I}_a \,, \quad \forall (a, b) \in A^2$$

$\mathsf{D} \subset \mathsf{P}$

From

$$\mathfrak{D}_{\mathsf{D}a} = \{\emptyset, \Omega\} \otimes \mathfrak{U}_{\mathsf{D}a} \subset \mathfrak{I}_a \subset \mathfrak{F} \otimes \mathfrak{U}_{\mathsf{P}a}$$

we conclude that

$$\mathsf{D}a \subset \mathsf{P}a \;, \quad \forall a \in A$$

or, equivalently, that

$$\mathsf{D} \subset \mathsf{P}$$

- When $b \, \mathsf{D} \, a$, we say that the action of agent b is remembered by or passed on to agent a, or that the action of agent b is embedded in the information of agent a
- If $b \, \mathsf{D} \, a$, the action made by agent b is passed on to agent a and, by the fact that $\mathsf{D} \subset \mathsf{P}$, b is a predecessor of a
- However, the agent b can be a predecessor of a, but its influence may happen without passing on its action to a

What land have we covered?
What comes next?

With these four relations

- precedence relation P
- subsystem relation S
- information-memory relation M
- action-memory relation D

we can provide a typology of systems (W-models), expanded from [Witsenhausen, 1975a]

Outline of the presentation

Static team

Static team

Static team [Witsenhausen, 1975a]

A static team is a subset C of A such that $\mathrm{P}C = \emptyset$, that is, agents in C have no predecessors

- A static team necessarily is a subset of the largest static team defined by

$$A_0 = \{a \in A \mid \mathfrak{I}_a \subset \mathfrak{F} \otimes \bigotimes_{b \in A} \{\emptyset, \mathcal{U}_b\}\} = \{a \in A \mid \mathrm{P}a = \emptyset\}$$

- When the whole set A of agents is a static team, any agent $a \in A$ has no predecessor: $\mathrm{P}a = \emptyset, \forall a \in A$
- A system is static if the set A of agents is a static team

Static team made of two agents

Two agents a, b form a static team iff

$$\mathfrak{I}_a \subset \mathfrak{F} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \{\emptyset, \mathcal{U}_b\} \ , \ \ \mathfrak{I}_b \subset \mathfrak{F} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \{\emptyset, \mathcal{U}_b\}$$

There is no interdependence between the actions of the agents, just a dependence upon states of Nature

Station and sequential system

Station

A station is a subset of agents such that the set of information fields of these agents is totally ordered under inclusion (i.e., nested)

Station [Witsenhausen, 1975a]

A subset C of agents in A is a station

- iff the information-memory relation M induces a total order on C (i.e., it consists of a chain of length $m = \mathrm{card}(C)$)
- iff there exists an ordering $(a_1, \ldots, a_m)$ of C such that

$$\mathfrak{I}_{a_1} \subset \cdots \subset \mathfrak{I}_{a_k} \subset \mathfrak{I}_{a_{k+1}} \subset \cdots \subset \mathfrak{I}_{a_m}$$

  or, equivalently, that

$$a_{k-1} \in \mathrm{M}a_k \ , \ \ \forall k = 2, \ldots, m$$

In other words, in a station,
the antecessor $k-1$ is necessarily an informer of k

A station with two agents

$$\mathfrak{I}_a = \{\emptyset, \Omega, \{\omega^1\}, \{\omega^2\}\} \times \{\emptyset, \mathcal{U}_a\} \otimes \{\emptyset, \mathcal{U}_b\}$$

$$\mathfrak{I}_b = \{\emptyset, \Omega, \{\omega^1\}, \{\omega^2\}\} \times \{\emptyset, \mathcal{U}_a, \{u_a^1\}, \{u_a^2\}\} \otimes \{\emptyset, \mathcal{U}_b\}$$

$\mathfrak{I}_a \subset \mathfrak{I}_b$ may be interpreted in different ways

- one may say that
  agent *a* communicates its own information to agent *b*.
- If agent *a* is an individual at time $t = 0$,
  while agent *b* is the same individual at time $t = 1$,
  one may say that the information is not forgotten with time
  (memory of past knowledge)

Sequential system

Sequential system [Witsenhausen, 1975a]

A system is sequential if there exists an ordering $(a_1, \ldots, a_{|A|})$ of A such that each agent a_k is influenced at most by the previous (former or antecessor) agents $a_1, \ldots, a_{k-1}$, that is,

$$\mathrm{P}a_1 = \emptyset \text{ and } \mathrm{P}a_k \subset \{a_1, \ldots, a_{k-1}\} \,, \ \forall k = 2, \ldots, |A|$$

In other words, in a sequential system, predecessors are necessarily antecessors

Example of sequential system with two agents

The set of agents $A = \{a, b\}$ with information fields given by

$$\mathfrak{I}_a = \mathfrak{F} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \{\emptyset, \mathcal{U}_b\} \ , \ \ \mathfrak{I}_b = \{\emptyset, \Omega\} \otimes \mathfrak{U}_a \otimes \{\emptyset, \mathcal{U}_b\}$$

forms a sequential system where

- agent a precedes agent b, because
  $\mathrm{P}a = \emptyset$ and $\mathrm{P}b = \{a\}$
- but $\mathfrak{I}_a$ and $\mathfrak{I}_b$ are not comparable:
  agent a observes only the state of Nature,
  whereas agent b observes only agent a's action

Example of sequential system with two agents

$$\mathfrak{I}_a = \{\emptyset, \Omega, \{\omega^1\}, \{\omega^2\}\} \times \{\emptyset, \mathcal{U}_a\} \otimes \{\emptyset, \mathcal{U}_b\}$$

$$\mathfrak{I}_b = \{\emptyset, \Omega, \{\omega^1\}, \{\omega^2\}\} \times \{\emptyset, \mathcal{U}_a, \{u_a^1\}, \{u_a^2\}\} \otimes \{\emptyset, \mathcal{U}_b\}$$

The system is sequential as

1. agent a observes the state of Nature
   and makes its action accordingly
2. agent b observes both agent a's action and the state of Nature
   and makes its action accordingly

Partially nested systems

Partially nested system

Partially nested system

A partially nested system is one for which
the precedence relation is included in the information-memory relation,
that is,

$$\mathrm{P} \subset \mathrm{M}$$

- In a partially nested system, if agent *a* is a predecessor of agent *b* — hence, *a* can influence *b* — then agent *b* knows what agent *a* knows
- In a partially nested system, any agent has access to the information of those agents who are its predecessors (and thus influence its own information)
- In other words, in a partially nested system, predecessors are necessarily informers

Quasiclassical system

Quasiclassical system [Witsenhausen, 1975a]

A system is quasiclassical

- iff it is sequential and partially nested
- iff there exists an ordering $(a_1, \ldots, a_{|A|})$ of A such that $\mathrm{P}a_1 = \emptyset$ and

$$\mathrm{P}a_k \subset \{a_1, \ldots, a_{k-1}\} \text{ and } \mathrm{P}a_k \subset \mathrm{M}a_k \ , \quad \forall k = 2, \ldots, |A|$$

In other words, in a quasiclassical system,
predecessors are necessarily antecessors and
predecessors are necessarily informers

Classical system

Classical system [Witsenhausen, 1975a]

A system is classical

- iff there exists an ordering $(a_1, \ldots, a_{|A|})$ of A for which it is both sequential and such that $\mathfrak{I}_{a_k} \subset \mathfrak{I}_{a_{k+1}}$ for $k = 1, \ldots, n-1$ (station property)
- iff there exists an ordering $(a_1, \ldots, a_{|A|})$ of A such that $\mathrm{P}a_1 = \emptyset$ and for $k = 2, \ldots, |A|$,

$$\mathrm{P}a_k \subset \{a_1, \ldots, a_{k-1}\} \subset \{a_1, \ldots, a_{k-1}, a_k\} \subset \mathrm{M}a_k$$

In other words, in a classical system,
predecessors are necessarily antecessors and
antecessors are necessarily informers

- A classical system is necessarily partially nested because $\mathrm{P}a_k \subset \mathrm{M}a_k$ for $k = 1, \ldots, n$
- Hence, a classical system is quasiclassical

A classical system with two agents

- The set of agents $A = \{a, b\}$ with information fields given by

$$\mathfrak{I}_a = \mathfrak{F} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \mathfrak{U}_b \ , \ \ \mathfrak{I}_b = \mathfrak{F} \otimes \{\emptyset, \mathcal{U}_a\} \otimes \{\emptyset, \mathcal{U}_b\}$$

  forms a classical system
- Indeed, first, the system is sequential as b precedes a because $\mathrm{P}b = \emptyset$ and $b \in \mathrm{P}a$:
  - agent b observes the state of Nature and makes its action accordingly
  - agent a observes both agent b's decision and the state of Nature and makes its action based on that information
- Second, one has that $\mathfrak{I}_b \subset \mathfrak{I}_a$ ($b \in \mathrm{M}a$): agent b communicates its own information to agent a

Subsystem inheritence

Theorem ([Witsenhausen, 1975a])

Any of the properties static team, sequentiality, quasiclassicality, classicality, causality of a system is shared by all its subsystems

Hierarchical and parallel systems

Hierarchical systems

Hierarchical system (based on Ho-Chu)

A system is hierarchical when the set A of agents can be partitioned in (nonempty) disjoint sets $A_0, \ldots, A_K$ as follows

$$A_0 = \{a \in A \mid \mathrm{P}a = \emptyset\}$$
$$A_1 = \{a \in A \mid a \notin A_0 \text{ and } \mathrm{P}a \subset A_0\}$$
$$A_{k+1} = \{a \in A \mid a \notin \bigcup_{i=1}^{k} A_j \text{ and } \mathrm{P}a \subset \bigcup_{i=1}^{k} A_j\}$$

for $k = 2, \ldots, K$

Agents in A_0 form the largest static team ($\mathrm{P}A_0 = \emptyset$)

Parallel coordinated systems

Parallel coordinated system

A system is parallel coordinated
when the set A of agents can be partitioned
in (nonempty) disjoint sets $A_0, A_1, \ldots, A_K$ as follows

- A_0 is the largest static team ($\mathrm{P}A_0 = \emptyset$)
- every subset $A_1 \cup A_0, \ldots, A_K \cup A_0$ is a subsystem

Outline of the presentation

Causal configuration orderings: "Alice and Bob"

- no Nature, two agents a (Alice) and b (Bob)
- two possible actions each $\mathcal{U}_a = \{u_a^+, u_a^-\}$, $\mathcal{U}_b = \{u_b^+, u_b^-\}$
- configuration space $\mathcal{H} = \{u_a^+, u_a^-\} \times \{u_b^+, u_b^-\}$ (4 elements)
- set of total orderings (2 elements: a plays first or b plays first)

$$\Sigma^2 = \left\{ (ab) = \begin{pmatrix} \sigma:\{1,2\}\to\{a,b\} \\ \sigma(1)=a \\ \sigma(2)=b \end{pmatrix}, (ba) = \begin{pmatrix} \sigma:\{1,2\}\to\{a,b\} \\ \sigma(1)=b \\ \sigma(2)=a \end{pmatrix} \right\}$$

Consider the following information structure:

- $\mathfrak{I}_b = \{\emptyset, \{u_a^+, u_a^-\}\} \otimes \{\emptyset, \{u_b^+, u_b^-\}\}$
  Bob knows nothing
- $\mathfrak{I}_a = \{\emptyset, \{u_a^+, u_a^-\}\} \otimes \{\emptyset, \{u_b^+\}, \{u_b^-\}, \{u_b^+, u_b^-\}\}$
  Alice knows what Bob does

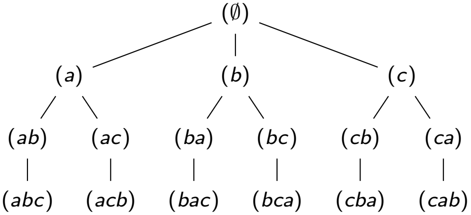
We say that the constant configuration-ordering

- $\varphi(h) = (ab)$, for all $h \in \mathcal{H}$ (a plays first) is noncausal
- $\varphi(h) = (ba)$, for all $h \in \mathcal{H}$ (b plays first) is causal

The tree of partial orderings

There is a natural order on the set $\Sigma = \bigcup_{k \in \mathbb{N}^*} \Sigma^k$ of partial orderings

$$(\emptyset) \succeq (a) \succeq (ab) \succeq (abc)$$



Configuration-orderings

The set of total orderings is

$$\Sigma^{|A|} = \left\{ \kappa : [\![1, |A|]\!] \to A \,\middle|\, \kappa \text{ is a bijection} \right\}$$

Configuration-ordering [Witsenhausen, 1975a]

A configuration-ordering is a mapping

$$\varphi : \underbrace{\mathcal{H}}_{\text{configurations}} \to \underbrace{\Sigma^{|A|}}_{\text{total orderings}}$$

$$\mathcal{H}^{\varphi}_{\kappa} = \left\{ h \in \mathcal{H} \,\middle|\, \underbrace{\psi_{|\kappa|}\big(\varphi(h)\big) = \kappa}_{\substack{\text{configuration } h \\ \text{is ordered by } \kappa}} \right\}$$

Causality (nonanticipativity)

Causal W-model [Witsenhausen, 1975a]

A W-model is causal if there exists (at least one) configuration-ordering $\varphi : \mathcal{H} \to \Sigma^{|A|}$ with the property that, for any $\kappa = (\kappa_-, \kappa_\star) \in \Sigma$

$$\forall G \in \mathfrak{I}_{\kappa_\star} \quad \text{then} \quad \overbrace{\underbrace{\mathcal{H}^{\varphi}_{\kappa}}_{\substack{\text{agents} \\ \text{ordered by } \kappa}} \cap G}^{\substack{\text{information of} \\ \text{the last agent } \kappa_\star}} \in \overbrace{\mathfrak{F} \otimes \mathfrak{U}_{\|\kappa_-\|}}^{\substack{\text{depends at most} \\ \text{on actions of agents} \\ \text{having lower rank}}}$$

We also say that $\varphi : \mathcal{H} \to \Sigma^{|A|}$ is a causal configuration-ordering

Information comes first,
(possible) causal ordering comes second

If a W-model has no nonempty static team, it cannot be causal

A causal but nonsequential system

- We consider a set of agents $A = \{a, b\}$ with

$$\mathcal{U}_a = \{u_a^1, u_a^2\} \ , \ \mathcal{U}_b = \{u_b^1, u_b^2\} \ , \ \Omega = \{\omega^1, \omega^2\}$$

- The agents' information fields are given by

$$\mathfrak{I}_a = \sigma(\{u_a^1, u_a^2\} \times \{u_b^1, u_b^2\} \times \{\omega^2\}, \{u_a^1, u_a^2\} \times \{u_b^1\} \times \{\omega^1\})$$
$$\mathfrak{I}_b = \sigma(\{u_a^1, u_a^2\} \times \{u_b^1, u_b^2\} \times \{\omega^1\}, \{u_a^1\} \times \{u_b^1, u_b^2\} \times \{\omega^2\})$$

- When the state of Nature is ω^2, agent a only sees ω^2, whereas agent b sees ω^2 and the action of a: thus a acts first, then b
- The reverse holds true when the state of Nature is ω^1
- A non constant configuration-ordering mapping $\varphi : \mathcal{H} \to \{(a, b), (b, a)\}$ is defined by (for any couple (u_a, u_b))

$$\varphi\Big((u_a, u_b, \omega^2)\Big) = (a, b) \text{ and } \varphi\Big((u_a, u_b, \omega^1)\Big) = (b, a)$$

- The system is causal but not sequential

Causality implies playability

Proposition [Witsenhausen, 1971]

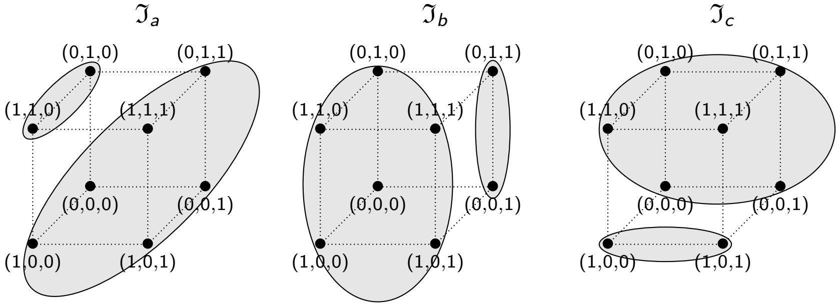
Causality implies (recursive) playability
with a measurable solution map

$$S_\lambda = \widetilde{S}_\lambda^{(|A|)} \circ \cdots \circ \widetilde{S}_\lambda^{(1)} \circ \widetilde{S}_\lambda^{(0)}$$

Kuhn's extensive form of a game encapsulates causality in the tree

Playable noncausal example [Witsenhausen, 1971]

- No Nature, $A = \{a, b, c\}$, $\mathcal{U}_a = \mathcal{U}_b = \mathcal{U}_c = \{0, 1\}$
- Set of configurations $\mathcal{H} = \{0, 1\}^3$, and information fields $\mathfrak{I}_a = \sigma(u_b(1 - u_c))$, $\mathfrak{I}_b = \sigma(u_c(1 - u_a))$, $\mathfrak{I}_c = \sigma(u_a(1 - u_b))$
- The "game" can be played but…cannot be started (no first agent)



Outline of the presentation

Optimization problem

- We consider a playable W-model
- We suppose given a measurable criterion (objective function)

$$j : \Omega \times \mathcal{U}_A \to \overline{\mathbb{R}}$$

- We consider the optimization problem

$$\min_{\lambda_A \in \Lambda_A} \mathbb{E}_{\mathbb{P}}\big[j \circ S_\lambda\big]$$

where $\mathbb{P}$ is a probability on $(\Omega, \mathfrak{F})$

Subsystem and strategy decomposition

- We consider a subsystem $B \subset A$ of agents
- We can decompose any W-strategy $\lambda \in \Lambda_A$ as

$$\lambda = (\lambda_B, \lambda_{A \setminus B}) \in \Lambda_B \times \Lambda_{A \setminus B}$$

where

$$\lambda_B \colon \underbrace{\Omega \times \overbrace{\cancel{\mathcal{U}_{A \setminus B}}}^{B \text{ subsystem}} \times \mathcal{U}_B}_{\equiv \, \Omega \times \mathcal{U}_B} \to \mathcal{U}_B$$

$$\lambda_{A \setminus B} \colon \Omega \times \mathcal{U}_{A \setminus B} \times \mathcal{U}_B \to \mathcal{U}_{A \setminus B}$$

- For instance, in stochastic control, with time either discrete, $t \in \mathbb{N} = A$, or continuous, $t \in [0, +\infty[= A$, the first agents $[0, t]$ form a subsystem of A, because they cannot make decisions based on the future agent's decisions

Subsystem and co-cycle property of the solution map

- We consider a subsystem $B \subset A$ of agents in a playable W-model
- For any admissible strategy $\lambda \in \Lambda_A$, there exist two partial solution maps

$$S_{\lambda_B} : \Omega \to \Omega \times \mathcal{U}_B$$

$$S_{\lambda_{A \setminus B}} : \Omega \times \mathcal{U}_B \to \Omega \times \mathcal{U}_{A \setminus B} \times \mathcal{U}_B$$

such that the solution map S_λ has the following co-cycle property

$$S_\lambda = S_{(\lambda_B, \lambda_{A \setminus B})} = S_{\lambda_{A \setminus B}} \circ S_{\lambda_B}$$

$$S_{(\lambda_B, \lambda_{A \setminus B})} : \Omega \xrightarrow{S_{\lambda_B}} \Omega \times \mathcal{U}_B \xrightarrow{S_{\lambda_{A \setminus B}}} \Omega \times \mathcal{U}_{A \setminus B} \times \mathcal{U}_B$$

Co-cycle property and dynamical equation in stochastic control

- Finite agents $A = [\![0, T]\!]$
- Decision of agent t is taken in a set $\mathcal{U}_t$
- Filtration $\{\mathfrak{F}_t\}_{t \in [\![0,T]\!]}$ of the sample space $(\Omega, \mathfrak{F})$
- Information of (nonanticipative) agent t is modeled as

$$\mathfrak{I}_t \subset \mathfrak{F}_t \otimes \bigotimes_{r<t} \mathfrak{U}_r \otimes \bigotimes_{s \geq t} \{\emptyset, \mathcal{U}_s\}$$

- "Dynamical equation"

$$\overbrace{S_{\lambda_{[\![0,T]\!]}}(\omega)}^{\text{final state}} = \overbrace{S_{\lambda_{[\![0,T]\!] \setminus [\![0,t]\!]}}}^{\text{dynamics}} \Big(\overbrace{S_{\lambda_{[\![0,t]\!]}}(\omega)}^{\text{current state}} \Big)$$

$$\underbrace{S_{\lambda_A}(\omega)}_{\text{"final state"}} = \underbrace{S_{\lambda_{A \setminus B}}}_{\text{"dynamics"}} \Big(\underbrace{S_{\lambda_B}(\omega)}_{\text{"current state"}} \Big)$$

Strategy independence of conditional expectation (SICE)

In the discrete case, Witsenhausen provides sufficient conditions, on the information structure, of the type

$$\mathfrak{I}_B \vee \mathfrak{U}_B \subset \mathfrak{I}_{A \setminus B}$$

to obtain SICE [Witsenhausen, 1975b]

Assumption SICE

1. There exists a probability $\mathbb{Q}_B$ on $\Omega \times \mathcal{U}_B$ such that

$$\mathbb{P} \circ S_{\lambda_B}^{-1} = T_{\lambda_B} \mathbb{Q}_B \text{ with } \mathbb{E}_{\mathbb{Q}_B}\big[T_{\lambda_B} \mid \mathfrak{I}_B \big] > 0 \ , \ \ \forall \lambda_B \in \Lambda_B$$

2. There exists a probability $\mathbb{Q}_A$ on $\Omega \times \mathcal{U}_A = \Omega \times \mathcal{U}_{A \setminus B} \times \mathcal{U}_B$ such that

$$\mathbb{P} \circ S_{\lambda_A}^{-1} = T_{\lambda_A} \mathbb{Q}_A \text{ with } \mathbb{E}_{\mathbb{Q}_A}\big[T_{\lambda_A} \mid \mathfrak{I}_A \big] > 0 \ , \ \ \forall \lambda_A \in \Lambda_A$$

Dynamic programming equation

$$V_A = \mathbb{E}_{\mathbb{Q}_A}\left[j \mid \mathfrak{J}_A\right]$$

$$V_B = \min_{\lambda_{A\setminus B} \in \Lambda_{A\setminus B}} \mathbb{E}_{\mathbb{Q}_B}\left[V_A \circ S_{\lambda_{A\setminus B}} \mid \mathfrak{J}_B\right]$$

$$V_\emptyset = \min_{\lambda_B \in \Lambda_B} \mathbb{E}_{\mathbb{P}}\left[V_B \circ S_{\lambda_B}\right]$$

State reduction

[Carpentier, Chancelier, De Lara, Martin, and Rigaut, 2023]

$$
\begin{tikzcd}
\mathcal{H} = (\Omega \times \mathcal{U}_{A \setminus B}) \times \mathcal{U}_B \arrow[r, "\mathit{Identity}"] \arrow[d, "\theta_A"] \arrow[d, "\mathit{Identity}"] & \mathcal{H} \arrow[d, "\theta_B"] \\
\mathcal{X}_A \times \mathcal{U}_B \arrow[r, "\mathit{Dynamics}"] & \mathcal{X}_B
\end{tikzcd}
$$

Two reduction mappings $\qquad \theta_A : \mathcal{H} \to \mathcal{X}_A \ , \ \ \theta_B : \mathcal{H} \to \mathcal{X}_B$

State reduction
and classical dynamic programming equation

Under assumptions of

- factorization

$$V_A = \mathbb{E}_{\mathbb{Q}_A}\big[j \mid \mathfrak{I}_A\big] = \widetilde{V}_A \circ \theta_A \text{ where } \widetilde{V}_A : \mathcal{X}_A \to \overline{\mathbb{R}}$$

- dynamics

$$\theta_A \circ S_{\lambda_A} = \mathrm{dynamics}\big(\theta_B \circ S_B, \lambda_{A \setminus B}\big)$$

we recover the classical dynamic programming equation

Conclusion

- a rich language
- a lot of open questions, and a lot of things not yet properly defined
- we are looking for feedback

Thank you :-)

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