Background in Optimization

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Outline of the presentation

Optimization problems, convex functions, local and global minima

Magic formulas in optimization

Lagrangian duality (the case of equality constraints) and Uzawa algorithm

More on convexity and duality

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More on convexity and duality

Here are the ingredients for a general abstract optimization problem

$$\inf_{w \in \mathcal{W}^{ad}} J(w)$$

- Optimization set $\mathcal{W} (= \mathbb{R}^N)$ containing optimization variables $w \in \mathcal{W}$
- A criterion $J : \mathcal{W} \to \mathbb{R} \cup \{+\infty\}$
- Constraints of the form $w \in \mathcal{W}^{ad} \subset \mathcal{W}$

Examples of classes of optimization problems

$$\inf_{w \in \mathcal{W}^{ad}} J(w)$$

- Linear programming
  - Optimization set $\mathcal{W} = \mathbb{R}^N$
  - Criterion J is linear (affine)
  - Constraints $\mathcal{W}^{ad}$ defined by
    a finite number of linear (affine) equalities and inequalities
- Convex optimization
  - Criterion J is a convex function
  - Constraints $\mathcal{W}^{ad}$ define a convex set
- Combinatorial optimization
  - Optimization set $\mathcal{W}$ is discrete (binary $\{0,1\}^N$, integer $\mathbb{Z}^N$, etc.)

Outline of the presentation

Convex sets

Let $N \in \mathbb{N}^*$. We consider subsets of the Euclidian space $\mathbb{R}^N$

- The subset $C \subset \mathbb{R}^N$ is convex if for any $x_1 \in C$, $x_2 \in C$ and $t \in [0, 1]$, we have that $tx_1 + (1 - t)x_2 \in C$
- An intersection of convex sets is convex
- A segment is convex
- A hyperplane is convex ($H \subset \mathbb{R}^N$ is a hyperplane if there exists $y \in \mathbb{R}^N \backslash \{0\}$ and $b \in \mathbb{R}$ such that $H = \left\{x \in \mathbb{R}^N \,\middle|\, \langle x, y\rangle + b = 0\right\}$)
- An affine subspace (intersection of hyperplanes) is convex

Linear and affine functions

Consider a function $f : \mathbb{R}^N \to \mathbb{R}$

- The function f is linear if, for any $x_1 \in \mathbb{R}^N$, $x_2 \in \mathbb{R}^N$ and $t_1 \in \mathbb{R}$, $t_2 \in \mathbb{R}$,

$$f(t_1 x_1 + t_2 x_2) = t_1 f(x_1) + t_2 f(x_2)$$

- The function f is affine if, for any $x_1 \in \mathbb{R}^N$, $x_2 \in \mathbb{R}^N$ and $t_1 \in \mathbb{R}$, $t_2 \in \mathbb{R}$ such that $t_1 + t_2 = 1$,

$$f(t_1 x_1 + t_2 x_2) = t_1 f(x_1) + t_2 f(x_2)$$

Exercise. Show that $f : \mathbb{R}^N \to \mathbb{R}$ is affine if and only if $g(x) = f(x) - f(0)$ is linear

Convex functions (definitions)

Let $C \subset \mathbb{R}^N$ be an nonempty convex subset of $\mathbb{R}^N$, where $N \in \mathbb{N}^*$, and $f : C \to \mathbb{R}$ be a function

- The function f is affine if, for any $x_1 \in C$, $x_2 \in C$ and $t \in \mathbb{R}$,

$$f(tx_1 + (1-t)x_2) = tf(x_1) + (1-t)f(x_2)$$

- The function f is convex if, for any $x_1 \in C$, $x_2 \in C$ and $t \in [0,1]$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

- The function f is strictly convex if,
  for any $x_1 \in C$, $x_2 \in C$, $x_1 \neq x_2$, and $t \in]0,1[$,

$$f(tx_1 + (1-t)x_2) < tf(x_1) + (1-t)f(x_2)$$

- The function f is strongly convex (of modulus $a > 0$) if,
  for any $x_1 \in C$, $x_2 \in C$ and $t \in [0,1]$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) - \frac{a}{2}t(1-t)\|x_1 - x_2\|^2$$

Exercises

Let $C \subset \mathbb{R}^N$ be an nonempty subset of $\mathbb{R}^N$, where $N \in \mathbb{N}^*$

- Show that both definitions of an affine function coincide when $C = \mathbb{R}^N$
- Show that a function $f : C \to \mathbb{R}$ is convex if and only if its epigraph is a convex set subset of $\mathbb{R}^N \times \mathbb{R}$
- Show that a function $f : C \to \mathbb{R}$ is strongly convex of modulus $a > 0$ if and only if $g(x) = f(x) - \frac{a}{2}\|x\|^2$ is convex
- If $f : C \to \mathbb{R}$ is convex, show that f is *not* strictly convex if and only there exists a nonempty convex subset $C' \subset C$ over which f is affine

Convex functions on the real line

Proposition

Let $I \subset \mathbb{R}$ be an nonempty interval

- *A C^1 function $f : I \to \mathbb{R}$ is convex if and only if f' is increasing on I*
- *A C^2 function $f : I \to \mathbb{R}$ is convex if and only if $f''(x) \geq 0$, for all $x \in I$*
- *Let $a > 0$. A C^2 function $f : I \to \mathbb{R}$ is a-strongly convex if and only if $f''(x) \geq a$, for all $x \in I$*
- *A C^1 function $f : I \to \mathbb{R}$ is strictly convex if and only if f is convex and the set $\{x \in I \mid f''(x) = 0\}$ is either empty or is a singleton*

Exercise. Study the family of functions $f_\alpha :]0, +\infty[\to \mathbb{R}$ given by $f_\alpha(x) = x^\alpha$. For which values of the parameter α is the function f_α convex? For a given $a > 0$, for which values of the parameter α is the function f_α strongly convex of modulus a? Provide an example of a strictly convex function which is not strongly convex.

Convexity for multivariate functions

The *Hessian* matrix $\mathcal{H}_f(x)$ of a C^2 function $f : \mathbb{R}^N \to \mathbb{R}$ is the $N \times N$ symmetric matrix given by

$$\mathcal{H}_f(x) = \left\{ \frac{\partial^2 f}{\partial x^i \partial x^j}(x) \right\}_{(i,j) \in \{1,\ldots,N\}^2}$$

Proposition

Let $C \subset \mathbb{R}^N$ be an nonempty convex subset of $\mathbb{R}^N$, where $N \in \mathbb{N}^*$

- *A C^2 function $f : \mathbb{R}^N \to \mathbb{R}$ is convex on C if and only if the symmetric Hessian matrix $\mathcal{H}_f(x)$ is positive for all $x \in C$*
- *A C^2 function $f : \mathbb{R}^N \to \mathbb{R}$ is strongly convex of modulus $a > 0$ on C if and only if the eigenvalues of the symmetric Hessian matrix $\mathcal{H}_f(x)$ are uniformly bounded below by $a > 0$ on C*

Exercise. Let Q be a $N \times N$ symmetric matrix and $f(x) = 1/2x'Qx$, where x' is the transpose of the vector x. Give conditions on the smallest eigenvalue of Q so that the function f is convex, or strictly convex, or strongly convex of modulus a.

Operations on functions preserving convexity

Proposition

Let $(f_i)_{i \in I}$ be a family of convex functions
Then $\sup_{i \in I} f_i$ is a convex function

Proposition

Let $(f_i)_{i=1,\ldots,n}$ be convex functions
Let $(\alpha_i)_{i=1,\ldots,n}$ be nonnegative numbers
Then $\sum_{i=1}^{m} \alpha_i f_i$ is a convex function

Proposition

Let $f : \mathbb{R}^N \to \mathbb{R}$ be convex
Let A be a $N \times M$ matrix and $b \in \mathbb{R}^N$
Then $y \in \mathbb{R}^M \mapsto f(Ay + b)$ is a convex function

Outline of the presentation

Coercivity

Definition

A function $f : \mathbb{R}^N \to \mathbb{R}$ is coercive if

$$\lim_{\|x\| \to +\infty} f(x) = +\infty$$

Proposition

A strongly convex function is coercive

Minimum

Definition

We say that $w^* \in \mathcal{W}$ is a (global) minimum of the optimization problem $\inf_{w \in \mathcal{W}^{ad}} J(w)$ if

$$w^* \in \mathcal{W}^{ad} \text{ and } J(w^*) \leq J(w) \ , \ \ \forall w \in \mathcal{W}^{ad}$$

In this case, we write

$$J(w^*) = \min_{w \in \mathcal{W}^{ad}} J(w)$$

Existence and uniqueness of a minimum

We consider the optimization problem

$$\inf_{w \in \mathcal{W}^{ad}} J(w) \text{ where } \mathcal{W}^{ad} \subset \mathcal{W} = \mathbb{R}^N$$

Proposition

If the criterion J is continuous and the constraint set $\mathcal{W}^{ad}$ is compact (bounded and closed), then there is a minimum

Proposition

If the constraint set $\mathcal{W}^{ad}$ is closed and the criterion J is continuous and coercive, then there is a minimum

Proposition

If the constraint set $\mathcal{W}^{ad}$ is convex, and if the criterion J is strictly convex,
a minimum is necessarily unique

Exercises

We consider the optimization problem

$$\inf_{w \in \mathcal{W}^{ad}} J(w)$$

Give an example

- of continuous criterion J and of constraint set $\mathcal{W}^{ad}$ for which there is no minimum
- of criterion J and of compact constraint set $\mathcal{W}^{ad}$ for which there is no minimum
- of continuous criterion J and of unbounded and closed constraint set $\mathcal{W}^{ad}$ for which there is no minimum
- of convex criterion J and of constraint set $\mathcal{W}^{ad}$ for which there is more than one minimum
- of strictly convex criterion J and of constraint set $\mathcal{W}^{ad}$ for which there is more than one minimum

Local minimum

Definition

We say that $w^* \in \mathcal{W}$ is a local minimum of the optimization problem $\inf_{w \in \mathcal{W}^{ad}} J(w)$ if there exists a neighborhood $\mathcal{V}$ of w^* in $\mathcal{W}^{ad}$ such that

$$w^* \in \mathcal{W}^{ad} \text{ and } J(w^*) \leq J(w) \ , \ \ \forall w \in \mathcal{V}$$

Proposition

If the constraint set $\mathcal{W}^{ad}$ is convex, and if the criterion J is convex, a local minimum is a global minimum

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More on convexity and duality

More on the magic formulas in the coming slides

tower formula

$$\inf_{(a,b)\in\mathcal{A}\times\mathcal{B}} h(a,b) = \inf_{a\in\mathcal{A},b\in\mathcal{B}} h(a,b) = \inf_{a\in\mathcal{A}} \big(\inf_{b\in\mathcal{B}} h(a,b) \big)$$

linearity formula

$$\inf_{a\in\mathcal{A}} \lambda f(a) = \lambda \inf_{a\in\mathcal{A}} f(a) \ , \ \ \forall \lambda \geq 0$$

independence formula

$$\inf_{(a,b)\in\mathcal{A}\times\mathcal{B}} \big(f(a) + g(b) \big) = \inf_{a\in\mathcal{A},b\in\mathcal{B}} \big(f(a) + g(b) \big) = \inf_{a\in\mathcal{A}} f(a) + \inf_{b\in\mathcal{B}} g(b)$$

Tower formula

For any function

$$h : \mathcal{A} \times \mathcal{B} \to]-\infty, +\infty]$$

we have

$$\inf_{a \in \mathcal{A}, b \in \mathcal{B}} h(a, b) = \inf_{a \in \mathcal{A}} \big(\inf_{b \in \mathcal{B}} h(a, b) \big)$$

and if $\mathcal{B}(a) \subset \mathcal{B}$, $\forall a \in \mathcal{A}$, we have

$$\inf_{a \in \mathcal{A}, b \in \mathcal{B}(a)} h(a, b) = \inf_{a \in \mathcal{A}} \big(\inf_{b \in \mathcal{B}(a)} h(a, b) \big)$$

Independence

- For any functions

$$f : \mathcal{A} \to]-\infty, +\infty] \;,\;\; g : \mathcal{B} \to]-\infty, +\infty]$$

we have

$$\inf_{a \in \mathcal{A}, b \in \mathcal{B}} \big(f(a) + g(b) \big) = \inf_{a \in \mathcal{A}} f(a) + \inf_{b \in \mathcal{B}} g(b)$$

- For any finite set $\mathbb{S}$, any functions $f_s : \mathcal{A}_s \to]-\infty, +\infty]$ and any nonnegative scalars $\pi_s \geq 0$, for $s \in \mathbb{S}$, we have

$$\inf_{\{a_s\}_{s \in \mathbb{S}} \in \prod_{s \in \mathbb{S}} \mathcal{A}_s} \sum_{s \in \mathbb{S}} \pi_s f_s(a_s) = \sum_{s \in \mathbb{S}} \pi_s \inf_{a_s \in \mathcal{A}_s} f_s(a_s)$$

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Outline of the presentation

Duality gap

Consider a function

$$\phi : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$$

Proposition

We have the inequality

$$\inf_x \sup_y \phi(x, y) \geq \sup_y \inf_x \phi(x, y)$$

Notice that we minimize in the first variable x (primal variable) and maximize in the second variable y (dual variable)

Definition

The duality gap is

$$\inf_x \sup_y \phi(x, y) - \sup_y \inf_x \phi(x, y) \geq 0$$

Saddle-point

Definition

We say that $(\bar{x}, \bar{y}) \in \mathcal{X} \times \mathcal{Y}$ is a saddle-point if

- $y \mapsto \phi(\bar{x}, y)$ achieves a *maximum* at $\bar{y}$
- $x \mapsto \phi(x, \bar{y})$ achieves a *minimum* at $\bar{x}$

or, equivalently

$$\phi(x, \bar{y}) \geq \phi(\bar{x}, \bar{y}) \geq \phi(\bar{x}, y)$$

Proposition

When there exists a saddle-point, there is no duality gap
(that is, the duality gap is zero)

Existence of a saddle point

Proposition

Suppose that $\phi : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$

- is continuous,
- convex-concave (convex in the variable x, concave in the variable y),
- there exists two convex closed sets $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$ such that
  - there exists a $\hat{x} \in X$ such that $\lim_{\|y\| \to +\infty} \phi(\hat{x}, y) = -\infty$, or the set Y is bounded,
  - there exists a $\hat{y} \in Y$ such that $\lim_{\|x\| \to +\infty} \phi(x, \hat{y}) = +\infty$, or the set X is bounded.

Then, there exists a saddle point for the function ϕ on $X \times Y$

Outline of the presentation

Optimization under equality constraints

- We consider the optimization problem

$$\inf_{w \in \mathbb{R}^N} J(w)$$

under the constraint

$$\Theta(w) = 0$$

where $\Theta = (\Theta_1, \ldots, \Theta_M) : \mathbb{R}^N \to \mathbb{R}^M$

- The Lagrangian $L : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ is defined by

$$L(w, \lambda) = J(w) + \langle \lambda, \Theta(w) \rangle = J(w) + \sum_{j=1}^{M} \lambda_j \Theta_j(w)$$

Primal problem

Definition

The primal optimization problem is

$$\inf_{w \in \mathbb{R}^N} \sup_{\lambda \in \mathbb{R}^M} L(w, \lambda) = \inf_{w \in \mathbb{R}^N} \sup_{\lambda \in \mathbb{R}^M} \left(J(w) + \sum_{j=1}^{M} \lambda_j \Theta_j(w) \right)$$

Proposition

The original and the primal optimization problems have the same solutions (in $w \in \mathbb{R}^N$)

Dual problem

Definition

The dual optimization problem is

$$\sup_{\lambda \in \mathbb{R}^M} \inf_{w \in \mathbb{R}^N} L(w, \lambda) = \sup_{\lambda \in \mathbb{R}^M} \inf_{w \in \mathbb{R}^N} \left(J(w) + \sum_{j=1}^{M} \lambda_j \Theta_j(w) \right)$$

Definition

The dual function is $\psi : \mathbb{R}^M \to \mathbb{R} \cup \{-\infty\}$ given by

$$\psi(\lambda) = \inf_{w \in \mathbb{R}^N} L(w, \lambda) = \inf_{w \in \mathbb{R}^N} \left(J(w) + \sum_{j=1}^{M} \lambda_j \Theta_j(w) \right),$$

hence is concave

Proposition

When there exists a saddle-point for the Lagrangian, primal and dual problems are equivalent

First-order optimality conditions and saddle point

Proposition

We suppose that

- the criterion J is differentiable and convex
- in the equality constraints $\Theta(w) = 0$, the function Θ is affine

Let $w^* \in \mathbb{R}^N$ be a *minimum* of J, among the w such that $\Theta(w) = 0$. Then, there exists a vector λ^* of $\mathbb{R}^M$ (Lagrange multiplier) such that (w^*, λ^*) is a saddle point of the Lagrangian L, that is,

$$w \mapsto L(w, \lambda^*)$$

achieves a minimum at w^*, and

$$\lambda \mapsto L(w^*, \lambda)$$

achieves a maximum at λ^*

Existence of a minimum and of a saddle point

Proposition

We suppose that

- the criterion J is differentiable and strongly convex
- in the equality constraints $\Theta(w) = 0$, the function Θ is affine

Then

- there exists a unique *minimum* $w^* \in \mathbb{R}^N$ of J among the w such that $\Theta(w) = 0$
- there exists a vector λ^* of $\mathbb{R}^M$ (Lagrange multiplier) such that (w^*, λ^*) is a saddle point of the Lagrangian L

Outline of the presentation

The Uzawa algorithm or dual gradient algorithm

We suppose that

- the criterion J is differentiable and a-strongly convex
- in the equality constraints $\Theta(w) = 0$,
  the function Θ is affine, with norm κ

Then, when $0 < \rho < 2a/\kappa^2$, the following algorithm converges towards the (unique) minimum of

$$\inf_{w \in \mathbb{R}^N} J(w) \ , \ \ \Theta(w) = 0$$

Data: Initial multiplier $\lambda^{(0)}$, step ρ
Result: minimum and multiplier;
repeat
  $u^{(k)} = \arg\min_{u \in \mathbb{R}^N} L(w, \lambda^{(k)})$ (minimization w.r.t. the first variable) ;
  $\lambda^{(k+1)} = \lambda^{(k)} + \rho\Theta(w^{(k)})$ (gradient step for the second variable) ;
until $\Theta(w^{(k)}) = 0$;

Algorithm 1: Dual Gradient Algorithm

Outline of the presentation

Optimization under equality constraints

We consider the optimization problem

$$\inf_{w \in \mathbb{R}^N} J(w)$$

under the constraint

$$\Theta(w) = 0$$

where Θ is a function with values in $\mathbb{R}^M$

$$\Theta = (\Theta_1, \ldots, \Theta_M) : \mathbb{R}^N \to \mathbb{R}^M$$

whose components are denoted by Θ_j, where j runs from 1 to M

Sufficient condition for qualification in case of equality constraints

Definition

Let $w^* \in \mathbb{R}^N$. The equality constraints $\Theta(w) = 0$ are said to be *regular* at w^* if, when $\Theta(w^*) = 0$, the function Θ is differentiable at w^* and the vectors $\nabla\Theta_j(w^*)$, $j \in \{1, \ldots, M\}$, are linearly independent

Let $w^* \in \mathbb{R}^N$. In case

- either the equality constraints $\Theta(w) = 0$ are regular at w^*
- or the function Θ is affine

we say that the equality constraints $\Theta(w) = 0$ are qualified at w^*

Lagrangian

Definition

The Lagrangian $L : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ is defined by

$$L(w, \lambda) = J(w) + \langle \Theta(w), \lambda \rangle = J(w) + \sum_{j=1}^{M} \lambda_j \Theta_j(w)$$

The variables λ are called (Lagrange) multipliers

First-order optimality conditions (necessary) Karush-Kuhn-Tucker (KKT) optimality conditions

Proposition

We suppose that the criterion J is differentiable. Let $w^* \in \mathbb{R}^N$.
If the equality constraints $\Theta(w) = 0$ are qualified at w^*,
then a *necessary condition* for w^* to be a *local minimum* of J,
among the w such that $\Theta(w) = 0$,
is that there exists a vector λ^* of $\mathbb{R}^M$ (Lagrange multiplier) such that

$$\frac{\partial L}{\partial w}(w^*, \lambda^*) = 0 \text{ and } \frac{\partial L}{\partial \lambda}(w^*, \lambda^*) = 0$$

expressing the first-order optimality conditions
(KKT optimality conditions)

First-order optimality conditions (sufficient)

Proposition

Let $w^* \in \mathbb{R}^N$. We suppose that

- the criterion J is differentiable and convex
- in the equality constraints $\Theta(w) = 0$, the function Θ is affine

Then a *sufficient condition* for w^* to be a *minimum* of J,
among the w such that $\Theta(w) = 0$,
is that there exists a vector λ^* of $\mathbb{R}^M$ (Lagrange multiplier) such that

$$\frac{\partial L}{\partial w}(w^*, \lambda^*) = 0 \text{ and } \frac{\partial L}{\partial \lambda}(w^*, \lambda^*) = 0$$

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Extended real valued functions

$$\overline{\mathbb{R}} = [-\infty, +\infty] = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$$

For any set $\mathcal{W}$ and extended real valued function $h : \mathcal{W} \to \overline{\mathbb{R}}$

- the epigraph is

$$\operatorname{epi} h = \left\{(w,t) \in \mathcal{W} \times \mathbb{R} \,\middle|\, h(w) \leq t\right\} \subset \mathcal{W} \times \mathbb{R}$$

- the effective domain is

$$\operatorname{dom} h = \left\{w \in \mathcal{W} \,\middle|\, h(w) < +\infty\right\} \subset \mathcal{W}$$

- the function is said to be proper
  if it never takes the value $-\infty$ and if it takes at least one finite value

$$h \text{ is proper} \iff -\infty < h \quad \text{and} \quad \operatorname{dom} h \neq \emptyset$$

Moreau additions, characteristic function and constraints in optimization

- The Moreau lower and upper addition extend the usual addition with

$$(+\infty) \dot{+} (-\infty) = (-\infty) \dot{+} (+\infty) = +\infty$$
$$(+\infty) \underset{\cdot}{+} (-\infty) = (-\infty) \underset{\cdot}{+} (+\infty) = -\infty$$

- For any subset $W \subset \mathcal{W}$, the indicator function $\iota_W : \mathcal{W} \to \overline{\mathbb{R}}$ is

$$\iota_W(w) = \begin{cases} 0 & \text{if } w \in W \\ +\infty & \text{if } w \notin W \end{cases}$$

and, for any function $h : \mathcal{W} \to \overline{\mathbb{R}}$, we have

$$\inf_{w \in W} h(w) = \inf_{w \in \mathcal{W}} \Big(h(w) \dot{+} \iota_W(w) \Big)$$

Duality in convex analysis

Extended real valued convex and lsc functions

Let $\mathcal{X}$ be a (real) vector space

Convex function

A function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is said to be convex
if its epigraph $\operatorname{epi} f$ is a convex subset of $\mathcal{X} \times \mathbb{R}$

Let $\mathcal{X}$ be a topological (real) vector space

Lower semi continuous (lsc) function = closed function

A function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is said to be lower semi continuous (lsc) or closed
if its epigraph $\operatorname{epi} f$ is a closed subset of $\mathcal{X} \times \mathbb{R}$

Bilinear duality, primal and dual spaces

- Let $\mathcal{X}$ and $\mathcal{Y}$ be two (real) vector spaces that are paired:
  - there exists a bilinear form $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ and locally convex topologies that are compatible in the sense
    that the continuous linear forms on $\mathcal{X}$ are the functions $x \in \mathcal{X} \mapsto \langle x, y \rangle$, for all $y \in \mathcal{Y}$,
    and that the continuous linear forms on $\mathcal{Y}$ are the functions $y \in \mathcal{Y} \mapsto \langle x, y \rangle$, for all $x \in \mathcal{X}$
- The space $\mathcal{X}$ is called the primal space
- The space $\mathcal{Y}$ is called the dual space

Subdifferential of a function

Subdifferential

The subdifferential of a function $f : \mathcal{X} \to \overline{\mathbb{R}}$ at $x \in \mathcal{X}$ is the subset

$$\partial f(x) = \left\{ y \in \mathcal{Y} \,\middle|\, f(x') - \langle x', y \rangle \geq f(x) - \langle x, y \rangle \;, \;\; \forall x' \in \mathcal{X} \right\}$$

$$y \in \partial f(x) \iff f(x') \geq \underbrace{f(x) + \langle x' - x, y \rangle}_{\substack{\text{affine function of } x' \\ \text{sharp at } x \in \mathcal{X}}} \;, \;\; \forall x' \in \mathcal{X}$$

Subdifferential and argmin

$$0 \in \partial f(\bar{x}) \iff \bar{x} \in \mathop{\arg\min}_{x \in \mathcal{X}} f(x)$$

The Fenchel conjugacy

- The Fenchel conjugacy $\star$ is defined,
  for any functions $f : \mathcal{X} \to \overline{\mathbb{R}}$ and $g : \mathcal{Y} \to \overline{\mathbb{R}}$, by

$$f^{\star}(y) = \sup_{x \in \mathcal{X}} \Big(\langle x , y \rangle - f(x) \Big) , \ \ \forall y \in \mathcal{Y}$$

$$g^{\star'}(x) = \sup_{y \in \mathcal{Y}} \Big(\langle x , y \rangle - g(y) \Big) , \ \ \forall x \in \mathcal{X}$$

$$f^{\star\star'}(x) = \sup_{y \in \mathcal{Y}} \Big(\langle x , y \rangle - f^{\star}(y) \Big) , \ \ \forall x \in \mathcal{X}$$

- The Fenchel biconjugate $f^{\star\star'}$ is closed convex and satisfies

$$f^{\star\star'} \leq f$$

(it is not necessarily the best closed convex lower approximation, as illustrated by closed convex *valley functions*)

The Fenchel conjugacy and closed convex functions

Theorem

- The Fenchel conjugacy induces a one-to-one correspondence between the proper closed convex functions on $\mathcal{X}$ and the proper closed convex functions on $\mathcal{Y}$
- For any function $f : \mathcal{X} \to [-\infty, +\infty]$, we have

$$f \text{ is closed convex proper or } f \equiv -\infty \text{ or } f \equiv +\infty \iff f^{\star\star'} = f$$

- For any function $f : \mathcal{X} \to]-\infty, +\infty]$, we have

$$f \text{ is closed convex } \iff f^{\star\star'} = f$$

Subdifferential and Fenchel conjugacy

$$y \in \partial f(x) \iff f(x) + f^{\star}(y) = \langle x,\, y \rangle$$

$$\partial f(x) \neq \emptyset \implies f^{\star\star'}(x) = f(x)$$

Outline of the presentation

Dual problems: perturbation scheme [Rockafellar, 1974]

sets	optimization set $\mathcal{W}$	primal space $\mathcal{X}$	pairing $\mathcal{X} \overset{\langle \cdot, \cdot \rangle}{\leftrightarrow} \mathcal{Y}$	dual space $\mathcal{Y}$
variables	decision $w \in \mathcal{W}$	perturbation $x \in \mathcal{X}$	$\langle x, y \rangle \in \mathbb{R}$	sensitivity $y \in \mathcal{Y}$
bivariate functions		Rockafellian $R : \mathcal{W} \times \mathcal{X} \to \overline{\mathbb{R}}$		Lagrangian $L : \mathcal{W} \times \mathcal{Y} \to \overline{\mathbb{R}}$
definition				$L(w, y) = \inf_{x \in \mathcal{X}} \{R(w, x) - \langle x, y \rangle\}$
property				$-L(w, \cdot) = \big(R(w, \cdot)\big)^{\star}$
property				$-L(w, \cdot)$ is $\star'$-convex (hence $L(w, \cdot)$ is concave usc)
univariate functions		perturbation function $\varphi : \mathcal{X} \to \overline{\mathbb{R}}$		dual function $\psi : \mathcal{Y} \to \overline{\mathbb{R}}$
definition		$\varphi(x) = \inf_{w \in \mathcal{W}} R(w, x)$		$\psi(y) = \inf_{w \in \mathcal{W}} L(w, y)$
property				$-\psi = \varphi^{\star}$

- Anchor $0 \in \mathcal{X}$ and dual maximization problem (weak duality) $\varphi^{\star\star'}(0) = \sup_{y \in \mathcal{Y}} \{-\psi(y)\} \leq \inf_{w \in \mathcal{W}} R(w, 0) = \varphi(0)$
- Strong duality iff φ is $\star$-convex at 0 iff $\varphi^{\star\star'}(0) = \varphi(0)$

Dual problems given by Fenchel conjugacy

- Set $\mathcal{W}$, function $h : \mathcal{W} \to \overline{\mathbb{R}}$ and original minimization problem

$$\inf_{w \in \mathcal{W}} h(w)$$

- Embedding/perturbation scheme given by a nonempty set $\mathcal{X}$, an anchor $\overline{x} \in \mathcal{X}$ and a Rockafellian $R : \mathcal{W} \times \mathcal{X} \to \overline{\mathbb{R}}$ such that

$$h(w) = R(w, \overline{x}) \ , \ \ \forall w \in \mathcal{W}$$

- Paired spaces $\mathcal{X}$ and $\mathcal{Y}$, and Lagrangian $L : \mathcal{W} \times \mathcal{Y} \to \overline{\mathbb{R}}$ given by

$$L(w, y) = \inf_{x \in \mathcal{X}} \left\{ R(w, x) - \langle x - \overline{x}, \, y \rangle \right\}$$

- Original minimization problem

$$\inf_{w \in \mathcal{W}} \sup_{y \in \mathcal{Y}} L(w, y) = \inf_{w \in \mathcal{W}} h(w)$$

Duality gap

- Dual maximization problem

$$\sup_{y \in \mathcal{Y}} \inf_{w \in \mathcal{W}} L(w, y)$$

- Weak duality always holds true

$$\sup_{y \in \mathcal{Y}} \inf_{w \in \mathcal{W}} L(w, y) \leq \inf_{w \in \mathcal{W}} h(w)$$

  When it exists, the duality gap is the nonnegative difference

- Strong duality holds true, or there is no duality gap, when

$$\sup_{y \in \mathcal{Y}} \inf_{w \in \mathcal{W}} L(w, y) = \inf_{w \in \mathcal{W}} h(w)$$

Abstract Karush-Kuhn-Tucker (KKT) condition

Karush-Kuhn-Tucker (KKT) condition

Abstract Karush-Kuhn-Tucker (KKT) condition

The couple $(\overline{w}, \overline{y}) \in \mathcal{W} \times \mathcal{Y}$ satisfies the KKT condition if $(\overline{w}, \overline{y})$ is a saddle point of the Lagrangian L, that is,

- the function $\mathcal{W} \ni w \mapsto L(w, y)$ achieves a *minimum* at $\overline{w}$
- the function $\mathcal{Y} \ni y \mapsto L(w, y)$ achieves a *maximum* at $\overline{y}$

Strong duality and KKT condition under convexity

Theorem [Rockafellar, 1974, Theorem 15, p. 40]

Suppose that the function $x \mapsto R(w, x)$ is closed convex

Then, the following conditions are equivalent

1. There is no duality gap
   and $\overline{w} \in \arg\min_{w \in \mathcal{W}} h(w)$
   and $\overline{y} \in \arg\max_{y \in \mathcal{Y}} \inf_{w \in \mathcal{W}} L(w, y)$
2. The couple $(\overline{w}, \overline{y}) \in \mathcal{W} \times \mathcal{Y}$ satisfies the KKT condition

Strong duality and KKT condition under convexity

Theorem [Rockafellar, 1974, Corollary 15A, p. 40]

Suppose that there is no duality gap
and $\overline{w} \in \arg\min_{w \in \mathcal{W}} h(w)$

Then, the following conditions are equivalent

1. $\overline{w} \in \arg\min_{w \in \mathcal{W}} h(w)$
2. there exists $\overline{y} \in \mathcal{Y}$ such that
   the couple $(\overline{w}, \overline{y}) \in \mathcal{W} \times \mathcal{Y}$ satisfies the KKT condition

Sensitivity analysis

Subdifferential of the perturbation function (sensitivity analysis)

The perturbation function is

$$\varphi(x) = \inf_{w \in \mathcal{W}} R(w, x) \; , \;\; \forall x \in \mathcal{X}$$

Theorem [Rockafellar, 1974, Theorem 16, p. 40]

For $\overline{y} \in \mathcal{Y}$, the following conditions are equivalent

1. $\overline{y} \in \arg\max_{y \in \mathcal{Y}} \inf_{w \in \mathcal{W}} L(w, y)$ and $\max_{y \in \mathcal{Y}} \inf_{w \in \mathcal{W}} L(w, y) = \inf_{w \in \mathcal{W}} h(w)$
2. $\overline{y} \in \partial\varphi(\overline{x})$
3. $\inf_{w \in \mathcal{W}} h(w) = \inf_{w \in \mathcal{W}} L(w, \overline{y})$

Subdifferential of the perturbation function (sensitivity analysis)

The convex case

Theorem [Rockafellar, 1974, Theorem 18, p. 41]

Suppose that

- the function $R : \mathcal{W} \times \mathcal{X} \to \overline{\mathbb{R}}$ is convex
- there exists $w \in \mathcal{W}$ such that the function $x \mapsto R(w, x)$ is bounded above in a neighborhood of $\overline{x}$

Then there exists $\overline{y} \in \mathcal{Y}$ such that

1. $\overline{y} \in \arg\max_{y \in \mathcal{Y}} \inf_{w \in \mathcal{W}} L(w, y)$ and $\max_{y \in \mathcal{Y}} \inf_{w \in \mathcal{W}} L(w, y) = \inf_{w \in \mathcal{W}} h(w)$
2. $\overline{y} \in \partial\varphi(\overline{x})$
3. $\inf_{w \in \mathcal{W}} h(w) = \inf_{w \in \mathcal{W}} L(w, \overline{y})$

Dual problems with general couplings

Dual problems: perturbation scheme [Rockafellar, 1974]

- Set $\mathcal{W}$, function $h : \mathcal{W} \to \overline{\mathbb{R}}$ and original minimization problem

$$\inf_{w \in \mathcal{W}} h(w)$$

- Embedding/perturbation scheme given by a nonempty set $\mathcal{X}$ (perturbations), an element $\overline{x} \in \mathcal{X}$ (anchor) and a function (Rockafellian) $R : \mathcal{W} \times \mathcal{X} \to \overline{\mathbb{R}}$ such that

$$h(w) = R(w, \overline{x})$$

- Perturbation function

$$\varphi(x) = \inf_{w \in \mathcal{W}} R(w, x)$$

- Original minimization problem

$$\varphi(\overline{x}) = \inf_{w \in \mathcal{W}} R(w, \overline{x}) = \inf_{w \in \mathcal{W}} h(w)$$

Dual problems: conjugacy, weak and strong duality

- Coupling $\mathcal{X} \overset{c}{\leftrightarrow} \mathcal{Y}$, and Lagrangian $L : \mathcal{W} \times \mathcal{Y} \to \overline{\mathbb{R}}$ given by

$$L(w,y) = \inf_{x \in \mathcal{X}} \left\{ R(w,x) \dot{+} \left(-c(x,y) \right) \right\}$$

- Dual function

$$\psi(y) = -\varphi^{c}(y) = \inf_{w \in \mathcal{W}} L(w,y)$$

- Dual maximization problem (weak duality)

$$\varphi^{cc'}(\overline{x}) = \sup_{y \in \mathcal{Y}} \left\{ c(\overline{x},y) \underset{\cdot}{+} \psi(y) \right\} \leq \inf_{w \in \mathcal{W}} h(w) = \varphi(\overline{x})$$

- Strong duality holds true when φ is c-convex at $\overline{x}$, that is,

$$\varphi^{cc'}(\overline{x}) = \sup_{y \in \mathcal{Y}} \left\{ c(\overline{x},y) \underset{\cdot}{+} \psi(y) \right\} = \inf_{w \in \mathcal{W}} h(w) = \varphi(\overline{x})$$

Dual problems: perturbation scheme [Rockafellar, 1974]

sets	optimization set $\mathcal{W}$	primal set $\mathcal{X}$	coupling $\mathcal{X} \overset{c}{\leftrightarrow} \mathcal{Y}$	dual set $\mathcal{Y}$
variables	decision $w \in \mathcal{W}$	perturbation $x \in \mathcal{X}$	$c(x,y) \in \overline{\mathbb{R}}$	sensitivity $y \in \mathcal{Y}$
bivariate functions		Rockafellian $R : \mathcal{W} \times \mathcal{X} \to \overline{\mathbb{R}}$		Lagrangian $L : \mathcal{W} \times \mathcal{Y} \to \overline{\mathbb{R}}$
definition				$L(w,y) = \inf_{x \in \mathcal{X}} \left\{ R(w,x) \dotplus (-c(x,y)) \right\}$
property				$-L(w,\cdot) = \big(R(w,\cdot)\big)^{c}$
property				$-L(w,\cdot)$ is c'-convex
univariate functions		perturbation function $\varphi : \mathcal{X} \to \overline{\mathbb{R}}$		dual function $\psi : \mathcal{Y} \to \overline{\mathbb{R}}$
definition		$\varphi(x) = \inf_{w \in \mathcal{W}} R(w,x)$		$\psi(y) = \inf_{w \in \mathcal{W}} L(w,y)$
property				$-\psi = \varphi^{c}$

- Anchor $\overline{x} \in \mathcal{X}$ and dual maximization problem (weak duality) $\varphi^{cc'}(\overline{x}) = \sup_{y \in \mathcal{Y}} \left\{ c(\overline{x}, y) \dotplus \psi(y) \right\} \leq \inf_{w \in \mathcal{W}} R(w, \overline{x}) = \varphi(\overline{x})$
- Strong duality iff φ is c-convex at $\overline{x}$ iff $\varphi^{cc'}(\overline{x}) = \varphi(\overline{x})$

Outline of the presentation

Classic Lagrangian duality

- Let $\theta = (\theta_1, \ldots, \theta_p) : \mathcal{W} \to \mathbb{R}^p$ be a mapping, and $\overline{x} \in \mathbb{R}^p$
- We consider the optimization problem

$$\min_{\theta(w) \leq \overline{x}} h(w) = \min_{\substack{\theta_1(w) \leq \overline{x}_1 \\ \cdots \\ \theta_p(w) \leq \overline{x}_p}} h(w)$$

- In that case, take the perturbation scheme with $\mathcal{X} = \mathbb{R}^p$ and

$$R(w, x) = h(w) \dotplus \iota_{\{\theta(w) \leq x\}} = h(w) \dotplus \sum_{j=1}^{p} \iota_{\{\theta_j(w) \leq x_j\}}$$

- which gives the Lagrangian $L : \mathcal{W} \times \mathcal{Y} \to \overline{\mathbb{R}}$, with $\mathcal{Y} = \mathbb{R}^p$ and

$$L(w, y) = h(w) + \langle \theta(w) - \overline{x}, \, y \rangle = h(w) + \sum_{j=1}^{p} y_j \big(\theta_j(w) - \overline{x}\big)$$

Slater qualification constraint

The convex case

Theorem [Rockafellar, 1974, p. 45]

Suppose that

- the functions h and $\theta_1, \ldots, \theta_p$ are is convex
- there exists $w \in \mathcal{W}$ such that

$$\theta_1(w) < \overline{x}_1, \ldots, \theta_p(w) < \overline{x}_p$$

Then there exists $\overline{y} \in \mathcal{Y}$ such that

1. $\overline{y} \in \arg\max_{y \in \mathcal{Y}} \inf_{w \in \mathcal{W}} L(w, y)$ and $\max_{y \in \mathcal{Y}} \inf_{w \in \mathcal{W}} L(w, y) = \inf_{w \in \mathcal{W}} h(w)$
2. $\overline{y} \in \partial\varphi(\overline{x})$
3. $\inf_{w \in \mathcal{W}} h(w) = \inf_{w \in \mathcal{W}} L(w, \overline{y})$

Jonathan M. Borwein and Adrian S. Lewis. *Convex analysis and nonlinear optimization*, volume 3 of *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC*. Springer, New York, second edition, 2006. ISBN 978-0387-29570-1; 0-387-29570-4. Theory and examples.

R. Tyrrell Rockafellar. *Conjugate Duality and Optimization*. CBMS-NSF Regional Conference Series in Applied Mathematics. Society for Industrial and Applied Mathematics, 1974.