Stochastic Process (ENPC)
Monday, 27th of January 2020 (2h30)

Vocabulary (english/*français*) : random walk =*marche aléatoire* ; distribution =*loi* ; positive = *strictement positif* ; interwining relationship =*relation d'entrelacement*.

We shall assume that all the random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Exercise 1 (Mean of some exit time)**.** Let $B = (B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion.

1. Prove that $M = (M_t = B_t^2 - t, t \geq 0)$ is a martingale.
2. Let $a > 0$ and set $\tau_a = \inf\{t \geq 0;\, B_t \notin [-a, a]\}$.
   (a) Prove that $\mathbb{E}[t \wedge \tau_a] = \mathbb{E}[B^2_{t\wedge\tau_a}]$.
   (b) Deduce that $\mathbb{E}[\tau_a]$ is finite and then compute $\mathbb{E}[\tau_a]$.
3. Let $a > 0$ and $b > 0$ and set $\tau_{a,b} = \inf\{t \geq 0;\, B_t \notin [-a, b]\}$.
   (a) Check that $\tau_{a,b}$ is a.s. finite. Using that B is a martingale, compute $\mathbb{P}(B_{\tau_{a,b}} = -a)$.
   (b) Deduce the value of $\mathbb{E}[\tau_{a,b}]$.

△

We write $\mathbb{N}^* = \mathbb{Z} \cap [1, +\infty)$ and $\mathbb{N} = \mathbb{N}^* \bigcup \{0\}$. For $x \in \mathbb{R}$, we set $x_+ = \max(x, 0)$ for the positive part of x. We recall the notation $\mathbb{P}(A|\, \mathcal{G}) = \mathbb{E}[\mathbf{1}_A|\, \mathcal{G}]$ for any $A \in \mathcal{F}$ and $\mathcal{G} \subset \mathcal{F}$ a σ-field.

Exercise 2 (When is a functional of a Markov chain again a Markov chain ?)**.** Let $p \in (0, 1)$ and $q = 1 - p$. Let ζ be a $\{-1, 1\}$-valued random variable such that :

$$\mathbb{P}(\zeta = 1) = p \quad \text{and} \quad \mathbb{P}(\zeta = -1) = 1 - p = q.$$

Let $(\zeta_n, n \in \mathbb{N}^*)$ be independent random variables distributed as ζ. We define the simple random walk $S = (S_n, n \in \mathbb{N})$ by $S_0 = 0$ and $S_n = S_{n-1} + \zeta_n$ for $n \in \mathbb{N}^*$. The process S is a Markov chain on $\mathbb{Z}$ with transition matrix $P = (P(s, t);\, s, t \in \mathbb{Z})$, see Figure 1, given by :

$$P(s, t) = p\mathbf{1}_{\{t=s+1\}} + q\mathbf{1}_{\{t=s-1\}}.$$

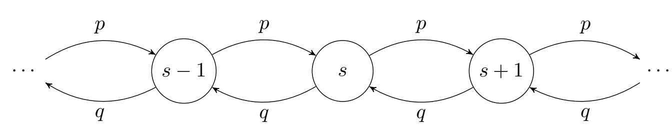

Figure 1 – Transition graph for the simple random walk S on $\mathbb{Z}$.

We consider the natural filtration $\mathbb{F} = (\mathcal{F}_n = \sigma(S_0, \ldots, S_n),\, n \in \mathbb{N})$ of S. We define the infimum process $(I_n, n \in \mathbb{N})$ associated to S by, for $n \in \mathbb{N}$:

$$I_n = \min\{S_k,\, 0 \leq k \leq n\}.$$

We define the processes $U = (U_n = S_n - I_n, n \in \mathbb{N})$ and $\tilde{S} = (\tilde{S}_n = (S_n, I_n), n \in \mathbb{N})$ the random walk completed with its infimum process taking values in $\mathbb{N}$. Notice that $\tilde{S}$ takes values in $E = \{(s, i) \in \mathbb{Z}^2,\, s \geq i \text{ and } 0 \geq i\}$. Figure 2 represents a simulation of the processes S, I and $U = S - I$.

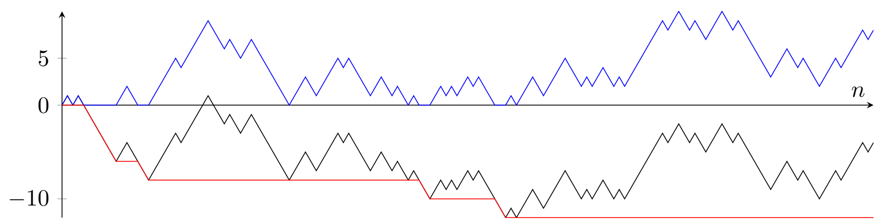

Figure 2 – A path simulation of S (black), I (red) and $U = S - I$ (blue) up to time $n = 150$ for $p = 0.55$.

I Some Markov chains related to S

1. Prove that $\tilde{S}$ is a E-valued Markov chain with respect to (w.r.t.) the filtration $\mathbb{F}$ and with transition matrix $\tilde{P} = (\tilde{P}((s,i),(t,j));\ (s,i),(t,j) \in E)$ given by :
$$\tilde{P}((s,i),(t,j)) = p\mathbf{1}_{\{t=s+1,j=i\}} + q\left(\mathbf{1}_{\{s>i,t=s-1,j=i\}} + \mathbf{1}_{\{s=i,t=s-1,j=i-1\}}\right).$$
2. Compute $\mathbb{P}(U_{n+1} = u |\, \mathcal{F}_n)$ for $u \in \mathbb{N}$.
3. Deduce U is a Markov chain w.r.t. the filtration $\mathbb{F}$ and give its transition matrix R.
4. Prove that $\mathbb{F}$ is also the natural filtration of U.
5. Prove that U is distributed as the reflected simple random walk $S^{\text{refl}} = (S_n^{\text{refl}}, n \in \mathbb{N})$ defined by $S_0^{\text{refl}} = 0$ and $S_n^{\text{refl}} = (S_{n-1}^{\text{refl}} + \zeta_n)_+$ for $n \in \mathbb{N}^*$.
6. **(Dynkin's criterion [1])**. For this question only : let $\tilde{S} = (\tilde{S}_n, n \in \mathbb{N})$ be a general Markov chain on a discrete state space E with transition matrix $\tilde{P}$; and φ be a function from E to $F = \varphi(E)$. Set $\Phi = (\Phi(\tilde{s},x);\ \tilde{s} \in E, x \in F)$ with $\Phi(\tilde{s},x) = \mathbf{1}_{\{\varphi(\tilde{s})=x\}}$. Assume the intertwining relation $\tilde{P}\Phi = \Phi R$ holds for some stochastic matrix R on F.
   (a) Prove that $\varphi(\tilde{S})$ is a Markov chain w.r.t. the natural filtration of $\tilde{S}$ and with transition matrix R. (Hint : check $\tilde{P}\tilde{g} = \tilde{P}\Phi g$, with g non-negative defined on F and $\tilde{g} = g \circ \varphi$.)
   (b) Explain how this result generalises Question 3.

II Infimum and excursion

We define the hitting time $\tau_a = \inf\{k \in \mathbb{N},\ S_k = -a\}$ of $-a$ for $a \geq 1$ (with the convention that $\inf \emptyset = +\infty$) and the infimum $I_\infty = \inf\{S_k,\ k \in \mathbb{N}\}$ of S. We set $\varphi(\lambda) = \mathbb{E}[\mathrm{e}^{-\lambda\tau_1}]$ for $\lambda > 0$.

1. Prove that τ_2 is distributed as $\tau_1 + \tau_1'$, where τ_1' is distributed as τ_1 and independent of τ_1.
2. Prove that $\varphi(\lambda) = p\,\mathrm{e}^{-\lambda}\,\varphi(\lambda)^2 + q\,\mathrm{e}^{-\lambda}$.
3. Compute $\varphi(\lambda)$ and deduce that $\mathbb{P}(\tau_1 < +\infty) = \min\left(1, \frac{q}{p}\right)$.
4. Check that $\{n \in \mathbb{N}^*,\ I_n = I_{n-1} - 1\} = \{n \in \mathbb{N}^*,\ (U_{n-1}, U_n) = (0,0)\}$. Using that U is a Markov chain, prove that $|I_\infty| + 1$ is geometric with parameter $\mathbb{P}(\tau_1 = +\infty)$ (with the convention that a geometric random variable with parameter 0 is a.s. infinite).
5. We write $\mathbb{P}_{(p)}$ for $\mathbb{P}$ to stress out that $\mathbb{P}(\zeta = 1) = p$. On $\{\tau_1 < +\infty\}$, we define $\mathcal{E} = (S_0, \ldots, S_{\tau_1 - 1})$ the finite excursion of S strictly above -1. Prove that $\mathcal{E}$ conditionally on $\{\tau_1 < +\infty\}$ has the same distribution under $\mathbb{P}_{(p)}$ and under $\mathbb{P}_{(q)}$.

$\triangle$

1. E. Dynkin. *Markov processes.* Vol. I. Springer, 1965. (See Section X.6.)

Correction

Exercise 1 Let $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ denote the Brownian filtration of B.

1. We have clearly that M is $\mathbb{F}$-adapted. Since $\mathbb{E}[B_t^2] = t$, we deduce that M_t is integrable that for all $t \geq 0$. We have for all $t \geq 0, s \geq 0$:
$$\begin{aligned}\mathbb{E}[B_{t+s}^2 | \mathcal{F}_t] &= \mathbb{E}[(B_{t+s} - B_t)^2 + B_t^2 + 2B_t(B_{t+s} - B_t) | \mathcal{F}_t] \\ &= \mathbb{E}[B_s^2] + B_t^2 + 2B_t\mathbb{E}[(B_{t+s} - B_t) | \mathcal{F}_t] \\ &= s + B_t^2,\end{aligned}$$
where we used that $B_{t+s} - B_t$ is independent of $\mathcal{F}_t$ and distributed as B_s for the second and third equalities. We deduce that $\mathbb{E}[M_{t+s} | \mathcal{F}_t] = M_t$. This gives that M is a martingale.

2. (a) Since $\{\tau_a > t\} = \bigcap_{s \in \mathbb{Q} \cap [0,t]} \{B_s \in [-a, a]\}$, we deduce that τ_a is a stopping time. By the optional stopping theorem, we get that $\mathbb{E}[M_{t \wedge \tau_a}] = \mathbb{E}[M_0] = 0$. This implies that for all $t \geq 0$:
$$\mathbb{E}[t \wedge \tau_a] = \mathbb{E}[B_{t \wedge \tau_a}^2].$$
(b) Since $B_{t \wedge \tau_a}^2 \leq a^2$, we get that $\mathbb{E}[t \wedge \tau_a] \leq a^2$. By monotone convergence, we deduce that $\mathbb{E}[\tau_a] \leq a^2$. In particular τ_a is a.s. finite. This implies that a.s. $\lim_{t \to +\infty} B_{t \wedge \tau_a}^2 = B_{\tau_a}^2 = a^2$. By dominated convergence, we get that $\lim_{t \to +\infty} \mathbb{E}[B_{t \wedge \tau_a}^2] = a^2$. By monotone convergence, we deduce that :
$$\mathbb{E}[\tau_a] = \lim_{t \to +\infty} \mathbb{E}[t \wedge \tau_a] = \lim_{t \to +\infty} \mathbb{E}[B_{t \wedge \tau_a}^2] = a^2.$$

3. (a) Since $\{\tau_{a,b} > t\} = \bigcap_{s \in \mathbb{Q} \cap [0,t]} \{B_s \in [-a, b]\}$, we deduce that $\tau_{a,b}$ is a stopping time. Since $\tau_{a,b} \leq \tau_a + \tau_b$, we deduce from the answer to Question 2 that $\tau_{a,b}$ is a.s. finite. Since B is a martingale, by the optional stopping theorem, we get $\mathbb{E}[B_{t \wedge \tau_{a,b}}] = 0$ for all $t \geq 0$. Then use that a.s. $\lim_{t \to +\infty} B_{t \wedge \tau_{a,b}} = B_{\tau_{a,b}}$ and that $|B_{t \wedge \tau_{a,b}}| \leq a + b$ to get by dominated convergence that $\mathbb{E}[B_{\tau_{a,b}}] = 0$. Since $B_{\tau_{a,b}} \in \{-a, b\}$, we deduce that :
$$-a\mathbb{P}(B_{\tau_{a,b}} = -a) + b\mathbb{P}(B_{\tau_{a,b}} = b) = 0 \quad \text{and} \quad \mathbb{P}(B_{\tau_{a,b}} = -a) + \mathbb{P}(B_{\tau_{a,b}} = b) = 1.$$
This gives :
$$\mathbb{P}(B_{\tau_{a,b}} = -a) = \frac{b}{a+b} \quad \text{and} \quad \mathbb{P}(B_{\tau_{a,b}} = b) = \frac{a}{a+b}.$$
(b) Arguing as in the answer to Question 2, we get :
$$\mathbb{E}[\tau_{a,b}] = \lim_{t \to +\infty} \mathbb{E}[t \wedge \tau_{a,b}] = \lim_{t \to +\infty} \mathbb{E}[B_{t \wedge \tau_{a,b}}^2] = \mathbb{E}[B_{\tau_{a,b}}^2].$$
We deduce that :
$$\mathbb{E}[\tau_{a,b}] = \mathbb{E}[B_{\tau_{a,b}}^2] = a^2\mathbb{P}(B_{\tau_{a,b}} = -a) + b^2\mathbb{P}(B_{\tau_{a,b}} = b) = ab.$$

Exercise 2 **I Some Markov chains related to S**

1. We have that $I_{n+1} = I_n - \mathbf{1}_{\{\zeta_{n+1} = -1, S_n = I_n\}}$. We deduce that
$$\tilde{S}_{n+1} = (S_{n+1}, I_{n+1}) = (S_n + \zeta_{n+1},\, I_n - \mathbf{1}_{\{\zeta_{n+1} = -1, S_n = I_n\}}) = f(\tilde{S}_n, \zeta_{n+1}), \tag{1}$$

for some function f. Since $(\zeta_n, n \in \mathbb{N}^*)$ are independent identically distributed random variables independent of $\tilde{S}_0$, the process $\tilde{S}$ is a stochastic dynamical system and thus a Markov chain. Clearly $\tilde{S}$ takes values in E. The transition matrix is easily computed from (1) and the fact that $\mathbb{P}(\zeta_{n+1} = 1) = 1 - \mathbb{P}(\zeta_{n+1} = -1) = p$.

2. We have for $u \in \mathbb{N}^*$:

$$\{U_{n+1} = u\} = \{S_n - I_n = u - 1, \zeta_{n+1} = 1\} \cup \{S_n - I_n = u + 1, \zeta_{n+1} = -1\},$$

and for $u = 0$:

$$\{U_{n+1} = 0\} = \{S_n - I_n = 0, \zeta_{n+1} = -1\} \cup \{S_n - I_n = 1, \zeta_{n+1} = -1\},$$

where the unions are between disjoint sets. Since (S_n, I_n) is $\mathcal{F}_n$-measurable and ζ_{n+1} is independent from $\mathcal{F}_n$, we deduce that for $u \in \mathbb{N}^*$:

$$\mathbb{P}(U_{n+1} = u | \mathcal{F}_n) = p\mathbf{1}_{\{S_n - I_n = u-1\}} + q\mathbf{1}_{\{S_n - I_n = u+1\}} = p\mathbf{1}_{\{U_n = u-1\}} + q\mathbf{1}_{\{U_n = u+1\}},$$

and for $u = 0$:

$$\mathbb{P}(U_{n+1} = 0 | \mathcal{F}_n) = q\mathbf{1}_{\{S_n - I_n \in \{0,1\}\}} = q\mathbf{1}_{\{U_n \in \{0,1\}\}}.$$

3. For $u \in \mathbb{N}$, we get that $\mathbb{P}(U_{n+1} = u | \mathcal{F}_n) = R(U_n, u)$, where for $w \in \mathbb{N}$:

$$R(w, u) = p\mathbf{1}_{\{u \geq 1\}}\mathbf{1}_{\{w = u-1\}} + q\mathbf{1}_{\{w = u+1\}} + q\mathbf{1}_{\{u = 0\}}\mathbf{1}_{\{w = 0\}}. \tag{2}$$

This implies that U is a $\mathbb{N}$-valued Markov chain with respect to the filtration $\mathbb{F}$ and with transition matrix R.

4. Let $\mathbb{G} = (\mathcal{G}_n = \sigma(U_0, \ldots, U_n), , \in \mathbb{N})$ be the natural filtration of U. Since U is $\mathbb{F}$-adapted, we have $\mathcal{G}_n \subset \mathcal{F}_n$. For $n \in \mathbb{N}$, we have :

$$\zeta_n = (U_n - U_{n-1})\mathbf{1}_{\{U_n \neq U_{n-1}\}} - \mathbf{1}_{\{U_n = U_{n-1}\}}.$$

Thus the random variable ζ_n is $\mathcal{G}_n$-measurable. This implies that $\sigma(\zeta_1, \ldots, \zeta_n) \subset \mathcal{G}_n$. Then use that $\mathcal{F}_n = \sigma(S_0, \ldots, S_n) = \sigma(\zeta_1, \ldots, \zeta_n)$ to conclude that $\mathcal{G}_n = \mathcal{F}_n$.

5. Notice from the definition of R that $R(w, w+1) = p$ for $w \in \mathbb{N}$; $R(w, w-1) = q$ for $w \in \mathbb{N}^*$; and $R(0,0) = q$. It is clear from the definition of S^{refl} that S^{refl} is a Markov chain with transition matrix R.

6. Recall that functions are seen as column vectors. If M is a matrix (with non-negative entries) and v a column vectors (with non-negative entries), then we may also write $M[v]$ for the column vector Mv.

   (a) We have, for $n \in \mathbb{N}$ and g a non-negative function defined on F :

   $$\mathbb{E}[g \circ \varphi(\tilde{S}_{n+1}) | \mathcal{F}_n] = \tilde{P}[g \circ \varphi](\tilde{S}_n) = \tilde{P}\Phi[g](\tilde{S}_n) = \Phi R[g](\tilde{S}_n) = R[g](\varphi(\tilde{S}_n)),$$

   where we used that $\tilde{S}$ is a Markov chain with respect to the filtration $\mathbb{F}$ for the first equality. This readily implies that $\varphi(\tilde{S})$ is a Markov chain with respect to the filtration $\mathbb{F}$ and with transition matrix R.

(b) In the setting of Question 3, for $\tilde{s}=(s,i)$, we set $\varphi(\tilde{s}) = s-i$, so that $U=(U_n = \varphi(\tilde{S}_n), n\in\mathbb{N})$. We have $F=\mathbb{N}$. Let $(s,i)\in E$ and $u\in\mathbb{N}$. On the one hand, we have by definition of R given in (2) that :

$$\begin{aligned}\Phi R((s,i),u) &= \sum_{w\in F} \Phi((s,i),w)R(w,u)\\ &= R(s-i,u)\\ &= p\mathbf{1}_{\{u\geq 1\}}\mathbf{1}_{\{s-i=u-1\}} + q\mathbf{1}_{\{s-i=u+1\}} + q\mathbf{1}_{\{u=0\}}\mathbf{1}_{\{s-i=0\}}.\end{aligned}$$

On the other hand, we have :

$$\begin{aligned}\tilde{P}\Phi((s,i),u) &= \sum_{(t,j)\in E} \tilde{P}((s,i),(t,j))\Phi((t,j),u)\\ &= \sum_{(t,j)\in E} \left[p\mathbf{1}_{\{t=s+1,j=i\}} + q\left(\mathbf{1}_{\{s>i,t=s-1,j=i\}} + \mathbf{1}_{\{s=i,t=s-1,j=i-1\}}\right)\right]\mathbf{1}_{\{t-j=u\}}\\ &= p\mathbf{1}_{\{s-i=u-1\}} + q\mathbf{1}_{\{s>i,s-i=u+1\}} + q\mathbf{1}_{\{u=0\}}\mathbf{1}_{\{s-i=0\}}.\end{aligned}$$

Use that $s-i\geq 0$ and $u\geq 0$ to get that the intertwining relation relation $\tilde{P}\Phi = \Phi R$ holds. Since R is a stochastic matrix, Question 3 is a particular case of the Dynkin's criterion.

II Infimum and excursion

1. On $\{\tau_1=+\infty\}$, the statement is true. On $\{\tau_1<+\infty\}$, as $S_{\tau_1}=-1$, we get :

$$\tau_2 = \tau_1 + \inf\{k\in\mathbb{N},\, S_{\tau_1+k}=-2\} = \tau_1+\tau_1' \quad\text{with}\quad \tau_1' = \inf\Big\{k\in\mathbb{N},\, \sum_{i=1}^{k}\zeta_{\tau_1+i}=-1\Big\}.$$

By the strong Markov property for the sequence $(\zeta_n, n\in\mathbb{N})$, we get that $(\zeta_{\tau_1+n}, n\in\mathbb{N})$ is, on $\{\tau_1<+\infty\}$, independent of $(\zeta_k, 1\leq k\leq\tau_1)$ and distributed as $(\zeta_n,n\in\mathbb{N})$. This implies that τ_1 and τ_1' are independent and with the same distribution.

2. We have :

$$\begin{aligned}\varphi(\lambda) &= \mathbb{E}[\mathrm{e}^{-\lambda\tau_1}\,\mathbf{1}_{\{S_1=1\}}] + \mathbb{E}[\mathrm{e}^{-\lambda\tau_1}\,\mathbf{1}_{\{S_1=-1\}}]\\ &= \mathrm{e}^{-\lambda}\,\mathbb{E}[\mathrm{e}^{-\lambda\tau_2}] + q\,\mathrm{e}^{-\lambda}\\ &= \mathrm{e}^{-\lambda}\,\mathbb{E}[\mathrm{e}^{-\lambda\tau_1}]^2 + q\,\mathrm{e}^{-\lambda}\\ &= p\,\mathrm{e}^{-\lambda}\,\varphi(\lambda)^2 + q\,\mathrm{e}^{-\lambda},\end{aligned}$$

where we decomposed according to S_1 equal to 1 or -1 for the first equality, that S started at 1 is distributed as $S+1$ for the second, and the previous question for the decomposition of τ_2 for the third.

3. We get, as $\varphi(\lambda)\leq 1$:

$$\varphi(\lambda) = \frac{\mathrm{e}^{\lambda}}{2p}\left(1-\sqrt{1-4pq\,\mathrm{e}^{-2\lambda}}\right).$$

Since $\lim_{\lambda\to 0+}\mathrm{e}^{-\lambda\tau_1} = \mathbf{1}_{\{\tau_1<+\infty\}}$, we deduce by dominated convergence that $\mathbb{P}(\tau_1<+\infty) = \lim_{\lambda\to 0+}\varphi(\lambda)$, that is, as $q=1-p$:

$$\mathbb{P}(\tau_1<+\infty) = \frac{1}{2p}\left(1-\sqrt{1-4pq}\right) = \frac{1}{2p}(1-|1-2p|) = \min\left(1,\frac{q}{p}\right).$$

4. Since U is a Markov chain, we deduce that it is also a second order Markov chain, that is $\hat{U} = (\hat{U}_n = (U_{n-1}, U_n), n \in \mathbb{N})$, with the convention that $U_{-1} = 0$, is a Markov chain started at $\hat{U}_0 = (0,0)$. For $n \in \mathbb{N}^*$, clearly $I_n = I_{n-1} - 1$ if and only if $\hat{U}_n = (0,0)$. In particular $|I_\infty| + 1$ is equal to the number of visits of $(0,0)$ for $\hat{U}$ (this latter is the cardinal of $\{n \in \mathbb{N}, (U_{n-1}, U_n) = (0,0)\}$). The number of visit is geometric with parameter $\mathbb{P}_{(0,0)}(T^{(0,0)} = +\infty)$, where $T^{(0,0)} = \inf\{n \in \mathbb{N}^*, \hat{U}_n = (0,0)\}$ is the first return time to $(0,0)$ for $\hat{U}$. Then notice that $T^{(0,0)} = \tau_1$ to conclude that $|I_\infty| + 1$ is geometric with parameter $\mathbb{P}(\tau_1 = +\infty)$.
5. We define the set of excursions of S above -1 of length $n \in \mathbb{N}^*$ as :

$$\mathcal{S}_n = \Big\{ e = (s_0, \ldots, s_n);\ s_0 = s_n = 0, z_k^e \in \{-1, 1\} \text{ and } s_k \geq 0 \text{ for all } 1 \leq k \leq n \Big\},$$

where $z_k^e = s_k - s_{k-1}$ for $e = (s_0, \ldots, s_n)$. We set $n^e = n$ the length of $e \in \mathcal{S}_n$. Notice that $\mathcal{S}_n$ is empty if n is odd. The set of finite excursions is $\mathcal{S} = \bigcup_{n \in 2\mathbb{N}^*} \mathcal{S}_n$. Let $e \in \mathcal{S}$ be an excursion. We denote by $n_+^e = \sum_{k=1}^{n^e} \mathbf{1}_{\{z_k^e = 1\}}$ (resp. $n_-^e = \sum_{k=1}^{n^e} \mathbf{1}_{\{z_k^e = -1\}}$) the number of positive (resp. negative) increments of the path e. Notice that $n_+^e = n_-^e = n/2$. For $e \in \mathcal{S}$, we have, taking into account that the step just after time $\tau_1 - 1$ is negative :

$$\mathbb{P}_{(p)}(\mathcal{E} = e) = p^{n_+^e} q^{n_-^e + 1} = (pq)^{n_+^e} q.$$

Using Question 3, this implies that :

$$\mathbb{P}_{(p)}(\mathcal{E} = e | \tau_1 < +\infty) = (pq)^{n_+^e} \frac{q}{\min\left(1, \frac{q}{p}\right)} = (pq)^{n_+^e} \max(p, q).$$

By symmetry, we get that :

$$\mathbb{P}_{(q)}(\mathcal{E} = e | \tau_1 < +\infty) = (qp)^{n_+^e} \max(q, p) = \mathbb{P}_{(p)}(\mathcal{E} = e | \tau_1 < +\infty),$$

that is $\mathcal{E}$ conditionally on $\{\tau_1 < +\infty\}$ has the same distribution under $\mathbb{P}_{(p)}$ and under $\mathbb{P}_{(q)}$.

It is interesting to note that most of the results of this exercise can be extended to the Brownian motion with drift.