# Stochastic Process (ENPC)
# Monday, 21st of January 2019 (2h30)

Vocabulary (english/*français*) : house of cards =*château de cartes* ; distribution =*loi* ; positive = *strictement positif* ; Brownian bridge = *pont brownien* ; $(0,1]$ = *]0,1]*.

We write $\mathbb{N}^* = \mathbb{Z} \cap [1, +\infty[$ and $\mathbb{N} = \mathbb{N}^* \bigcup \{0\}$.

**Exercice 1** (House of cards)**.** Consider a kid building an house of cards and denote by $X_n \in \mathbb{N}$ the size (or number of cards) of the house at time $n \in \mathbb{N}$. When adding a new card to the house which contains already $k$ cards, then, with probability $p_k$, the house with $k+1$ cards is stable and with probability $1 - p_k$ the house collapses and the kid has to restart from scratch, see figure 1. The "house of cards" model [1] is also an elementary example of growth-collapse models [2]. The aim of this exercise is to study some asymptotic properties of the "house of cards" dynamic.

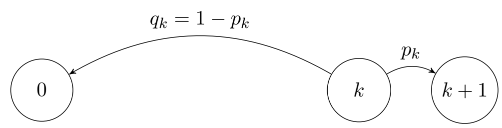


FIGURE 1 – Graph of transitions for the "house of cards" game (with $k \in \mathbb{N}$).

Let $p = (p_k,\, k \in \mathbb{N})$ be a sequence of elements of $(0,1]$ with $p_0 = 1$, and set $q_k = 1 - p_k$ for $k \in \mathbb{N}$. Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space. The "house of cards" is a Markov chain which is modelled by the following stochastic dynamical system $X = (X_n,\, n \in \mathbb{N})$ on the state space $\mathbb{N}$ :

$$X_{n+1} = (X_n + 1)\mathbf{1}_{\{U_{n+1} \leq p_{X_n}\}} \quad \text{for } n \in \mathbb{N},$$

where $(U_n, n \in \mathbb{N}^*)$ is a sequence of independent random variables uniformly distributed over $[0,1]$ which is independent of the $\mathbb{N}$-valued random variable $X_0$. We consider $\mathcal{F} = (\mathcal{F}_n = \sigma(X_0, \ldots, X_n),\, n \in \mathbb{N})$ the natural filtration of $X$. We write $\mathbb{P}_k$ and $\mathbb{E}_k$ the probability measure and corresponding expectation when the Markov chain $X$ starts with $X_0 = k \in \mathbb{N}$.

1. Give the transition matrix $P$ of the Markov chain $X$.
2. Give a necessary and sufficient condition on $p$ for $X$ to be irreducible.
   (*Hint. First check that* $\mathbb{P}_0(X_k = k) > 0$ *for all* $k \in \mathbb{N}^*$*.*)

From now on, we assume that $X$ is irreducible. We set $\Delta_0 = 1$, $\Delta_n = \prod_{k=0}^{n-1} p_k$ for $n \in \mathbb{N}^*$ and :

$$S_2 = \sum_{k=0}^{\infty} q_k \quad \text{and} \quad S_1 = \sum_{k=0}^{\infty} \Delta_k.$$

Let $\tau_k$ be the return time to $k \in \mathbb{N}$ given by $\tau_k = \inf\{n \geq 1;\, X_n = k\}$.

3. We study the transience and recurrence of $X$.
   (a) Prove that $\mathbb{P}_0(\tau_0 > n) = \Delta_n$ for all $n \in \mathbb{N}$. Deduce that $\mathbb{E}_0[\tau_0] = S_1$.
   (b) Prove that $\lim_{n\to\infty} \Delta_n = 0$ if and only if $S_2 = +\infty$.

1. *An introduction to probability theory and its applications. Vol. I.* Third edition. John Wiley & Sons, 1968. (See pages 381-382, 390, 398, 403 and 408.)

2. T. Huillet. On a Markov chain model for population growth subject to rare catastrophic events. *Physica A. Statistical Mechanics and its Applications*, 390(23-24) :4073-4086, 2011.

(c) Characterise the transience, the null recurrence and the positive recurrence of $X$ in terms of $S_2$ and $S_1$ being finite or not.

4. We study the invariant measures $\pi = (\pi_k, k \in \mathbb{N})$ of $X$.

   (a) Compute the invariant probability measure $\pi$ when $X$ is positive recurrent. (*Hint. Check that* $\pi_k = \Delta_k \pi_0$ *for* $k \in \mathbb{N}$.)

   A measure $\pi$ is invariant for $X$ if $\pi_k \in (0, +\infty)$ for all $k \in \mathbb{N}$ and $\pi = \pi P$.

   (b) Compute an invariant measure $\pi$ for $X$ when $X$ is null recurrent and check this measure is unique (up to a multiplicative factor) and that $\sum_{k=0}^{\infty} \pi_k = +\infty$. (*Hint. Check that* $\sum_{k=0}^{n} q_k \Delta_k = 1 - \Delta_{n+1}$ *for* $n \in \mathbb{N}$.)

   (c) Prove there exists no invariant measure for $X$ when $X$ is transient.

5. We shall compute $\mathbb{P}_k(\tau_\ell < \tau_0)$ for $0 < k < \ell$. We define the function $\varphi = (\varphi(k), k \in \mathbb{N})$ by $\varphi(0) = 0$ and $\varphi(k) = 1/\Delta_k$ for $k \in \mathbb{N}^*$. We consider the random process $M = (M_n, n \in \mathbb{N})$ defined by $M_n = \varphi(X_{n \wedge \tau_0})$.

   (a) Prove that $\mathbb{E}\left[\mathbf{1}_{\{\tau_0 > n+1\}} \mid \mathcal{F}_n\right] = p_{X_n} \mathbf{1}_{\{\tau_0 > n\}}$ for $n \in \mathbb{N}$.

   (b) Prove that $M$ is a martingale under $\mathbb{P}_k$ for $k \in \mathbb{N}^*$. (*Hint. Check that* $M_n = \Delta_{X_n}^{-1} \mathbf{1}_{\{\tau_0 > n\}}$ *for* $n \in \mathbb{N}$.)

   (c) For $0 < k < \ell$, compute $\mathbb{E}_k\left[M_{\tau_\ell \wedge n}\right]$ for $n \in \mathbb{N}$ and deduce the value of $\mathbb{E}_k\left[M_{\tau_\ell}\right]$.

   (d) Deduce the value of $\mathbb{P}_k(\tau_\ell < \tau_0)$ for $0 < k < \ell$.

6. We shall compute $\mathbb{E}_k[\tau_\ell]$ for $k, \ell \in \mathbb{N}$. We define the random process $V = (V_n, n \in \mathbb{N})$ by $V_0 = X_0$ and for $n \in \mathbb{N}$ :

$$V_{n+1} = \frac{\mathbf{1}_{\{X_{n+1} \neq 0\}}}{p_{X_n}} (1 + V_n).$$

   (a) Prove that $Q = (Q_n = V_n - n,\ n \in \mathbb{N})$ is a martingale as soon as $\mathbb{E}[X_0]$ is finite.

   (b) Prove that $\mathbb{E}_0[\tau_\ell \wedge n] = \mathbb{E}_0\left[V_{\tau_\ell \wedge n}\right]$ for $\ell \in \mathbb{N}^*$ and $n \in \mathbb{N}$.

   (c) For $\ell \in \mathbb{N}^*$, compute $V_{\tau_\ell}$ under $\mathbb{P}_0$ and deduce the value of $\mathbb{E}_0\left[\tau_\ell\right]$.

   (d) Use the strong Markov property to prove that $\mathbb{E}_k\left[\tau_\ell\right] = \mathbb{E}_0\left[\tau_\ell\right] - \mathbb{E}_0\left[\tau_k\right]$ for $0 < k < \ell$.

   (e) Assume that $X$ is recurrent positive. Give the value of $\mathbb{E}_\ell[\tau_\ell]$ and prove that $\mathbb{E}_\ell[\tau_0] = \mathbb{E}_\ell[\tau_\ell] - \mathbb{E}_0[\tau_\ell]$ and $\mathbb{E}_\ell\left[\tau_k\right] = \mathbb{E}_\ell[\tau_\ell] - \mathbb{E}_0[\tau_\ell] + \mathbb{E}_0[\tau_k]$ for $0 < k < \ell$.

△

**Exercice 2** (Yet another representation of the Brownian bridge)**.** Let $B = (B_t,\ t \in \mathbb{R}_+)$ be a standard Brownian motion. The distribution of the Brownian bridge is the distribution of the Gaussian process $W = (W_t = B_t - tB_1,\ t \in [0, 1])$.

1. Check that $X = (X_t = W_{1-t},\ t \in [0, 1])$ is a Brownian bridge.
2. Prove that $Y = (Y_t = (1-t)B_{t/(1-t)},\ t \in [0, 1])$, with the convention that $Y_1 = 0$, is a Brownian bridge.
3. Deduce that $Z = (Z_t = tB_{(1-t)/t},\ t \in [0, 1])$, with the convention that $Z_0 = 0$, is a Brownian bridge.

△

## Correction

*Exercice 1* 1. We have $P(k,k+1)=p_k$, $P(k,0)=q_k$ and $P(k,\ell)=0$ for $\ell\notin\{0,k+1\}$.

2. The Markov chain is irreducible if and only if the set $\{k\in\mathbb{N};\, p_k<1\}$ is infinite.

3. (a) The result is clear for $n=0$. Assume $n\in\mathbb{N}^*$. The event $\{\tau_0>n, X_0=0\}$ is equal to $\{X_k=k \text{ for } 0\leq k\leq n\}$. Under $\mathbb{P}_0$, this latter event has probability $\prod_{k=0}^{n-1}P(k,k+1)=\prod_{k=0}^{n-1}p_k=\Delta_n$. Use that $\mathbb{E}[Y]=\sum_{k=0}^\infty \mathbb{P}(Y>k)$ for any $\mathbb{N}$-valued random variable $Y$ to get that $\mathbb{E}[\tau_0]=S_1$.

   (b) The sums $S_2$ and $S_3=-\sum_{k=0}^\infty \log(1-q_k)$ are of the same nature. Conclude using that $\lim_{n\to\infty}\Delta_n=\mathrm{e}^{-S_3}$.

   (c) The chain $X$ is transient if and only if $\mathbb{P}_0(\tau_0=\infty)>0$, which is equivalent to $\lim_{n\to\infty}\mathbb{P}_0(\tau_0>n)>0$ that is $S_2<+\infty$. The chain $X$ is null recurrent if and only if $\mathbb{P}_0(\tau_0=\infty)=0$ and $\mathbb{E}_0[\tau_0]=+\infty$ that is $S_2=S_1=+\infty$. The chain $X$ is positive recurrent if and only if $\mathbb{E}_0[\tau_0]<+\infty$ that is $S_1<+\infty$ (which implies that $S_2=+\infty$).

4. (a) We deduce from the equation $\pi P=\pi$ that

$$\pi_0=\sum_{k=0}^\infty q_k\pi_k, \tag{1}$$

$$\pi_k=p_{k-1}\pi_{k-1} \quad \text{for } k\in\mathbb{N}^*. \tag{2}$$

   Equality (2) implies that $\pi_k=\Delta_k\pi_0$ for $k\in\mathbb{N}^*$. This equality trivially holds also for $k=0$. Then use that $\pi$ is a probability to get :

$$1=\sum_{k=0}^\infty \pi_k=\pi_0\sum_{k=0}^\infty\Delta_k=\pi_0 S_1.$$

   Since $X$ is positive recurrent, we have that $S_1<+\infty$ and thus $\pi_k=\Delta_k/S_1$ for $k\in\mathbb{N}$. Now, we check that (1) holds. Indeed, as $S_1$ is finite, we have :

$$\sum_{k=0}^\infty q_k\pi_k=\frac{1}{S_1}\sum_{k=0}^\infty(1-p_k)\Delta_k=\frac{1}{S_1}\sum_{k=0}^\infty\Delta_k-\frac{1}{S_1}\sum_{k=1}^\infty\Delta_k=\frac{\Delta_0}{S_1}=\pi_0.$$

   (b) We deduce from the equation $\pi P=\pi$ that (1) and (2) hold. Using (2), we get that $\pi_k=\Delta_k\pi_0$ for $k\in\mathbb{N}$. We shall check that (1) holds. We have :

$$\sum_{k=0}^n q_k\Delta_k=\sum_{k=0}^n(1-p_k)\Delta_k=\sum_{k=0}^n\Delta_k-\sum_{k=1}^{n+1}\Delta_k=\Delta_0-\Delta_{n+1}=1-\Delta_{n+1}.$$

   We deduce that :

$$\sum_{k=0}^\infty q_k\pi_k=\lim_{n\to\infty}\sum_{k=0}^n q_k\pi_k=\lim_{n\to\infty}\pi_0\sum_{k=0}^n q_k\Delta_k=\pi_0(1-\lim_{n\to\infty}\Delta_n)=\pi_0,$$

   where we used for the last equality that $\lim_{n\to\infty}\Delta_n=0$ as $X$ is null recurrent. We deduce that $(\pi_0\Delta_k, k\in\mathbb{N})$ is the unique invariant measure up to the multiplicative factor $\pi_0\in(0,+\infty)$.

(c) If $X$ is transient then $\lim_{n\to\infty} \Delta_n > 0$ and, arguing as in the previous question, we get that $\pi_k = \Delta_k \pi_0$ for $k \in \mathbb{N}$ thanks to (2) and then $\sum_{k=0}^{\infty} q_k \pi_k = \pi_0(1 - \lim_{n\to\infty} \Delta_n) < \pi_0$. Thus (1) can not be satisfied. Hence there exists no invariant measure.

5. (a) Let $n \in \mathbb{N}$. We have :

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\{\tau_0 > n+1\}} \mid \mathcal{F}_n\right] &= \mathbf{1}_{\{\tau_0 > n\}} \mathbb{E}\left[\mathbf{1}_{\{X_{n+1} = X_n + 1\}} \mid \mathcal{F}_n\right] \\
&= \mathbf{1}_{\{\tau_0 > n\}} \mathbb{E}\left[\mathbf{1}_{\{X_{n+1} = X_n + 1\}} \mid X_n\right] \\
&= p_{X_n} \mathbf{1}_{\{\tau_0 > n\}},
\end{aligned}$$

where we used that $\{\tau_0 > n+1\} = \{\tau_0 > n\} \bigcap \{X_{n+1} = X_n + 1\}$ and $\{\tau_0 > n\} \in \mathcal{F}_n$ as $\tau_0$ is a $\mathcal{F}$-stopping time for the first equality, the Markov property of $X$ for the second and that the transition probability from $k$ to $k+1$ is $p_k$ for the last.

(b) Let $n \in \mathbb{N}$. On $\{\tau_0 \leq n\}$, we have $\varphi(X_{n\wedge\tau_0}) = \varphi(0) = 0$. We deduce that $M_n = \Delta_{X_n}^{-1} \mathbf{1}_{\{\tau_0 > n\}}$. Since $\tau_0$ is a $\mathcal{F}$-stopping time, we get that $\{\tau_0 > n\} \in \mathcal{F}_n$ and thus $M$ is $\mathcal{F}$-adapted. Since $M$ is non-negative, we can compute $\mathbb{E}[M_{n+1} | \mathcal{F}_n]$. We have :

$$\begin{aligned}
\mathbb{E}[M_{n+1} | \mathcal{F}_n] &= \mathbb{E}\left[\frac{1}{\Delta_{X_n} p_{X_n}} \mathbf{1}_{\{\tau_0 > n+1\}} \mid \mathcal{F}_n\right] \\
&= \frac{1}{\Delta_{X_n} p_{X_n}} \mathbb{E}\left[\mathbf{1}_{\{\tau_0 > n+1\}} \mid \mathcal{F}_n\right] \\
&= \frac{1}{\Delta_{X_n}} \mathbf{1}_{\{\tau_0 > n\}},
\end{aligned}$$

where we used that $X_{n+1} = X_n + 1$, and thus $\Delta_{X_{n+1}} = \Delta_{X_n} p_{X_n}$, on $\{\tau_0 > n+1\}$ for the first equality and the previous question for the last. We deduce that $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n$. Taking the expectation, we deduce that $\mathbb{E}[M_{n+1}] = \mathbb{E}[M_n]$ and by induction $\mathbb{E}[M_{n+1}] = \mathbb{E}[M_0]$. Thus, if $\mathbb{E}[M_0]$ is finite, then $M_n$ is integrable for all $n \in \mathbb{N}$ and since $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n$ for all $n \in \mathbb{N}$, we get that $M$ is a martingale. Notice that $\mathbb{E}_k[M_0] = \varphi(k) < +\infty$ to conclude.

(c) Notice that $\mathbb{E}_k[M_0] = \varphi(k)$ is finite, so $M$ is a martingale under $\mathbb{P}_k$. Since $\tau_\ell$ is a stopping time, we get by the stopping time theorem that $\mathbb{E}_k\left[M_{\tau_\ell \wedge n}\right] = \varphi(k)$. In both the transient case and recurrent case, we obtain that $\tau_\ell$ is a.s. finite. Since the sequence $(M_{\tau_\ell \wedge n},\, n \in \mathbb{N})$ is non-negative, bounded from above by $\varphi(\ell)$ and converges $\mathbb{P}_k$-a.s. to $M_{\tau_\ell}$, we deduce from the dominated convergence theorem that :

$$\mathbb{E}_k\left[M_{\tau_\ell}\right] = \lim_{n\to\infty} \mathbb{E}_k\left[M_{\tau_\ell \wedge n}\right] = \varphi(k).$$

(d) Since $M_{\tau_\ell}$ is equal to 0 on $\{\tau_0 < \tau_\ell\}$, to $\varphi(\ell)$ on $\{\tau_\ell < \tau_0\}$, and that a.s. $\tau_0 \neq \tau_\ell$, we get that :

$$\mathbb{E}_k\left[M_{\tau_\ell}\right] = \varphi(\ell) \mathbb{P}_k(\tau_\ell < \tau_0).$$

Use the previous question to get that $\mathbb{P}_k(\tau_\ell < \tau_0) = \Delta_\ell / \Delta_k$ for $0 < k < \ell$. (Notice this result could have been seen directly as there is only one path starting from $k$ to $\ell$ which avoids 0 ; it corresponds to the event $\{X_j = k + j,\, j \in \{0, k - \ell\}\}$ which has probability $\prod_{r=k}^{\ell-1} p_r$.)

6. (a) By an easy induction, we get that $V$ and $Q$ are $\mathcal{F}$-adapted. Since $V$ is non-negative, we can compute $\mathbb{E}[V_{n+1}|\,\mathcal{F}_n]$. We have :

$$\begin{aligned}\mathbb{E}\left[V_{n+1} \mid \mathcal{F}_n\right] &= \frac{1}{p_{X_n}}(1+V_n)\mathbb{E}\left[\mathbf{1}_{\{X_{n+1}\neq 0\}} \mid \mathcal{F}_n\right] \\ &= \frac{1}{p_{X_n}}(1+V_n)\mathbb{E}\left[\mathbf{1}_{\{X_{n+1}\neq 0\}} \mid X_n\right] \\ &= \frac{1}{p_{X_n}}(1+V_n)\mathbb{E}\left[\mathbf{1}_{\{X_{n+1}=X_n+1\}} \mid X_n\right] \\ &= 1+V_n,\end{aligned}$$

where we used the Markov property of $X$ for the second equality, that $\{X_{n+1}\neq 0\} = \{X_{n+1} = X_n+1\}$ for the third and that the transition probability from $k$ to $k+1$ is $p_k$ for the last. We deduce that $\mathbb{E}[V_{n+1}|\,\mathcal{F}_n] = V_n + 1$. Taking the expectation, we deduce that $\mathbb{E}[V_{n+1}] = \mathbb{E}[V_n] + 1$ and by induction $\mathbb{E}[V_{n+1}] = \mathbb{E}[X_0] + n$ which is finite by assumption. Thus $V_n$ and $Q_n$ are integrable for all $n \in \mathbb{N}$ and since $\mathbb{E}[Q_{n+1}|\,\mathcal{F}_n] = \mathbb{E}[V_{n+1}|\,\mathcal{F}_n] - n - 1 = Q_n$ for all $n \in \mathbb{N}$, we get that $Q$ is a martingale.

(b) Notice that $\mathbb{E}_0[X_0] = 0$ is finite, so $Q$ is a martingale under $\mathbb{P}_0$. Since $\tau_\ell$ is a stopping time, we get by the stopping time theorem that $\mathbb{E}_0\left[Q_{\tau_\ell\wedge n}\right] = 0$. This implies that :

$$\mathbb{E}_0\left[\tau_\ell \wedge n\right] = \mathbb{E}_0\left[V_{\tau_\ell\wedge n}\right] \quad \text{for } \ell \in \mathbb{N}^* \text{ and } n \in \mathbb{N}. \tag{3}$$

(c) By monotone convergence, we have that $\mathbb{E}_0\left[\tau_\ell\right] = \lim_{n\to\infty}\mathbb{E}_0\left[\tau_\ell\wedge n\right]$. As $\ell > 0$, in both the transient case and recurrent case, we obtain that $\tau_\ell$ is a.s. finite. The sequence $(V_{\tau_\ell\wedge n},\, n\in\mathbb{N})$ is non-negative, and using an elementary induction, it is bounded from above by $V_{\tau_\ell}$ which is $\mathbb{P}_0$-a.s. equal to $\Delta_\ell^{-1}\sum_{k=0}^{\ell-1}\Delta_k$. We deduce from the dominated convergence theorem that :

$$\lim_{n\to\infty}\mathbb{E}_0\left[V_{\tau_\ell\wedge n}\right] = \mathbb{E}_0\left[V_{\tau_\ell}\right] = \frac{1}{\Delta_\ell}\sum_{k=0}^{\ell-1}\Delta_k.$$

Taking the limit in (3), we deduce that $\mathbb{E}_0[\tau_\ell] = \Delta_\ell^{-1}\sum_{k=0}^{\ell-1}\Delta_k$.

(d) Let $0 < k < \ell$. Under $\mathbb{P}_0$, we have that $\tau_\ell = \tau_k + \tau'_\ell$, where :

$$\tau'_\ell = \inf\{n \geq 1;\, Y_n = \ell\} \quad \text{with} \quad Y_n = X_{n+\tau_k} \quad \text{for } n \in \mathbb{N}.$$

Recall that $\mathbb{P}_0$-a.s. $\tau_k$ is finite and thus $\mathbb{P}_0$-a.s. $X_{\tau_k} = k$. By the strong Markov property at time $\tau_k$, we get that $Y = (Y_n, n \in \mathbb{N})$ is distributed as $X$ under $\mathbb{P}_k$. This implies that $\mathbb{E}_0\left[\tau'_\ell\right] = \mathbb{E}_k[\tau_\ell]$, and thus :

$$\mathbb{E}_k[\tau_\ell] = \mathbb{E}_0\left[\tau'_\ell\right] = \mathbb{E}_0\left[\tau_\ell\right] - \mathbb{E}_0\left[\tau_k\right].$$

(e) Since $X$ is recurrent positive, we get $\mathbb{E}_\ell[\tau_\ell] = 1/\pi_\ell = S_1/\Delta_\ell$. Let $0 \leq k < \ell$. Under $\mathbb{P}_\ell$, to return to $\ell$, one has first to pass to 0 and thus to $k$. Thus, under $\mathbb{P}_\ell$, we have that $\tau_\ell = \tau_k + \tau'_\ell$, where :

$$\tau'_\ell = \inf\{n \geq 1;\, Y_n = \ell\} \quad \text{with} \quad Y_n = X_{n+\tau_k} \quad \text{for } n \in \mathbb{N}.$$

Recall that $\mathbb{P}_\ell$-a.s. $\tau_k$ is finite and thus $\mathbb{P}_\ell$-a.s. $X_{\tau_k} = k$. By the strong Markov property at time $\tau_k$, we get that $Y = (Y_n, n \in \mathbb{N})$ is distributed as $X$ under $\mathbb{P}_k$. This implies that $\mathbb{E}_\ell\left[\tau'_\ell\right] = \mathbb{E}_k[\tau_\ell]$ and thus for $0 \leq k < \ell$ :

$$\mathbb{E}_\ell[\tau_k] = \mathbb{E}_\ell[\tau_\ell] - \mathbb{E}_\ell[\tau'_\ell] = \mathbb{E}_\ell[\tau_\ell] - \mathbb{E}_k[\tau_\ell].$$

Then, use Question 6(c) to conclude when $k > 0$.

*Exercice 2* The Gaussian process $W$ is centerd with covariance kernel $K_W = (K_W(s,t);\, s,t \in [0,1])$, where $K_W(s,t) = s(1-t)$ for $0 \leq s \leq t \leq 1$.

1. By construction $X$ is a centered Gaussian process. We remark its covariance kernel $K_X = (K_X(s,t);\, s,t \in [0,1])$ is given by, for $0 \leq s \leq t \leq 1$ :

$$K_X(s,t) = K_W(1-s, 1-t) = (1-t)(1-(1-s)) = K_W(s,t).$$

   Thus, we get $K_X = K_W$ (as the covariance kernel is symmetric). We deduce that $X$ and $W$ have the same distribution.

2. By construction $Y$ is a centered Gaussian process. We remark its covariance kernel $K_Y = (K_Y(s,t);\, s,t \in [0,1])$ is given by, for $0 \leq s \leq t < 1$ :

$$K_Y(s,t) = (1-t)(1-s)K_W(s/(1-s), t/(1-t)) = (1-t)s = K_W(s,t),$$

   as $s/(1-s) \leq t/(1-t)$. If $0 \leq s \leq t = 1$, then as $Y_1 = 0$, we get $K_Y(s,1) = 0 = K_W(s,1)$. Thus, we get $K_Y = K_W$ (as the covariance kernel is symmetric). We deduce that $Y$ and $W$ have the same distribution.

3. Notice that $Z_t = Y_{1-t}$ for $t \in [0,1]$. Since $Y$ is distributed as $W$ according to Question 2, we deduce that $Z$ is distributed as $X$ and thus as $W$ according to Question 1.