Stochastic Process (ENPC)
Monday, 24th of January 2022 (2h30)

Vocabulary (english/*français*): positive = *strictement positif*; shift= *décalage*.

Exercise 1 (Martingale and law of large numbers)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_n, n \in \mathbb{N})$. Let $M = (M_n, n \in \mathbb{N})$ be a martingale, $\alpha > 0$ and c finite such that $\mathbb{E}[M_n^2] \leq c\, n^\alpha$ for all $n \in \mathbb{N}$. Let $\beta > \alpha/2$. We shall prove that:

$$\text{a.s.} \quad \lim_{n\to\infty} n^{-\beta} M_n = 0. \tag{1}$$

For $\lambda > 0$ and $n \in \mathbb{N}$, we consider the event $A_n(\lambda) = \{\max_{2^n \leq k < 2^{n+1}} k^{-\beta} |M_k| \geq \lambda\}$. We also recall the maximal inequality $\mathbb{E}\left[\sup_{0\leq k\leq n} |M_k|^p\right] \leq C_p \mathbb{E}\left[|M_n|^p\right]$ which holds for all $p > 1$ and $n \in \mathbb{N}$ with $C_p = (p/(p-1))^p$.

1. Using the maximal inequality, prove that $\mathbb{P}\left(A_n(\lambda)\right) \leq (\lambda 2^{\beta n})^{-2}\, C_2\, \mathbb{E}\left[M^2_{2^{n+1}}\right]$.
2. Prove that $\mathbb{E}\left[\sum_{n=1}^\infty \mathbf{1}_{A_n(1/n)}\right]$ is finite.
3. Deduce that Equation (1) holds.
4. Let $(Y_n, n \in \mathbb{N}^*)$ be independent identically distributed real-valued random variables with finite variance. Using the martingale $(\sum_{k=1}^n Y_k - \mathbb{E}[Y_k], n \in \mathbb{N})$, recover the strong law of large numbers.

△

Exercise 2 (Random walk in random environment)**.** We shall study the velocity of the simple random walk in a random environment[1] and prove that it is slower than the velocity of the simple random walk in a constant deterministic environment with the same mean drift. This phenomenon appears in more general physical models (disordered media, DNA unzipping, ...). For the proof, we shall adopt in the last question the point of view of the random environment seen from the random walk[2].

Let $\varepsilon_0 \in (0, 1/2)$ and $E = [\varepsilon_0, 1-\varepsilon_0]^{\mathbb{Z}}$ be the set of $[\varepsilon_0, 1-\varepsilon_0]$-valued sequences indexed by $\mathbb{Z}$ (the product space E is endowed with the product σ-field). An environment is a sequence $p = (p(k), k \in \mathbb{Z}) \in E$, and we shall set $\boxed{q(k) = 1 - p(k)}$ for $k \in \mathbb{Z}$. A random walk $X = (X_n, n \in \mathbb{N})$ in the given environment p is, under the probability measure $\mathbf{P}_p$, an inhomogeneous Markov chain such that $X_0 = 0$ and for all $n \in \mathbb{N}$:

$$\mathbf{P}_p(X_{n+1} = X_n + 1 |\, \mathcal{F}_n) = p(X_n) = 1 - \mathbf{P}_p(X_{n+1} = X_n - 1 |\, \mathcal{F}_n),$$

where $\mathbb{F} = (\mathcal{F}_n = \sigma(X_0, \ldots, X_n), n \in \mathbb{N})$ is the natural filtration of the process X, see also Figure 1 for a representation of the transition probabilities.

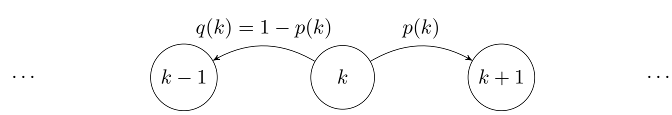

Figure 1: Transition probabilities for the simple random walk X on $\mathbb{Z}$ in the environment p.

[1] F. Solomon. Random walks in random environment. *Ann. Probab.*, 3:1-31, 1975.

[2] A.-S. Sznitman. Topics in random walks in random environment. *School and Conference on Probability Theory, ICTP Lect. Notes.* XVII: 203-266, 2004.

1. We set $M_0 = 0$ and $M_n = X_n - X_0 - \sum_{k=0}^{n-1}(p(X_k) - q(X_k))$ for $n \in \mathbb{N}^*$.

   (a) Prove that $(M_n, n \in \mathbb{N})$ is a martingale in L^2.

   (b) Check that $\mathbb{E}[M_n^2] = \mathbb{E}[(M_n - M_{n-1})^2] + \mathbb{E}[M_{n-1}^2]$ and deduce that $\mathbb{E}[M_n^2] \leq 4n$.

   (c) Deduce from Equation (1) of Exercise 1 that:

$$\mathbf{P}_p\text{-a.s.} \quad \lim_{n\to\infty} \frac{X_n}{n} - \frac{1}{n}\sum_{k=1}^{n}(p(X_k) - q(X_k)) = 0. \tag{2}$$

2. For $i \in \mathbb{Z}$, we set $\Pi_{i,i}(p) = 1$ and for $j > i$:

$$\Pi_{i,j}(p) = \prod_{r=i+1}^{j} \frac{q(r)}{p(r)}. \tag{3}$$

   We consider the measurable function h defined on $\mathbb{Z} \times E$ by $h(0,p) = 0$ and for $k \neq 0$:

$$h(k,p) = -\sum_{\ell=0}^{k-1} \Pi_{0,\ell}(p) \quad \text{if } k > 0, \quad \text{and} \quad h(k,p) = \sum_{\ell=k}^{-1} \frac{1}{\Pi_{\ell,0}(p)} \quad \text{if } k < 0.$$

   (a) Check that $h(1,p) = -1$, $h(-1,p) = p(0)/q(0)$, and that $k \mapsto h(k,p)$ is strictly decreasing on $\mathbb{Z}$.

   (b) Prove that $N = (N_n = h(X_n, p), n \in \mathbb{N})$ is a martingale.

   (c) Assume that:

$$c(p) := -\lim_{k\to\infty} h(k,p) < +\infty \quad \text{and} \quad \lim_{k\to-\infty} h(k,p) = +\infty. \tag{4}$$

   Using the non-negative martingale $N + c(p)$, prove that $\mathbf{P}_p$-a.s. $\lim_{n\to\infty} X_n = +\infty$.

From now on, we assume that, under the probability measure $\mathbb{P}$, $p = (p(k), k \in \mathbb{Z})$ is a sequence of independent $[\varepsilon_0, 1 - \varepsilon_0]$-valued random variable with the same distribution. The probability measure $\mathbf{P}_p$ can then be seen as $\mathbb{P}$ conditionally on the environment p being given. We assume that $\mathbb{E}[\log(R)] < 0$, where:

$$R = \frac{q(0)}{p(0)}, \quad \text{and we set} \quad v = \frac{1 - \mathbb{E}[R]}{1 + \mathbb{E}[R]}.$$

3. We shall prove that $\mathbb{P}$-a.s. $\lim_{n\to\infty} X_n = +\infty$.

   (a) Prove that $\mathbb{P}$-a.s. $\lim_{\ell\to\infty} \frac{1}{\ell}\log(\Pi_{0,\ell}(p)) = \lim_{\ell\to-\infty} \frac{1}{\ell}\log\left(\Pi_{\ell,0}(p)^{-1}\right) = \mathbb{E}[\log(R)]$.

   (b) Deduce that $\mathbb{P}$-a.s. the two convergences of Equation (4) hold.

   (c) Deduce from Question 2c that $\mathbb{P}$-a.s. $\lim_{n\to\infty} X_n = +\infty$.

4. We introduce some notations and give some independent results.

   (a) Check that $\mathbb{E}[R] < 1$, so that $v > 0$. Prove that $\mathbb{E}[1/p(0)] \geq 1/\mathbb{E}[p(0)]$ and then that $v \leq \mathbb{E}[p(0)] - \mathbb{E}[q(0)]$, with a strict inequality if $p(0)$ is not a.s. equal to a constant.

We set:

$$g(p) = v\,(1+R)\,c(p) \quad \text{with} \quad c(p) = \sum_{\ell=0}^{\infty} \Pi_{0,\ell}(p),$$

where $\Pi_{0,\ell}(p)$ is defined in Equation (3). Recall that $c(p)$ is finite $\mathbb{P}$-a.s., see Question 3b.

(b) Prove that $g(p)$ is positive and that $\mathbb{E}[g(p)] = 1$.

For $\ell \in \mathbb{Z}$, we define the "shift by ℓ" function, S_ℓ, from E to itself by:

$$S_\ell(\eta) = (\eta(k+\ell), k \in \mathbb{Z}) \quad \text{where} \quad \eta = (\eta(k), k \in \mathbb{Z}) \in E.$$

(c) Check that p and $S_1(p)$ have the same distribution.

(d) Let f be a non-negative measurable function defined on E. Prove that:

$$\begin{aligned} \mathbb{E}\left[c(p)\, f \circ S_1(p)\right] &= \mathbb{E}\left[(1 + Rc(p))\, f(p)\right], \\ \mathbb{E}\left[R\, c(p)\, f \circ S_{-1}(p)\right] &= \mathbb{E}\left[(c(p) - 1)\, f(p)\right]. \end{aligned} \tag{5}$$

5. We consider the environment seen by the random walk as the E-valued process $Z = (Z_n = S_{X_n}(p), n \in \mathbb{N})$, so that $Z_0 = p$, and its natural filtration $\mathbb{G} = (\mathcal{G}_n, n \in \mathbb{N})$, with $\mathcal{G}_n = \sigma(Z_0, \ldots, Z_n) = \sigma(p, X_1, \ldots, X_n)$.

   (a) Let f be a bounded measurable function defined on E. Prove that for all $n \in \mathbb{N}$:

   $$\mathbb{P}\text{-a.s.}, \quad \mathbb{E}\left[f(Z_{n+1})|\, \mathcal{G}_n\right] = Z_n(0) f \circ S_1(Z_n) + (1 - Z_n(0)) f \circ S_{-1}(Z_n). \tag{6}$$

   (In particular, Z is a E-valued homogeneous Markov chain under $\mathbb{P}$.)

   According to Question 4b, we can define the probability measure $\tilde{\mathbb{P}}$ by: $\tilde{\mathbb{E}}[W] = \mathbb{E}[g(p)\, W]$, where W is any non-negative random variable.

   (b) Let f be a non-negative measurable function defined on E. Using Equation (5), prove that $\tilde{\mathbb{E}}[f(Z_1)] = \tilde{\mathbb{E}}[f(Z_0)]$. (In particular, since Equation (6) also holds with $\mathbb{E}$ replaced by $\tilde{\mathbb{E}}$, we get that Z is a E-valued stationary homogeneous Markov chain under $\tilde{\mathbb{P}}$.)

   One can prove that the ergodic theorem holds (which is outside the scope of this course as E is uncountable), that is, if f is a bounded measurable function defined on E, then:

   $$\mathbb{P}\text{-a.s.}, \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(Z_k) = \tilde{\mathbb{E}}[f(p)] = \mathbb{E}[g(p)\, f(p)].$$

   (c) Deduce from this ergodic theorem that $\mathbb{P}$-a.s. $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} Z_k(0) = 1/(1 + \mathbb{E}[R])$.

   (d) Deduce from Equation (2) that $\mathbb{P}$-a.s. $\lim_{n\to\infty} X_n/n = v$.

In conclusion, the asymptotic velocity v of the random walk in the random environment p (see Question 5d), when $\mathbb{E}[\log(R)] < 0$ and $p(0)$ not a.s. constant, is strictly slower (see Question 4a) than the asymptotic velocity of the random walk in the constant deterministic environment $\bar{p} \equiv \mathbb{E}[p(0)]$ which is given by the mean drift $\mathbb{E}[X_1] = \mathbb{E}[p(0)] - \mathbb{E}[q(0)] > 0$.

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Correction

Exercise 1 1. We have:

$$\begin{aligned}\mathbb{P}\left(A_n(\lambda)\right) &\leq \mathbb{P}\left(\max_{2^n\leq k<2^{n+1}} |M_k| \geq 2^{\beta n}\lambda\right)\\ &\leq \mathbb{P}\left(\max_{0\leq k\leq 2^{n+1}} |M_k| \geq 2^{\beta n}\lambda\right)\\ &\leq (\lambda 2^{\beta n})^{-2}\, \mathbb{E}\left[\max_{0\leq k\leq 2^{n+1}} |M_k|^2\right]\\ &\leq (\lambda 2^{\beta n})^{-2}\, C_2\, \mathbb{E}\left[M_{2^{n+1}}^2\right],\end{aligned}$$

where we used the maximal inequality for the last inequality.

2. We have:

$$\mathbb{E}\left[\sum_{n=1}^{\infty} \mathbf{1}_{A_n(1/n)}\right] = \sum_{n=1}^{\infty} \mathbb{P}(A_n(1/n)) \leq c\, C_2 \sum_{n=1}^{\infty} n^2 2^{-2\beta n} 2^{(n+1)\alpha} < +\infty,$$

where we used Fubini's theorem for the equality, the previous question for the first inequality, and that $\alpha - 2\beta > 0$ for the last.

3. Since the expectation of the non-negative random variable $X = \sum_{n=1}^{\infty} \mathbf{1}_{A_n(1/n)}$ is finite, it implies that X is a.s. finite, and thus a.s. all the indicators $\mathbf{1}_{A_n(1/n)}$ are zero for n large enough. This readily implies that Equation (1) holds.

4. Set $M_n = \sum_{k=1}^{n} Y_k - \mathbb{E}[Y_k]$, so that $(M_n, n \in \mathbb{N})$ is indeed a martingale. Notice that, as the random variables $(Y_n, n \in \mathbb{N}^*)$ are independent, $\mathbb{E}[M_n^2] = \mathrm{Var}(\sum_{k=1}^{n} Y_k) = n\sigma^2$, with σ^2 the variance of Y_1. We deduce from Equation (1) with $\alpha = \beta = 1$ that a.s $\lim_{n\to\infty} n^{-1} M_n = 0$ and thus $\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} Y_k = \mathbb{E}[Y_1]$, which is the strong law of large numbers.

Exercise 2 1. On the process $M = (M_n, n \in \mathbb{N})$.

(a) Since $|X_{n+1} - X_n| = 1$, we deduce that $|M_n - M_{n-1}| \leq 2$ and thus $|M_n| \leq 2n$, so that M_n belongs to L^2. Clearly M_n, as a measurable function of $(X_0, \ldots, X_n)$, is $\mathcal{F}_n$-measurable, and thus M is $\mathbb{F}$-adapted. We now compute for all $n \in \mathbb{N}$:

$$\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n + \mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] - (p(X_n) - q(X_n)) = M_n.$$

Thus M is a martingale.

(b) We have for $n \in \mathbb{N}^*$:

$$\mathbb{E}[M_n^2] = \mathbb{E}[(M_n - M_{n-1})^2] + \mathbb{E}[M_{n-1}^2] + 2\mathbb{E}[(M_n - M_{n-1})M_{n-1}].$$

Furthermore, by conditioning with respect to $\mathcal{F}_{n-1}$, we get:

$$\mathbb{E}[(M_n - M_{n-1})M_{n-1}] = \mathbb{E}\left[\mathbb{E}[M_n - M_{n-1} | \mathcal{F}_{n-1}] M_{n-1}\right] = 0.$$

This yields $\mathbb{E}[M_n^2] = \mathbb{E}[(M_n - M_{n-1})^2] + \mathbb{E}[M_{n-1}^2]$. Since $|M_n - M_{n-1}| \leq 2$ (see the answer to the previous question), we get that $\mathbb{E}[M_n^2] \leq 4 + \mathbb{E}[M_{n-1}^2]$ and thus, by recursion (as $\mathbb{E}[M_0^2] = 0$), $\mathbb{E}[M_n^2] \leq 4n$.

(c) We get from Equation (1) of Exercise 1, with $\alpha = \beta = 1$, that a.s. $\lim_{n\to\infty} M_n/n = 0$ and thus a.s.:

$$\lim_{n\to\infty} \frac{X_n}{n} - \frac{1}{n}\sum_{k=0}^{n-1}(p(X_k) - q(X_k)) = 0.$$

Then, as $p(X_0)-q(X_0)$ and $p(X_n)-q(X_n)$ belong to $[-1,1]$, we get that Equation (2) holds.

2. On the process $N = (N_n, n \in \mathbb{N})$.

(a) Since $\Pi_{0,0}(p) = 1$, we get $h(1,p) = -1$ and $h(-1,p) = p(0)/q(0)$. For $k \geq 1$, as $\Pi_{0,k}(p) > 0$, we get that $h(k+1,p) < h(k,p)$. Similarly, we have $h(k+1,p) < h(k,p)$ for $k \leq -1$. Since $h(0,p) = 0$, we get that $k \mapsto h(k,p)$ is strictly decreasing on $\mathbb{Z}$.

(b) Since $|X_n| \leq n$, we get that $|N_n| \leq h(-n,p) - h(n,p)$ and thus N_n is integrable. It is clearly $\mathcal{F}_n$-measurable, so that N is adapted. We have:

$$\mathbb{E}[N_{n+1}|\,\mathcal{F}_n] = \mathbb{E}[h(X_{n+1},p)|\,\mathcal{F}_n] = J(X_n),$$

with $J(k) = p(k)h(k+1,p) + q(k)h(k-1,p)$. To conclude that N is a martingale, it is enough to prove that $J(k) = h(k)$ for all $k \in \mathbb{Z}$. For $k = 0$, we get:

$$J(0) = -p(0) + q(0)\frac{p(0)}{q(0)} = 0 = h(0).$$

For $k \geq 1$, we get:

$$\begin{aligned} J(k) &= p(k)\,(h(k) - \Pi_{0,k}(p)) + q(k)\,(h(k) + \Pi_{0,k-1}(p)) \\ &= h(k) - p(k)\frac{q(k)}{p(k)}\Pi_{0,k-1}(p) + q(k)\Pi_{0,k-1}(p) = h(k). \end{aligned}$$

For $k \leq -1$, we get:

$$\begin{aligned} J(k) &= p(k)\left(h(k) - \Pi_{k,0}(p)^{-1}\right) + q(k)\left(h(k) + \Pi_{k-1,0}(p)^{-1}\right) \\ &= h(k) - p(k)\Pi_{k,0}(p)^{-1} + q(k)\frac{p(k)}{q(k)}\Pi_{k,0}(p)^{-1} = h(k). \end{aligned}$$

(c) Since h is decreasing, we get that $N + c(p)$ is a nonnegative martingale, and thus a.s. it converges to a limit, say Y, which is integrable. This gives that $h(X_n,p)$ converges a.s. to $Y - c(p)$ which is integrable. Since h is strictly decreasing and the process X is $\mathbb{Z}$-valued, this implies that a.s. X converges to $+\infty$ or a.s. X converges to $-\infty$. In the latter case, we would get $Y = h(-\infty,p) + c(p) = +\infty$ which is not integrable. So, we deduce that a.s. X converges to $+\infty$ (and $Y = 0$.)

3. Convergence of X.

(a) We have for $\ell \geq 1$:

$$\frac{1}{\ell}\log(\Pi_{0,\ell}(p)) = \frac{1}{\ell}\sum_{r=1}^{\ell}\log\left(\frac{q(r)}{p(r)}\right).$$

Since the random variables $(q(r)/p(r), r \in \mathbb{Z})$ are independent and distributed as R, we deduce from the strong law of large number that $\mathbb{P}$-a.s. $\lim_{\ell\to\infty} \ell^{-1}\log(\Pi_{0,\ell}(p)) = \mathbb{E}[\log(R)] < 0$. The other result is proved similarly.

(b) We deduce that as ℓ goes to infinity, $\Pi_{0,\ell}(p) = \mathrm{e}^{\ell(\mathbb{E}[\log(R)]+o(1))}$ and thus the series $\sum_{\ell=0}^{\infty} \Pi_{0,\ell}(p)$ converges to a finite random variable, say $c(p)$, as $\mathbb{E}[\log(R)]$ is negative. This gives that the first part of Equation (4) holds.
As ℓ goes to $-\infty$, $1/\Pi_{\ell,0}(p) = \mathrm{e}^{-\ell(\mathbb{E}[\log(R)]+o(1))}$ and thus the series $\sum_{\ell=-\infty}^{-1} 1/\Pi_{\ell,0}(p)$ converges to $+\infty$ as $\mathbb{E}[\log(R)]$ is negative. This gives that the second part of Equation (4) holds.

(c) For $\mathbb{P}$-a.s. all p, we get that Equation (4) holds, and we deduce from Question 2c that $\mathbf{P}_p$-a.s. $\lim_{n\to\infty} X_n = +\infty$. Thus, we get (by integrating with respect to the distribution of p) that $\mathbb{P}$-a.s. $\lim_{n\to\infty} X_n = +\infty$.

In what follows, for simplicity, we shall write for $k \in \mathbb{Z}$:

$$\rho(k) = \frac{q(k)}{p(k)}\cdot$$

4. Some independent results.

(a) We have by Jensen inequality (as the exponential function is convex):

$$\mathbb{E}[R] = \mathbb{E}\left[\mathrm{e}^{\log(R)}\right] \leq \mathrm{e}^{\mathbb{E}[\log(R)]} < 1.$$

This also gives that $v > 0$. By Cauchy-Schwarz inequality, we get that $1 = \mathbb{E}[1] \leq \mathbb{E}[p(0)]\mathbb{E}[1/p(0)]$. This implies that:

$$v = \frac{1-\mathbb{E}[R]}{1+\mathbb{E}[R]} = \frac{2}{\mathbb{E}[1/p(0)]} - 1 \leq 2\mathbb{E}[p(0)] - 1 = \mathbb{E}[p(0)] - \mathbb{E}[q(0)].$$

Now, the Cauchy-Schwarz inequality is an equality if and only if $p(0)$ is a.s. equal to a constant time $1/p(0)$, that is if and only if $p(0)$ is a.s. equal to a constant. Otherwise, the inequality is strict, which implies then that $v < \mathbb{E}[p(0)] - \mathbb{E}[q(0)]$.

(b) The function g is indeed positive on E as $c(p)$ and v are positive. We have:

$$\mathbb{E}[g(p)] = v\sum_{\ell=0}^{\infty} \mathbb{E}\left[\frac{1}{p(0)}\,\Pi_{0,\ell}(p)\right] = v\mathbb{E}\left[\frac{1}{p(0)}\right]\sum_{\ell=0}^{\infty}\prod_{r=1}^{\ell}\mathbb{E}\left[\frac{q(r)}{p(r)}\right]$$
$$= v(\mathbb{E}[R]+1)\sum_{\ell=0}^{\infty}\mathbb{E}[R]^{\ell} = 1,$$

where we used the definition of $c(p)$ for the first equality, the independence of $(p(r), r \in \mathbb{Z})$ for the second, and that they have the same law for the third.

(c) Since $p = (p(k), k \in \mathbb{Z})$ is a sequence of independent random variables distributed as $p(0)$, we deduce that $S_1(p) = (p(k+1), k \in \mathbb{Z})$ is also a sequence of independent random variables distributed as $p(0)$, that is as $p(1) = S_1(p)(0)$. Thus p and $S_1(p)$ have the same distribution.

(d) From the previous question, we get that $\mathbb{E}[c(p)\, f \circ S_1(p)] = \mathbb{E}[c(S_{-1}(p))\, f(p)]$. Then, to get the first equality of (5), use that:

$$c(S_{-1}(p)) = \sum_{\ell=0}^{\infty}\prod_{r=1}^{\ell}\rho(r-1) = \sum_{\ell=0}^{\infty}\prod_{r=0}^{\ell-1}\rho(r) = 1 + \rho(0)c(\rho) = 1 + Rc(p).$$

From the previous question, we get $\mathbb{E}\left[R\, c(p)\, f \circ S_{-1}(p)\right] = \mathbb{E}\left[\rho(1)\, c(S_1(p))\, f(p)\right]$. Then, to get the second equality of (5), use that:

$$\rho(1)\, c(S_1(p)) = \rho(1) \sum_{\ell=0}^{\infty} \prod_{r=1}^{\ell} \rho(r+1) = \rho(1) \sum_{\ell=1}^{\infty} \prod_{r=2}^{\ell} \rho(r) = c(p) - 1.$$

5. On the process Z.

   (a) We have:

$$\begin{aligned} \mathbb{E}\left[f(Z_{n+1}) \middle|\, \mathcal{G}_n\right] &= \mathbb{E}\left[f(S_{X_{n+1}}(p)) \middle|\, \mathcal{G}_n\right] \\ &= \mathbf{E}_p\left[f(S_{X_{n+1}}(p)) \middle| \mathcal{F}_n\right] \\ &= p(X_n) f(S_{X_n+1}(p)) + q(X_n) f(S_{X_n-1}(p)) \\ &= Z_n(0) f(S_1(S_{X_n}(p))) + (1 - Z_n(0)) f(S_{-1}(S_{X_n}(p))) \\ &= Z_n(0) f \circ S_1(Z_n) + (1 - Z_n(0)) f \circ S_{-1}(Z_n), \end{aligned}$$

   where we used the Markov property of X under $\mathbf{E}_p$ for the third equality and that $Z_n(0) = p(X_n) = 1 - q(X_n)$ and $S_{i+j} = S_i \circ S_j$ for the fourth.

   (b) Thanks to the Markov property of Z and since $1 + R = 1/p(0)$, we have:

$$\begin{aligned} \tilde{\mathbb{E}}[f(Z_1)] = \mathbb{E}[g(Z_0) f(Z_1)] &= \mathbb{E}[g(Z_0) \mathbb{E}[f(Z_1) \,|\, \mathcal{G}_0]] \\ &= \mathbb{E}\left[g(p)\Big(p(0)\, f \circ S_1(p) + q(0)\, f \circ S_{-1}(p)\Big)\right] \\ &= v\mathbb{E}\left[c(p)\, f \circ S_1(p) + Rc(p)\, f \circ S_{-1}(p)\right] \\ &= v\mathbb{E}\left[\Big(1 + Rc(p) + c(p) - 1\Big) f(p)\right] \\ &= \mathbb{E}[g(p) f(p)], \end{aligned}$$

   where we used (5) for the fourth equality. Then use that $\mathbb{E}[g(p) f(p)] = \tilde{\mathbb{E}}[f(Z_0)]$ to conclude.

   For $A \in \mathcal{G}_n$ and W a non-negative random variable, we have:

$$\tilde{\mathbb{E}}[W \mathbf{1}_A] = \mathbb{E}[W g(Z_0) \mathbf{1}_A] = \mathbb{E}[g(Z_0) \mathbf{1}_A \mathbb{E}[W \,|\, \mathcal{G}_n]] = \tilde{\mathbb{E}} \mathbf{1}_A \mathbb{E}[W \,|\, \mathcal{G}_n]].$$

   By the characterization of the conditional expectation, we get that a.s. $\tilde{\mathbb{E}}[W \,|\, \mathcal{G}_n] = \mathbb{E}[W \,|\, \mathcal{G}_n]$. Thus, Equation (6) holds indeed with $\mathbb{E}$ replaced by $\tilde{\mathbb{E}}$.

   (c) Notice the function $\eta \mapsto \eta(0)$ defined on E is measurable, by definition of the product σ-field. By the ergodic theorem, the limit of $n^{-1} \sum_{k=1}^{n} Z_k(0)$ a.s. exists and is given by $\tilde{\mathbb{E}}[p(0)] = \mathbb{E}[g(p)p(0)]$. By definition of $g(p)$, we have that $\mathbb{E}[g(p)p(0)] = v\mathbb{E}[c(p)]$. Arguing as for Question 4b, we get:

$$\mathbb{E}[c(p)] = \sum_{\ell=0}^{\infty} \mathbb{E}[\Pi_{0,\ell}(p)] = \sum_{\ell=0}^{\infty} \mathbb{E}[R]^{\ell} = \frac{1}{1 - \mathbb{E}[R]} \cdot$$

   This gives $\mathbb{E}[g(p)p(0)] = 1/(1 + \mathbb{E}[R])$.

   (d) We deduce from Equation (2) (which holds $\mathbf{P}_p$-a.s. for all p, and thus $\mathbb{P}$-a.s. also) that $\mathbb{P}$-a.s.:

$$\lim_{n\to\infty} \frac{X_n}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (Z_k(0) - (1 - Z_k(0))) = -1 + \frac{2}{1 + \mathbb{E}[R]} = v.$$