Two Players Game Theory with Information: Introducing the Witsenhausen Intrinsic Model

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Outline of the presentation

Introduction to games in intrinsic form

Witsenhausen intrinsic model and game theory with information

Nash equilibrium with information

Witsenhausen intrinsic model and principal-agent models

Games solvable by dynamic programming

Open questions

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Open questions

Information plays a crucial role in competition

- Information — who knows what and when — plays a crucial role in competitive contexts
- Concealing, dissimulation, cheating, lying, deception are effective strategies

Our goals are to

1. introduce the notion of game in intrinsic form
2. contribute to the analysis of decentralized, non-cooperative decision settings
3. provide a (very) general mathematical language for mechanism design

We will distinguish an individual from an agent

- An individual who makes a first, followed by a second decision, is represented by two agents (two decision makers)
- An individual who makes a sequence of decisions — one for each period $t = 0, 1, 2, \ldots, T-1$ — is represented by T agents, labelled $t = 0, 1, 2, \ldots, T-1$
- N individuals — each i of whom makes a sequence of decisions, one for each period $t = 0, 1, 2, \ldots, T_i - 1$ — is represented by $\prod_{i=1}^{N} T_i$ agents, labelled by

$$(i, t) \in \bigcup_{j=1}^{N} \{j\} \times \{0, 1, 2, \ldots, T_j - 1\}$$

What is a game in intrinsic form?

- Nature, the source of all randomness, or *states of Nature*
- Agents, who
  - hold *information*
  - make *decisions*, by means of *admissible strategies*, those fueled by information
- Players, who
  - hold *beliefs* about states of Nature
  - hold a *subset of agents* under their exclusive control (*executives*)
  - hold *objectives*, that they achieve by selecting proper admissible strategies for the agents under their control

What is a game in intrinsic form?

- Nature, the source of all randomness —
  a set Ω equipped with a σ-field $\mathcal{F}$
- Agents, who hold *information* and make *decisions* — a set A
  - for each agent $a \in A$, an action set $\mathbb{U}_a$ equipped with a σ-field $\mathcal{U}_a$
  - for each agent $a \in A$, an information field

$$\mathcal{I}_a \subset \mathcal{H} = \mathcal{U}_A \otimes \mathcal{F} = \bigotimes_{b \in A} \mathcal{U}_b \otimes \mathcal{F}$$

- Players, who hold *objectives* and *beliefs* —
  a partition $(A_p)_{p \in P}$ of the set A of agents
  - for each player $p \in P$, a criterion

$$j_p : \mathbb{H} = \mathbb{U}_A \times \Omega = \prod_{b \in A} \mathbb{U}_b \times \Omega \to \mathbb{R}$$

  - for each player $p \in P$, a probability $\mathcal{P}_p$ over $(\Omega, \mathcal{F})$

Outline of the presentation

Witsenhausen intrinsic model with Nature and two players, each made of a single agent

We lay out

- basic sets
  - decision sets
  - states of Nature
  - history set

  and their σ-fields
- objective functions
- beliefs
- information σ-fields, admissible strategies and predecessors

Nature's moves and agents decisions

- Let Ω be a measurable set equipped with a σ-field $\mathcal{F}$ which represents all uncertainties:
  any $\omega \in \Omega$ is called a state of Nature
- The agent a makes one decision $u_a \in \mathbb{U}_a$
  where the decision set $\mathbb{U}_a$ is equipped with a σ-field $\mathcal{U}_a$
- The agent b makes one decision $u_b \in \mathbb{U}_b$
  where the decision set $\mathbb{U}_b$ is equipped with a σ-field $\mathcal{U}_b$

History space

The history space is the product space

$$\mathbb{H} = \mathbb{U}_a \times \mathbb{U}_b \times \Omega$$

equipped with the product history field

$$\mathcal{H} = \mathcal{U}_a \otimes \mathcal{U}_b \otimes \mathcal{F}$$

Players, criteria and beliefs

- From now on, we consider the partition $\{a\}, \{b\}$ of players, and we identify player $\{a\}$ with agent a, and player $\{b\}$ with agent b
- The two players a, b have a criterion,

$$j_a : \mathbb{U}_a \times \mathbb{U}_b \times \Omega \to \mathbb{R} \; , \;\; j_b : \mathbb{U}_a \times \mathbb{U}_b \times \Omega \to \mathbb{R}$$

  that are measurable functions over history $\mathbb{H}$
- The two players a, b have a belief,

$$\mathcal{P}_a : \mathcal{F} \to [0,1] \; , \;\; \mathcal{P}_b : \mathcal{F} \to [0,1]$$

  that are probability distributions over $(\Omega, \mathcal{F})$

Information and predecessors

Information

- When making a decision,
  agent a and agent b can make use of information,
  materialized under the form of σ-fields
- The information field $\mathcal{I}_a$ of the agent a
  is a subfield of the history field $\mathcal{H}$

$$\mathcal{I}_a \subset \mathcal{U}_a \otimes \mathcal{U}_b \otimes \mathcal{F}$$

- The information field $\mathcal{I}_b$ of the agent b
  is a subfield of the history field $\mathcal{H}$

$$\mathcal{I}_b \subset \mathcal{U}_a \otimes \mathcal{U}_b \otimes \mathcal{F}$$

Absence of "self-information"

- The information fields $\mathcal{I}_a$ and $\mathcal{I}_b$ display the absence of "self-information" when

$$\mathcal{I}_a \subset \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{U}_b \otimes \mathcal{F}$$

$$\mathcal{I}_b \subset \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$$

- In what follows, we always assume absence of "self-information" (otherwise, we would be led to paradoxes)

Classical information patterns in game theory

Two agents: the principal $\mathtt{Pr}$ (leader) and the agent $\mathtt{Ag}$ (follower)

- *Moral hazard* (the insurance company cannot observe if the insured plays with matches at home)

$$\mathcal{I}_{\mathtt{Pr}} \subset \{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \mathcal{F}$$

- *Stackelberg leadership model*

$$\mathcal{I}_{\mathtt{Ag}} \subset \{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \mathcal{U}_{\mathtt{Pr}} \otimes \mathcal{F} \ , \quad \mathcal{I}_{\mathtt{Pr}} \subset \{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \mathcal{F}$$

- *Adverse selection* (the insurance company cannot observe if the insured has good health)

$$\{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \mathcal{F} \subset \mathcal{I}_{\mathtt{Ag}} \ , \quad \mathcal{I}_{\mathtt{Pr}} \subset \mathcal{U}_{\mathtt{Ag}} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \{\emptyset, \Omega\}$$

- *Signaling*

$$\{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \mathcal{F} \subset \mathcal{I}_{\mathtt{Ag}} \ , \quad \mathcal{I}_{\mathtt{Pr}} = \mathcal{U}_{\mathtt{Ag}} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \{\emptyset, \Omega\}$$

Cylindric subfields

- Information only carried by the moves of Nature

$$\mathcal{H}_{\emptyset} = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$$

- Information only carried by the moves of Nature and by the decisions of agent a

$$\mathcal{H}_{\{a\}} = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$$

- Information only carried by the moves of Nature and by the decisions of agent b

$$\mathcal{H}_{\{b\}} = \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{U}_b \otimes \mathcal{F}$$

- Information carried by the moves of Nature and by the decisions of agents a and b

$$\mathcal{H}_{\{a,b\}} = \mathcal{U}_a \otimes \mathcal{U}_b \otimes \mathcal{F} = \mathcal{H}$$

Definition of predecessor, excluding Nature

Consider a subset B of $\{a, b\}$ — $B \in \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ — and define

$$\mathcal{H}_B = \prod_{c \in B} \mathcal{U}_c \otimes \prod_{c \notin B} \{\emptyset, \mathbb{U}_c\} \otimes \mathcal{F}$$

Predecessor

For any agent $c \in \{a, b\}$, we define $\langle c \rangle_{\mathfrak{P}}$
as the intersection of all subsets B of $\{a, b\}$ such that $\mathcal{I}_c \subset \mathcal{H}_B$

$$\langle c \rangle_{\mathfrak{P}} = \bigcap_{B,\ \mathcal{I}_c \subset \mathcal{H}_B} B$$

When non empty, an element of $\langle c \rangle_{\mathfrak{P}}$ is called a predecessor of c

- Nature has no predecessor: Nature plays before the agents (but is not necessarily revealed to the agents)
- As an illustration, absence of "self-information" is equivalent to $c \notin \langle c \rangle_{\mathfrak{P}}$, for any $c \in \{a, b\}$

Sequential and non-sequential information patterns

- Sequential patterns
  - When $\langle a \rangle_{\mathfrak{P}} = \emptyset$ and $\langle b \rangle_{\mathfrak{P}} = \emptyset$,
    agent a and agent b both play first (static team)
  - When $\langle a \rangle_{\mathfrak{P}} = \emptyset$ and $\langle b \rangle_{\mathfrak{P}} = \{a\}$,
    agent a plays first, agent b plays second
  - When $\langle a \rangle_{\mathfrak{P}} = \{b\}$ and $\langle b \rangle_{\mathfrak{P}} = \emptyset$,
    agent b plays first, agent a plays second
- Non-sequential pattern
  - When $\langle a \rangle_{\mathfrak{P}} = \{b\}$ and $\langle b \rangle_{\mathfrak{P}} = \{a\}$,
    agent a and agent b
    - can be in a deadlock (non causal system)
    - or can be first and second agents depending on Nature's move (causal system)

Strategies and admissible strategies

Pure strategies

- A (pure) strategy the agent a is a measurable mapping

$$\lambda_a : \mathbb{U}_a \times \mathbb{U}_b \times \Omega \to \mathbb{U}_a \ , \quad \lambda_a^{-1}(\mathcal{U}_a) \subset \mathcal{H}$$

  and the set of strategies of agent a is

$$\Lambda_a = \left\{ \lambda_a : (\mathbb{H}, \mathcal{H}) \to (\mathbb{U}_a, \mathcal{U}_a) \mid \lambda_a^{-1}(\mathcal{U}_a) \subset \mathcal{H} \right\}$$

- A (pure) strategy of agent b is a measurable mapping

$$\lambda_b : \mathbb{U}_a \times \mathbb{U}_b \times \Omega \to \mathbb{U}_b \ , \quad \lambda_b^{-1}(\mathcal{U}_b) \subset \mathcal{H}$$

  and the set of strategies of agent b is

$$\Lambda_a = \left\{ \lambda_a : (\mathbb{H}, \mathcal{H}) \to (\mathbb{U}_a, \mathcal{U}_a) \mid \lambda_b^{-1}(\mathcal{U}_a) \subset \mathcal{H} \right\}$$

- We denote the set of strategies of all agents in A by

$$\Lambda_A = \Lambda_a \times \Lambda_b$$

Mixed strategies

- A mixed strategy (or randomized strategy) for agent a is an element of $\Delta(\Lambda_a)$, the set of probability distributions over the set of strategies of agent a
- A mixed strategy (or randomized strategy) for agent b is an element of $\Delta(\Lambda_b)$, the set of probability distributions over the set of strategies of agent b
- We denote the set of mixed strategies of *players* by

$$\Delta(\Lambda_a) \times \Delta(\Lambda_b) \subset \Delta(\Lambda_a \times \Lambda_b)$$

We introduce admissible strategies to account for the interplay between decision and information

- Information is the fuel of strategies

Admissible strategy

An admissible strategy of the agent $c \in \{a, b\}$ is a mapping

$$\lambda_c : \mathbb{U}_a \times \mathbb{U}_b \times \Omega \to \mathbb{U}_c \text{ such that } \lambda_c^{-1}(\mathcal{U}_c) \subset \mathcal{I}_c$$

- The set of admissible strategies of the agent $c \in \{a, b\}$ is

$$\Lambda_c^{ad} = \{\lambda_c \mid \mathbb{U}_a \times \mathbb{U}_b \times \Omega \to \mathbb{U}_c \ , \ \lambda_c^{-1}(\mathcal{U}_c) \subset \mathcal{I}_c\}$$

- The set of admissible strategies is

$$\Lambda^{ad} = \Lambda_a^{ad} \times \Lambda_b^{ad}$$

- The set of mixed admissible strategies is

$$\Delta(\Lambda_a^{ad}) \times \Delta(\Lambda_b^{ad}) \subset \Delta(\Lambda_a^{ad} \times \Lambda_b^{ad})$$

Absence of "self-information" and structure of admissible strategies

- The information fields $\mathfrak{I}_a$ and $\mathfrak{I}_b$ display the absence of "self-information" when

$$\mathfrak{I}_a \subset \{\emptyset, \mathbb{U}_a\} \otimes \mathfrak{U}_b \otimes \mathfrak{F} \iff a \notin \langle a \rangle_{\mathfrak{P}}$$

$$\mathfrak{I}_b \subset \mathfrak{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathfrak{F} \iff b \notin \langle b \rangle_{\mathfrak{P}}$$

- When σ-fields include singletons and we exclude "self-information", then, for any admissible strategy λ_c of the agent $c \in \{a, b\}$, we have that the expression $\lambda_c(u_a, u_b, \omega)$ does not depend on u_c:

$$\lambda_a(\cancel{u_a}, u_b, \omega) = \widetilde{\lambda_a}(u_b, \omega) \; , \; \lambda_b(u_a, \cancel{u_b}, \omega) = \widetilde{\lambda_b}(u_a, \omega)$$

Sequential patterns and structure of admissible strategies

- When $\langle a \rangle_{\mathfrak{P}} = \emptyset$ and $\langle b \rangle_{\mathfrak{P}} = \emptyset$

$$\lambda_a(\cancel{u_a}, \cancel{u_b}, \omega) = \widetilde{\lambda_a}(\omega) \ , \ \ \lambda_b(\cancel{u_a}, \cancel{u_b}, \omega) = \widetilde{\lambda_b}(\omega)$$

- When $\langle a \rangle_{\mathfrak{P}} = \emptyset$ and $\langle b \rangle_{\mathfrak{P}} = \{a\}$

$$\lambda_a(\cancel{u_a}, \cancel{u_b}, \omega) = \widetilde{\lambda_a}(\omega) \ , \ \ \lambda_b(u_a, \cancel{u_b}, \omega) = \widetilde{\lambda_b}(u_a, \omega)$$

- When $\langle a \rangle_{\mathfrak{P}} = \{b\}$ and $\langle b \rangle_{\mathfrak{P}} = \emptyset$

$$\lambda_a(\cancel{u_a}, u_b, \omega) = \widetilde{\lambda_a}(u_b, \omega) \ , \ \ \lambda_b(\cancel{u_a}, \cancel{u_b}, \omega) = \widetilde{\lambda_b}(\omega)$$

Non-sequential information patterns and structure of admissible strategies

When $\langle a \rangle_{\mathfrak{P}} = \{b\}$ and $\langle b \rangle_{\mathfrak{P}} = \{a\}$, agent a and agent b

- can be in a deadlock

$$\lambda_a(\cancel{u_a}, u_b, \omega) = \widetilde{\lambda_a}(u_b, \omega) \ , \ \ \lambda_b(u_a, \cancel{u_b}, \omega) = \widetilde{\lambda_b}(u_a, \omega)$$

- or can be first and second agents depending on Nature's move
  - when Nature's move is ω^+, agent a plays first, agent b plays second

$$\lambda_a(\cancel{u_a}, \cancel{u_b}, \omega^+) = \widetilde{\lambda_a}(\omega^+) \ , \ \ \lambda_b(u_a, \cancel{u_b}, \omega^+) = \widetilde{\lambda_b}(u_a, \omega^+)$$

  - when Nature's move is ω^-, agent b plays first, agent a plays second

$$\lambda_a(\cancel{u_a}, u_b, \omega^-) = \widetilde{\lambda_a}(u_b, \omega^-) \ , \ \ \lambda_b(\cancel{u_a}, \cancel{u_b}, \omega^-) = \widetilde{\lambda_b}(\omega^-)$$

Solvability property

The information fields $\mathcal{I}_a$ and $\mathcal{I}_b$ display the solvability property when,

- for any couple $(\lambda_a, \lambda_b) \in \Lambda_a^{ad} \times \Lambda_b^{ad}$ of admissible strategies and any state of Nature $\omega \in \Omega$,
- there exists one, and only one, couple $(u_a, u_b) \in \mathbb{U}_a \times \mathbb{U}_b$ of decisions such that

$$u_a = \lambda_a(u_a, u_b, \omega)$$

$$u_b = \lambda_b(u_a, u_b, \omega)$$

Solvability property and solution map

Solution map

In case of solvability, we can define $S_{(\lambda_a,\lambda_b)}(\omega)$, for any $\omega \in \Omega$, by

$$S_{(\lambda_a,\lambda_b)}(\omega) = (u_a, u_b, \omega) \iff \begin{cases} u_a = \lambda_a(u_a, u_b, \omega) \\ u_b = \lambda_b(u_a, u_b, \omega) \end{cases}$$

Hence, we obtain a mapping called the solution map

$$S_{(\lambda_a,\lambda_b)} : \Omega \to \mathbb{U}_a \times \mathbb{U}_b \times \Omega$$

- The solvability property holds true in the sequential cases
- The graph of $S_{(\lambda_a,\lambda_b)}$ belongs to $\mathcal{I}_a \vee \mathcal{U}_a \vee \mathcal{I}_b \vee \mathcal{U}_b$.

Co-cycle property of the solution map (I)

- We suppose that $\langle a \rangle_{\mathfrak{P}} = \{b\}$ and $\langle b \rangle_{\mathfrak{P}} = \emptyset$, that is, agent b plays first, agent a plays second
- We consider a couple $(\lambda_a, \lambda_b) \in \Lambda_a^{ad} \times \Lambda_b^{ad}$ of admissible strategies

Co-cycle property of the solution map

We have that

- the strategy λ_b can be identified with $\lambda_b : \Omega \to \mathbb{U}_b$ and the partial solution map $S_{\lambda_b} : \Omega \to \mathbb{U}_b \times \Omega$ is such that $S_{\lambda_b}(\omega) = (\lambda_b(\omega), \omega)$
- the strategy λ_a can be identified with $\lambda_a : \mathbb{U}_b \times \Omega \to \mathbb{U}_a$
- the solution map has the following co-cycle property

$$S_{(\lambda_a, \lambda_b)} = (\lambda_a \circ S_{\lambda_b}, S_{\lambda_b}) : \Omega \to \mathbb{U}_a \times \big(\mathbb{U}_b \times \Omega\big)$$

$$S_{(\lambda_a, \lambda_b)}(\omega) = \Big(\lambda_a\Big(\lambda_b(\omega), \omega\Big), \lambda_b(\omega), \omega\Big) \; , \quad \forall \omega \in \Omega$$

Co-cycle property of the solution map (II)

The co-cycle property

$$S_{(\lambda_a,\lambda_b)} = (\lambda_a \circ S_{\lambda_b}, S_{\lambda_b})$$

is equivalent to

$$S_{(\lambda_a,\lambda_b)}(\omega) = (u_a, u_b, \omega) \iff \begin{cases} (u_b, \omega) & = S_{\lambda_b}(\omega) \\ u_a & = \lambda_a(u_b, \omega) \end{cases}$$

Outline of the presentation

Criteria composed with solution map

- Costs or payoffs are

$$j_a : \mathbb{U}_a \times \mathbb{U}_b \times \Omega \to \mathbb{R}$$

$$j_b : \mathbb{U}_a \times \mathbb{U}_b \times \Omega \to \mathbb{R}$$

- Solution map is

$$S_{(\lambda_a,\lambda_b)} : \Omega \to \mathbb{U}_a \times \mathbb{U}_b \times \Omega$$

- The composition of criteria with the solution map provides random variables

$$j_a \circ S_{(\lambda_a,\lambda_b)} : \Omega \to \mathbb{R}$$

$$j_b \circ S_{(\lambda_a,\lambda_b)} : \Omega \to \mathbb{R}$$

Pure Bayesian Nash equilibrium

We recall that player a has belief $\mathcal{P}_a$ and player b has belief $\mathcal{P}_b$

Bayesian Nash equilibrium

We say that the couple $(\overline{\lambda}_a, \overline{\lambda}_b) \in \Lambda_a^{ad} \times \Lambda_b^{ad}$ of admissible strategies is a Bayesian Nash equilibrium if (in case of payoffs)

$$\mathcal{E}_{\mathcal{P}_a}\left[j_a \circ S_{(\overline{\lambda}_a, \overline{\lambda}_b)}\right] \geq \mathcal{E}_{\mathcal{P}_a}\left[j_a \circ S_{(\lambda_a, \overline{\lambda}_b)}\right] , \quad \forall \lambda_a \in \Lambda_a^{ad}$$

$$\mathcal{E}_{\mathcal{P}_b}\left[j_b \circ S_{(\overline{\lambda}_a, \overline{\lambda}_b)}\right] \geq \mathcal{E}_{\mathcal{P}_b}\left[j_b \circ S_{(\overline{\lambda}_a, \lambda_b)}\right] , \quad \forall \lambda_b \in \Lambda_b^{ad}$$

Mixed Bayesian Nash equilibrium

We say that the couple of mixed admissible strategies

$$(\overline{\mu}_a, \overline{\mu}_b) \in \Delta(\Lambda_a^{ad}) \times \Delta(\Lambda_b^{ad})$$

is a Bayesian Nash equilibrium if (in case of payoffs)

$$\int_{\Lambda_a^{ad} \times \Lambda_b^{ad}} \overline{\mu}_a(d\lambda_a) \otimes \overline{\mu}_b(d\lambda_b) \mathcal{E}_{\mathcal{P}_a}\left[j_a \circ S_{(\lambda_a, \lambda_b)}\right] \geq$$

$$\int_{\Lambda_a^{ad} \times \Lambda_b^{ad}} \mu_a(d\lambda_a) \otimes \overline{\mu}_b(d\lambda_b) \mathcal{E}_{\mathcal{P}_a}\left[j_a \circ S_{(\lambda_a, \lambda_b)}\right] , \ \forall \mu_a \in \Delta(\Lambda_a^{ad})$$

$$\int_{\Lambda_a^{ad} \times \Lambda_b^{ad}} \overline{\mu}_a(d\lambda_a) \otimes \overline{\mu}_b(d\lambda_b) \mathcal{E}_{\mathcal{P}_b}\left[j_b \circ S_{(\lambda_a, \lambda_b)}\right] \geq$$

$$\int_{\Lambda_a^{ad} \times \Lambda_b^{ad}} \overline{\mu}_a(d\lambda_a) \otimes \mu_b(d\lambda_b) \mathcal{E}_{\mathcal{P}_b}\left[j_b \circ S_{(\overline{\lambda}_a, \lambda_b)}\right] , \ \forall \mu_a \in \Delta(\Lambda_b^{ad})$$

Outline of the presentation

Principal-agent models with two players

- A branch of Economics studies so-called principal-agent models
- Principal-agent models display a general information structure, which can be transparently expressed thanks to Witsenhausen intrinsic model
- The model exhibits two players
  - the principal $\mathtt{Pr}$ (leader), makes decisions $u_{\mathtt{Pr}} \in \mathbb{U}_{\mathtt{Pr}}$, where the set of decisions is equipped with a σ-field $\mathcal{U}_{\mathtt{Pr}}$
  - the agent $\mathtt{Ag}$ (follower) makes decisions $u_{\mathtt{Ag}} \in \mathbb{U}_{\mathtt{Ag}}$, where the set of decisions is equipped with a σ-field $\mathcal{U}_{\mathtt{Ag}}$
- and Nature, corresponding to private information (or type) of the agent $\mathtt{Ag}$
  - Nature selects $\omega \in \Omega$, where Ω is equipped with a σ-field $\mathcal{F}$

Here is the most general information structure of principal-agent models

$$\mathfrak{I}_{\mathtt{Pr}} \subset \mathfrak{U}_{\mathtt{Ag}} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \mathfrak{F}$$

$$\mathfrak{I}_{\mathtt{Ag}} \subset \{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \mathfrak{U}_{\mathtt{Pr}} \otimes \mathfrak{F}$$

- By these expressions of the information fields
  - $\mathfrak{I}_{\mathtt{Pr}}$ of the principal $\mathtt{Pr}$ (leader)
  - $\mathfrak{I}_{\mathtt{Ag}}$ of the agent $\mathtt{Ag}$ (follower)
- we have excluded self-information, that is, we suppose that the information of a player cannot be influenced by his actions

Classical information patterns in game theory

Now, we will make the information structure more specific

- Stackelberg leadership model
- Moral hazard
- Adverse selection
- Signaling

Stackelberg leadership model

- In the Stackelberg leadership model of game theory,
- the follower $\mathtt{Ag}$ may partly observe the action of the leader $\mathtt{Pr}$

$$\mathcal{I}_{\mathtt{Ag}} \subset \{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \mathcal{U}_{\mathtt{Pr}} \otimes \mathcal{F}$$

- whereas the leader $\mathtt{Pr}$ observes at most the state of Nature

$$\mathcal{I}_{\mathtt{Pr}} \subset \{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \mathcal{F}$$

- As a consequence, the system is sequential
  - with the principal $\mathtt{Pr}$ as first player (leader)
  - and the agent $\mathtt{Ag}$ as second player (follower)
- Stackelberg games can be solved by bi-level optimization, for some information structures, like when

$$\mathcal{I}_{\mathtt{Pr}} \vee \{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \mathcal{U}_{\mathtt{Pr}} \otimes \{\emptyset, \Omega\} \subset \mathcal{I}_{\mathtt{Ag}}$$

Moral hazard

- An insurance company (the principal $\mathtt{Pr}$) cannot observe the efforts of the insured (the agent $\mathtt{Ag}$) to avoid risky behavior
- The firm faces the hazard that insured persons behave "immorally" (playing with matches at home)
- Moral hazard (hidden action) occurs when the decisions of the agent $\mathtt{Ag}$ are hidden to the principal $\mathtt{Pr}$

$$\mathcal{I}_{\mathtt{Pr}} \subset \{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \mathcal{F}$$

- In case of moral hazard, the system is sequential with the principal as first player, (which does not preclude to choose the agent as first player in some special cases, as in a static team situation)
- Moral hazard games can be solved by bi-level optimization, for some information structures

Adverse selection

- In the absence of observable information on potential customers (the agent $\mathtt{Ag}$), an insurance company (the principal $\mathtt{Pr}$) offers a unique price for a contract hence screens and selects the "bad" ones
- Adverse selection occurs when
  - the agent $\mathtt{Ag}$ knows the state of nature (his type, or private information)

$$\{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \mathcal{F} \subset \mathcal{I}_{\mathtt{Ag}}$$

    (the agent $\mathtt{Ag}$ can possibly observe the principal $\mathtt{Pr}$ action)
  - but the principal $\mathtt{Pr}$ does not know the state of nature

$$\mathcal{I}_{\mathtt{Pr}} \subset \mathcal{U}_{\mathtt{Ag}} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \{\emptyset, \Omega\}$$

    (the principal $\mathtt{Pr}$ can possibly observe the agent $\mathtt{Ag}$ action)
- In case of adverse selection, the system may or may not be sequential

Signaling

- In biology, a peacock signals its "good genes" (genotype) by its lavish tail (phenotype)
- In economics, a worker signals his working ability (productivity) by his educational level (diplomas)
- There is room for signaling
  - when the agent $\mathtt{Ag}$ knows the state of nature (his type)

$$\{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \mathcal{F} \subset \mathfrak{I}_{\mathtt{Ag}}$$

    (the agent $\mathtt{Ag}$ can possibly observe the principal $\mathtt{Pr}$ action)
  - whereas the principal $\mathtt{Pr}$ does not know the state of nature, but the principal $\mathtt{Pr}$ observes the agent $\mathtt{Ag}$ action

$$\mathfrak{I}_{\mathtt{Pr}} = \mathfrak{U}_{\mathtt{Ag}} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \{\emptyset, \Omega\}$$

  as the agent $\mathtt{Ag}$ may reveal his type by his decision which is observable by the principal $\mathtt{Pr}$

Signaling

- The system is sequential (with the agent as first player) when

$$\mathcal{I}_{\mathtt{Ag}} = \{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \mathcal{F}$$

- The system is non causal when

$$\{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \mathcal{F} \subsetneq \mathcal{I}_{\mathtt{Ag}} \subset \{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \mathcal{U}_{\mathtt{Pr}} \otimes \mathcal{F}$$

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Open questions

Stackelberg leadership model

- In the Stackelberg leadership model of game theory, we consider a leader $\mathtt{Pr}$ (principal) and a follower $\mathtt{Ag}$ (agent)
- We suppose that $\langle \mathtt{Pr} \rangle_{\mathfrak{P}} = \emptyset$, that is, leader $\mathtt{Pr}$ plays first,

$$\mathcal{I}_{\mathtt{Pr}} \subset \{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \mathcal{F}$$

- and that $\langle \mathtt{Ag} \rangle_{\mathfrak{P}} \subset \{\mathtt{Pr}\}$, that is, follower $\mathtt{Ag}$ plays second

$$\mathcal{I}_{\mathtt{Ag}} \subset \{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \mathcal{U}_{\mathtt{Pr}} \otimes \mathcal{F}$$

We work on a reduced history space

- As both information fields — $\mathfrak{I}_{\mathtt{Pr}} \subset \{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \mathcal{F}$ and $\mathfrak{I}_{\mathtt{Ag}} \subset \{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \mathcal{U}_{\mathtt{Pr}} \otimes \mathcal{F}$ — do not depend on $\mathcal{U}_{\mathtt{Ag}}$, the actions of the follower Ag (agent) do not fuel strategies (via information), so that we introduce
- the reduced history space $\widetilde{\mathbb{H}}$ (without the actions of the follower Ag) equipped with the reduced history field $\widetilde{\mathcal{H}}$

$$\widetilde{\mathbb{H}} = \mathbb{U}_{\mathtt{Pr}} \times \Omega \,, \quad \widetilde{\mathcal{H}} = \mathcal{U}_{\mathtt{Pr}} \otimes \mathcal{F}$$

- and the reduced information fields $\widetilde{\mathfrak{I}}_{\mathtt{Pr}}$ and $\widetilde{\mathfrak{I}}_{\mathtt{Ag}}$ defined by

$$\begin{aligned} \mathfrak{I}_{\mathtt{Pr}} &= \{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \widetilde{\mathfrak{I}}_{\mathtt{Pr}} \quad \text{with } \widetilde{\mathfrak{I}}_{\mathtt{Pr}} \subset \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \mathcal{F} \subset \widetilde{\mathcal{H}} \\ \mathfrak{I}_{\mathtt{Ag}} &= \{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \widetilde{\mathfrak{I}}_{\mathtt{Ag}} \quad \text{with } \widetilde{\mathfrak{I}}_{\mathtt{Ag}} \subset \mathcal{U}_{\mathtt{Pr}} \otimes \mathcal{F} = \widetilde{\mathcal{H}} \end{aligned}$$

Here is what become the admissible strategies on the reduced history space

We consider a couple $(\lambda_{\mathtt{Ag}}, \lambda_{\mathtt{Pr}}) \in \Lambda^{ad}_{\mathtt{Ag}} \times \Lambda^{ad}_{\mathtt{Pr}}$ of admissible strategies

- As $\mathcal{I}_{\mathtt{Pr}} = \{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \widetilde{\mathcal{I}}_{\mathtt{Pr}}$ with $\widetilde{\mathcal{I}}_{\mathtt{Pr}} \subset \{\emptyset, \mathbb{U}_{\mathtt{Pr}}\} \otimes \mathcal{F}$, the strategy $\lambda_{\mathtt{Pr}}$ of the leader $\mathtt{Pr}$ can be identified with

$$\widetilde{\lambda}_{\mathtt{Pr}} : \Omega \to \mathbb{U}_{\mathtt{Pr}}$$

  (indeed, the strategies of the leader $\mathtt{Pr}$ depend at most upon Nature)

- As $\mathcal{I}_{\mathtt{Ag}} = \{\emptyset, \mathbb{U}_{\mathtt{Ag}}\} \otimes \widetilde{\mathcal{I}}_{\mathtt{Ag}}$, with $\widetilde{\mathcal{I}}_{\mathtt{Ag}} \subset \mathcal{U}_{\mathtt{Pr}} \otimes \mathcal{F}$, the strategy $\lambda_{\mathtt{Ag}}$ of the follower $\mathtt{Ag}$ can be identified with

$$\widetilde{\lambda}_{\mathtt{Ag}} : \mathbb{U}_{\mathtt{Pr}} \times \Omega \to \mathbb{U}_{\mathtt{Ag}}$$

Therefore, we can work with reduced admissible strategies

$$(\widetilde{\lambda}_{\mathtt{Ag}}, \widetilde{\lambda}_{\mathtt{Pr}}) \in \widetilde{\Lambda}^{ad}_{\mathtt{Ag}} \times \widetilde{\Lambda}^{ad}_{\mathtt{Pr}}$$

Strategy independence of conditional expectation (SICE)

Assumption SICE

There exists a probability $\mathbb{Q}$ on $\widetilde{\mathbb{H}} = \mathbb{U}_{\mathtt{Pr}} \times \Omega$ such that

$$\mathcal{P}_{\mathtt{Ag}} \circ S_{\widetilde{\lambda}_{\mathtt{Pr}}}^{-1} = T_{\widetilde{\lambda}_{\mathtt{Pr}}} \mathbb{Q} \text{ with } \mathcal{E}_{\mathbb{Q}}\left[T_{\widetilde{\lambda}_{\mathtt{Pr}}} \mid \widetilde{\mathfrak{I}}_{\mathtt{Ag}} \right] > 0 \,, \quad \forall \widetilde{\lambda}_{\mathtt{Pr}} \in \widetilde{\Lambda}_{\mathtt{Pr}}^{ad}$$

and that, under $\mathbb{Q}$, the conditional expected gain of the follower $\mathtt{Ag}$ does not change when one adds to his information both the actions and the information available of the leader $\mathtt{Pr}$, namely

$$\mathcal{E}_{\mathbb{Q}}\left[j_{\mathtt{Ag}}(u_{\mathtt{Ag}}, \cdot) \mid \widetilde{\mathfrak{I}}_{\mathtt{Ag}} \right] = \mathcal{E}_{\mathbb{Q}}\left[j_{\mathtt{Ag}}(u_{\mathtt{Ag}}, \cdot) \mid \widetilde{\mathfrak{I}}_{\mathtt{Ag}} \vee \widetilde{\mathfrak{I}}_{\mathtt{Pr}} \vee \widetilde{\mathfrak{D}}_{\mathtt{Pr}} \right] \,, \quad \forall u_{\mathtt{Ag}} \in \mathbb{U}_{\mathtt{Ag}}$$

Bayesian Nash equilibria can be obtained by bi-level optimization under assumption SICE

Suppose assumption SICE holds true

- The (upper level) optimization problem for the follower $\mathtt{Ag}$

$$\min_{u_{\mathtt{Ag}} \in \mathbb{U}_{\mathtt{Ag}}} \mathcal{E}_{\mathbb{Q}} \left[j_{\mathtt{Ag}}(u_{\mathtt{Ag}}, \cdot) \mid \widetilde{\mathfrak{I}}_{\mathtt{Ag}} \right]$$

  provides (under technical assumptions, by a measurable selection theorem) an $\widetilde{\mathfrak{I}}_{\mathtt{Ag}}$-measurable solution

$$\overline{\overline{\lambda}}_{\mathtt{Ag}} : \mathbb{U}_{\mathtt{Pr}} \times \Omega \to \mathbb{U}_{\mathtt{Ag}} \ , \quad \sigma(\overline{\overline{\lambda}}_{\mathtt{Ag}}) \subset \widetilde{\mathfrak{I}}_{\mathtt{Ag}}$$

- Then, the (lower level) optimization problem for the leader $\mathtt{Pr}$ is

$$\min_{\widetilde{\lambda}_{\mathtt{Pr}} \in \widetilde{\Lambda}^{ad}_{\mathtt{Pr}}} \mathcal{E}_{\mathcal{P}_{\mathtt{Pr}}} \left[j_{\mathtt{Pr}} \circ S_{(\overline{\overline{\lambda}}_{\mathtt{Ag}}, \widetilde{\lambda}_{\mathtt{Pr}})} \right]$$

Here is what becomes the solution map on the reduced history space

- By sequentiality, the solution map $S_{(\lambda_{\mathtt{Ag}},\lambda_{\mathtt{Pr}})}$ satisfies the co-cycle property

$$S_{(\lambda_{\mathtt{Ag}},\lambda_{\mathtt{Pr}})} = (\lambda_{\mathtt{Ag}} \circ S_{\lambda_{\mathtt{Pr}}}, S_{\lambda_{\mathtt{Pr}}}) = (\lambda_{\mathtt{Ag}}, \mathrm{Id}_{\mathbb{U}_{\mathtt{Pr}}\times\Omega}) \circ S_{\lambda_{\mathtt{Pr}}}$$

- If we introduce a reduced solution map $S_{\widetilde{\lambda}_{\mathtt{Pr}}} = \big(\widetilde{\lambda}_{\mathtt{Pr}}, \mathrm{Id}_{\Omega}\big)$

$$\Omega \xrightarrow{S_{\widetilde{\lambda}_{\mathtt{Pr}}}} \mathbb{U}_{\mathtt{Pr}} \times \Omega \, , \quad \omega \mapsto \big(\widetilde{\lambda}_{\mathtt{Pr}}(\omega), \omega\big) \, ,$$

we can now write $S_{(\lambda_{\mathtt{Ag}},\lambda_{\mathtt{Pr}})} = \big(\widetilde{\lambda}_{\mathtt{Ag}}, \mathrm{Id}_{\mathbb{U}_{\mathtt{Pr}}\times\Omega}\big) \circ S_{\widetilde{\lambda}_{\mathtt{Pr}}}$, that is,

$$S_{(\lambda_{\mathtt{Ag}},\lambda_{\mathtt{Pr}})} : \Omega \xrightarrow{S_{\widetilde{\lambda}_{\mathtt{Pr}}}} \mathbb{U}_{\mathtt{Pr}} \times \Omega \xrightarrow{(\widetilde{\lambda}_{\mathtt{Ag}}, \mathrm{Id}_{\mathbb{U}_{\mathtt{Pr}}\times\Omega})} \mathbb{U}_{\mathtt{Ag}} \times \mathbb{U}_{\mathtt{Pr}} \times \Omega$$

that is,

$$S_{(\lambda_{\mathtt{Ag}},\lambda_{\mathtt{Pr}})} : \omega \mapsto \big(\widetilde{\lambda}_{\mathtt{Pr}}(\omega), \omega\big) \mapsto \Big(\widetilde{\lambda}_{\mathtt{Ag}}\big(\widetilde{\lambda}_{\mathtt{Pr}}(\omega), \omega\big), \widetilde{\lambda}_{\mathtt{Pr}}(\omega), \omega\Big)$$

Strategy independence of conditional expectation (SICE)

Assumption SICE

There exists a probability $\mathbb{Q}$ on $\widetilde{\mathbb{H}} = \mathbb{U}_{\mathtt{Pr}} \times \Omega$ such that

$$\mathcal{P}_{\mathtt{Ag}} \circ S_{\widetilde{\lambda}_{\mathtt{Pr}}}^{-1} = T_{\widetilde{\lambda}_{\mathtt{Pr}}} \mathbb{Q} \text{ with } \mathcal{E}_{\mathbb{Q}}\big[T_{\widetilde{\lambda}_{\mathtt{Pr}}} \mid \widetilde{\mathfrak{I}}_{\mathtt{Ag}}\big] > 0 \,, \quad \forall \widetilde{\lambda}_{\mathtt{Pr}} \in \widetilde{\Lambda}_{\mathtt{Pr}}^{ad}$$

and that

$$\mathcal{E}_{\mathbb{Q}}\big[j_{\mathtt{Ag}}(u_{\mathtt{Ag}}, \cdot) \mid \widetilde{\mathfrak{I}}_{\mathtt{Ag}}\big] = \mathcal{E}_{\mathbb{Q}}\big[j_{\mathtt{Ag}}(u_{\mathtt{Ag}}, \cdot) \mid \widetilde{\mathfrak{I}}_{\mathtt{Ag}} \vee \widetilde{\mathfrak{I}}_{\mathtt{Pr}} \vee \widetilde{\mathfrak{D}}_{\mathtt{Pr}}\big] \,, \quad \forall u_{\mathtt{Ag}} \in \mathbb{U}_{\mathtt{Ag}}$$

Under assumption SICE, we have that

$$\begin{aligned}
\mathcal{E}_{\mathcal{P}_a}\Big[j_a \circ S_{(\lambda_a, \lambda_b)}\Big] &= \mathcal{E}_{\mathcal{P}_a}\Big[j_a \circ (\widetilde{\lambda}_{\mathtt{Ag}}, \mathrm{Id}_{\mathbb{U}_{\mathtt{Pr}} \times \Omega}) \circ S_{\widetilde{\lambda}_{\mathtt{Pr}}}\Big] \\
&= \mathcal{E}_{\mathcal{P}_{\mathtt{Ag}} \circ S_{\widetilde{\lambda}_{\mathtt{Pr}}}^{-1}}\Big[j_a \circ (\widetilde{\lambda}_{\mathtt{Ag}}, \mathrm{Id}_{\mathbb{U}_{\mathtt{Pr}} \times \Omega})\Big] \\
&= \mathcal{E}_{\mathbb{Q}}\Big[T_{\widetilde{\lambda}_{\mathtt{Pr}}} j_a \circ (\widetilde{\lambda}_{\mathtt{Ag}}, \mathrm{Id}_{\mathbb{U}_{\mathtt{Pr}} \times \Omega})\Big]
\end{aligned}$$

Bayesian Nash equilibrium under assumption SICE

Bayesian Nash equilibrium

Under assumption SICE,
the couple $(\overline{\widetilde{\lambda}}_{\mathtt{Ag}}, \overline{\widetilde{\lambda}}_{\mathtt{Pr}}) \in \widetilde{\Lambda}^{ad}_{\mathtt{Ag}} \times \widetilde{\Lambda}^{ad}_{\mathtt{Pr}}$ of reduced admissible strategies
is a Bayesian Nash equilibrium if (in case of payoffs)

$$\begin{aligned}
\mathcal{E}_{\mathbb{Q}}\left[j_{\mathtt{Ag}} \circ (\overline{\widetilde{\lambda}}_{\mathtt{Ag}}, \mathrm{Id}_{\mathbb{U}_{\mathtt{Pr}} \times \Omega})\right] \geq \quad & \mathcal{E}_{\mathbb{Q}}\left[j_{\mathtt{Ag}} \circ (\widetilde{\lambda}_{\mathtt{Ag}}, \mathrm{Id}_{\mathbb{U}_{\mathtt{Pr}} \times \Omega})\right] \\
& \forall \widetilde{\lambda}_{\mathtt{Ag}} \in \widetilde{\Lambda}^{ad}_{\mathtt{Ag}}
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_{\mathcal{P}_{\mathtt{Pr}}}\left[j_{\mathtt{Pr}} \circ S_{(\overline{\widetilde{\lambda}}_{\mathtt{Ag}}, \overline{\widetilde{\lambda}}_{\mathtt{Pr}})}\right] \geq \quad & \mathcal{E}_{\mathcal{P}_{\mathtt{Pr}}}\left[j_{\mathtt{Pr}} \circ S_{(\overline{\widetilde{\lambda}}_{\mathtt{Ag}}, \widetilde{\lambda}_{\mathtt{Pr}})}\right] \\
& \forall \widetilde{\lambda}_{\mathtt{Pr}} \in \widetilde{\Lambda}^{ad}_{\mathtt{Pr}}
\end{aligned}$$

There exists an optimal strategy of the follower $\mathtt{Ag}$ that does not depend on the leader $\mathtt{Pr}$ strategy

$$\min_{\widetilde{\lambda}_{\mathtt{Ag}} \in \widetilde{\Lambda}^{ad}_{\mathtt{Ag}}} \mathcal{E}_{\mathbb{Q}} \Big[j_{\mathtt{Ag}} \circ (\widetilde{\lambda}_{\mathtt{Ag}}, \mathrm{Id}_{\mathbb{U}_{\mathtt{Pr}} \times \Omega}) \Big] = \min_{\widetilde{\lambda}_{\mathtt{Ag}} \,,\; \widetilde{\lambda}_{\mathtt{Ag}}^{-1}(\mathcal{U}_{\mathtt{Ag}}) \subset \widetilde{\mathcal{I}}_{\mathtt{Ag}}} \mathcal{E}_{\mathbb{Q}} \Big[j_{\mathtt{Ag}} \circ (\widetilde{\lambda}_{\mathtt{Ag}}, \mathrm{Id}_{\mathbb{U}_{\mathtt{Pr}} \times \Omega}) \Big]$$

$$= \mathcal{E}_{\mathbb{Q}} \Big[\min_{u_{\mathtt{Ag}} \in \mathbb{U}_{\mathtt{Ag}}} \mathcal{E}_{\mathbb{Q}} \big[j_{\mathtt{Ag}}(u_{\mathtt{Ag}}, \cdot) \mid \widetilde{\mathcal{I}}_{\mathtt{Ag}} \big] \Big]$$

Outline of the presentation

Introduction to games in intrinsic form

Witsenhausen intrinsic model and game theory with information

Nash equilibrium with information

Witsenhausen intrinsic model and principal-agent models

Games solvable by dynamic programming

Open questions

Research questions

- How should we talk about games using WIM?
  - Can we extend the Bayesian Nash Equilibrium concept to general risk measures?
  - How does the notion of subgame perfect Nash equilibrium translate within this framework?
- WIM: game theoretical results
  - What would a Nash theorem be in the WIM setting?
  - When do we have a generalized "backward induction" mechanism?
  - Under proper sufficient conditions on the information structure (extension of perfect recall), can we restrict the search among behavioral strategies instead of mixed strategies?
- Applications of WIM
  - Can we re-organize the games bestiary using WIM?
  - Can we use the WIM framework for mechanism design?

We obtain a Nash theorem in the WIM setting

Theorem

Any finite, solvable, Witsenhausen game has a mixed NE

Proof

- The set of policies is finite, as policies map the finite history set towards finite decision sets
- To each policy profile, we associate a payoff vector
- We thus obtain a matrix game and we can apply Nash theorem

Generalized existence result

Step one, discretization

- We introduce $g_a^{(n)}$ the injection from $\mathbb{U}_a^{(n)}$ into $\mathbb{U}_a$

$$g_a^{(n)} : \mathbb{U}_a^{(n)} \hookrightarrow \mathbb{U}_a$$

- We introduce $h^{(n)}$ that maps $\mathbb{H}$ into $\mathbb{H}^{(n)}$ with $h^{(n)}_{\mathbb{H}^{(n)}} = Id_{\mathbb{H}^{(n)}}$
- $(\lambda_a^{ad})^{(n)} = \{\lambda_a \in \lambda_a^{(n)}, \sigma(g_a^{(n)} \circ \lambda_a \circ h^{(n)}) \subseteq \mathcal{I}_a\}$

Current difficulties:

- Definition of the discretization, in particular $h^{(n)}$, to obtain a limit
- Continuity of the solution map

$$\Lambda_A \times \Omega \to \mathbb{H}\ , \ \ (\lambda, \omega) \mapsto S_\lambda(\omega)$$

Behavioral vs mixed strategies

- Mixed strategies are
$$\prod_{p\in P} \Delta\Big(\prod_{a\in A_p} \Lambda_a^{ad}\Big)$$
and reflect the synchronization of his agents by the player
- Behavioral strategies are
$$\prod_{p\in P} \prod_{a\in A_p} \Delta\big(\Lambda_a^{ad}\big)$$
and they do not require any correlating procedure
- Under proper sufficient conditions on the information structure, we expect to prove that some games can be solved over the smaller set of behavioral strategies instead of the large set of mixed strategies
$$\underbrace{\prod_{p\in P} \prod_{a\in A_p} \Delta\big(\Lambda_a^{ad}\big)}_{behavioral} \subset \underbrace{\prod_{p\in P} \Delta\Big(\prod_{a\in A_p} \Lambda_a^{ad}\Big)}_{mixed}$$

Applications

- The WIM is of particular interest for non sequential games
- In particular we envision applications for networks, auctions and decentralized energy systems

Mechanism design presented in the intrinsic framework

- The designer (= principal) can extend the natural history set, by offering new decisions to every agent (messages)
- He is free to extend the information fields of the agents as he wishes
- He can partly shape the objective functions of the players

Thank you :-)