Spatial Decomposition/Coordination Methods for Stochastic Optimal Control Problems

Practical aspects and theoretical questions

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Large scale storage systems stand as powerful motivation

- The Optimization and Systems team was created in 2000 at École des Ponts ParisTech with emphasis on stochastic optimization.

- Since 2011, we witness a growing demand from (small and large) energy firms for stochastic optimization, fueled by a deep and fast transformation of power systems.
Something is changing in power systems

- Renewable energies penetration, telecommunication technologies and markets remold power systems and challenge optimization
- More renewable energies $\rightarrow$ more unpredictability + more variability $\rightarrow$
  - more storage $\rightarrow$ more dynamic optimization, optimal control
  - more stochastic optimization
- hence, stochastic optimal control (SOC)
Lecture outline

1. Decomposition and coordination
   - A bird’s eye view of decomposition methods
   - A brief insight into Progressive Hedging
   - Spatial decomposition methods in the deterministic case
   - The stochastic case raises specific obstacles

2. Dual approximate dynamic programming (DADP)
   - Problem statement
   - DADP principle and implementation
   - Numerical results on a small size problem

3. Theoretical questions
   - Existence of a saddle point
   - Convergence of the Uzawa algorithm
   - Convergence w.r.t. information

4. Summary and research agenda
A long-term effort in our group


Let us fix problem and notations

\[
\min_{u, x} \mathbb{E} \left( \sum_{i=1}^{N} \left( \sum_{t=0}^{T-1} L_t^i(x_t^i, u_t^i, w_{t+1}) + K_i^i(x_T^i) \right) \right)
\]

subject to dynamics constraints

\[
x_{t+1}^i = f_t^i(x_t^i, u_t^i, w_{t+1}), \quad x_0^i = f_{-1}^{-1}(w_0)
\]

subject to measurability constraints on the control \(u_t^i\)

\[
u_t^i \leq \sigma(w_0, \ldots, w_t) \iff u_t^i = \mathbb{E} \left( u_t^i \middle| w_0, \ldots, w_t \right)
\]

and to instantaneous coupling constraints

\[
\sum_{i=1}^{N} \theta_{t}^i(x_t^i, u_t^i) = 0
\]

(The letter \(u\) stands for the Russian word for control: \textit{upravlenie})
Decomposition and coordination
A bird’s eye view of decomposition methods

Couplings for stochastic problems

\[
\min \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L^i_t(x^i_t, u^i_t, w_{t+1})
\]

M. De Lara (École des Ponts ParisTech)
Couplings for stochastic problems: in time

\[
\min \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L_{t}^{i}(x_{t}^{i}, u_{t}^{i}, w_{t+1})
\]

s.t. \( x_{t+1}^{i} = f_{t}^{i}(x_{t}^{i}, u_{t}^{i}, w_{t+1}) \)
Couplings for stochastic problems: in uncertainty

\[
\begin{align*}
\min & \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L^i_t(x^i_t, u^i_t, w_{t+1}) \\
\text{s.t.} & \quad x^{i+1}_{t} = f^i_t(x^i_t, u^i_t, w_{t+1}) \\
& \quad u^i_t = E\left(u^i_t \mid w_0, \ldots, w_t\right)
\end{align*}
\]
Couplings for stochastic problems: in space

\[
\min \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L_{t}^{i}(x_{t}^{i}, u_{t}^{i}, w_{t+1})
\]

\[
\text{s.t. } x_{t+1}^{i} = f_{t}^{i}(x_{t}^{i}, u_{t}^{i}, w_{t+1})
\]

\[
u_{t}^{i} = \mathbb{E} \left( u_{t}^{i} \mid w_{0}, \ldots, w_{t} \right)
\]

\[
\sum_{i} \Theta_{t}^{i}(x_{t}^{i}, u_{t}^{i}) = 0
\]
Can we decouple stochastic problems?

\[
\begin{align*}
\min & \sum_\omega \sum_i \sum_t \pi_\omega L_t^i(x_t^i, u_t^i, w_{t+1}) \\
\text{s.t.} & x_{t+1}^i = f_t^i(x_t^i, u_t^i, w_{t+1}) \\
& u_t^i = E \left( u_t^i \ \middle| \ w_0, \ldots, w_t \right) \\
& \sum_i \Theta_t^i(x_t^i, u_t^i) = 0
\end{align*}
\]
Decompositions for stochastic problems: in time

\[ \min \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L_{t}^{i}(x_{t}^{i}, u_{t}^{i}, w_{t+1}) \]

s.t. \[ x_{t+1}^{i} = f_{t}^{i}(x_{t}^{i}, u_{t}^{i}, w_{t+1}) \]

\[ u_{t}^{i} = \mathbb{E} \left( u_{t}^{i} \bigg| w_0, \ldots, w_t \right) \]

\[ \sum_{i} \Theta_{t}^{i}(x_{t}^{i}, u_{t}^{i}) = 0 \]

Dynamic Programming
Bellman (56)
Decompositions for stochastic problems: in uncertainty

$$\min \sum_{\omega} \sum_{i} \sum_{t} \pi_{\omega} L_{t}^{i}(x_{t}^{i}, u_{t}^{i}, w_{t+1})$$

s.t. $$x_{t+1}^{i} = f_{t}^{i}(x_{t}^{i}, u_{t}^{i}, w_{t+1})$$

$$u_{t}^{i} = \mathbb{E}\left( u_{t}^{i} \mid w_{0}, \ldots, w_{t} \right)$$

$$\sum_{i} \Theta_{t}^{i}(x_{t}^{i}, u_{t}^{i}) = 0$$

Progressive Hedging
Rockafellar - Wets (91)
Decompositions for stochastic problems: in space

\[
\min \sum_{\omega} \sum_{i} \sum_{t} \pi_\omega \lambda_t^i(x_t^i, u_t^i, w_{t+1})
\]

s.t. \( x_{t+1}^i = f_t^i(x_t^i, u_t^i, w_{t+1}) \)

\[
u_t^i = \mathbb{E}\left( u_t^i \Bigg| w_0, \ldots, w_t \right)
\]

\[
\sum_i \Theta_t^i(x_t^i, u_t^i) = 0
\]

Dual Approximate Dynamic Programming
Outline of the presentation

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4. Summary and research agenda
Non-anticipativity constraints are linear

- From tree to scenarios (comb)
- Equivalent formulations of the non-anticipativity constraints
  - pairwise equalities
  - all equal to their mathematical expectation
- Linear structure

\[
\mathbf{u}_t = \mathbb{E} \left( \mathbf{u}_t \mid \mathbf{w}_0, \ldots, \mathbf{w}_t \right)
\]
Progressive Hedging stands as a scenario decomposition method

We dualize the non-anticipativity constraints

- When the criterion is strongly convex, we use a Lagrangian relaxation (algorithm “à la Uzawa”) to obtain a scenario decomposition
- When the criterion is linear, Rockafellar - Wets (91) propose to use an augmented Lagrangian, and obtain the Progressive Hedging algorithm
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4 Summary and research agenda
Decomposition and coordination

- The system to be optimized consists of interconnected subsystems
- We want to use this structure to formulate optimization subproblems of reasonable complexity
- But the presence of interactions requires a level of coordination
- Coordination iteratively provides a local model of the interactions for each subproblem
- We expect to obtain the solution of the overall problem by concatenation of the solutions of the subproblems
Example: the “flower model”

Unit Commitment Problem

\[
\min_u \sum_{i=1}^{N} J_i(u_i) \\
\text{s.t.} \sum_{i=1}^{N} \theta_i(u_i) = 0
\]
Intuition of spatial decomposition

- **Purpose:** satisfy a demand with \( N \) production units, at minimal cost
- **Price decomposition**
  - the coordinator sets a price \( \lambda \)
  - the units send their optimal decision \( u_i \)
  - the coordinator compares total production \( \sum_{i=1}^{N} \theta_i(u_i) \) and demand, and then updates the price accordingly
  - and so on...
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  - and so on...

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Spatial decomposition methods in the deterministic case

Decomposition and coordination

Unit 1

$\lambda^{(1)}$

Coordinator

$\lambda^{(1)}$

Unit 2

Unit 3

M. De Lara (École des Ponts ParisTech)

CMM, Santiago de Chile, December 2014
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Intuition of spatial decomposition

- Purpose: satisfy a demand with $N$ production units, at minimal cost
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  - the coordinator sets a price $\lambda$
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  - the coordinator compares total production $\sum_{i=1}^{N} \theta_i(u_i)$ and demand, and then updates the price accordingly
  - and so on...
Price decomposition relies on dualization

\[
\min_{u_i \in U_i, i = 1 \ldots N} \sum_{i=1}^{N} J_i(u_i) \quad \text{subject to} \quad \sum_{i=1}^{N} \theta_i(u_i) = 0
\]

1. Form the \textbf{Lagrangian} and assume that a saddle point exists

\[
\max_{\lambda \in \mathcal{V}} \min_{u_i \in U_i, i = 1 \ldots N} \sum_{i=1}^{N} \left( J_i(u_i) + \langle \lambda, \theta_i(u_i) \rangle \right)
\]

2. Solve this problem by the \textbf{dual gradient algorithm “à la Uzawa”}

\[
u^{(k+1)}_i \in \arg \min_{u_i \in U_i} J_i(u_i) + \langle \lambda^{(k)}, \theta_i(u_i) \rangle, \quad i = 1 \ldots, N
\]

\[
\lambda^{(k+1)} = \lambda^{(k)} + \rho \sum_{i=1}^{N} \theta_i(u^{(k+1)}_i)
\]
Remarks on decomposition methods

- The theory is available for infinite dimensional Hilbert spaces, and thus applies in the stochastic framework, that is, when the $U_i$ are spaces of random variables.

- The minimization algorithm used for solving the subproblems is not specified in the decomposition process.

- New variables $\lambda^{(k)}$ appear in the subproblems arising at iteration $k$ of the optimization process:

  $$\min_{u_i \in U_i} J_i(u_i) + \langle \lambda^{(k)}, \theta_i(u_i) \rangle$$

- These variables are fixed when solving the subproblems, and do not cause any difficulty, at least in the deterministic case.
Price decomposition applies to various couplings
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4 **Summary and research agenda**
**Stochastic optimal control (SOC) problem formulation**

Consider the following SOC problem

\[
\min_{u, x} \mathbb{E}\left( \sum_{i=1}^{N} \left( \sum_{t=0}^{T-1} L^i_t(x^i_t, u^i_t, w_{t+1}) + K^i(x^i_T) \right) \right)
\]

subject to the constraints

\[
x^i_0 = f^i_1(w_0), \quad i = 1\ldots N
\]
\[
x^i_{t+1} = f^i_t(x^i_t, u^i_t, w_{t+1}), \quad t = 0\ldots T-1, \quad i = 1\ldots N
\]
\[
u^i_t \leq F_t = \sigma(w_0, \ldots, w_t), \quad t = 0\ldots T-1, \quad i = 1\ldots N
\]
\[
\sum_{i=1}^{N} \theta^i_t(x^i_t, u^i_t) = 0, \quad t = 0\ldots T-1
\]
Stochastic optimal control (SOC) problem formulation

Consider the following SOC problem

$$\min_{u,x} \sum_{i=1}^{N} \left( \mathbb{E}\left( \sum_{t=0}^{T-1} L_t^i(x_t^i, u_t^i, w_{t+1}) + K_i^i(x_T^i) \right) \right)$$

subject to the constraints

$$x_0^i = f_1^i(w_0), \quad i = 1 \ldots N$$
$$x_t^i = f_t^i(x_t^i, u_t^i, w_{t+1}), \quad t = 0 \ldots T-1, \quad i = 1 \ldots N$$

$$u_t^i \leq \mathcal{F}_t = \sigma(w_0, \ldots, w_t), \quad t = 0 \ldots T-1, \quad i = 1 \ldots N$$

$$\sum_{i=1}^{N} \theta_t^i(x_t^i, u_t^i) = 0, \quad t = 0 \ldots T-1$$
Dynamic programming yields centralized controls

- As we want to solve this SOC problem using dynamic programming (DP), we suppose to be in the Markovian setting, that is, \( w_0, \ldots, w_T \) are a white noise
- The system is made of \( N \) interconnected subsystems, with the control \( u^i_t \) and the state \( x^i_t \) of subsystem \( i \) at time \( t \)
- The optimal control \( u^i_t \) of subsystem \( i \) is a function of the whole system state \( (x^1_t, \ldots, x^N_t) \)
  \[ u^i_t = \gamma^i_t(x^1_t, \ldots, x^N_t) \]

**Naive decomposition should lead to decentralized feedbacks**

\[ u^i_t = \hat{\gamma}^i_t(x^i_t) \]

which are, in most cases, far from being optimal...
The stochastic case raises specific obstacles

The crucial point is that the **optimal feedback** of a subsystem a priori depends on the state of all other subsystems, so that using a decomposition scheme by subsystems is not obvious.

As far as we have to deal with **dynamic programming**, the central concern for decomposition/coordination purpose boils down to

- how to decompose a feedback $\gamma_t$ w.r.t. its domain $X_t$ rather than its range $U_t$?

**And the answer is**

- impossible in the general case!
Price decomposition and dynamic programming

When applying price decomposition to the problem by dualizing the (almost sure) coupling constraint \( \sum_i \theta_t^i(x_t^i, u_t^i) = 0 \), multipliers \( \Lambda_t^{(k)} \) appear in the subproblems arising at iteration \( k \)

\[
\min_{u_t^i, x_t^i} \mathbb{E} \left( \sum_t L_t^i(x_t^i, u_t^i, w_{t+1}) + \Lambda_t^{(k)} \cdot \theta_t^i(x_t^i, u_t^i) \right)
\]

- The variables \( \Lambda_t^{(k)} \) are fixed random variables, so that the random process \( \Lambda^{(k)} \) acts as an additional input noise in the subproblems
- But this process may be correlated in time, so that the white noise assumption has no reason to be fulfilled
- DP cannot be applied in a straightforward manner!

**Question:** how to handle the coordination instruments \( \Lambda_t^{(k)} \) to obtain (an approximation of) the overall optimum?
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3. Theoretical questions
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4. Summary and research agenda
Optimization problem

The SOC problem under consideration reads

\[
\min_{u,x} \mathbb{E} \left( \sum_{i=1}^{N} \left( \sum_{t=0}^{T-1} L_t^i(x_t^i, u_t^i, w_{t+1}) + K_i^i(x_T^i) \right) \right)
\]

subject to dynamics constraints

\[
\begin{align*}
  x_0^i &= f_{-1}^i(w_0) \\
  x_{t+1}^i &= f_t^i(x_t^i, u_t^i, w_{t+1})
\end{align*}
\]

to measurability constraints

\[
  u_t^i \preceq \sigma(w_0, \ldots, w_t)
\]

and to instantaneous coupling constraints

\[
\sum_{i=1}^{N} \theta_t^i(x_t^i, u_t^i) = 0
\]

Constraints to be dualized
Assumptions

Assumption 1 (White noise)
Noises \( \mathbf{w}_0, \ldots, \mathbf{w}_T \) are independent over time

Hence dynamic programming applies: there is no optimality loss
to look after the controls \( \mathbf{u}_t^i \) as functions of the state at time \( t \)

Assumption 2 (Constraint qualification)
A saddle point of the Lagrangian \( \mathcal{L} \) exists (more on that later)

\[
\mathcal{L}(\mathbf{x}, \mathbf{u}, \Lambda) = \mathbb{E}\left( \sum_{i=1}^{N} \left( \sum_{t=0}^{T-1} L_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i, \mathbf{w}_{t+1}) + K^i(\mathbf{x}_T^i) + \sum_{t=0}^{T-1} \Lambda_t \cdot \theta_t^i(\mathbf{x}_t^i, \mathbf{u}_t^i) \right) \right)
\]

where the \( \Lambda_t \) are \( \sigma(\mathbf{w}_0, \ldots, \mathbf{w}_t) \)-measurable random variables

Assumption 3 (Dual gradient algorithm)
Uzawa algorithm applies... (more on that later)
Uzawa algorithm

At iteration $k$ of the algorithm,

1. **Solve** Subproblem $i$, $i = 1, \ldots, N$, with $\Lambda^{(k)}$ fixed

   \[
   \min_{u^i, x^i} \mathbb{E} \left( \sum_{t=0}^{T-1} \left( L_t^i(x_t^i, u_t^i, w_{t+1}) + \Lambda_t^{(k)} \cdot \theta_t^i(x_t^i, u_t^i) \right) + K^i(x_T^i) \right)
   \]

   subject to

   \[
   x_{t+1}^i = f_t^i(x_t^i, u_t^i, w_{t+1}) \\
   u_t^i \leq \sigma(w_0, \ldots, w_t)
   \]

   whose solution is denoted $(u^i,(k+1), x^i,(k+1))$

2. **Update** the multipliers $\Lambda_t$

   \[
   \Lambda_t^{(k+1)} = \Lambda_t^{(k)} + \rho_t \left( \sum_{i=1}^{N} \theta_t^i(x_t^i,(k+1), u_t^i,(k+1)) \right)
   \]
Structure of a subproblem

- Subproblem $i$ reads

$$\min_{u^i_t, x^i_t} \mathbb{E} \left( \sum_{t=0}^{T-1} \left( L_t^i(x^i_t, u^i_t, w_{t+1}) + \Lambda_t^{(k)} \cdot \theta_t^i(x^i_t, u^i_t) \right) \right)$$

subject to

$$x^i_{t+1} = f^i_t(x^i_t, u^i_t, w_{t+1})$$
$$u^i_t \preceq \sigma(w_0, \ldots, w_t)$$

- Without some knowledge of the process $\Lambda^{(k)}$ (we just know that $\Lambda_t^{(k)} \preceq (w_0, \ldots, w_t)$), the informational state of this subproblem $i$ at time $t$ cannot be summarized by the physical state $x^i_t$.
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4. Summary and research agenda
We outline the main idea in DADP

- **The core idea** of DADP is to replace the multiplier $\Lambda_t^{(k)}$ at iteration $k$ by its conditional expectation $\mathbb{E}(\Lambda_t^{(k)} | y_t)$

- where we introduce a new adapted “information” process $y = (y_0, \ldots, y_{T-1})$

- (More on the interpretation later)
Let us go on with our “trick”

- Using this idea, we replace Subproblem $i$ by

$$
\min_{u^i, x^i} \mathbb{E} \left( \sum_{t=0}^{T-1} \left( L_t^i(x_t^i, u_t^i, w_{t+1}) + \mathbb{E}(\Lambda_t^{(k)} | y_t) \cdot \theta_t^i(x_t^i, u_t^i) \right) + K_i(x_T^i) \right)
$$

subject to

$$
x_{t+1}^i = f_t^i(x_t^i, u_t^i, w_{t+1})
$$

$$
u_t^i \preceq \sigma(w_0, \ldots, w_t)
$$

- The conditional expectation $\mathbb{E}(\Lambda_t^{(k)} | y_t)$ is an (updated) function of the variable $y_t$
- so that Subproblem $i$ involves the two noises processes $w$ and $y$

If $y$ follows a dynamical equation, DP applies
We obtain a dynamic programming equation by subsystem

Assuming a non-controlled dynamics $y_{t+1}^{i} = h_{t}^{i}(y_{t}, w_{t+1})$ for the information process $y$, the DP equation writes

$$
V_{T}^{i}(x, y) = K^{i}(x)
$$

$$
V_{t}^{i}(x, y) = \min_{u} \mathbb{E} \left( L_{t}^{i}(x, u, w_{t+1}) \right) + \mathbb{E} \left( \Lambda_{t}^{(k)} \mid y_{t} = y \right) \cdot \theta_{t}^{i}(x, u) + V_{t+1}^{i}(x_{t+1}^{i}, y_{t+1})
$$

subject to the extended dynamics

$$
x_{t+1}^{i} = f_{t}^{i}(x, u, w_{t+1})$$

$$
y_{t+1}^{i} = h_{t}^{i}(y, w_{t+1})$$
What have we done?

- Trick: DADP as an approximation of the optimal multiplier
  \[ \lambda_t \rightsquigarrow \mathbb{E}(\lambda_t \mid y_t) \]

- Interpretation: DADP as a decision-rule approach in the dual
  \[
  \max_{\lambda} \min_u L(\lambda, u) \rightsquigarrow \max_{\lambda_t \leq y_t} \min_u L(\lambda, u)
  \]

- Interpretation: DADP as a constraint relaxation in the primal
  \[
  \sum_{i=1}^{n} \theta_t^i(x_t^i, u_t^i) = 0 \rightsquigarrow \mathbb{E}\left(\sum_{i=1}^{n} \theta_t^i(x_t^i, u_t^i) \mid y_t\right) = 0
  \]
A bunch of practical questions remains open

★ How to choose the information variables $y_t$?

- Perfect memory: $y_t = (w_0, \ldots, w_t)$
- Minimal information: $y_t \equiv \text{cste}$
- Static information: $y_t = h_t(w_t)$
- Dynamic information: $y_{t+1} = h_t(y_t, w_{t+1})$

★ How to obtain a feasible solution from the relaxed problem?

- Use an appropriate heuristic!

★ How to accelerate the gradient algorithm?

- Augmented Lagrangian
- More sophisticated gradient methods

★ How to handle more complex structures than the flower model?
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4. Summary and research agenda
We consider 3 dams in a row, amenable to DP
Problem specification

- We consider a 3 dam problem, over 12 time steps
- We relax each constraint with a given information process \( y^i \) that depends on the constraint
- All random variable are discrete (noise, control, state)

- We test the following information processes
  - **Constant information**: equivalent to replace each a.s. constraint by the expected constraint
  - **Part of noise**: the information process depends on the constraint and is the inflow of the above dam \( y^i_t = w^{i-1}_t \)
  - **Phantom state**: the information process mimicks the optimal trajectory of the state of the first dam (by statistical regression over the known optimal trajectory in this case)
Numerical results are encouraging

<table>
<thead>
<tr>
<th></th>
<th>DADP - E</th>
<th>DADP - $w^{j-1}$</th>
<th>DADP - dyn.</th>
<th>DP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nb of iterations</td>
<td>165</td>
<td>170</td>
<td>25</td>
<td>1</td>
</tr>
<tr>
<td>Time (min)</td>
<td>2</td>
<td>3</td>
<td>67</td>
<td>41</td>
</tr>
<tr>
<td>Lower Bound</td>
<td>$-1.386 \times 10^6$</td>
<td>$-1.379 \times 10^6$</td>
<td>$-1.373 \times 10^6$</td>
<td></td>
</tr>
<tr>
<td>Final Value</td>
<td>$-1.335 \times 10^6$</td>
<td>$-1.321 \times 10^6$</td>
<td>$-1.344 \times 10^6$</td>
<td>$-1.366 \times 10^6$</td>
</tr>
<tr>
<td>Loss</td>
<td>$-2.3%$</td>
<td>$-3.3%$</td>
<td>$-1.6%$</td>
<td>ref.</td>
</tr>
</tbody>
</table>

$\sim \sim$ PhD thesis of J.-C. Alais
Choose an information process $y$ following $y_{t+1} = \tilde{f}_t(y_t, w_{t+1})$

Relax the almost sure coupling constraint into a conditional expectation

Then apply a price decomposition scheme to the relaxed problem

The subproblems can be solved by dynamic programming with the modest state $(x^i_t, y_t)$

In the theoretical part, we give

- Conditions for the existence of an $L^1$ multiplier
- Convergence of the algorithm (fixed information process)
- Consistency result (family of information process)
1. Decomposition and coordination

2. Dual approximate dynamic programming (DADP)

3. Theoretical questions

4. Summary and research agenda
What are the issues to consider?

- We treat the coupling constraints in a stochastic optimization problem by duality methods.
- Uzawa algorithm is a dual method which is naturally described in an Hilbert space, but we cannot guarantee the existence of an optimal multiplier in the space $L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$!
- Consequently, we extend the algorithm to the non-reflexive Banach space $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$, by giving a set of conditions ensuring the existence of a $L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ optimal multiplier, and by providing a convergence result of the algorithm.

- We also have to deal with the approximation induced by the information variable: we give an epi-convergence result related to such an approximation.

\[ \rightsquigarrow \textit{PhD thesis of V. Leclère} \]
Abstract formulation of the problem

We consider the following abstract optimization problem

\[(P) \quad \min_{u \in U^{ad}} J(u) \quad \text{s.t.} \quad \Theta(u) \in -C\]

where \(U\) and \(V\) are two Banach spaces, and
- \(J : U \to \mathbb{R}\) is the objective function
- \(U^{ad}\) is the admissible set
- \(\Theta : U \to V\) is the constraint function to be dualized
- \(C \subset V\) is the cone of constraint

Here, \(U\) is a space of random variables, and \(J\) is defined by

\[J(u) = \mathbb{E}(j(u, w))\]
1 Decomposition and coordination
   - A bird’s eye view of decomposition methods
   - A brief insight into Progressive Hedging
   - Spatial decomposition methods in the deterministic case
   - The stochastic case raises specific obstacles

2 Dual approximate dynamic programming (DADP)
   - Problem statement
   - DADP principle and implementation
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3 Theoretical questions
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4 Summary and research agenda
Standard duality in $L^2$ spaces (I)

Assume that $\mathcal{U} = L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ and $\mathcal{V} = L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)$.

The standard sufficient constraint qualification condition

$$0 \in \text{ri} \left( \Theta(\mathcal{U}^{ad} \cap \text{dom}(J)) + C \right)$$

is scarcely satisfied in such a stochastic setting.

**Proposition 1**

*If the $\sigma$-algebra $\mathcal{F}$ is not finite modulo $\mathbb{P}$,\(^a\) then for any subset $\mathcal{U}^{ad} \subset \mathbb{R}^n$ that is not an affine subspace, the set*

$$\mathcal{U}^{ad} = \left\{ u \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n) \mid u \in \mathcal{U}^{ad} \quad \mathbb{P} - \text{a.s.} \right\}$$

*has an empty relative interior in $L^p$, for any $p < +\infty$*

\(^a\)If the $\sigma$-algebra is finite modulo $\mathbb{P}$, $\mathcal{U}$ and $\mathcal{V}$ are finite dimensional spaces.
Consider the following optimization problem
(a variation on a linear example given by R. Wets)

$$\inf_{u_0, u_1} u_0^2 + \mathbb{E}((u_1 + \alpha)^2)$$

s.t. $u_0 \geq a$

$u_1 \geq 0$

$u_0 - u_1 \geq w$

where $w$ is a random variable uniform on $[1, 2]$

For $a < 2$, we exhibit a maximizing sequence of multipliers for the dual problem that does not converge in $L^2$.
(We are in the so-called non relatively complete recourse case, that is, the case where the constraints on $u_1$ induce a stronger constraint on $u_0$)

The optimal multiplier is not in $L^2$, but in $(L^\infty)^*$
Constraint qualification in \((L^\infty, L^1)\)

From now on, we assume that

\[
U = L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n) \\
V = L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m) \\
C = \{0\}
\]

where the \(\sigma\)-algebra \(\mathcal{F}\) is not finite modulo \(\mathbb{P}\).

We consider the pairing \((L^\infty, L^1)\) with the following topologies:

- \(\sigma(L^\infty, L^1)\): weak* topology on \(L^\infty\) (coarsest topology such that all the \(L^1\)-linear forms are continuous),

- \(\tau(L^\infty, L^1)\): Mackey-topology on \(L^\infty\) (finest topology such that the continuous linear forms are only the \(L^1\)-linear forms)
Weak* closedness of linear subspaces of $L^\infty$

**Proposition 2**

Let $\Theta : L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n) \to L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)$ be a linear operator, and assume that there exists a linear operator $\Theta^\dagger : L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m) \to L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n)$ such that:

$$\langle v, \Theta(u) \rangle = \langle \Theta^\dagger(v), u \rangle, \quad \forall u, \forall v$$

Then the linear operator $\Theta$ is weak* continuous

**Applications**

- $\Theta(u) = u - \mathbb{E}(u \mid \mathcal{B})$: non-anticipativity constraints
- $\Theta(u) = Au$ with $A \in \mathcal{M}_{m,n}(\mathbb{R})$: finite number of constraints
A duality theorem

\[(\mathcal{P}) \quad \min_{u \in \mathcal{U}} J(u) \quad \text{s.t.} \quad \Theta(u) = 0 \]

with \( J(u) = \mathbb{E}(j(u, w)) \)

Theorem 1

Assume that \( j \) is a convex normal integrand, that \( J \) is continuous in the Mackey topology at some point \( u_0 \) such that \( \Theta(u_0) = 0 \), and that \( \Theta \) is weak\(^*\) continuous on \( L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n) \).

Then, \( u^* \in \mathcal{U} \) is an optimal solution of Problem \( (\mathcal{P}) \) if and only if there exists \( \lambda^* \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m) \) such that

\[ u^* \in \text{arg min}_{u \in \mathcal{U}} \mathbb{E}(j(u, w) + \lambda^* \cdot \Theta(u)) \]

\[ \Theta(u^*) = 0 \]

Extension of a result given by R. Wets for non-anticipativity constraints
1. Decomposition and coordination
   - A bird’s eye view of decomposition methods
   - A brief insight into Progressive Hedging
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2. Dual approximate dynamic programming (DADP)
   - Problem statement
   - DADP principle and implementation
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3. Theoretical questions
   - Existence of a saddle point
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   - Convergence w.r.t. information

4. Summary and research agenda
Uzawa algorithm

\[(P) \quad \min_{u \in U} J(u) \quad \text{s.t.} \quad \Theta(u) = 0\]

with \(J(u) = \mathbb{E}(j(u, w))\)

The standard Uzawa algorithm makes sense

\[
\begin{align*}
\mathbf{u}^{(k+1)} & \in \arg \min_{u \in U^{\text{ad}}} J(u) + \langle \lambda^{(k)}, \Theta(u) \rangle \\
\lambda^{(k+1)} & = \lambda^{(k)} + \rho \Theta(\mathbf{u}^{(k+1)})
\end{align*}
\]

Note that all the multipliers \(\lambda^{(k)}\) belong to \(L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)\), as soon as the initial multiplier \(\lambda^{(0)} \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)\)
Convergence result

Theorem 2

Assume that

1. $J : \mathcal{U} \rightarrow \overline{\mathbb{R}}$ is proper, weak* l.s.c., differentiable and $a$-convex
2. $\Theta : \mathcal{U} \rightarrow \mathcal{V}$ is affine, weak* continuous and $\kappa$-Lipschitz
3. $\mathcal{U}^{ad}$ is weak* closed and convex,
4. an admissible $u_0 \in \text{dom} J \cap \Theta^{-1}(0) \cap \mathcal{U}^{ad}$ exists
5. an optimal $L^1$-multiplier to the constraint $\Theta(u) = 0$ exists
6. the step $\rho$ is such that $0 < \rho < \frac{2a}{\kappa}$

Then, there exists a subsequence $\{u^{(n_k)}\}_{k \in \mathbb{N}}$ of the sequence generated by the Uzawa algorithm converging in $L^\infty$ toward the optimal solution $u^*$ of the primal problem.
Discussion

- The result is not as good as expected (convergence of a subsequence)
- Improvements and extensions (inequality constraint) needed
- The Mackey-continuity assumption forbids the use of extended functions
  - In order to deal with almost sure bound constraints, we can turn towards the work of T. Rockafellar and R. Wets
  - In a series of 4 papers (stochastic convex programming), they have detailed the duality theory on two-stage and multistage problems, with the focus on non-anticipativity constraints
  - These papers require
    - a strict feasibility assumption
    - a relatively complete recourse assumption
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4. Summary and research agenda
Relaxed problems

Following the interpretation of DADP in terms of a relaxation of the original problem, and given a sequence \( \{ \mathcal{F}_n \}_{n \in \mathbb{N}} \) of subfields of the \( \sigma \)-field \( \mathcal{F} \), we replace the abstract problem

\[
(P) \quad \min_{u \in U} J(u) \quad \text{s.t.} \quad \Theta(u) = 0
\]

by the sequence of approximated problems:

\[
(P_n) \quad \min_{u \in U} J(u) \quad \text{s.t.} \quad \mathbb{E}(\Theta(u) \mid \mathcal{F}_n) = 0
\]

We assume the Kudo convergence of \( \{ \mathcal{F}_n \}_{n \in \mathbb{N}} \) toward \( \mathcal{F} \):

\[
\mathcal{F}_n \longrightarrow \mathcal{F} \quad \iff \quad \mathbb{E}(z \mid \mathcal{F}_n) \overset{L^1}{\longrightarrow} \mathbb{E}(z \mid \mathcal{F}) \quad , \quad \forall z \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})
\]
Convergence result

Theorem 3

Assume that

- \( \mathcal{U} \) is a topological space
- \( \mathcal{V} = L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m) \) with \( p \in [1, +\infty) \)
- \( J \) and \( \Theta \) are continuous operators
- \( \{\mathcal{F}_n\}_{n \in \mathbb{N}} \) Kudo converges toward \( \mathcal{F} \)

Then the sequence \( \{\tilde{J}_n\}_{n \in \mathbb{N}} \) epi-converges toward \( \tilde{J} \), with

\[
\tilde{J}_n = \begin{cases} 
J(u) & \text{if } u \text{ satisfies the constraints of } (\mathcal{P}_n) \\
+\infty & \text{otherwise}
\end{cases}
\]
Summing up theoretical questions

- Conditions for the existence of an $L^1$ multiplier
- Convergence of the algorithm (fixed information process)
- Consistency result (family of information process)
1. Decomposition and coordination

2. Dual approximate dynamic programming (DADP)

3. Theoretical questions

4. Summary and research agenda
Discussing DADP

- **DADP** (Dual Approximate Dynamic Programming) is a method to design **stochastic price signals** allowing **decentralized** agents to act as a **team**
- Hence, **DADP** is especially adapted to tackle **large-scale** stochastic optimal control problems, such as those found in energy management
- A host of **theoretical and practical questions** remains open
- We would like to test **DADP** on “network models” (smart grids) extending the works already made on “flower models” (unit commitment problem) and on “chained models” (hydraulic valley management)
Let us move to broader stochastic optimization challenges

- **Stochastic optimization** requires to make risk attitudes explicit
  - robust, worst case, risk measures, in probability, almost surely, etc.
- **Stochastic dynamic optimization** requires to make online information explicit
  - State-based functional approach
  - Scenario-based measurability approach

**Numerical walls**

- in dynamic programming, the bottleneck is the dimension of the state
- in stochastic programming, the bottleneck is the number of stages
Here is our research agenda for stochastic decomposition

- Combining different decomposition methods
  - time: dynamic programming
  - scenario: progressive hedging
  - space: dual approximate dynamic programming

- Designing risk criterion compatible with decomposition (time-consistent dynamic risk measures)

- Mixing decomposition with analytical properties (convexity, linearity) on costs, constraints and dynamics functions