Tutorial

The Perturbation-Duality Scheme in Optimization

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Introduction

Duality is widely used in optimization

- linear programming
  (Lagrangian duality, including optimal transport, etc.)
- convex programming
  (Lagrangian duality in mathematical programming, minimal cost flow on a graph etc.)
- conic programming, semidefinite programming, etc.

Introduction

The perturbation-duality scheme (PDS)

- Introduced in Rockafellar [1974]
- Goal: systematically produce dual optimization problems
  from a given optimization problem
  by perturbation followed by conjugate duality

Outline

PDS for linear programming (LP)

PDS for convex programming

PDS for pure integer linear programming (PILP)

Conclusion

Outline of the presentation

PDS for linear programming (LP)

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Conclusion

Outline of the presentation

Primal and dual problems in standard LP

$$\overbrace{\sup_{\substack{\lambda \in \mathbb{R}^m \\ \lambda^T A \leq k}} \langle b, \lambda \rangle}^{\text{dual problem}} \underbrace{\leq}_{\text{weak duality}} \overbrace{\inf_{\substack{x \in \mathbb{R}^n \\ Ax = b \\ x \geq 0}} \langle x, k \rangle}^{\text{primal problem}}$$

Strong duality for LP

Adapted from Conforti, Cornuéjols, and Zambelli [2014, Theorem 3.7,Proposition 3.9]

Theorem

Given a matrix $A \in \mathbb{R}^{m \times n}$ and vectors $k \in \mathbb{R}^n, b \in \mathbb{R}^m$,
if $\left\{x \in \mathbb{R}^n \,\middle|\, Ax = b, x \geq 0\right\} \neq \emptyset$ or $\left\{\lambda \in \mathbb{R}^m \,\middle|\, \lambda^T A \leq k\right\} \neq \emptyset$,
(that is, if the primal or the dual problem is feasible)

then we have

$$\sup_{\substack{\lambda \in \mathbb{R}^m \\ \lambda^T A \leq k}} \langle b, \lambda \rangle \underbrace{=}_{\text{strong duality}} \inf_{\substack{x \in \mathbb{R}^n \\ Ax = b \\ x \geq 0}} \langle x, k \rangle$$

Sketch of the proof

1. Introduce the Lagrangian $\mathcal{L} : \mathbb{R}^n_+ \times \mathbb{R}^m \to \mathbb{R}$ by

$$\mathcal{L}(x,\lambda) = \langle x,\ k\rangle + \langle b - Ax,\ \lambda\rangle\ ,\ \ \forall x \in \mathbb{R}^n_+, \lambda \in \mathbb{R}^m$$

2. Use sup-inf inversion inequality to get weak duality

$$\underbrace{\sup_{\lambda\in\mathbb{R}^m}\ \inf_{x\in\mathbb{R}^n_+} \mathcal{L}(x,\lambda)}_{\text{dual problem}} \le \underbrace{\inf_{x\in\mathbb{R}^n_+}\ \sup_{\lambda\in\mathbb{R}^m} \mathcal{L}(x,\lambda)}_{\text{primal problem}}$$

3. Find a saddle-point $(\overline{x},\overline{\lambda}) \in \mathbb{R}^n_+ \times \mathbb{R}^m$

$$\mathcal{L}(\overline{x},\lambda) \le \mathcal{L}(\overline{x},\overline{\lambda}) \le \mathcal{L}(x,\overline{\lambda})\ ,\ \ \forall x \in \mathbb{R}^n_+, \lambda \in \mathbb{R}^m$$

to prove strong duality, i.e.

$$\underbrace{\sup_{\lambda\in\mathbb{R}^m}\ \inf_{x\in\mathbb{R}^n_+} \mathcal{L}(x,\lambda)}_{\text{dual problem}} = \underbrace{\inf_{x\in\mathbb{R}^n_+}\ \sup_{\lambda\in\mathbb{R}^m} \mathcal{L}(x,\lambda)}_{\text{primal problem}}$$

Now, let us deduce the previous duality results from a perturbation duality scheme (PDS)

Outline of the presentation

PDS for linear programming (LP)
- Fundamental LP duality theorem
- LP weak duality through PDS
- LP strong duality through PDS
- Summary

PDS for convex programming

PDS for pure integer linear programming (PILP)

Conclusion

Steps of the perturbation-duality scheme

Rockafellar [1974]

1. Perturb a minimization problem
   with a perturbation (primal) variable
   belonging to a vector space,
   and a Rockafellian function
2. Pair the (primal) perturbation space with a dual space
   by means of a bilinear form $\langle \, , \rangle$
3. Biconjugate the perturbation function, and get
   - a dual problem
   - weak duality
4. Deduce conditions for strong duality by means of
   either global or local properties of the perturbation function

Illustration of the scheme in Linear Programming (LP)

- Constraint matrix $A \in \mathbb{R}^{m \times n}$
- Cost vector $k \in \mathbb{R}^n$
- Anchor $\bar{b} \in \mathbb{R}^m$

Initial/original minimization problem

$$\inf_{\substack{x \in \mathbb{R}^n \\ Ax = \bar{b} \\ x \geq 0}} \langle x, k \rangle$$

Step 1. Perturbation of the initial minimization problem

- Introduce a perturbation space, $\mathbb{R}^m$, and embed the original problem into a family of minimization problems (more on the Rockafellian later)
- Introduce the perturbation function $\varphi\colon \mathbb{R}^m \to \overline{\mathbb{R}} = \mathbb{R} \cup \underbrace{\{-\infty\}}_{\text{unbounded}} \cup \underbrace{\{+\infty\}}_{\text{unfeasible}}$

$$\forall b \in \mathbb{R}^m \,, \;\; \varphi(b) = \inf_{\substack{x \in \mathbb{R}^n \\ Ax = b \\ x \geq 0}} \langle x, \, k \rangle$$

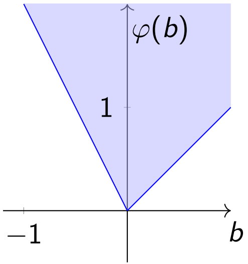
- The value of the original problem is then $\varphi(\bar{b})$

Example of perturbation function's epigraph for LP

Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be defined as

$$\forall b \in \mathbb{R}\,, \quad \varphi(b) = \inf_{\substack{x \in \mathbb{R}^2 \\ x_1 - x_2 = b \\ x \geq 0}} x_1 + 2x_2$$

Then $\varphi(b) = \max\{-2b, b\}\,, \quad \forall b \in \mathbb{R}$



Step 2. Dual space, coupling and conjugate function

- Perturbation space: $\mathbb{R}^m$ — dual space: $\mathbb{R}^m$ (linear functions)
- Introduce the bilinear coupling

$$\langle \, , \rangle \; : \quad \overbrace{\mathbb{R}^m}^{\substack{\text{perturbation} \\ \text{space}}} \quad \times \quad \overbrace{\mathbb{R}^m}^{\text{dual space}} \quad \to \mathbb{R}$$

- Deduce the conjugate function $\varphi^\star \colon \mathbb{R}^m \to \overline{\mathbb{R}}$ of the perturbation function

$$\forall \lambda \in \mathbb{R}^m \, , \; \varphi^\star(\lambda) = \sup_{b \in \mathbb{R}^m} \big\{ \langle b, \lambda \rangle - \varphi(b) \big\}$$

Conjugate function and Lagrangian

$$
\begin{aligned}
\varphi^\star(\lambda) &= \sup_{b \in \mathbb{R}^m} \left\{ \langle b,\, \lambda \rangle - \varphi(b) \right\} \\
&= \sup_{b \in \mathbb{R}^m} \left\{ \langle b,\, \lambda \rangle - \inf_{\substack{Ax=b \\ x \geq 0}} \langle x,\, k \rangle \right\} \\
&= \sup_{b \in \mathbb{R}^m} \left\{ \langle b,\, \lambda \rangle + \sup_{\substack{Ax=b \\ x \geq 0}} \langle -x,\, k \rangle \right\} \\
&= \sup_{x \geq 0} \left\{ \sup_{\substack{Ax=b \\ b \in \mathbb{R}^m}} \langle b,\, \lambda \rangle - \langle x,\, k \rangle \right\} \\
&= \sup_{x \geq 0} \left\{ \langle Ax,\, \lambda \rangle - \langle x,\, k \rangle \right\} \\
&= \langle \bar{b},\, \lambda \rangle - \inf_{x \geq 0} \underbrace{\left\{ \langle x,\, k \rangle + \langle \bar{b} - Ax,\, \lambda \rangle \right\}}_{\text{Lagrangian } \mathcal{L}(x,\lambda)}
\end{aligned}
$$

Step 3. Biconjugate and weak duality

- Biconjugate function $\varphi^{\star\star'} : \mathbb{R}^m \to \overline{\mathbb{R}}$

$$\forall b \in \mathbb{R}^m \;, \;\; \varphi^{\star\star'}(b) = \sup_{\lambda \in \mathbb{R}^m} \left\{ \langle b, \, \lambda \rangle - \varphi^{\star}(\lambda) \right\}$$

- We obtain weak duality for all $b \in \mathbb{R}^m$

$$\underbrace{\begin{array}{cc} \sup & \langle b, \, \lambda \rangle \\ \lambda \in \mathbb{R}^m & \\ \lambda^T A \leq k & \end{array}}_{\text{dual problem}} = \varphi^{\star\star'}(b) \leq \varphi(b) = \begin{array}{cc} \inf & \langle x, \, k \rangle \\ x \in \mathbb{R}^n & \\ Ax = b & \\ x \geq 0 & \end{array}$$

- At the anchor $\overline{b}$

$$\varphi^{\star\star'}(\overline{b}) = \sup_{\lambda \in \mathbb{R}^m} \{ \underbrace{\langle \overline{b}, \, \lambda \rangle - \varphi^{\star}(\lambda)}_{\inf_{x \geq 0} \mathcal{L}(x, \lambda)} \}$$

We have obtained the LP weak duality result
What about the LP strong duality result?

Outline of the presentation

Step 4. Conditions for strong duality

Proposition

Let $A \in \mathbb{R}^{m \times n}$ and $k \in \mathbb{R}^n$
If $-\infty < \varphi(0)$, that is, the corresponding LP is bounded
then, for all $b \in \mathbb{R}^m$,

$$\left(\sup_{\substack{\lambda \in \mathbb{R}^m \\ \lambda^T A \leq k}} \langle b,\ \lambda \rangle = \right) \underbrace{\varphi^{\star\star'}(b) = \varphi(b)}_{\text{strong duality}} \left(= \inf_{\substack{x \in \mathbb{R}^n \\ Ax = b \\ x \geq 0}} \langle x,\ k \rangle \right)$$

Remark

This result is true even if $b \in \mathbb{R}^m$ is such that $\varphi(b) = +\infty$, meaning for any unfeasible LPs

Proof of strong duality for LP. Sketch of the proof

adapted from Rockafellar [1974, p.24]

(a) We show that if the LP $\varphi(0)$ is bounded, then every feasible LP is bounded

$$-\infty < \varphi(0) \implies \varphi \text{ is proper}$$

(b) We show that $\mathrm{epi}\ \varphi$ is a closed convex set (by showing that $\mathrm{epi}\ \varphi$ is a polyhedron)

$$\varphi \text{ is a closed convex function}$$

(c) We apply Fenchel-Moreau Theorem to get strong duality

$$\varphi^{\star\star'} = \varphi$$

Strong duality for LP. Step (a) Proper functions

Definition

Let $f : \mathbb{R}^m \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$

- $\mathrm{dom}\ f = \{b \in \mathbb{R}^m : f(b) < +\infty\}$
- The function f is said to be proper
  if $\mathrm{dom}\ f \neq \emptyset$ and $-\infty < f(b)$, $\forall b \in \mathbb{R}^m$

Lemma

If $-\infty < \varphi(0)$ (the corresponding LP is bounded)
then the value function φ is proper (all feasible LPs are bounded)

Idea of the proof

The recession cone of $\{x \in \mathbb{R}^n \colon Ax = b\}$
is given by $\{r \in \mathbb{R}^n \colon Ax = 0, r \geq 0\}$

Conforti, Cornuéjols, and Zambelli [2014, Proposition 3.15]

Strong duality for LP. Step (b) Closed convex epigraph

Definition

Let $f : \mathbb{R}^m \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$
The epigraph of the function f is defined by

$$\mathrm{epi}\ f = \left\{(b, t) \in \mathbb{R}^m \times \mathbb{R} \colon f(b) \leq t\right\}$$

Proposition

Let $A \in \mathbb{R}^{m \times n}$ and $k \in \mathbb{R}^n$ define
the value function $\varphi \colon \mathbb{R}^m \to \overline{\mathbb{R}}$ by

$$\forall b \in \mathbb{R}^m\ , \quad \varphi(b) = \inf_{\substack{x \in \mathbb{R}^n \\ Ax = b \\ x \geq 0}} \langle x,\ k \rangle$$

Then $\mathrm{epi}\ \varphi$ is a polyhedron

Proof that $\operatorname{epi}\varphi$ is a polyhedron

$A \in \mathbb{R}^{m\times n}$, $k \in \mathbb{R}^n$

Let $b \in \mathbb{R}^m$, we assume that $-\infty < \varphi(b)$

$$
\begin{aligned}
&\varphi(b) \leq t \\
&\quad\iff \inf_{\substack{x\in\mathbb{R}^n \\ Ax=b \\ x\geq 0}} \langle x,\, k\rangle \leq t \\
&\quad\iff \min_{\substack{x\in\mathbb{R}^n \\ Ax=b \\ x\geq 0}} \langle x,\, k\rangle \leq t \quad \text{(as bounded feasible LPs are attained)} \\
&\quad\iff \exists x \in \mathbb{R}^n \text{ s.t. } Ax = b\,,\ \ x \geq 0\,,\ \ \langle x,\, k\rangle - t \leq 0 \\
&\quad\iff \operatorname{epi}\varphi = \pi_{(b,t)}\left\{(b,t,x) \in \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n :\ \begin{cases} Ax = b \\ x \geq 0 \\ \langle x,\, k\rangle - t \leq 0 \end{cases}\right\}
\end{aligned}
$$

Thus $\operatorname{epi}\varphi$ is the projection of a polyhedron

So, $\operatorname{epi}\varphi$ is a polyhedron

Rockafellar [1970, Theorem 19.3]

Strong duality for LP. Step (c) Fenchel-Moreau theorem

Definition

A function $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ is said to be closed convex Rockafellar [1974]
if EITHER [f is proper AND $\mathrm{epi}\, f$ is a closed convex set]
OR $f \equiv +\infty$ OR $f \equiv -\infty$

Theorem

[Fenchel-Moreau Theorem]
A function $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ is closed convex if and only if $f^{\star\star'} = f$

So we have strong duality

$$\varphi^{\star\star'} = \underbrace{\varphi}$$

as Steps (a) and (b) imply
that φ is a closed function

About closed convex functions: the case of valley functions

Definition

Let $C \subset \mathbb{R}^n$ be a closed convex set
Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a function
We say that f is a valley function if

$$f(u) = \begin{cases} -\infty \text{ if } u \in C \\ +\infty \text{ otherwise} \end{cases}$$

Remark

Valley functions have a closed convex epigraph
BUT are not closed convex functions
(except the cases $f \equiv -\infty$ or $f \equiv +\infty$)

Example when strong duality is not achieved for LP

$$-\infty = \begin{pmatrix} \sup\limits_{\substack{\lambda \in \mathbb{R}^2 \\ \lambda_1 + \lambda_2 \leq -1 \\ -\lambda_1 - \lambda_2 \leq 0}} & \lambda_1 \end{pmatrix}$$

$$= \varphi^{\star\star'}((1,0)) < \varphi((1,0)) =$$

$$\begin{pmatrix} \inf\limits_{\substack{x \in \mathbb{R}^2 \\ x_1 - x_2 = 1 \\ x_1 - x_2 = 0 \\ x \geq 0}} & -x_1 \end{pmatrix} = +\infty$$

Outline of the presentation

Summary of the perturbation-duality scheme for LP

Rockafellar [1974]

1. We perturb a minimization problem

$$\forall b \in \mathbb{R}^m \; , \;\; \varphi(b) = \inf_{\substack{x \in \mathbb{R}^n \\ Ax = b \\ x \geq 0}} \langle x, \, k \rangle$$

2. We pair the primal space $\mathbb{R}^m$ and a dual space $\mathbb{R}^m$

$$\langle \, , \rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$$

3. We biconjugate the perturbation function φ

$$\Bigg(\sup_{\substack{\lambda \in \mathbb{R}^m \\ \lambda^T A \leq k}} \langle b, \, \lambda \rangle = \Bigg) \underbrace{\varphi^{\star\star'}(b) \leq \varphi(b) \; , \;\; \forall b \in \mathbb{R}^m}_{\text{weak duality is guaranteed}}$$

4. Under suitable assumptions, strong duality by polyhedral property of the epigraph of the perturbation function

Indicator function of a subset

For any subset $X \subset \mathcal{X}$, its indicator function ι_X is

$$\iota_X(x) = \begin{cases} 0 & \text{if } x \in X \\ +\infty & \text{if } x \notin X \end{cases}$$

Summary (bis) of the perturbation-duality scheme for LP

1. We perturb a minimization problem

$$\varphi(b) = \inf_{x \in \mathbb{R}^n} \mathcal{R}(x, b) \ , \ \ \forall b \in \mathbb{R}^m$$

   where the Rockafellian $\mathcal{R} \colon \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ is defined by $\mathcal{R}(x, b) = \langle x, \ k \rangle + \iota_{\mathbb{R}^n_+}(u) + \iota_{\{0\}}(Ax - b)$

2. We pair the primal space $\mathbb{R}^m$ and a dual space $\mathbb{R}^m$

$$\langle \, , \rangle \colon \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$$

3. We biconjugate the perturbation function φ

$$\Big(\sup_{\substack{\lambda \in \mathbb{R}^m \\ \lambda^T A \leq k}} \langle b, \ \lambda \rangle = \Big) \underbrace{\varphi^{\star\star'}(b) \leq \varphi(b) \ , \ \ \forall b \in \mathbb{R}^m}_{\text{weak duality is guaranteed}}$$

4. Under suitable assumptions, strong duality by polyhedral property of the epigraph of the perturbation function

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PDS for linear programming (LP)

PDS for convex programming

- Background on duality in convex analysis
- PDS with Fenchel duality
- Examples of PDS for convex programs
- Generalized perturbation duality scheme

PDS for pure integer linear programming (PILP)

Conclusion

The Fenchel conjugacy

Definition

Two vector spaces $\mathcal{U}$ and $\mathcal{V}$, paired by a bilinear form $\langle \, , \rangle$ (in the sense of convex analysis),
give rise to the classic Fenchel conjugacy between $\overline{\mathbb{R}}^{\mathcal{U}}$ and $\overline{\mathbb{R}}^{\mathcal{V}}$

With any function $f : \mathcal{U} \to \overline{\mathbb{R}}$, we associate
the function $f^{\star} : \mathcal{V} \to \overline{\mathbb{R}}$ defined by

$$f^{\star}(v) = \sup_{u \in \mathcal{U}} \Big\{ \langle u, v \rangle - f(u) \Big\} , \ \forall v \in \mathcal{V}$$

The biconjugate function is a minorant of the function

Definition

Let $f : \mathcal{U} \to \overline{\mathbb{R}}$ be a function
Its biconjugate $f^{\star\star'} : \mathcal{U} \to \overline{\mathbb{R}}$ is defined by

$$f^{\star\star'}(u) = \sup_{v \in \mathcal{V}} \left\{ \langle u, v \rangle - f^{\star}(v) \right\}$$

The inequality below is instrumental in obtaining weak duality

Proposition

For any function $f : \mathcal{U} \to \overline{\mathbb{R}}$, we have that

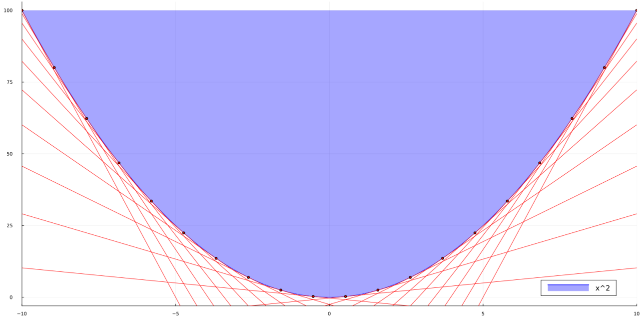
$$f^{\star\star'} \leq f$$

Fenchel-Moreau Theorem

The equality below is instrumental in obtaining strong duality

Theorem

[Fenchel-Moreau]
The function $f : \mathcal{U} \to \overline{\mathbb{R}}$ is closed convex if and only if $f^{\star\star'} = f$

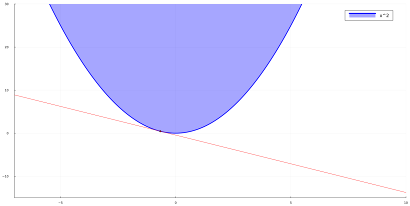


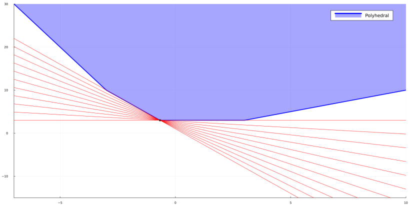
Moreau-Rockafellar subdifferential

Definition

Let $f : \mathcal{U} \to \overline{\mathbb{R}}$ be a function
Its subdifferential $\partial f(u) \subset \mathcal{V}$ at any $u \in \mathcal{U}$ such that $f(u) \in \mathbb{R}$, is defined by

$$v \in \partial f(u) \iff \langle u', v \rangle - f(u') \leq \langle u, v \rangle - f(u) \ , \ \forall u' \in \mathcal{U}$$





Outline of the presentation

Steps of the perturbation-duality scheme

Rockafellar [1974]

1. Perturb a minimization problem
   with a perturbation (primal) variable
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2. Pair the (primal) perturbation space with a dual space
   by means of a bilinear form $\langle \, , \rangle$
3. Biconjugate the perturbation function, and get
   - a dual problem
   - weak duality
4. Deduce conditions for strong duality by means of
   either global or local properties of the perturbation function

Perturbation duality scheme Rockafellar [1974]

sets	optimization set $\mathcal{X}$	primal space $\mathcal{U}$	pairing $\mathcal{U} \overset{\langle , \rangle}{\leftrightarrow} \mathcal{V}$	dual space $\mathcal{V}$
variables	decision $x \in \mathcal{X}$	perturbation $u \in \mathcal{U}$	$\langle u, \ v \rangle \in \mathbb{R}$	sensitivity $v \in \mathcal{V}$
bivariate functions		Rockafellian $\mathcal{R} \colon \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$		Lagrangian $\mathcal{L} \colon \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$
definition				$\mathcal{L}(x, v) = \inf_{u \in \mathcal{U}} \{ \mathcal{R}(x, u) - \langle u, \ v \rangle \}$
property				$-\mathcal{L}(x, \cdot) = \big(\mathcal{R}(x, \cdot)\big)^{\star}$
property				$-\mathcal{L}(x, \cdot)$ is $\star'$-convex (hence $\mathcal{L}(x, \cdot)$ is concave usc)
univariate functions		perturbation function $\varphi \colon \mathcal{U} \to \overline{\mathbb{R}}$		dual function $\psi \colon \mathcal{V} \to \overline{\mathbb{R}}$
definition		$\varphi(u) = \inf_{x \in \mathcal{X}} \mathcal{R}(x, u)$		$\psi(v) = \inf_{x \in \mathcal{X}} \mathcal{L}(x, v)$
property				$-\psi = \varphi^{\star}$

Weak duality

Perturbation/Rockafellian (Step 1)

Data: set $\mathcal{X}$, function $h \colon \mathcal{X} \to \overline{\mathbb{R}}$ and

$$\text{original minimization problem} \qquad \inf_{x \in \mathcal{X}} h(x)$$

- Embedding/perturbation scheme given by a vector space $\mathcal{U}$, and a Rockafellian $\mathcal{R} \colon \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$ such that

$$h(x) = \mathcal{R}(x, 0) \ , \ \ \forall x \in \mathcal{X}$$

- The perturbation function $\varphi \colon \mathcal{U} \to \overline{\mathbb{R}}$ is defined by

$$\varphi(u) = \inf_{x \in \mathcal{X}} \mathcal{R}(x, u)$$

$$\text{original minimization problem} \qquad \varphi(0) = \inf_{x \in \mathcal{X}} h(x)$$

Duality/Fenchel conjugacy (Steps 2,3)

- Dual vector space $\mathcal{V}$ paired to $\mathcal{U}$ by a bilinear form $\langle \,, \rangle$

We obtain weak duality

$$\varphi^{\star\star\prime}(0) = \overbrace{\sup_{v \in \mathcal{V}} \left\{ -\varphi^{\star}(v) \right\}}^{\text{dual problem}}$$

$$\leq$$

$$\varphi(0) = \underbrace{\inf_{x \in \mathcal{X}} h(x)}_{\text{original problem}}$$

Weak duality and Lagrangian

Lagrangian

- Lagrangian $\mathcal{L}\colon \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$ defined by

$$\mathcal{L}(x,v) = \inf_{u \in \mathcal{U}} \Big\{ \underbrace{\mathcal{R}(x,u)}_{\text{Rockafellian}} - \langle u \,, v \rangle \Big\} \,, \;\; \forall (x,v) \in \mathcal{X} \times \mathcal{V}$$

- As $\mathcal{L}(x,v) \leq \mathcal{R}(x,0) - \langle 0 \,, v \rangle = h(x)$, we get that

$$\sup_{v \in \mathcal{V}} \mathcal{L}(x,v) \leq h(x)$$

hence that

$$\text{original minimization problem} \qquad \inf_{x \in \mathcal{X}} \sup_{v \in \mathcal{V}} \mathcal{L}(x,v) \leq \inf_{x \in \mathcal{X}} h(x)$$

Dual function

- The dual function $\psi\colon \mathcal{V} \to \overline{\mathbb{R}}$ is defined by

$$\psi(v) = \inf_{x \in \mathcal{X}} \mathcal{L}(x, v) \,, \quad \forall v \in \mathcal{V}$$

- and the dual problem is

$$\varphi^{\star\star'}(0) = \sup_{v \in \mathcal{V}} \left\{ \langle 0,\, v \rangle - \varphi^{\star}(v) \right\} = \overbrace{\sup_{v \in \mathcal{V}} \psi(v)}^{\text{dual problem}}$$

as

$$\begin{aligned} -\varphi^{\star}(v) &= -\big(\inf_{x \in \mathcal{X}} \mathcal{R}(x, \cdot)\big)^{\star}(v) \\ &= -\sup_{x \in \mathcal{X}} \left\{ \sup_{u \in \mathcal{U}} \left\{ \langle u,\, v \rangle - \mathcal{R}(x, u) \right\} \right\} \\ &= -\sup_{x \in \mathcal{X}} \Big\{ \underbrace{-\inf_{u \in \mathcal{U}} \left\{ -\langle u,\, v \rangle + \mathcal{R}(x, u) \right\}}_{\text{Lagrangian}} \Big\} \\ &= \inf_{x \in \mathcal{X}} \mathcal{L}(x, v) = \psi(v) \end{aligned}$$

Weak duality with Lagrangian

$$
\begin{aligned}
&\varphi^{\star\star\prime}(0) \\
&= \sup_{v \in \mathcal{V}} \left\{ -\varphi^{\star}(v) \right\} \\
&= \underbrace{\sup_{v \in \mathcal{V}} \inf_{x \in \mathcal{X}} \mathcal{L}(x, v)}_{\text{dual problem}} \\
&\leq \\
&\inf_{x \in \mathcal{X}} \sup_{v \in \mathcal{V}} \mathcal{L}(x, v) \\
&\leq \inf_{x \in \mathcal{X}} h(x) \\
&= \varphi(0)
\end{aligned}
$$

Strong duality

Strong duality

$$\overbrace{\varphi^{\star\star'}(0)}^{\text{dual problem}} \underbrace{\leq}_{\text{weak duality}} \overbrace{\varphi(0)}^{\text{original problem}}$$

Definition

$$\text{Strong duality} \iff \varphi^{\star\star'}(0) = \varphi(0) \iff \varphi \text{ is } \star\text{-convex at } 0$$

Paths to strong duality in the convex case

- Suppose that the Rockafellian $\mathcal{R}\colon \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$ is a jointly convex function
- Then, the perturbation function $\varphi\colon \mathcal{U} \to \overline{\mathbb{R}}$ is convex as the marginal function $\varphi(u) = \inf_{x \in \mathcal{X}} \mathcal{R}(x, u)$
- If, in addition,
  - either (global property) the function φ is proper and lower semicontinuous, and then $\varphi^{\star\star'} = \varphi$ by the Fenchel-Moreau Theorem,
  - or (local property) the subdifferential $\partial\varphi(0) \neq \emptyset$, and then the function φ is $\star$-convex at 0,

  and we get strong duality $\underbrace{\varphi^{\star\star'}(0)}_{\text{dual problem}} = \underbrace{\varphi(0)}_{\text{original problem}}$

Outline of the presentation

Lagrangian duality (the case of inequality constraints)

Classic Lagrangian duality (the case of inequality constraints)

- Optimization set $\mathcal{X}$
- Objective function $h \colon \mathcal{X} \to]-\infty, +\infty]$
- Mapping $\theta = (\theta_1, \ldots, \theta_p) : \mathcal{X} \to \mathbb{R}^p$, and $\overline{u} \in \mathbb{R}^p$

We consider the optimization problem

$$\min_{\theta(x) \leq \overline{u}} h(x) = \min_{\substack{\theta_1(x) \leq \overline{u}_1 \\ \vdots \\ \theta_p(x) \leq \overline{u}_p}} h(x)$$

Perturbation and Rockafellian

- Perturbation space $\mathcal{U} = \mathbb{R}^p$
- Rockafellian $\mathcal{R}\colon \mathcal{X} \times \mathbb{R}^p \to \overline{\mathbb{R}}$

$$\mathcal{R}(x, u) = h(x) + \iota_{\{\theta(x) - \bar{u} \le u\}} = h(x) + \sum_{j=1}^{p} \iota_{\{\theta_j(x) - \bar{u}_j \le u_j\}}$$

Duality, Lagrangian and dual function

- Dual space $\mathcal{V} = \mathbb{R}^p$
- We deduce the Lagrangian $\mathcal{L} \colon \mathcal{X} \times \mathbb{R}^p \to \overline{\mathbb{R}}$

$$\mathcal{L}(x, v) = h(x) + \langle \theta(x) - \overline{u},\ v \rangle = h(x) + \sum_{j=1}^{p} v_j \big(\theta_j(x) - \overline{u} \big)$$

- We deduce the dual function $\psi \colon \mathbb{R}^p \to \overline{\mathbb{R}}$

$$\psi(v) = \inf_{x \in \mathcal{X}} \mathcal{L}(x, v) = \inf_{x \in \mathcal{X}} \Big\{ h(x) + \sum_{j=1}^{p} v_j \big(\theta_j(x) - \overline{u} \big) \Big\}$$

which is concave upper semicontinuous,
as the supremum of affine functions

Paths to strong duality in the convex case

- Suppose that
  - the optimization set $\mathcal{X}$ is a vector space
  - the objective function $h\colon \mathcal{X} \to]-\infty, +\infty]$ is convex
  - each component of the mapping $\theta = (\theta_1, \ldots, \theta_p)\colon \mathcal{X} \to \mathbb{R}^p$ is a convex function
- Then, the perturbation function $\varphi\colon \mathbb{R}^p \to \overline{\mathbb{R}}$ is a convex function as the marginal

$$\varphi(u) = \inf_{x \in \mathcal{X}} \Big\{ h(x) + \sum_{j=1}^{p} v_j \big(\theta_j(x) - u \big) \Big\}$$

- If, in addition,
  - either the function φ is proper and lower semicontinuous
  - or its subdifferential $\partial\varphi(0) \neq \emptyset$

  then we get strong duality

Fenchel-Rockafellar duality

The Fenchel-Rockafellar dual problem

Proposition

adapted from Rockafellar [1970, Corollary 31.2.1]

Let $f, g\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions and let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping

$$\sup_{v\in\mathbb{R}^m} \{-g^\star(v) - f^\star(L^T v)\} \leq \inf_{u\in\mathbb{R}^n} \{f(u) + g(Lu)\}$$

Furthermore, equality is achieved if either

- $\exists u \in \mathrm{ri}(\mathrm{dom} f)$ s.t. $Lu \in \mathrm{ri}(\mathrm{dom} g)$
- $\exists v \in \mathrm{ri}(\mathrm{dom} g^\star)$ s.t. $L^T v \in \mathrm{ri}(\mathrm{dom} f^\star)$

Perturbation and Rockafellian TODO

- Perturbation space $\mathcal{U} = \mathbb{R}^p$
- Rockafellian $\mathcal{R} \colon \mathcal{X} \times \mathbb{R}^p \to \overline{\mathbb{R}}$

$$\mathcal{R}(x, u) = h(x) + \iota_{\{\theta(x) - \overline{u} \leq u\}} = h(x) + \sum_{j=1}^{p} \iota_{\{\theta_j(x) - \overline{u}_j \leq u_j\}}$$

Duality, Lagrangian and dual function TODO

- Dual space $\mathcal{V} = \mathbb{R}^p$
- We deduce the Lagrangian $\mathcal{L} \colon \mathcal{X} \times \mathbb{R}^p \to \overline{\mathbb{R}}$

$$\mathcal{L}(x, v) = h(x) + \langle \theta(x) - \overline{u}, \, v \rangle = h(x) + \sum_{j=1}^{p} v_j \big(\theta_j(x) - \overline{u} \big)$$

- We deduce the dual function $\psi \colon \mathbb{R}^p \to \overline{\mathbb{R}}$

$$\psi(v) = \inf_{x \in \mathcal{X}} \mathcal{L}(x, v) = \inf_{x \in \mathcal{X}} \Big\{ h(x) + \sum_{j=1}^{p} v_j \big(\theta_j(x) - \overline{u} \big) \Big\}$$

which is concave upper semicontinuous,
as the supremum of affine functions

Application to regularized problem

For a linear mapping $L\colon \mathbb{R}^n \to \mathbb{R}^m$ and a proper convex function $f\colon \mathbb{R}^m \to \overline{\mathbb{R}}$ and suitable assumptions

$$\sup_{v \in \mathbb{R}^m} -\left\|L^T v\right\|^2 - f^\star(v) = \inf_{u \in \mathbb{R}^n} f(Lu) + \frac{1}{2}\|u\|^2$$

Can be useful for computation if $m < n$ and $f^\star$ easy to compute

Semidefinite programming dual problem

adapted from Calafiore and El Ghaoui [2014, Chapter 11]

Let $\mathbb{S}^n$ be the set of $n \times n$ symmetric matrices
Let $\mathbb{S}^n_+ \subset \mathbb{S}^n$ be the set of $n \times n$ semidefinite matrices

Proposition

Let $K, A_1, \ldots, A_m \in \mathbb{S}^n$ and $b \in \mathbb{R}^m$
Then, we have

$$\sup_{\substack{v \in \mathbb{R}^m \\ K - \sum_{j=1}^m v_j A_j \succeq 0}} \langle b,\, v \rangle \leq \inf_{\substack{X \in \mathbb{S}^n \\ \mathrm{trace} A_j X = b_j\ ,\ \ j=1,\ldots,m \\ X \succeq 0}} \mathrm{trace} KX$$

Furthermore, equality is achieved
if some Slater's condition is satisfied

Perturbation and Rockafellian TODO

- Perturbation space $\mathcal{U} = \mathbb{R}^p$
- Rockafellian $\mathcal{R} \colon \mathcal{X} \times \mathbb{R}^p \to \overline{\mathbb{R}}$

$$\mathcal{R}(x, u) = h(x) + \iota_{\{\theta(x) - \overline{u} \leq u\}} = h(x) + \sum_{j=1}^{p} \iota_{\{\theta_j(x) - \overline{u}_j \leq u_j\}}$$

Duality, Lagrangian and dual function TODO

- Dual space $\mathcal{V} = \mathbb{R}^p$
- We deduce the Lagrangian $\mathcal{L} \colon \mathcal{X} \times \mathbb{R}^p \to \overline{\mathbb{R}}$

$$\mathcal{L}(x, v) = h(x) + \langle \theta(x) - \overline{u}, \, v \rangle = h(x) + \sum_{j=1}^{p} v_j \big(\theta_j(x) - \overline{u} \big)$$

- We deduce the dual function $\psi \colon \mathbb{R}^p \to \overline{\mathbb{R}}$

$$\psi(v) = \inf_{x \in \mathcal{X}} \mathcal{L}(x, v) = \inf_{x \in \mathcal{X}} \Big\{ h(x) + \sum_{j=1}^{p} v_j \big(\theta_j(x) - \overline{u} \big) \Big\}$$

which is concave upper semicontinuous,
as the supremum of affine functions

Outline of the presentation

Introducing generalized convexity

Fenchel conjugate	c-conjugate
$f^{\star}(v) = \sup_{u \in \mathbb{R}^m} \langle u, v \rangle - f(u)$	$g^{c}(v) = \sup_{u \in U} c(u,v) \dot{+} \left(-g(u)\right)$
Fenchel biconjugate	**c-biconjugate**
$f^{\star\star'}(u) = \sup_{v \in \mathbb{R}^m} \langle u, v \rangle - f^{\star}(v)$	$g^{cc'}(u) = \sup_{v \in V} c(u,v) \dot{+} \left(-g^{c}(v)\right)$
$\star -$ convex functions	**c-convex functions**
$\iff f = f^{\star\star'}$	$\iff g = g^{cc'}$

with the Moreau lower and upper additions

$$(+\infty) \dot{+} (-\infty) = (-\infty) \dot{+} (+\infty) = -\infty$$
$$(+\infty) \dot{+} (-\infty) = (-\infty) \dot{+} (+\infty) = +\infty$$

Generalized perturbation-duality scheme Balder [1977]

sets	optimization set $\mathcal{X}$	primal set $\mathcal{U}$	coupling $\mathcal{U} \overset{c}{\leftrightarrow} \mathcal{V}$	dual set $\mathcal{V}$
variables	decision $x \in \mathcal{X}$	perturbation $u \in \mathcal{U}$	$c(u,v) \in \overline{\mathbb{R}}$	sensitivity $v \in \mathcal{V}$
bivariate functions		Rockafellian $\mathcal{R}: \mathcal{X} \times \mathcal{U} \to \overline{\mathbb{R}}$		Lagrangian $\mathcal{L}: \mathcal{X} \times \mathcal{V} \to \overline{\mathbb{R}}$
definition				$\mathcal{L}(x,v) = \inf_{u \in \mathcal{U}} \Big\{ \mathcal{R}(x,u) \dotplus \big(-c(u,v)\big) \Big\}$
property				$-\mathcal{L}(x,\cdot) = \big(\mathcal{R}(x,\cdot)\big)^{c}$
property				$-\mathcal{L}(x,\cdot)$ is c'-convex
univariate functions		perturbation function $\varphi: \mathcal{U} \to \overline{\mathbb{R}}$		dual function $\psi: \mathcal{V} \to \overline{\mathbb{R}}$
definition		$\varphi(u) = \inf_{x \in \mathcal{X}} \mathcal{R}(x,u)$		$\psi(v) = \inf_{x \in \mathcal{X}} \mathcal{L}(x,v)$
property				$-\psi = \varphi^{c}$

- Anchor $\overline{u} \in \mathcal{U}$ and dual maximization problem (weak duality) $\varphi^{cc'}(\overline{u}) = \sup_{v \in \mathcal{V}} \big\{ c(\overline{u}, v) \dotplus \psi(v) \big\} \leq \inf_{x \in \mathcal{X}} h(x) = \varphi(\overline{u})$
- Strong duality iff φ is c-convex at $\overline{u}$ iff $\varphi^{cc'}(\overline{u}) = \varphi(\overline{u})$

Case of evaluation couplings (developped later)

- Given a primal set $\mathcal{U}$ and a function set $\mathcal{F} \subset \{F : \mathcal{U} \to \overline{\mathbb{R}}\}$, the evaluation coupling $c_{\mathcal{F}} : \mathcal{U} \times \mathcal{F} \to \overline{\mathbb{R}}$ is defined by

$$c_{\mathcal{F}}(u, F) = F(u) \; , \quad \forall u \in \mathcal{U}, F \in \mathcal{F}$$

- For a given (perturbation) function $\varphi : \mathcal{U} \to \overline{\mathbb{R}}$, weak duality is always achieved

$$\varphi^{c_{\mathcal{F}} c_{\mathcal{F}}'} \leq \varphi$$

- Sufficient condition for strong duality

$$\varphi \in \mathcal{F} \implies \varphi^{c_{\mathcal{F}} c_{\mathcal{F}}'} = \varphi$$

- Two trivial cases of strong duality
  1. $\mathcal{F} = \{F : \mathcal{U} \to \overline{\mathbb{R}}\} = \overline{\mathbb{R}}^{\mathcal{U}}$
  2. $\mathcal{F} = \{\varphi\}$

Outline of the presentation

PDS for linear programming (LP)

PDS for convex programming

PDS for pure integer linear programming (PILP)

Conclusion

Pure integer linear program in standard form

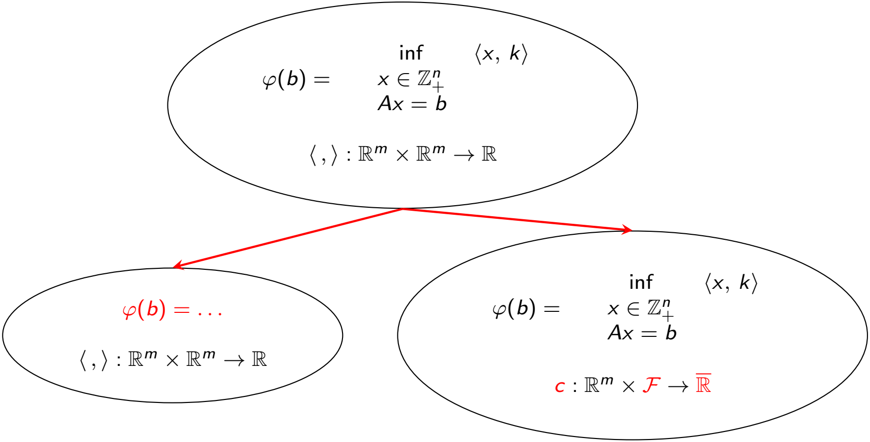
$$\begin{aligned} \inf \quad & \langle x, k \rangle \\ x &\in \mathbb{Z}^n \\ Ax &= b \\ x &\geq 0 \end{aligned}$$

Usual continuous dual in PILP

$$\overbrace{\sup_{\substack{\lambda \in \mathbb{R}^m \\ A^T\lambda \leq k}} \langle b, \lambda \rangle}^{\text{dual (continuous) problem}} \underbrace{\leq}_{\text{weak duality}} \overbrace{\inf_{\substack{x \in \mathbb{Z}^n \\ Ax = b \\ x \geq 0}} \langle x, k \rangle}^{\text{primal (integer) problem}}$$

- Right-hand side b perturbation and scalar product coupling
- Usually strong duality is not achieved
- Can we design tighter dual problems?
  (Useful for Branch-and-bound like methods)

Changing the perturbation/changing the coupling



Outline of the presentation

PDS for linear programming (LP)

PDS for convex programming

PDS for pure integer linear programming (PILP)

- (Geoffrion) Lagrangian relaxation
- Evaluation coupling in PDS
- Subadditive dual problem for PILP
- Chvàtal strong duality for rational PILP

Conclusion

(Geoffrion) Lagrangian relaxation Geoffrion [1974]

1. We partially perturb

$$\forall b^1 \in \mathbb{R}^{m_1}, \quad \varphi(b^1) = \inf_{\substack{x \\ A^1 x = b^1 \\ A^2 x = b^2 \\ x \geq 0 \\ x \in \mathbb{Z}^n}} \langle x , k \rangle$$

2. We pair the primal space $\mathbb{R}^{m_1}$ and a dual space $\mathbb{R}^{m_1}$

$$\langle \, , \rangle : \mathbb{R}^{m_1} \times \mathbb{R}^{m_1} \to \mathbb{R}$$

3. We biconjugate the perturbation function φ

$$\varphi^{\star\star'}(b^1) = \sup_{\lambda \in \mathbb{R}^{m_1}} \underbrace{\inf_{\substack{A^2 x = b^2 \\ x \geq 0 \\ x \in \mathbb{Z}^n}} \langle x , k \rangle + \langle b^1 - A^1 x , \lambda \rangle}_{g(\lambda)}$$

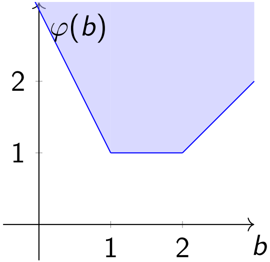
Example of perturbation function's epigraph for LP

Example

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined as

$$\varphi(b) = \inf_{\substack{x \in \mathbb{R}^3 \\ x_1 + x_2 + 3x_3 = 1 \\ x_1 + 2x_2 + 4x_3 = b \\ x \geq 0}} x_1 + x_2 + x_3$$

Then $\varphi(b) = \max\{3 - 2b, 1, b - 1\}$, $\forall b \in \mathbb{R}$



Condition for tighter gap than continuous dual problem

adapted from Conforti, Cornuéjols, and Zambelli [2014, Corollary 8.4.]

Proposition

If A^2 and b^2 are rational then

$$\sup_{\substack{\lambda \in \mathbb{R}^m \\ A^T\lambda \leq k}} \langle b,\, \lambda \rangle \leq \varphi^{\star\star\prime}(b^1) = \sup_{\lambda \in \mathbb{R}^{m_1}} g(\lambda)$$

where

$$g(\lambda) = \inf_{\substack{A^2 x = b^2 \\ x \geq 0 \\ x \in \mathbb{Z}^n}} \langle x,\, k \rangle + \langle b^1 - A^1 x,\, \lambda \rangle$$

and $A = \begin{pmatrix} A^1 \\ A^2 \end{pmatrix}$, $b = \begin{pmatrix} b^1 \\ b^2 \end{pmatrix}$, $m = m_1 + m_2$

Outline of the presentation

Evaluation coupling

Definition

Let $\mathcal{F} \subset \{F \colon \mathbb{R}^m \to \overline{\mathbb{R}}\}$ be a set of functions
We call $c_{\mathcal{F}} \colon \mathbb{R}^m \times \mathcal{F} \to \overline{\mathbb{R}}$ defined by

$$c_{\mathcal{F}}(b, F) = F(b) \; , \; \; \forall b \in \mathbb{R}^m, \forall F \in \mathcal{F}$$

the evaluation coupling of $\mathcal{F}$

Remark

- *Here the dual variables are functions*
- *If* $\mathcal{F} = \left\{F \colon \mathbb{R}^m \to \overline{\mathbb{R}} \,\middle|\, \text{is linear}\right\}$, *then* $c_{\mathcal{F}} = \langle \, , \rangle$

Resulting evaluation dual problem

also see Tind and Wolsey [1981, Sect. 6]

Consider the perturbation function defined by

$$\forall b \in \mathbb{R}^m \,, \quad \varphi(b) = \inf_{\substack{x \\ Ax = b \\ x \geq 0 \\ x \in \mathbb{Z}^n}} \langle x \,, \, k \rangle$$

Proposition

Let $\mathcal{F} \subset \{F \colon \mathbb{R}^m \to \overline{\mathbb{R}}\}$ be a set of functions

Then, for any $b \in \mathbb{R}^m$

$$\varphi^{c_{\mathcal{F}} c_{\mathcal{F}}'}(b) = \sup_{F \in \mathcal{F}} \Big\{ F(b) \dotplus \inf_{x \in \mathbb{Z}^n_+} \big\{ \langle x \,, \, k \rangle - F(Ax) \big\} \Big\}$$

$$\varphi^{c_{\mathcal{F}} c_{\mathcal{F}}'}(b) \underbrace{\leq}_{\text{weak duality}} \varphi(b)$$

Proof. Compute first conjugate

$$\begin{aligned}
\varphi^c(F) &= \sup_{b\in\mathbb{R}^m} \left\{ c(b,F) \dot{+} \left(-\varphi(b)\right) \right\} \\
&= \sup_{b\in\mathbb{R}^m} \left\{ c(b,F) \dot{+} \left(- \inf_{\substack{x\in\mathbb{Z}^n_+ \\ Ax=b}} \langle x,\ k\rangle \right) \right\} \\
&= \sup_{b\in\mathbb{R}^m} \left\{ c(b,F) \dot{+} \sup_{\substack{x\in\mathbb{Z}^n_+ \\ Ax=b}} -\langle x,\ k\rangle \right\} \\
&= \sup_{x\in\mathbb{Z}^n_+} \left\{ -\langle x,\ k\rangle + \sup_{\substack{b\in\mathbb{R}^m \\ Ax=b}} c(b,F) \right\} \\
&= \sup_{x\in\mathbb{Z}^n_+} \left\{ -\langle x,\ k\rangle + c(Ax,F) \right\} \\
&= - \inf_{x\in\mathbb{Z}^n_+} \left\{ \langle x,\ k\rangle - F(Ax) \right\}
\end{aligned}$$

Revisiting the Fenchel coupling with evaluation coupling

- Bilinear coupling $\langle\,,\rangle : \overbrace{\mathbb{R}^m}^{\text{perturbation space}} \times \overbrace{\mathbb{R}^m}^{\text{dual space}} \to \mathbb{R}$
- $\mathbb{R}^m$ can be identified to the functional space $\Lambda = \{F \colon \mathbb{R}^m \to \mathbb{R} | F \text{ is linear}\}$

$$\lambda \in \mathbb{R}^m \leftrightarrow F \in \Lambda$$
$$\langle b,\, \lambda\rangle \leftrightarrow c_\Lambda(b, F)$$

- Thus, the resulting dual problem

$$\varphi^{c_\Lambda c_\Lambda{}'}(b) = \sup_{\lambda \in \mathbb{R}^m} \Big\{ \langle b,\, \lambda\rangle + \underbrace{\inf_{x \in \mathbb{Z}_+^n} \{\langle x,\, k\rangle - \langle Ax,\, \lambda\rangle\}}_{\iota_{A^T\lambda \leq k}} \Big\}$$

$$= \sup_{\substack{\lambda \in \mathbb{R}^m \\ A^T\lambda \leq k}} \langle b,\, \lambda\rangle$$

Inclusion of functional sets

Consider the perturbation function defined by

$$\forall b \in \mathbb{R}^m , \quad \varphi(b) = \inf_{\substack{x \\ Ax = b \\ x \geq 0 \\ x \in \mathbb{Z}^n}} \langle x, k \rangle$$

Proposition

If $\mathcal{F}^1 \subset \ldots \subset \mathcal{F}^J \subset \{F : \mathbb{R}^m \to \overline{\mathbb{R}}\}$, then

$$\varphi^{c_{\mathcal{F}^1} c_{\mathcal{F}^1}{}'} \leq \ldots \leq \varphi^{c_{\mathcal{F}^J} c_{\mathcal{F}^J}{}'} \leq \varphi$$

The larger the set of dual functions, the tighter the gap!

Characterization of strong duality for evaluation coupling

Proposition

Tind and Wolsey [1981, Proposition 6.8]

Let $\mathcal{F} \subset \{F : \mathbb{R}^m \to \overline{\mathbb{R}}\}$ be a set of functions and $f : \mathbb{R}^m \to \overline{\mathbb{R}}$

Then

$$f^{c_{\mathcal{F}} c_{\mathcal{F}}'} = f \iff \exists \{f_i\}_{i \in I} \subset \mathcal{F} \text{ s.t. } f = \sup_{i \in I} f_i$$

Remark

Cases when the equality is trivially true

- $\mathcal{F}$ *is too "general":* $\mathcal{F} = \{F : \mathbb{R}^m \to \overline{\mathbb{R}}\}$
- $\mathcal{F}$ *is too "specific":* $\mathcal{F} = \{f\}$

Outline of the presentation

PDS for linear programming (LP)

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- (Geoffrion) Lagrangian relaxation
- Evaluation coupling in PDS
- Subadditive dual problem for PILP
- Chvàtal strong duality for rational PILP

Conclusion

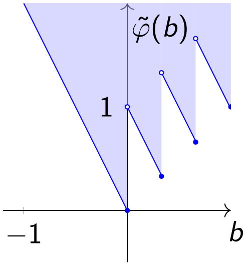
Example of perturbation function's epigraph for PILP

Example

Let $\varphi \colon \mathbb{R} \to \mathbb{R}$ be defined as

$$\forall b \in \mathbb{R}\ , \quad \varphi(b) = \inf_{\substack{x \in \mathbb{Z}^2 \\ x_1 - x_2 = b \\ x \geq 0}} x_1 + 2x_2$$

Then φ coincides on its domain with
$\tilde{\varphi}(b) = \max\{-2b, \lceil 3b \rceil - 2b\}\ , \ \forall b \in \mathbb{R}$



Summary of the perturbation-duality scheme for PILP

1. We perturb a minimization problem

$$\forall b \in \mathbb{R}^m \,, \quad \varphi(b) = \inf_{\substack{x \\ Ax = b \\ x \in \mathbb{Z}^n_+}} \langle x , \, k \rangle$$

2. We pair the primal space $\mathbb{R}^m$ and a function set $\mathcal{F}$

$$c_{\mathcal{F}} \colon \mathbb{R}^m \times \mathcal{F} \to \overline{\mathbb{R}}$$
$$c_{\mathcal{F}}(b, F) = F(b)$$

3. We biconjugate the perturbation function φ

$$\overbrace{\varphi^{cc'}(b) \leq \varphi(b) \,, \quad \forall b \in \mathbb{R}^m}^{\text{weak duality is guaranteed}}$$
$$\varphi^{cc'}(b) = \sup_{F \in \mathcal{F}} \Big\{ F(b) \dot{+} \inf_{x \in \mathbb{Z}^n_+} \big\{ \langle x , \, k \rangle - F(Ax) \big\} \Big\}$$

Strong duality for the subadditive dual problem

Let $\mathcal{S}$ be the set of subadditive functions

$$\mathcal{S} = \{F \colon \mathbb{R}^m \to \overline{\mathbb{R}} \text{ s.t. } F(b^1 + b^2) \leq F(b^1) \dotplus F(b^2) \ , \ \forall b^1, b^2\}$$

$$(-\infty) \dotplus (+\infty) = (+\infty) \dotplus (-\infty) = +\infty$$

Proposition

Let $A \in \mathbb{R}^{m \times n}$ and $k \in \mathbb{R}^n$
Let $c_{\mathcal{S}} \colon \mathbb{R}^m \times \mathcal{S} \to \overline{\mathbb{R}}$ be the evaluation coupling of $\mathcal{S}$
Then, for all $b \in \mathbb{R}^m$

$$\varphi^{c_{\mathcal{S}} c_{\mathcal{S}}'}(b) = \varphi(b)$$

Proof of subadditive strong duality. φ is subadditive

(i) Let $b^1, b^2 \in \mathbb{R}^m$

(ii)
- Assume there are $x^1, x^2 \in \mathbb{Z}^n_+$ s.t.

$$Ax^1 = b^1 \; , \;\; Ax^2 = b^2$$

  Then $A\big(x^1 + x^2\big) = b^1 + b^2$

- For all such x^1, x^2 and by definition of φ

$$\varphi(b^1 + b^2) \leq \langle k,\, x^1 + x^2 \rangle = \langle k,\, x^1 \rangle + \langle k,\, x^2 \rangle$$

- Going to the infimum in x^1, then in x^2

$$\varphi(b^1 + b^2) \leq \varphi(b^1) + \varphi(b^2)$$

(iii) If there is no such $x^1, x^2 \in \mathbb{Z}^n_+$
Then $\varphi(b^1) = +\infty$ or $\varphi(b^2) = +\infty$
Thus, by definition of $\dot{+}$

$$\varphi(b^1 + b^2) \leq \varphi(b^1) \dot{+} \varphi(b^2)$$

Previous work on PILP duality

- Surveys of previous works
  Tind and Wolsey [1981], Güzelsoy, Ralphs, and Cochran [2010]
- Some pioneer papers on strong duality in the rational case (when $A \in \mathbb{Q}^{m \times n}, k \in \mathbb{Q}^n$)
  Johnson [1973], Jeroslow [1979],Wolsey [1981], Blair and Jeroslow [1982]
- Computation of optimal dual functions
  Wolsey [1981], Klabjan [2007]

Subadditive dual problem in Jeroslow [1979]

$$\begin{aligned} \sup_{F:\mathbb{R}^m \to \mathbb{R}} \quad & F(\bar{b}) \\ F(A_j) &\leq k_j \\ F(0) &\leq 0 \\ F &\text{ is subadditive} \end{aligned}$$

Outline of the presentation

Somewhere in between the linear and the subadditive

$$\underbrace{\Lambda}_{\text{linear functions}} \subset {\color{red}\mathcal{C}} \subset \overbrace{\mathcal{S}}^{\text{subadditive functions}}$$

Definition of Chvátal functions

Definition

The class of Chvátal functions $\mathcal{C}$
is the smallest class of functions $\mathcal{F} \subset \{F : \mathbb{R}^m \to \mathbb{R}\}$ such that

$$b \in \mathbb{R}^m \mapsto \langle b, \lambda \rangle \in \mathcal{F}\ , \ \ \forall \lambda \in \mathbb{Q}^m \qquad \text{(linear functions)}$$

$$\alpha F_1 + \beta F_2 \in \mathcal{F}\ , \ \ \forall F_1, F_2 \in \mathcal{F}\ , \ \ \alpha, \beta \in \mathbb{Q}_+ \qquad \text{(conic combination)}$$

$$\lceil F \rceil \in \mathcal{F}\ , \ \ \forall F \in \mathcal{F} \qquad \text{(round-up)}$$

Examples in 1D

- $b \mapsto \frac{3}{4} b$
- $b \mapsto \lceil b \rceil$
- $b \mapsto \frac{3}{4} b + \frac{7}{10} \lceil b \rceil$
- $b \mapsto 15 b + \frac{39}{22} \lceil \frac{3}{4} b + \frac{7}{10} \lceil b \rceil \rceil + \lceil 16 b \rceil$

Strong duality with Chvátal functions

adapted from Blair and Jeroslow [1982]

We define a perturbation function

$$\forall b \in \mathbb{R}^m , \;\; \varphi(b) = \inf_{\substack{x \\ Ax = b \\ x \in \mathbb{Z}^n_+}} \langle x , \, k \rangle$$

Proposition

We remind that $c_{\mathcal{C}}$ is the evaluation coupling of the Chvátal functions
If $A \in \mathbb{Q}^{m \times n}$ and $k \in \mathbb{Q}^n$ then

$$\varphi^{c_{\mathcal{C}} c_{\mathcal{C}}'}(b) = \varphi(b) \; , \;\; \forall b \in \mathrm{dom}\varphi$$

Remark

The perturbation function φ *is defined on* $\mathbb{R}^m$ *but* $\mathrm{dom}\varphi \subset \mathbb{Q}^m$

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PDS for pure integer linear programming (PILP)

Conclusion

Steps of the perturbation-duality scheme

Rockafellar [1974]

1. Perturb a minimization problem
   with a perturbation (primal) variable
   belonging to a vector space,
   and a Rockafellian function
2. Pair the (primal) perturbation space with a dual space
   by means of a bilinear form $\langle\ ,\ \rangle$
3. Biconjugate the perturbation function, and get
   - a dual problem
   - weak duality
4. Deduce conditions for strong duality by means of
   either global or local properties of the perturbation functio

Branching out: rank restricted Chvàtal functions

Rank restricted Chvàtal functions

Let $F \in \mathcal{C}$ be a Chvàtal function

Definition

The rank of F is defined *unformally* as
the smallest number of $\lceil \cdot \rceil$ needed to encode F
We denote by $\mathcal{C}^r \subset \mathcal{C}$ the Chvàtal function
of rank not greater than $r \in \mathbb{N}$

- Inclusion of function sets

$$\underbrace{\mathcal{C}^0}_{\substack{\text{linear functions} \\ \text{with rational } \lambda}} \subset \mathcal{C}^1 \subset \cdots \subset \mathcal{C}^m \subset \cdots \subset \mathcal{S}$$

- Weak duality chain

$$\varphi^{c_{\mathcal{C}^0} c_{\mathcal{C}^0}{}'} \leq \varphi^{c_{\mathcal{C}^1} c_{\mathcal{C}^1}{}'} \leq \ldots \varphi^{c_{\mathcal{C}^r} c_{\mathcal{C}^r}{}'} \leq \cdots \leq \varphi^{c_{\mathcal{S}} c_{\mathcal{S}}{}'} \leq \varphi$$

Example of partially perturbed restricted Chvàtal scheme

- We define a perturbation function

$$\forall b \in \mathbb{R}^{m_1}\ ,\ \ \varphi(b^1) = \inf_{\substack{x \\ A^1 x = b^1 \\ A^2 x = b^2 \\ x \geq 0 \\ x \in \mathbb{Z}^n}} \langle x,\ k \rangle$$

- We define a coupling between primal and dual space

$$c : \mathbb{R}^{m_1} \times \mathbb{Q}^{m_1} \times \mathbb{Q}_+ \to \mathbb{R}$$

$$c(b^1, (\lambda, \alpha)) = \lambda^T b^1 + \alpha \left\lceil \beta^T b^1 \right\rceil\ ,$$

$$\forall b^1 \in \mathbb{R}^{m_1}\ ,\ \ \forall (\lambda, \alpha) \in \mathbb{Q}^{m_1} \times \mathbb{Q}_+$$

- We biconjugate the perturbation functions

$$\underbrace{\varphi^{cc'}(b^1) \leq \varphi(b^1)\ ,\ \ \forall b \in \mathbb{R}^{m_1}}_{\text{weak duality}}$$

Resulting dual problems yields a tighter gap

$$\varphi^{cc'}(b) = \underbrace{\sup_{(\lambda,\alpha)\in\mathbb{Q}^m\times\mathbb{Q}_+} \left\{\lambda^T b^1 + \alpha\left\lceil \beta^T b^1 \right\rceil + \inf_{\substack{A^2x=b^2\\ x\geq 0\\ x\in\mathbb{Z}^n}} \left\{k^T x + \lambda^T A^1 x + \alpha\left\lceil \beta^T A^1 x\right\rceil\right\}\right\}}_{\tilde{g}(\lambda,\alpha)}$$

We have a tighter gap

$$\underbrace{\sup_{\lambda\in\mathbb{Q}^{m_1}} g(\lambda)}_{\text{Lagrangian relaxation}} \leq \sup_{(\lambda,\alpha)\in\mathbb{Q}^{m_1}\times\mathbb{Q}_+} \tilde{g}(\lambda,\alpha) \leq \underbrace{\varphi(b)}_{\text{original problem}}$$

Thank you for your attention !

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