Comparison of viability kernels or generalized monotone controlled systems and applications to biological control

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Outline of the talk

Mathematical control of Wolbachia bacteria (I)

Comparison of flows under conic order

Comparison of viability kernels

Mathematical control of Wolbachia bacteria (II)

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Mathematical control of Wolbachia bacteria (II)

The Wolbachia bacterium

- The following model represents
  the dynamics of a mosquito population
  infected with the Wolbachia bacterium
- The Wolbachia bacterium is an inhibitor
  of the capacity of the mosquitoes to transmit diseases,
  such as dengue, in human populations
- Liberation of infected mosquitoes
  is used as a control method for the disease

Dynamics of a mosquito population infected with the Wolbachia bacterium

[Bliman, Aronna, Coelho, and da Silva, 2017]

$$
\begin{aligned}
\dot{L_U} &= F_L(L_U, A_U, L_W, A_W) = \alpha_U A_U \frac{A_U}{A_U + A_W} - \nu L_U - \mu\left(1 + k\left(L_U + L_W\right)\right) L_U \\
\dot{A_U} &= F_A(L_U, A_U, L_W, A_W) = \nu L_U - \mu_U A_U \\
\dot{L_W} &= G_L(L_U, A_U, L_W, A_W) = \alpha_W A_W - \nu L_W - \mu\left(1 + k\left(L_U + L_W\right)\right) L_W \\
\dot{A_W} &= G_A(L_U, A_U, L_W, A_W) = \nu L_W - \mu_W A_W
\end{aligned}
$$

- L_U, A_U (larva and adults respectively): uninfested mosquitoes abundances
- L_W, A_W (larva and adults respectively): infested mosquitoes abundances

Control of introduction of infected larvae in the population

We consider the following dynamical controlled system

$$\dot{x} = f(x, u(t))$$

where $x \in \mathbb{R}^4$ represents the same population compartments as in the Wolbachia model, and where

$$u(\cdot) \in \mathcal{U} = \{u(\cdot) : [0, +\infty) \to [0, u_{max}] \text{ is a measurable function}\}$$

represents a policy of introduction of infected larvae in the population, and where

$$f(x, u) = \begin{cases} F_L(L_U, A_U, L_W, A_W) \\ F_A(L_U, A_U, L_W, A_W) \\ G_L(L_U, A_U, L_W, A_W) + u(t) \\ G_A(L_U, A_U, L_W, A_W) \end{cases}$$

Controlling the system to remain above or below sustainable thresholds

[Barrios, Gajardo, and Vasilieva, 2018]

The goal of the control policy is to have, permanently,

- the infested population of mosquitoes to be above both $\overline{A_W}, \overline{L_W}$
- the uninfested population of mosquitoes to be below both $\underline{L_U}, \underline{A_U}$

We will show that the corresponding viability kernel can be computed by means of a single control policy, identically equal to u_{max}, instead of a family of controls

What is the generality behind such property?

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Comparison theorems

- In ordinary differential equations
- In monotone differential systems: [Smith, 1995], [Hirsch and Smith, 2003], [Hirsch and Smith, 2004]
- In monotone controlled differential systems: [Angeli and Sontag, 2003]
- In monotone controlled differential or discrete-time systems with state constraints (viability kernels): [De Lara, Doyen, Guilbaud, and Rochet, 2007], [De Lara, Gajardo, and Ramirez, 2011], [De Lara and Sepulveda Salcedo, 2016]
- Our contribution: an umbrella framework for comparison of viability kernels with conic orders

Conic order induced by a convex cone

If $K \subset \mathbb{R}^n$ is a closed convex cone,
then such cone induces a *pre-order* $\preceq_K$ on $\mathbb{R}^n$ by

$$x \preceq_K y \iff y - x \in K$$

For example, if $K = \mathbb{R}^n_+$,
then $\preceq_K$ is the usual componentwise order

K-quasimonotone mapping

[Smith, 1995]

Definition

We say that a mapping $h : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ is *K-quasimonotone* if

$$x \preceq_{K \cap \{x^*\}^\perp} y \implies h(x,t) \preceq_{\{x^*\}^+} h(y,t)$$

$$\forall x, y \in \mathbb{R}^n \ , \ \ \forall x^* \in K^+ \ , \ \ \forall t \in [0, +\infty)$$

where K^+ is the positive polar cone associated with K

.

Example of K-quasimonotonicity

- A useful example in $\mathbb{R}^n$ is the case of the cone generated by the vector set $\{(-1)^{m_j} e_j\}_{j=1}^{n}$, where e_j is the j-th element of the canonic base in $\mathbb{R}^n$ and $m_j \in \{0, 1\}$
- In such case, the mapping $h = (h_1, \dots, h_n)$ is K-quasimonotone if and only if

$$(-1)^{m_j+m_i} \frac{\partial h_j(x,t)}{\partial x_i} \geq 0$$

$$\forall (x,t) \in \mathbb{R}^n \times [0, +\infty) \quad \forall i, j \in \{1, \cdots, n\}, i \neq j$$

Controlled dynamical system

- State $x \in \mathbb{R}^n$
- Control $u \in \mathbb{U} \subset \mathbb{R}^m$
- Dynamics $f : \mathbb{R}^n \times \mathbb{U} \subset \mathbb{R}^m \to \mathbb{R}^n$
- Control paths

$$\mathcal{U} = \{u(\cdot) : [0, +\infty) \to \mathbb{U} \mid u(\cdot) \text{ is a measurable mapping}\}$$

Flow of a controlled dynamical system

- For the initial state $x_0 \in \mathbb{R}^n$ and the control path $u(\cdot) \in \mathcal{U}$, we denote by $\Psi_f^{u(\cdot)}(t, x_0)$ the *flow*, (unique) solution of

$$\dot{x} = f(x, u(t))$$

- We assume that, for any initial condition and control path, the flow is well defined on $[0, +\infty)$, that is, the solution exists and is unique

We introduce the notion of K-reduction

We propose the following two hypothesis

K-quasimonotonicity + K-reduction

- (K-QM) $\forall u(\cdot) \in \mathcal{U}$, the mapping $(x,t) \to f(x, u(t))$ is K-quasimonotone
- (K-R) there exists a set-valued mapping $\phi : \mathbb{U} \rightrightarrows \mathbb{U}$, satisfying, $\forall u(\cdot) \in \mathcal{U}$, there exists $v(\cdot) \in \mathcal{U}$, such that $v(t) \in \phi(u(t))$, $\forall t \in [0, +\infty)$, and

$$f(x, u(t)) \preceq_K f(x, v(t)) \quad \forall (x,t) \in \mathbb{R}^n \times [0, +\infty)$$

We call the set-valued mapping ϕ a K-reduction for the dynamics f

We provide a comparison result for flows

Proposition

Suppose that hypothesis K-quasimonotonicity (K-QM) and K-reduction (K-R) hold true for the controlled dynamics f with the cone K

Then, the following flows satisfy

$$x_0 \preceq_K y_0 \Rightarrow \Psi_f^{u(\cdot)}(t, x_0) \preceq_K \Psi_f^{v(\cdot)}(t, y_0) \qquad \forall\ t \in [0, +\infty)$$

where, with any control path $u(\cdot) \in \mathcal{U}$,
we associate the control path $v(\cdot) \in \mathcal{U}$, denoted $v_\phi(\cdot)$,
thanks to the K-reduction ϕ provided by hypothesis (K-R)

Outline of the talk

Viability kernel

[Aubin, 1990], [Aubin, Bayen, and Saint-Pierre, 2011]

Definition

Given the controlled dynamical system

$$\dot{x} = f(x, u(t))$$

and what we call a *desirable set*

$$\mathbb{D} \subset \mathbb{R}^n \times \mathbb{U}$$

the associated *viability kernel* $\mathbb{V}(f, \mathbb{D})$ is

$$\begin{aligned} \mathbb{V}(f, \mathbb{D}) = \quad & \big\{ x_0 \in \mathbb{R}^n \mid \exists u(\cdot) \in \mathcal{U} \\ & \big(\Psi_f^{u(\cdot)}(t, x_0), u(t) \big) \in \mathbb{D} \ , \quad \forall t \in [0, +\infty) \big\} \end{aligned}$$

Preparation for viability kernels comparison

Under hypothesis *K-quasimonotonicity* (K-QM) and *K-reduction* (K-R) with K-reduction ϕ, we introduce the following

- alternative desirable set

$$\mathbb{D}_{K,\phi} = \bigcup_{(x,u)\in\mathbb{D}} (x+K) \times \phi(u)$$

- alternative dynamics

$$f_\phi(x,u) = f\left(x, v_\phi(u)\right)$$

We provide a viability kernels comparison result

Theorem

Under hypothesis K-quasimonotonicity (K-QM)
and K-reduction (K-R) with K-reduction ϕ,
one has the following inclusion between viability kernels

$$\mathbb{V}(f, \mathbb{D}) \subset \mathbb{V}(f_\phi, \mathbb{D}_{K,\phi})$$

If, in addition, one has that

$$\mathbb{D}_{K,\phi} \subset \mathbb{D}$$

then one has the following equality between viability kernels

$$\mathbb{V}(f, \mathbb{D}) = \mathbb{V}(f_\phi, \mathbb{D}_{K,\phi})$$

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A conic order suitable for the Wolbachia bacterium model

We introduce the cone

$$K = \mathbb{R}_- \times \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_+$$

so that we have

$$\begin{aligned} & (x_1, x_2, x_3, x_4) \preceq_K (y_1, y_2, y_3, y_4) \\ \iff & \\ & x_1 \geq y_1 \ , \ \ x_2 \geq y_2 \ , \ \ x_3 \leq y_3 \ , \ \ x_4 \leq y_4 \end{aligned}$$

The Wolbachia bacterium controlled model is K-quasimonotone

The Wolbachia bacterium controlled model is K-quasimonotone, because

- $\frac{\partial F_L}{\partial A_U} \geq 0$, $\frac{\partial F_L}{\partial L_W} \leq 0$, $\frac{\partial F_L}{\partial A_W} \leq 0$
- $\frac{\partial F_A}{\partial L_U} \geq 0$, $\frac{\partial F_A}{\partial L_W} \leq 0$, $\frac{\partial F_A}{\partial A_W} \leq 0$
- $\frac{\partial G_L}{\partial A_W} \geq 0$, $\frac{\partial G_L}{\partial L_U} \leq 0$, $\frac{\partial G_L}{\partial A_U} \leq 0$
- $\frac{\partial G_A}{\partial L_W} \geq 0$, $\frac{\partial G_A}{\partial L_U} \leq 0$, $\frac{\partial G_A}{\partial A_U} \leq 0$

for all $L_U, A_U, L_W, A_W > 0$

Existence of a K-reduction

- We define the mapping $\phi : [0, u_{max}] \to [0, u_{max}]$ by

$$\phi(u) = u_{max}$$

- The mapping ϕ is a K-reduction,
  satisfying hypothesis (K-R), because one has, $\forall t \in [0, +\infty)$

$$G_L(L_U, A_U, L_W, A_W) + u(t) \leq G_L(L_U, A_U, L_W, A_W) + u_{max}$$

Desirable set and its alternative

The desirable set is

$$\begin{aligned}\mathbb{D} =&\{(L_U, A_U, L_W, A_W, u) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times [0, u_{max}] \mid \\ &L_U \leq \underline{L_U}, A_U \leq \underline{A_U}, L_W \geq \overline{L_W}, A_W \geq \overline{A_W}\}\end{aligned}$$

and the alternative desirable set is

$$\begin{aligned}\mathbb{D}_{K,\phi} =&\{(L_U, A_U, L_W, A_W, u) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \{u_{max}\} \mid \\ &L_U \leq \underline{L_U}, A_U \leq \underline{A_U}, L_W \geq \overline{L_W}, A_W \geq \overline{A_W}\}\end{aligned}$$

so that

$$\mathbb{D}_{K,\phi} \subset \mathbb{D}$$

Viability kernels equality

Using the Theorem, we obtain the equality

$$\mathbb{V}(f, \mathbb{D}) = \mathbb{V}(f_\phi, \mathbb{D}_{K,\phi})$$

where f_ϕ denotes the dynamic mapping
with the control u_{max} applied

Therefore, computing the viability kernel has been reduced to compute the viability kernel for a single constant control policy, instead of for a family of controls,
which is an easier problem to handle

Conclusion

- Some (but not all) natural resource management problems display monotonicity properties:
  e.g. the more you harvest, the less abundance
- These properties can help simplify the analysis,
  here the computation of viability kernels
- By using conic orders, we have provided
  an umbrella framework for this purpose,
  relying on K-quasimonotonicity and K-reduction

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