Witsenhausen intrinsic model for stochastic control

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Outline of the presentation

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H. S. Witsenhausen. On information structures, feedback and causality. *SIAM J. Control*, 9(2):149–160, May 1971.

Sequentiality and perfect memory are tacit assumptions in control-oriented works on dynamic games

In control-oriented works on dynamic games (in particular, stochastic control problems) one usually finds a "dynamic equation" describing the evolution of a "state" in response to decision (control) variables of the players and to random variables. One also finds "output equations" which define output variables for a player as functions of the state, decision and random variables. Then the information structure is defined by allowing each decision variable to be any desired (measurable) function of the output variables generated for that player up to that time. Such a setup assumes that the time order in which the various decisions variables are selected is fixed in advance. It assumes that each player acts as if he had responsability only for one station. It assumes that this station has perfect memory.

Going beyond sequentiality and perfect memory

For large complex systems these tacit assumptions are unlikely to hold. (...) The order in which the various agents of the various organizations will have to act cannot always be predicted, and the information available to different agents, even of the same organization, may be noncomparable in the sense that, of two agents, neither one knows everything his colleague knows.

Kuhn's answer: games in extensive form

These difficulties in specifying the information structure of a game were faced and overcome in the early days of game theory

- Von Neumann and Morgenstern (1944)
  - fixed sequencing of decisions
  - variables range over finite sets
- Kuhn (1953)
  - removes the restriction of fixed sequencing of decisions
  - variables range over finite sets
- Aumann (1964)
  - fixed sequencing of decisions
  - variables range over measurable sets

Witsenhausen's answer: games as multiple feedback loops

> *The decision process is considered as a feedback loop and the game is characterized by its interaction with the policies of the agents, without prejudging questions of chronological order.*

In the Kuhn formulation,

> *the tree describing the game is an expression of the general solution of the closed loop relations. (These relations map information into decisions by the policies, and decisions into information by the rules of the game). For any combination of policies one can find the corresponding outcome by following the tree along selected branches, and this is an explicit procedure. Thus the difficulties that might arise in solving the loop have been eliminated by defining the game in terms of a general unique solution which must be found before the model can be set up.*

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Outline of the presentation

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We will distinguish an individual from an agent

- An individual who makes a first, followed by a second decision, is represented by two agents (two decision makers)
- An individual who makes a sequence of decisions — one for each period $t = 0, 1, 2, \ldots, T-1$ — is represented by T agents, labelled $t = 0, 1, 2, \ldots, T-1$
- N individuals — each i of whom makes a sequence of decisions, one for each period $t = 0, 1, 2, \ldots, T_i - 1$ — is represented by $\prod_{i=1}^{N} T_i$ agents, labelled by

$$(i, t) \in \bigcup_{j=1}^{N} \{j\} \times \{0, 1, 2, \ldots, T_j - 1\}$$

Agents and decisions

- Let $\mathbb{A}$ be a finite set, whose elements are called agents (or decision-makers)
- Each agent $a \in \mathbb{A}$ is supposed to make one decision

$$u_a \in \mathbb{U}_a$$

- where $\mathbb{U}_a$ is the set of decisions for agent a
- and is equipped with a σ-field $\mathcal{U}_a$

Decision space

- We define the decision space as the product set

$$\mathbb{U}_{\mathbb{A}} = \prod_{b \in \mathbb{A}} \mathbb{U}_b$$

- equipped with the product decision field

$$\mathcal{U}_{\mathbb{A}} = \bigotimes_{b \in \mathbb{A}} \mathcal{U}_b$$

States of Nature

- A state of Nature (or uncertainty, or scenario) is

$$\omega \in \Omega$$

- where Ω is a measurable set, the sample space,
- equipped with a σ-field $\mathcal{F}$
  (at this stage of the presentation, we do not need probability distribution, as we focus only on information)

History space

- The history space is the product space

$$\mathbb{H} = \mathbb{U}_{\mathbb{A}} \times \Omega = \prod_{b \in \mathbb{A}} \mathbb{U}_b \times \Omega$$

- equipped with the product history field

$$\mathcal{H} = \mathcal{U}_{\mathbb{A}} \otimes \mathcal{F} = \bigotimes_{b \in \mathbb{A}} \mathcal{U}_b \otimes \mathcal{F}$$

One agent, two possible decisions, two states of Nature

- Agents
$$\mathbb{A} = \{a\}$$
- Decision set and field
$$\mathbb{U}_a = \{u_a^1, u_a^2\} \ , \ \ \mathcal{U}_a = \{\emptyset, \{u_a^1, u_a^2\}, \{u_a^1\}, \{u_a^2\}\}$$
- Sample space and field
$$\Omega = \{\omega^1, \omega^2\} \ , \ \ \mathcal{F} = \{\emptyset, \{\omega^1, \omega^2\}, \{\omega^1\}, \{\omega^2\}\}$$
- History space and field
$$\mathbb{H} = \mathbb{U}_a \times \Omega = \{u_a^1, u_a^2\} \times \{\omega^1, \omega^2\} \ , \ \ \mathcal{H} = 2^{\mathbb{H}}$$

Two agents, two possible decisions, two states of Nature

- Agents

$$\mathbb{A} = \{a, b\}$$

- Decision sets and fields

$$\mathbb{U}_a = \{u_a^1, u_a^2\} \ , \ \ \mathcal{U}_a = \{\emptyset, \{u_a^1, u_a^2\}, \{u_a^1\}, \{u_a^2\}\}$$

  and

$$\mathbb{U}_b = \{u_b^1, u_b^2\} \ , \ \ \mathcal{U}_b = \{\emptyset, \{u_b^1, u_b^2\}, \{u_b^1\}, \{u_b^2\}\}$$

- Sample space and field

$$\Omega = \{\omega^1, \omega^2\} \ , \ \ \mathcal{F} = \{\emptyset, \{\omega^1, \omega^2\}, \{\omega^1\}, \{\omega^2\}\}$$

- History space and field

$$\mathbb{H} = \mathbb{U}_a \times \mathbb{U}_b \times \Omega = \{u_a^1, u_a^2\} \times \{u_b^1, u_b^2\} \times \{\omega^1, \omega^2\} \ , \ \ \mathcal{H} = 2^{\mathbb{H}}$$

Outline of the presentation

Information fields

- The information field of agent $a \in \mathbb{A}$ is a σ-field

$$\mathcal{I}_a \subset \mathcal{H}$$

- In this representation, $\mathcal{I}_a$ is a subfield of the history field $\mathcal{H}$ which represents the information available to agent a when he makes a decision
- Therefore, the information of agent a may depend
  - on the states of Nature
  - and on other agents' decisions

Stochastic system

Stochastic system

A stochastic system is a collection consisting of

- a finite set $\mathbb{A}$ of agents,
- states of Nature $(\Omega, \mathcal{F})$,
- decision sets, fields and information fields

$$\{\mathbb{U}_a, \mathcal{U}_a, \mathcal{I}_a\}_{a \in \mathbb{A}}$$

One agent, two possible decisions, two states of Nature

- History space and field

$$\mathbb{H} = \mathbb{U}_a \times \Omega = \{u_a^1, u_a^2\} \times \{\omega^1, \omega^2\}\ , \ \ \mathcal{H} = 2^{\mathbb{H}}$$

- Agent a knows nothing

$$\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \Omega\} = \{\emptyset, \{u_a^1, u_a^2\}\} \otimes \{\emptyset, \{\omega^1, \omega^2\}\}$$

- Agent a knows the state of Nature

$$\begin{aligned}\mathcal{I}_a =& \{\emptyset, \mathbb{U}_a\} \otimes 2^{\Omega} \\ =& \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \{\omega^1, \omega^2\}, \{\omega^1\}, \{\omega^2\}\} \\ =& \underbrace{\{\emptyset, \{u_a^1, u_a^2\}\}}_{\text{undistinguishable}} \otimes \underbrace{\{\emptyset, \{\omega^1, \omega^2\}, \{\omega^1\}, \{\omega^2\}\}}_{\text{distinguishable}}\end{aligned}$$

Two agents, two possible decisions, two states of Nature
Nested information fields

- History space and field

$$\mathbb{H} = \mathbb{U}_a \times \mathbb{U}_b \times \Omega = \{u_a^1, u_a^2\} \times \{u_b^1, u_b^2\} \times \{\omega^1, \omega^2\} \ , \ \ \mathcal{H} = 2^{\mathbb{H}}$$

- Agent a knows the state of Nature

$$\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \times \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \{\omega^1, \omega^2\}, \{\omega^1\}, \{\omega^2\}\}$$

  and agent b knows the state of Nature and what agent a does

$$\mathcal{I}_b = \{\emptyset, \{u_a^1, u_a^2\}, \{u_a^1\}, \{u_a^2\}\} \times \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \{\omega^1, \omega^2\}, \{\omega^1\}, \{\omega^2\}\}$$

- In this example, information fields are nested

$$\mathcal{I}_a \subset \mathcal{I}_b$$

  meaning that agent b knows what agent a knows

Two agents, two decisions, two states of Nature
Non nested information fields

- History space and field

$$\mathbb{H} = \mathbb{U}_a \times \mathbb{U}_b \times \Omega = \{u_a^1, u_a^2\} \times \{u_b^1, u_b^2\} \times \{\omega^1, \omega^2\} \ , \ \ \mathcal{H} = 2^{\mathbb{H}}$$

- Agent a only knows the state of Nature

$$\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \times \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \{\omega^1, \omega^2\}, \{\omega^1\}, \{\omega^2\}\}$$

  and agent b only knows what agent a does

$$\mathcal{I}_b = \{\emptyset, \{u_a^1, u_a^2\}, \{u_a^1\}, \{u_a^2\}\} \times \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \{\omega^1, \omega^2\}\}$$

- Information fields are not nested,
  as they cannot be compared by inclusion

Outline of the presentation

Handling subgroups of agents by means of cylindric extensions

- Let $C \subset \mathbb{A}$ be a subset of agents
- We introduce the subfield $\mathcal{U}_C$ of the decision field $\mathcal{U}_{\mathbb{A}}$

$$\mathcal{U}_C = \bigotimes_{c \in C} \mathcal{U}_c \otimes \bigotimes_{b \notin C} \{\emptyset, \mathbb{U}_b\} \subset \mathcal{U}_{\mathbb{A}}$$

- and the subfield $\mathcal{D}_C$ of the history field $\mathcal{H}$

$$\mathcal{D}_C = \mathcal{U}_C \otimes \{\emptyset, \Omega\} = \bigotimes_{c \in C} \mathcal{U}_c \otimes \bigotimes_{b \notin C} \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \Omega\} \subset \mathcal{U}_{\mathbb{A}} \otimes \mathcal{F} = \mathcal{H}$$

which contains the information provided by the decisions of the agents in the subset C

We will consider stochastic systems that display absence of self-information

Absence of self-information

A stochastic system displays absence of self-information when

$$\mathcal{I}_a \subset \mathcal{U}_{\mathbb{A}\setminus\{a\}} \otimes \mathcal{F}$$

for any agent $a \in \mathbb{A}$

- Absence of self-information means that the information of agent a may depend on the states of Nature and on all the other agents' decisions but not on his own decision
- Absence of self-information makes sense
  once we have distinguished an individual from an agent
  (else, it would lead to paradoxes)

Expressing dependencies

- The condition

  $$\mathcal{I}_b \subset \mathcal{I}_a$$

  formally expresses that what agent b knows is also known to agent a
- The condition

  $$\mathcal{D}_b \subset \mathcal{I}_a$$

  formally expresses that what does agent b is known to agent a

Outline of the presentation

Why the Witsenhausen intrinsic model?

Ingredients of Witsenhausen intrinsic model

Strategies, solvability and causality

Binary relations between agents

Typology of systems

Outline of the presentation

Strategies, policies, control laws, control designs

Strategy

A strategy (or policy, control law, control design) for agent a is a measurable mapping

$$\lambda_a : (\mathbb{H}, \mathcal{H}) \to (\mathbb{U}_a, \mathcal{U}_a)$$

that maps histories into decisions of agent a

We denote the set of strategies of agent a by

$$\Lambda_a = \left\{ \lambda_a : (\mathbb{H}, \mathcal{H}) \to (\mathbb{U}_a, \mathcal{U}_a) \;\middle|\; \lambda_a^{-1}(\mathcal{U}_a) \subset \mathcal{H} \right\}$$

and the set of strategies of all agents is

$$\Lambda_{\mathbb{A}} = \prod_{a \in \mathbb{A}} \Lambda_a$$

Information fuels admissible strategies

Admissible strategy

An admissible strategy for agent a is a mapping

$$\lambda_a : (\mathbb{H}, \mathcal{H}) \to (\mathbb{U}_a, \mathcal{U}_a)$$

which is measurable w.r.t. the information field $\mathcal{I}_a$ of agent a, that is,

$$\lambda_a^{-1}(\mathcal{U}_a) \subset \mathcal{I}_a$$

This condition expresses the property that
an admissible strategy for agent a
may only depend upon the information $\mathcal{I}_a$ available to him

Set of admissible strategies

We denote the set of admissible strategies of agent a by

$$\Lambda_a^{ad} = \left\{ \lambda_a : (\mathbb{H}, \mathcal{H}) \to (\mathbb{U}_a, \mathcal{U}_a) \;\middle|\; \lambda_a^{-1}(\mathcal{U}_a) \subset \mathcal{I}_a \right\}$$

and the set of admissible strategies of all agents is

$$\Lambda_{\mathbb{A}}^{ad} = \prod_{a \in \mathbb{A}} \Lambda_a^{ad}$$

Pure and mixed strategies

- What we have called a strategy,
  game theorists would call a pure strategy
- A mixed strategy (or randomized strategy) for agent a is an element of $\Delta(\Lambda_a)$, the set of probability distributions over the set of strategies of agent a
- A mixed admissible strategy (or randomized admissible strategy) for agent a is an element of $\Delta(\Lambda_a^{ad})$, the set of probability distributions over the set of admissible strategies of agent a

Outline of the presentation

Solvability (I)

- In the Witsenhausen's intrinsic model,
  agents make decisions in an order which is not fixed in advance
- Briefly speaking, solvability is the property that,
  for each state of Nature, the agents' decisions
  are uniquely determined by their admissible strategies
- The solvability property is crucial to develop Witsenhausen's theory:
  without the solvability property,
  we would not be able to determine the agents decisions

Solvability (II)

The solvability problem consists in finding

- for any collection $\lambda = \{\lambda_a\}_{a \in \mathbb{A}} \in \Lambda^{ad}_{\mathbb{A}}$ of admissible policies
- for any state of Nature $\omega \in \Omega$
- decisions $u \in \mathbb{U}_{\mathbb{A}}$ satisfying the implicit ("closed loop") equation

$$u = \lambda(u, \omega)$$

or, equivalently,

$$u_a = \lambda_a(\{u_b\}_{b \in \mathbb{A}}, \omega) \ , \quad \forall a \in \mathbb{A}$$

Solvability and information patterns (I)

- Existence and uniqueness of the solutions of $u = \lambda(u, \omega)$ are related to information patterns
- To illustrate the point, consider a stochastic system with two agents a and b, and displaying absence of self-information
- Assuming that the σ-fields $\mathcal{U}_a$, $\mathcal{U}_b$ and $\mathcal{F}$ contain singletons, admissible strategies λ_a and λ_b have the form

$$\lambda_a(u, \omega) = \widetilde{\lambda}_a(u_b, \omega) \;, \;\; \lambda_b(u, \omega) = \widetilde{\lambda}_b(u_a, \omega)$$

- Then, the equation $u = \lambda(u, \omega)$ becomes

$$u_a = \widetilde{\lambda}_a(u_b, \omega) \;, \;\; u_b = \widetilde{\lambda}_b(u_a, \omega)$$

  which may display
  - zero solution
  - one solution (solvability)
  - multiple solutions (undeterminacy)

Solvability and information patterns (II)

- Deadlock

$$\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{U}_b \otimes \{\emptyset, \Omega\} \ , \ \ \mathcal{I}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \Omega\}$$

in which case

$$u_a = \widetilde{\lambda}_a(u_b) \ , \ \ u_b = \widetilde{\lambda}_b(u_a)$$

may display zero solutions, one solution or multiple solutions, depending on the functional properties of $\widetilde{\lambda}_a$ and $\widetilde{\lambda}_b$

- Sequential

$$\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F} \ , \ \ \mathcal{I}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$$

in which case

$$u_a = \widetilde{\lambda}_a(\omega) \ , \ \ u_b = \widetilde{\lambda}_b(u_a, \omega)$$

always displays a unique solution (u_a, u_b), whatever $\omega \in \Omega$ and $\widetilde{\lambda}_a$ and $\widetilde{\lambda}_b$

Now, we define the solvability property

Solvability property

A stochastic system displays the solvability property when,
for any collection $\lambda \in \Lambda_{\mathbb{A}}^{ad}$ of admissible strategies,
and for any state of Nature $\omega \in \Omega$,
there exists one, and only one, decision $u \in \mathbb{U}_{\mathbb{A}}$
satisfying the implicit ("closed loop") equation

$$u = \lambda(u, \omega)$$

Solvability makes it possible to define a solution map (I)

- Without the solvability property,
  we would not be able to determine the agents decisions
- When the solvability property holds true,
  we denote by $M_\lambda(\omega)$ the unique $u \in \mathbb{U}_\mathbb{A}$ such that

$$u = \lambda(u, \omega) \iff u = M_\lambda(\omega)$$

- We thus obtain a mapping, called pre-solution map,

$$M_\lambda : \Omega \to \mathbb{U}_\mathbb{A}$$

- The solvability/measurability property holds true
  when the pre-solution map $M_\lambda : \Omega \to \mathbb{U}_\mathbb{A}$
  is measurable from $(\Omega, \mathcal{F})$ to $(\mathbb{U}_\mathbb{A}, \mathcal{U}_\mathbb{A})$, that is,

$$M_\lambda^{-1}(\mathcal{U}_\mathbb{A}) \subset \mathcal{F}$$

Solvability makes it possible to define a solution map (II)

Solution map

Suppose that the solvability property holds true.
Thanks to the pre-solution map M_λ, we define the solution map

$$S_\lambda : \Omega \to \mathbb{H}$$

that maps states of Nature towards histories, by

$$S_\lambda(\omega) = \big(M_\lambda(\omega), \omega\big) \; , \;\; \forall \omega \in \Omega \; ,$$

that is,

$$(u, \omega) = S_\lambda(\omega) \iff u = \lambda(u, \omega) \; , \;\; \forall (u, \omega) \in \mathbb{U}_{\mathbb{A}} \times \Omega$$

We include the state of Nature ω in the image of $S_\lambda(\omega)$, so that we map the set Ω towards the history space $\mathbb{H}$, making it possible to interpret $S_\lambda(\omega)$ as a history driven by the admissible strategy λ
(in classical control theory, a state trajectory is produced by a policy)

Outline of the presentation

Causality

In a causal system, agents are ordered, one playing after the other with available information depending only on agents acting earlier, but the order may depend upon the history

We lay out mathematical ingredients to define causality: Orderings and partial orderings

- Let $\mathbb{O}$ denote the set of total orderings of agents in $\mathbb{A}$, that is, injective mappings from $\{1, \ldots, A\}$ to $\mathbb{A}$, where $A = \mathrm{card}(\mathbb{A})$
- For $k \in \{1, \ldots, A\}$, let $\mathbb{O}_k$ denote the set of k-orderings, that is, injective mappings from $\{1, \ldots, k\}$ to $\mathbb{A}$ (thus $\mathbb{O} = \mathbb{O}_A$)
- There is a natural mapping ψ_k from $\mathbb{O}$ to $\mathbb{O}_k$, the restriction of any ordering of $\mathbb{A}$ to the domain set $\{1, \ldots, k\}$

We lay out mathematical ingredients to define causality: History-orderings

- Define a history-ordering as a mapping $\varphi : \mathbb{H} \to \mathbb{O}$ from histories towards orderings
- Along each history $h \in \mathbb{H}$, the agents are ordered by $\varphi(h) \in \mathbb{O}$
- With any $k \in \{1, \ldots, A\}$ and k-ordering $\rho_k \in \mathbb{O}_k$, we associate the set $\mathbb{H}^{\varphi}_{k,\rho_k}$ of histories that induce the same order than ρ_k for the agents having a rank smaller or equal to k, that is,

$$\mathbb{H}^{\varphi}_{k,\rho_k} = \{h \in \mathbb{H} \mid \psi_k(\varphi(h)) = \rho_k\}$$

Now, we define causality

Causality

A stochastic system is causal
if there exists (at least one) history-ordering φ from $\mathbb{H}$ towards $\mathbb{O}$, with the property that for any $k \in \{1, \ldots, A\}$ and $\rho_k \in \mathbb{O}_k$, the set $\mathbb{H}^{\varphi}_{k,\rho_k}$ satisfies

$$\mathbb{H}^{\varphi}_{k,\rho_k} \cap G \in \mathcal{U}_{\{\rho_k(1),\ldots,\rho_k(k-1)\}} \otimes \mathcal{F} \;, \quad \forall G \in \mathcal{I}_{\rho_k(k)}$$

- In other words, when the first k agents are known and given by $(\rho_k(1), \ldots, \rho_k(k))$, the information $\mathcal{I}_{\rho_k(k)}$ of the agent $\rho_k(k)$ with rank k depends at most on the decisions of agents $\rho_k(1)$, ..., $\rho_k(k-1)$
with rank stricly less than k
- We say that a stochastic system is sequential if it is causal with a constant history-ordering

Causality implies solvability

Proposition

Causality implies (recursive) solvability with a measurable solution map

A causal but non sequential system

- We consider a set of agents $\mathbb{A} = \{a, b\}$ with

$$\mathbb{U}_a = \{u_a^1, u_a^2\} \ , \ \ \mathbb{U}_b = \{u_b^1, u_b^2\} \ , \ \ \Omega = \{\omega^1, \omega^2\}$$

- The agents' information fields are given by

$$\mathcal{I}_a = \sigma(\{u_a^1, u_a^2\} \times \{u_b^1, u_b^2\} \times \{\omega^2\}, \{u_a^1, u_a^2\} \times \{u_b^1\} \times \{\omega^1\})$$
$$\mathcal{I}_b = \sigma(\{u_a^1, u_a^2\} \times \{u_b^1, u_b^2\} \times \{\omega^1\}, \{u_a^1\} \times \{u_b^1, u_b^2\} \times \{\omega^2\})$$

- When the state of Nature is ω^2, agent a only sees ω^2, whereas agent b sees ω^2 and the decision of a: thus a acts first, then b
- The reverse holds true when the state of Nature is ω^1
- Thus, there are history-ordering mappings φ from $\mathbb{H}$ towards $\{(a, b), (b, a)\}$, but they differ according to history:

$$\varphi\Big((u_a, u_b, \omega^2)\Big) = (a, b) \ \text{ and } \ \varphi\Big((u_a, u_b, \omega^1)\Big) = (b, a)$$

- The system is causal but not sequential

Outline of the presentation

Outline of the presentation

What are the agents whose decisions might affect the information of a focal agent?

- The precedence binary relation identifies the agents whose decisions influence the observations of a given agent
- For a given agent $a \in \mathbb{A}$, we consider the set $\mathcal{P}_a \subset 2^{\mathbb{A}}$ of subsets $C \subset \mathbb{A}$ of agents such that

$$\mathcal{I}_a \subset \mathcal{U}_C \otimes \mathcal{F} = \bigotimes_{c \in C} \mathcal{U}_c \otimes \bigotimes_{b \notin C} \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$$

- Any subset $C \in \mathcal{P}_a$ contains agents whose decisions affect the information $\mathcal{I}_a$ available to the focal agent a
- As the set $\mathcal{P}_a$ is stable under intersection, the following definition makes sense

The precedence relation $\mathfrak{P}$

Precedence binary relation

1. For any agent $a \in \mathbb{A}$, we define the subset $\langle a \rangle_{\mathfrak{P}} \subset \mathbb{A}$ of agents as the intersection of subsets $C \subset \mathbb{A}$ of agents such that

$$\mathfrak{I}_a \subset \mathfrak{U}_C \otimes \mathfrak{F}$$

2. We define a precedence binary relation $\mathfrak{P}$ on $\mathbb{A}$ by

$$b \, \mathfrak{P} \, a \iff b \in \langle a \rangle_{\mathfrak{P}}$$

   and we say that b is a predecessor of a (or a precedent of a)

In other words, the decisions of any predecessor of an agent affect the information of this agent: any agent is influenced by his predecessors (when they exist, because $\langle a \rangle_{\mathfrak{P}}$ might be empty)

Characterization of the predecessors of a focal agent

- For any agent $a \in \mathbb{A}$, the subset $\langle a \rangle_{\mathfrak{P}}$ of agents is the smallest subset $C \subset \mathbb{A}$ such that

$$\mathcal{I}_a \subset \mathcal{U}_C \otimes \mathcal{F}$$

- In other words, $\langle a \rangle_{\mathfrak{P}}$ is characterized by

$$\mathcal{I}_a \subset \mathcal{U}_{\langle a \rangle_{\mathfrak{P}}} \otimes \mathcal{F} \text{ and } \left(\mathcal{I}_a \subset \mathcal{U}_C \otimes \mathcal{F} \Rightarrow \langle a \rangle_{\mathfrak{P}} \subset C \right)$$

Potential for signaling

- Whenever $\langle a \rangle_{\mathfrak{P}} \neq \emptyset$, there is a potential for signaling, that is, for information transmission
- Indeed, any agent b in $\langle a \rangle_{\mathfrak{P}}$ influences the information $\mathcal{I}_a$ upon which agent a bases his decisions
- Therefore, whenever agent b is a predecessor of agent a, the former can, by means of his decisions, send a signal to the latter
- In case $\langle a \rangle_{\mathfrak{P}} = \emptyset$, the decisions of agent a depend, at most, on the state of Nature, and there is no room for signaling

Iterated predecessors

- Let $C \subset \mathbb{A}$ be a subset of agents
- We introduce the following subsets of agents

$$\langle C \rangle_{\mathfrak{P}} = \bigcup_{b \in C} \langle b \rangle_{\mathfrak{P}} \ , \ \ \langle C \rangle_{\mathfrak{P}}^{0} = C \ \text{ and } \ \langle C \rangle_{\mathfrak{P}}^{n+1} = \Big\langle \langle C \rangle_{\mathfrak{P}}^{n} \Big\rangle_{\mathfrak{P}} \ , \ \ \forall n \in \mathbb{N}$$

  that correspond to the iterated predecessors of the agents in C
- When C is a singleton $\{a\}$, we denote $\langle a \rangle_{\mathfrak{P}}^{n}$ for $\langle \{a\} \rangle_{\mathfrak{P}}^{n}$

Successor relation $\mathfrak{P}^{-1}$

- The converse of the precedence relation $\mathfrak{P}$
  is the successor relation $\mathfrak{P}^{-1}$ characterized by

$$b \, \mathfrak{P}^{-1} \, a \iff a \, \mathfrak{P} \, b$$

- Quite naturally, b is a successor of a iff a is a predecessor of b

Outline of the presentation

A subsystem is a subset of agents closed w.r.t. information

We define the information $\mathcal{I}_C \subset \mathcal{H}$ of the subset $C \subset \mathbb{A}$ of agents by

$$\mathcal{I}_C = \bigvee_{b \in C} \mathcal{I}_b$$

that is, the smallest σ-fields that contains all the σ-fields $\mathcal{I}_b$, for $b \in C$

Subsystem

A nonempty subset C of agents in $\mathbb{A}$ is a subsystem if the information field $\mathcal{I}_C$ at most depends on the decisions of the agents in C, that is,

$$\mathcal{I}_C \subset \mathcal{U}_C \otimes \mathcal{F}$$

Thus, the information received by agents in C depends upon states of Nature and decisions of members of C only

Generated subsystem

- The subsystem $\overline{C}$ generated by a nonempty subset C of agents in $\mathbb{A}$ is the intersection of all subsystems that contain C,
  that is, the smallest subsystem that contain C
- A subset $C \subset \mathbb{A}$ is a subsystem iff
  it coincides with the generated subsystem, that is,

$$C \text{ is a subsystem} \iff C = \overline{C}$$

The subsystem relation $\mathfrak{S}$

Precedence binary relation

We define the subsystem relation $\mathfrak{S}$ on $\mathbb{A}$ by

$$b \, \mathfrak{S} \, a \iff \overline{\{b\}} \subset \overline{\{a\}} \, , \quad \forall (a, b) \in \mathbb{A}^2$$

Therefore, $b \, \mathfrak{S} \, a$ means that

- agent b belongs to the subsystem generated by agent a
- or, equivalently, that the subsystem generated by agent a contains the one generated by agent b

The subsystem relation $\mathfrak{S}$ is a pre-order

Proposition

The subsystem relation $\mathfrak{S}$ is a pre-order, namely it is reflexive and transitive

Proposition

1. *A subset $C \subset \mathbb{A}$ is a subsystem iff $\langle C \rangle_{\mathfrak{P}} \subset C$, that is, iff the predecessors of agents in C belong to C:*

$$C \text{ is a subsystem} \iff \overline{C} = C \iff \langle C \rangle_{\mathfrak{P}} \subset C$$

2. *For any agent $a \in \mathbb{A}$, the subsystem generated by agent a is the union of $\{a\}$ and of all his iterated predecessors, that is,*

$$\overline{\{a\}} = \bigcup_{n \in \mathbb{N}} \langle a \rangle_{\mathfrak{P}}^{n}$$

Subsystem and co-cycle property of the pre-solution map

- We suppose that the stochastic system $\{\mathbb{U}_a, \mathfrak{U}_a, \mathfrak{I}_a\}_{a\in\mathbb{A}}$ displays the solvability property
- We consider a partition $\mathbb{A} = B \cup C$ and write, for an admissible strategy $\lambda \in \Lambda^{ad}_{\mathbb{A}}$,

$\lambda = (\lambda_B, \lambda_C)$ where $\lambda_B : \mathbb{U}_B \times \mathbb{U}_C \times \Omega \to \mathbb{U}_B$, $\lambda_C : \mathbb{U}_B \times \mathbb{U}_C \times \Omega \to \mathbb{U}_C$

Proposition

If B is a subsystem, the strategy λ_B can be identified with

$$\lambda_B : \mathbb{U}_B \times \Omega \to \mathbb{U}_B$$

and the pre-solution map has the following co-cycle property

$$M_{(\lambda_B,\lambda_C)}(\omega) = \Big(M_{\lambda_B}(\omega), M_{\lambda_C\left(M_{\lambda_B}(\omega),\cdot\right)}(\omega) \Big) \ , \quad \forall \omega \in \Omega$$

like a flow property, in the sense that

$$M_{(\lambda_B,\lambda_C)}(\omega) = (u_B, u_C) \iff \begin{cases} u_B & = M_{\lambda_B}(\omega) \\ u_C & = \lambda_C\big(u_B, u_C, \omega\big) \end{cases}$$

Outline of the presentation

The information-memory relation $\mathfrak{M}$

Information-memory binary relation $\mathfrak{M}$

1. With any agent $a \in \mathbb{A}$, we associate
   the subset $\langle a \rangle_{\mathfrak{M}}$ of agents who pass on their information to a, that is,
   $$\langle a \rangle_{\mathfrak{M}} = \{ b \in \mathbb{A} \mid \mathfrak{I}_b \subset \mathfrak{I}_a \}$$
2. We define an information memory binary relation $\mathfrak{M}$ on $\mathbb{A}$ by
   $$b \, \mathfrak{M} \, a \iff b \in \langle a \rangle_{\mathfrak{M}} \iff \mathfrak{I}_b \subset \mathfrak{I}_a \ , \quad \forall (a, b) \in \mathbb{A}^2$$

- When $b \, \mathfrak{M} \, a$, we say that
  agent b information is remembered by or passed on to agent a,
  or that
  the information of agent b is embedded in the information of agent a
- When agent b belongs to $\langle a \rangle_{\mathfrak{M}}$,
  the information available to b is also available to agent a

The information memory relation $\mathfrak{M}$ is a pre-order

Proposition

The information memory relation $\mathfrak{M}$ is a pre-order, namely $\mathfrak{M}$ is reflexive and transitive

Outline of the presentation

The decision-memory relation $\mathfrak{D}$

We recall that the decision subfield $\mathcal{D}_b$ is

$$\mathcal{D}_b = \mathcal{U}_b \otimes \bigotimes_{c \neq b} \{\emptyset, \mathbb{U}_c\} \otimes \{\emptyset, \Omega\}$$

Decision-memory binary relation

1. With any agent $a \in \mathbb{A}$, we associate

$$\langle a \rangle_{\mathfrak{D}} = \{ b \in \mathbb{A} \mid \mathcal{D}_b \subset \mathcal{I}_a \}$$

   the subset of agents b whose decision is passed on to a
2. We define a decision-memory binary relation $\mathfrak{D}$ on $\mathbb{A}$ by

$$b \, \mathfrak{D} \, a \iff b \in \langle a \rangle_{\mathfrak{D}} \iff \mathcal{D}_b \subset \mathcal{I}_a \,, \quad \forall (a, b) \in \mathbb{A}^2$$

$\mathfrak{D} \subset \mathfrak{P}$

From

$$\mathcal{D}_{\langle a \rangle_{\mathfrak{D}}} = \mathcal{U}_{\langle a \rangle_{\mathfrak{D}}} \otimes \{\emptyset, \Omega\} \subset \mathcal{I}_a \subset \mathcal{U}_{\langle a \rangle_{\mathfrak{P}}} \otimes \mathcal{F}$$

we conclude that

$$\langle a \rangle_{\mathfrak{D}} \subset \langle a \rangle_{\mathfrak{P}} \ , \quad \forall a \in \mathbb{A}$$

or, equivalently, that

$$\mathfrak{D} \subset \mathfrak{P}$$

- When $b \, \mathfrak{D} \, a$, we say that the decision of agent b is remembered by or passed on to agent a, or that the decision of agent b is embedded in the information of agent a
- If $b \, \mathfrak{D} \, a$, the decision made by agent b is passed on to agent a and, by the fact that $\mathfrak{D} \subset \mathfrak{P}$, b is a predecessor of a
- However, the agent b can be a predecessor of a, but his influence may happen without passing on his decision to a

Conclusion

With these four relations

- Precedence relation $\mathfrak{P}$
- Subsystem relation $\mathfrak{S}$
- Information-memory relation $\mathfrak{M}$
- Decision-memory relation $\mathfrak{D}$

we can provide a typology of systems

Outline of the presentation

Outline of the presentation

Static team

Static team

A static team is a subset C of $\mathbb{A}$ such that $\langle C \rangle_{\mathfrak{P}} = \emptyset$, that is, agents in C have no predecessors

- A static team necessarily is a subset of the largest static team defined by

$$\mathbb{A}_0 = \{a \in \mathbb{A} \mid \mathcal{I}_a \subset \bigotimes_{b \in \mathbb{A}} \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}\} = \{a \in \mathbb{A} \mid \langle a \rangle_{\mathfrak{P}} = \emptyset\}$$

- When the whole set $\mathbb{A}$ of agents is a static team, any agent $a \in \mathbb{A}$ has no predecessor: $\langle a \rangle_{\mathfrak{P}} = \emptyset$, $\forall a \in \mathbb{A}$
- A system is static if the set $\mathbb{A}$ of agents is a static team

Static team made of two agents

Two agents a, b form a static team iff

$$\mathcal{I}_a \subset \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F} \;, \;\; \mathcal{I}_b \subset \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$$

There is no interdependence between the decisions of the agents, just a dependence upon states of Nature

Outline of the presentation

Station

A station is a subset of agents such that the set of information fields of these agents is totally ordered under inclusion (i.e., nested)

Station

A subset C of agents in $\mathbb{A}$ is a station

- iff the information-memory relation $\mathfrak{M}$ induces a total order on C (i.e., it consists of a chain of length $m = \mathrm{card}(C)$)
- iff there exists an ordering $(a_1, \ldots, a_m)$ of C such that

$$\mathcal{I}_{a_1} \subset \cdots \subset \mathcal{I}_{a_k} \subset \mathcal{I}_{a_{k+1}} \subset \cdots \subset \mathcal{I}_{a_m}$$

  or, equivalently, that

$$a_{k-1} \in \langle a_k \rangle_{\mathfrak{M}} \ , \quad \forall k = 2, \ldots, m$$

In other words, in a station,
the antecessor $k-1$ is necessarily a predecessor of k

A station with two agents

$$\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \Omega, \{\omega^1\}, \{\omega^2\}\}$$

$$\mathcal{I}_b = \{\emptyset, \mathbb{U}_a, \{u_a^1\}, \{u_a^2\}\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \Omega, \{\omega^1\}, \{\omega^2\}\}.$$

$\mathcal{I}_a \subset \mathcal{I}_b$ may be interpreted in different ways

- one may say that
  agent a communicates his own information to agent b.
- If agent a is an individual at time $t = 0$,
  while agent b is the same individual at time $t = 1$,
  one may say that the information is not forgotten with time
  (memory of past knowledge)

Sequential system

Sequential system

A system is sequential if there exists an ordering $(a_1, \ldots, a_A)$ of $\mathbb{A}$ such that each agent a_k is influenced at most by the previous (former or antecessor) agents $a_1, \ldots, a_{k-1}$, that is,

$$\langle a_1 \rangle_{\mathfrak{P}} = \emptyset \text{ and } \langle a_k \rangle_{\mathfrak{P}} \subset \{a_1, \ldots, a_{k-1}\} , \quad \forall k = 2, \ldots, A$$

In other words, in a sequential system, predecessors are necessarily antecessors

Example of sequential system with two agents

The set of agents $\mathbb{A} = \{a, b\}$ with information fields given by

$$\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F} \; , \; \; \mathcal{I}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \Omega\}$$

forms a sequential system where

- agent a precedes agent b, because $\langle a \rangle_{\mathfrak{P}} = \emptyset$ and $\langle b \rangle_{\mathfrak{P}} = \{a\}$
- but $\mathcal{I}_a$ and $\mathcal{I}_b$ are not comparable: agent a observes only the state of Nature, whereas agent b observes only agent a's decision

Example of sequential system with two agents

$$\mathcal{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \Omega, \{\omega^1\}, \{\omega^2\}\}$$

$$\mathcal{I}_b = \{\emptyset, \mathbb{U}_a, \{u_a^1\}, \{u_a^2\}\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \Omega, \{\omega^1\}, \{\omega^2\}\}.$$

The system is sequential:

1. agent a observes the state of Nature
   and makes his decision accordingly
2. agent b observes both agent a's decision and the state of Nature
   and makes his decision accordingly

Outline of the presentation

Partially nested system

Partially nested system

A partially nested system is one for which
the precedence relation is included in the information-memory relation,
that is,

$$\mathfrak{P} \subset \mathfrak{M}$$

- In a partially nested system, if agent a is a predecessor of agent b — hence, a can influence b — then agent b knows what agent a knows
- In a partially nested system, any agent has access to the information of those agents who are his predecessors (and thus influence his own information)
- In other words, in a partially nested system, predecessors are necessarily informers

Quasiclassical system

Quasiclassical system

A system is quasiclassical

- iff it is sequential and partially nested
- iff there exists an ordering $(a_1, \ldots, a_A)$ of $\mathbb{A}$ such that $\langle a_1 \rangle_{\mathfrak{P}} = \emptyset$ and

$$\langle a_k \rangle_{\mathfrak{P}} \subset \{a_1, \ldots, a_{k-1}\} \text{ and } \langle a_k \rangle_{\mathfrak{P}} \subset \langle a_k \rangle_{\mathfrak{M}} \ , \ \forall k = 2, \ldots, A$$

In other words, in a quasiclassical system,
predecessors are necessarily antecessors and
predecessors are necessarily informers

Classical system

Classical system

A system is classical

- iff there exists an ordering $(a_1, \ldots, a_A)$ of $\mathbb{A}$ for which it is both sequential and such that $\mathcal{I}_{a_k} \subset \mathcal{I}_{a_{k+1}}$ for $k = 1, \ldots, n-1$ (station property)
- iff there exists an ordering $(a_1, \ldots, a_A)$ of $\mathbb{A}$ such that $\langle a_1 \rangle_{\mathfrak{P}} = \emptyset$ and

$$\langle a_k \rangle_{\mathfrak{P}} \subset \{a_1, \ldots, a_{k-1}\} \subset \{a_1, \ldots, a_{k-1}, a_k\} \subset \langle a_k \rangle_{\mathfrak{M}} \ , \quad \forall k = 2, \ldots, A$$

In other words, in a classical system, predecessors are necessarily antecessors and antecessors are necessarily informers

- A classical system is necessarily partially nested because $\langle a_k \rangle_{\mathfrak{P}} \subset \langle a_k \rangle_{\mathfrak{M}}$ for $k = 1, \ldots, n$
- Hence, a classical system is quasiclassical

A classical system with two agents

- The set of agents $\mathbb{A} = \{a, b\}$ with information fields given by

$$\mathfrak{I}_a = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F} \; , \;\; \mathfrak{I}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \mathcal{F}$$

  forms a classical system
- Indeed, first, the system is sequential as a precedes b because $\langle a \rangle_{\mathfrak{P}} = \emptyset$ and $a \in \langle b \rangle_{\mathfrak{P}}$:
  - agent a observes the state of Nature and makes his decision accordingly
  - agent b observes both agent a's decision and the state of Nature and makes his decision based on that information
- Second, one has that $\mathfrak{I}_a \subset \mathfrak{I}_b$ ($a \in \langle b \rangle_{\mathfrak{M}}$): agent a communicates his own information to agent b

Subsystem inheritence

Theorem

Any of the properties static team, sequentiality, quasiclassicality, classicality, causality of a system is shared by all its subsystems

Outline of the presentation

Hierarchical systems

Hierarchical system

A system is hierarchical when the set $\mathbb{A}$ of agents can be partitioned in (nonempty) disjoint sets $\mathbb{A}_0, \ldots, \mathbb{A}_K$ as follows

$$\mathbb{A}_0 = \{a \in \mathbb{A} \mid \langle a \rangle_{\mathfrak{P}} = \emptyset\}$$

$$\mathbb{A}_1 = \{a \in \mathbb{A} \mid a \notin \mathbb{A}_0 \text{ and } \langle a \rangle_{\mathfrak{P}} \subset \mathbb{A}_0\}$$

$$\mathbb{A}_{k+1} = \{a \in \mathbb{A} \mid a \notin \bigcup_{i=1}^{k} \mathbb{A}_j \text{ and } \langle a \rangle_{\mathfrak{P}} \subset \bigcup_{i=1}^{k} \mathbb{A}_j\}\ , \ \forall k = 2, \ldots, K$$

for $k = 2, \ldots, K$

Agents in $\mathbb{A}_0$ form the largest static team ($\langle \mathbb{A}_0 \rangle_{\mathfrak{P}} = \emptyset$)

Parallel coordinated systems

We consider the case when the set $\mathbb{A}$ of agents can be partitioned in (nonempty) disjoint sets $\mathbb{A}_0, \mathbb{A}_1, \ldots, \mathbb{A}_K$ as follows

- $\mathbb{A}_0$ is the largest static team ($\langle \mathbb{A}_0 \rangle_{\mathfrak{P}} = \emptyset$)
- every subset $\mathbb{A}_1 \cup \mathbb{A}_0, \ldots, \mathbb{A}_K \cup \mathbb{A}_0$ is a subsystem