Stochastic Process (ENPC)
Monday, 23rd of January 2023 (2h30)

Vocabulary (english/*français*): positive = *strictement positif*; stationary = *stationnaire*.

Questions 1, 2, 3, 4 and 6 are largely independent. Question 5 depends heavily on Question 4.

Problem (Exchangeability)**.** In this problem, all the random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider a (infinite) sequence $X = (X_n, n \in \mathbb{N}^*)$ of random variables, and the corresponding future σ-fields $\mathcal{G}_n$ for $n \in \mathbb{N}^*$ and tail-σ-field $\mathcal{G}_\infty$ defined by:

$$\mathcal{G}_n = \sigma(X_k, k \geq n) \quad \text{and} \quad \mathcal{G}_\infty = \bigcap_{n \in \mathbb{N}} \mathcal{G}_n.$$

For example, when the random variables takes values in $\mathbb{R}$, the event $\{\lim_{n\to\infty} \bar{X}_n \text{ exists}\}$ is $\mathcal{G}_\infty$-measurable, where for $n \in \mathbb{N}^*$:

$$\bar{X}_n = \frac{1}{n} \sum_{k=1}^{n} X_k.$$

We say that conditionally on $\mathcal{G}_\infty$, the random variables $(X_n, n \in \mathbb{N}^*)$ are independent and equally distributed if and only if for all $n \in \mathbb{N}^*$ and any measurable bounded real-valued functions $\varphi_1, \ldots, \varphi_n$, we have:

$$\mathbb{E}\left[\prod_{k=1}^{n} \varphi_k(X_k) \,\middle|\, \mathcal{G}_\infty\right] = \prod_{k=1}^{n} \mathbb{E}\left[\varphi_k(X_1) \,|\, \mathcal{G}_\infty\right].$$

We say the sequence X is *exchangeable* if for all $n \in \mathbb{N}^*$, for all (deterministic) permutation $\pi \in \mathcal{S}_n$ on $\{1, \ldots, n\}$, the random vectors $(X_{\pi(1)}, \ldots, X_{\pi(n)})$ and $(X_1, \ldots, X_n)$ have the same distribution. The aim of this exercise is to prove de Finetti's theorem[1] (1931).

Theorem (de Finetti's theorem)**.** *If the sequence X is exchangeable, then conditionally on $\mathcal{G}_\infty$, the random variables $(X_n, n \in \mathbb{N}^*)$ are independent and equally distributed.*

I Examples

1. Assume that $(X_n, n \in \mathbb{N}^*)$ are independent random variables identically distributed. Set $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ for $n \in \mathbb{N}^*$. Let $A \in \mathcal{G}_\infty$.
   (a) Prove that X is exchangeable.
   (b) Prove that $M = (M_n = \mathbb{E}[\mathbf{1}_A \,|\, \mathcal{F}_n], n \in \mathbb{N}^*)$ is a martingale.
   (c) Prove that M converges a.s. to $\mathbf{1}_A$.
   (d) Prove that $\mathbb{E}[\mathbf{1}_A \mathbf{1}_B] = \mathbb{P}(A)\mathbb{P}(B)$ for all $B \in \mathcal{F}_n$ and $n \in \mathbb{N}^*$.
   (e) Deduce that $\mathbb{P}(A)$ is equal to 0 or 1, which means that the tail σ-field $\mathcal{G}_\infty$ is trivial, and thus that conditionally on $\mathcal{G}_\infty$, the random variables $(X_n, n \in \mathbb{N}^*)$ are independent and equally distributed.
2. Let $X = (X_n, n \in \mathbb{N}^*)$ be a centered Gaussian process with covariance kernel $K = (K(n,k), (n,k) \in \mathbb{N}^2)$ and assume X is exchangeable.

[1] J. F. C. Kingman. Uses of exchangeability. *Ann. Probab.*, 6: 183-197, 1978.

(a) Prove there exists $\sigma \geq 0$ and $\rho \in [-\sigma^2, \sigma^2]$ such that for all $n, k \in \mathbb{N}^*$:

$$K(n,k) = \rho \mathbf{1}_{\{n \neq k\}} + \sigma^2 \mathbf{1}_{\{n=k\}}.$$

(b) Compute $\mathbb{E}[\bar{X}_n^2]$ for $n \in \mathbb{N}^*$ and deduce that $\rho \geq 0$.

Let $(Z_n, n \in \mathbb{N})$ be independent centered reduced Gaussian random variables and $\alpha, \beta \in \mathbb{R}$. Set $X'_n = \alpha Z_0 + \beta Z_n$ for all $n \in \mathbb{N}^*$.

(c) Find α and β such that X is distributed as X'.

(d) Deduce that a.s. $\bar{X}_\infty := \lim_{n\to\infty} \bar{X}_n$ exists and is $\mathcal{G}_\infty$-measurable.

(e) Check that $(X_n - \bar{X}_\infty, n \in \mathbb{N}^*)$ are independent and identically distributed, and deduce from the first part of Question 1e that $\mathcal{G}_\infty$ and $\sigma(\bar{X}_\infty)$ coincide up to negligeable events, so that conditionally on $\mathcal{G}_\infty$, the random variables $(X_n, n \in \mathbb{N}^*)$ are i.i.d..

II Reversed martingales

Let $\mathbb{G}' = (\mathcal{G}'_n, n \in \mathbb{N}^*)$ be a sequence of sub-σ-fields of $\mathcal{F}$ which is non-increasing, that is, $\mathcal{G}'_{n+1} \subset \mathcal{G}'_n$ for all $n \in \mathbb{N}^*$, and set $\mathcal{G}'_\infty = \bigcap_{n\in\mathbb{N}^*} \mathcal{G}'_n$. Let $M = (M_n, n \in \mathbb{N}^*)$ be a sequence of real-valued random variables. We say that M is a reversed martingale with respect to $\mathbb{G}'$ if M_n is integrable and $\mathcal{G}'_n$ measurable, and a.s. $\mathbb{E}[M_n \,|\, \mathcal{G}'_{n+1}] = M_{n+1}$ for all $n \in \mathbb{N}^*$. We admit[2] the following result on reversed martingales:

Theorem (Convergence for reversed martingales)**.** *If M a reversed martingale with respect to $\mathbb{G}'$, then it converges a.s. and in L^1 to $M_\infty := \mathbb{E}[M_1|\mathcal{G}'_\infty]$.*

3. We give an application of the theorem on the convergence for reversed martingales. Assume that X is a sequence of identically distributed integrable independent random variables. Let $\mathcal{G}'_n = \sigma(\bar{X}_n) \vee \mathcal{G}_{n+1} = \sigma(\bar{X}_n, X_{n+1}, X_{n+2}, \ldots)$.

   (a) Check that $\mathbb{G}' = (\mathcal{G}'_n, n \in \mathbb{N}^*)$ is a non-increasing sequence of σ-fields.

   (b) Prove that $(\bar{X}_n, n \in \mathbb{N}^*)$ is a reversed martingale with respect to $\mathbb{G}'$. (Hint. Check that $\mathbb{E}[X_k \,|\, \mathcal{G}'_n] = \mathbb{E}[X_n \,|\, \mathcal{G}'_n]$ for all $k \in \{1, \ldots, n\}$.)

   (c) Deduce that a.s. $\bar{X}_\infty := \lim_{n\to\infty} \bar{X}_n$ exists.

   (d) Prove the strong law of large numbers using the first part of Question I.1e.

4. We consider the following technical properties. Let $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ be two σ-fields and $V \in L^2$.

   (a) Assume that $\mathbb{E}[V \,|\, \mathcal{H}]$ and $\mathbb{E}[V \,|\, \mathcal{G}]$ have the same distribution. After computing $\mathbb{E}\left[(\mathbb{E}[V \,|\, \mathcal{H}] - \mathbb{E}[V \,|\, \mathcal{G}])^2\right]$, deduce that a.s. $\mathbb{E}[V \,|\, \mathcal{H}] = \mathbb{E}[V \,|\, \mathcal{G}]$.

   (b) Let (V, Y) and (V', Y') be random variables with the same distribution. We recall that a.s. $\mathbb{E}[V \,|\, Y] = \varphi(Y)$ for some real-valued measurable function φ. Prove that, for all measurable sets A, $\mathbb{E}[V' \mathbf{1}_{\{Y' \in A\}}] = \mathbb{E}[\varphi(Y')\mathbf{1}_{\{Y' \in A\}}]$, and deduce that $\mathbb{E}[V \,|\, Y]$ and $\mathbb{E}[V' \,|\, Y']$ have the same distribution.

   (c) Let Y be a random variable. We say that Y is independent of $\mathcal{H}$ conditionally on $\mathcal{G}_\infty$ if for all bounded real-valued measurable functions φ and all $B \in \mathcal{H}$, a.s. we have:

$$\mathbb{E}\left[\varphi(Y)\mathbf{1}_B \,|\, \mathcal{G}_\infty\right] = \mathbb{E}\left[\varphi(Y) \,|\, \mathcal{G}_\infty\right] \mathbb{E}\left[\mathbf{1}_B \,|\, \mathcal{G}_\infty\right].$$

[2] The proof of convergence for reversed martingales is similar to the proof of convergence for martingales.

Prove that Y is independent of $\mathcal{H}$ conditionally on $\mathcal{G}_\infty$ if for all bounded real-valued measurable function φ, a.s. we have:

$$\mathbb{E}\left[\varphi(Y) \middle| \mathcal{H} \vee \mathcal{G}_\infty\right] = \mathbb{E}\left[\varphi(Y) \,\middle|\, \mathcal{G}_\infty\right].$$

5. We shall prove de Finetti's theorem[3]. Assume that X is exchangeable. Let φ be a bounded real-valued measurable function and set for $n \in \mathbb{N}^* \cup \{\infty\}$:

$$M_n = \mathbb{E}[\varphi(X_1) \,|\, \mathcal{G}_n].$$

   (a) Using reversed martingales, prove that $(M_n, n \in \mathbb{N}^*)$ converges a.s. to M_∞.
   (b) Using Question 4b, prove that the random variables $(M_n, 2 \leq n < \infty)$ have the same distribution and then that $\mathbb{E}[\varphi(X_1) \,|\, \mathcal{G}_2]$ and $\mathbb{E}[\varphi(X_1) \,|\, \mathcal{G}_\infty]$ have the same distribution.
   (c) Using Questions 4a and 4c, prove that X_1 is independent of $\mathcal{G}_2$ conditionally on $\mathcal{G}_\infty$.
   (d) Prove de Finetti's theorem.

III Pólya urn [4]

Consider an urn at time $n = 0$ with $r \in \mathbb{N}^*$ red balls and $b \in \mathbb{N}^*$ blue balls. At time $n \in \mathbb{N}^*$, draw a ball uniformly at random from the urn and then return it to the urn, and add an additional ball of the same color. Set $X_n = 1$ if the new ball added at time $n \in \mathbb{N}^*$ is red and 0 otherwise. So at time n there are $r + b + n$ balls in the urn and $R_n = r + \sum_{k=1}^n X_k$ among them are red.

6. We shall determine limit of the fraction of red balls when the Pólya urn is filled.

   (a) Let $n \in \mathbb{N}^*$. Prove that:

$$\mathbb{P}\left(X_{n+1} = 1 \,|\, X_1, \ldots, X_n\right) = \mathbb{P}\left(X_{n+1} = 1 \,|\, R_1, \ldots, R_n\right) = \frac{R_n}{r + b + n}.$$

   (b) For $n \in \mathbb{N}^*$ and $(x_1, \ldots, x_n) \in \{0, 1\}^n$, prove that:

$$\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n) = \frac{\prod_{k=1}^{s_n}(r + k - 1) \prod_{k=1}^{n-s_n}(b + k - 1)}{\prod_{k=1}^{n}(r + b + k - 1)},$$

   with $s_n = \sum_{k=1}^n x_k$ and the convention $\prod_{k=1}^0 = 1$.
   (c) Deduce that X is exchangeable.
   (d) Prove that conditionally on $\mathcal{G}_\infty$, the random variables $(X_n, n \in \mathbb{N}^*)$ are independent Bernoulli random variables with $\mathcal{G}_\infty$-measurable random parameter $U \in [0, 1]$.

   We recall that a $[0, 1]$-valued random variable V has the $\beta(r, b)$ distribution if and only if for all $n \in \mathbb{N}^*$:

$$\mathbb{E}[V^n] = \frac{(r + n - 1)!}{(r - 1)!} \, \frac{(r + b - 1)!}{(r + b + n - 1)!}.$$

   (e) Prove that $\mathbb{P}(X_1 = 1, \ldots, X_n = 1) = \mathbb{E}\left[U^n\right]$, and identify the distribution of U.
   (f) Give the law of R_n conditionally on U, and the a.s. limit of the fraction of red balls $\lim_{n \to \infty} R_n/(r + b + n)$.

△

[3] D. Aldous. Exchangeability and related topics. *École d'été de probabilités de Saint-Flour, XIII-1983.* Lecture Notes in Math., Springer, 1117: 1-198, 1985.

[4] N. Johnson and S. Kotz. *Urn models and their application.* Wiley, 1977.

Correction

Problem

I Examples

1. (a) Clearly $(X_{\pi(1)}, \dots, X_{\pi(n)})$ is a sequence of independent identically distributed random variable, so with the same distribution as $(X_1, \dots, X_n)$.

   (b) This is a closed martingale.

   (c) As M is a closed martingale, it converges a.s. to $M_\infty = \mathbb{E}[\mathbf{1}_A \,|\, \mathcal{F}_\infty] = \mathbf{1}_A$, where for the last equality, we used that A belongs to $\mathcal{G}_\infty \subset \mathcal{F}_\infty$.

   (d) Since $A \in \mathcal{G}_{n+1}$ and $B \in \mathcal{F}_n$, and $\mathcal{G}_{n+1}$ is independent of $\mathcal{F}_n$ (as $(X_n + k,\, k \in \mathbb{N}^*)$ is independent of $(X_1, \dots, X_n)$), we have:

   $$\mathbb{E}[\mathbf{1}_A \mathbf{1}_B] = \mathbb{E}[\mathbf{1}_A]\, \mathbb{E}[\mathbf{1}_B] = \mathbb{P}(A)\mathbb{P}(B).$$

   (e) We deduce that:

   $$\mathbb{E}\left[M_n \mathbf{1}_B\right] = \mathbb{E}\left[\mathbb{E}[M_\infty \,|\, \mathcal{F}_n] \mathbf{1}_B\right] = \mathbb{E}\left[M_\infty \mathbf{1}_B\right] = \mathbb{P}(A)\mathbb{P}(B).$$

   This implies that a.s. $M_n = \mathbb{P}(A)$ and thus a.s. $\mathbf{1}_A = \mathbb{P}(A)$, that is, $\mathbb{P}(A)$ is equal to 0 or 1.

2. (a) By exchangeability, we deduce that for $n \neq k$ the random variables X_n and X_1 as well as (X_n, X_k) and (X_1, X_2) have the same distribution. This implies that $K(n, n)$ does not depend on n, and $K(n, k)$ does not depend on $n \neq k$. To conclude, deduce by Cauchy-Schwarz that:

   $$\rho^2 = \mathbb{E}[X_1 X_2]^2 \leq \mathbb{E}[X_1^2]\mathbb{E}[X_2^2] = \sigma^4.$$

   (b) We have:

   $$\mathbb{E}[\bar{X}_n^2] = \frac{n-1}{n}\rho + \frac{1}{n}\sigma^2.$$

   Since $\mathbb{E}[\bar{X}_n^2] \geq 0$, we deduce that $\rho \geq -\sigma^2/(n-1)$ for all $n \geq 2$. Letting n goes to infinity, we get that $\rho \geq 0$.

   (c) The process X' is Gaussian, centered with covariance process K' with, for $n \in \mathbb{N}^*$:

   $$K'(n, n) = \mathbb{E}[(\alpha Z_0 + \beta Z_n)^2] = \alpha^2 + \beta^2,$$

   and for $n \neq k \in \mathbb{N}^*$:

   $$K'(n, k) = \mathbb{E}[(\alpha Z_0 + \beta Z_n)(\alpha Z_0 + \beta Z_k)] = \alpha^2.$$

   We deduce that for $\alpha = \sigma$ and $\beta = \sqrt{\sigma^2 - \rho}$ the covariance process K' and K are equal. Since centered Gaussian process are characterized by they covariance process, we deduce that X' and X have the same distribution.

   (d) By the strong law of large number, we get that a.s. $\lim_{n\to\infty} \bar{X}'_n = X'_0$. Since X is distributed as X', we deduce that $\lim_{n\to\infty} \bar{X}_n$ a.s. exists. Let us denote it by X_∞. Let $n_0 \in \mathbb{N}^*$. We also have that a.s. $\bar{X}_\infty = \lim_{n\to\infty}(n + n_0)^{-1} \sum_{k=n_0+1}^{n} X_k$, so that $\bar{X}_\infty$ is $\mathcal{G}_{n_0}$ measurable. Since n_0 is arbitrary, we deduce that $\bar{X}_\infty$ is $\mathcal{G}_\infty$-measurable.

(e) Set $Y = (X_n - \bar{X}_\infty, n \in \mathbb{N}^*)$ and $Y' = (X'_n - \bar{X}'_\infty = Z_n, n \in \mathbb{N}^*)$. Since X and X' have the same distribution, we deduce that $(Y, \bar{X}_\infty)$ and $(Y', \bar{X}'_\infty = Z_0)$ have the same distribution. This implies that $(X_n - \bar{X}_\infty, n \in \mathbb{N}^*)$ are independent random centered Gaussian variables with variance σ^2, which are independent from $\bar{X}_\infty$. Since $\bar{X}_\infty$ is $\mathcal{G}_\infty$ measurable, we deduce that $\mathcal{G}_n = \sigma(\bar{X}_\infty) \vee \mathcal{H}_n$, where $\mathcal{H}_n = \sigma(X_k - \bar{X}_\infty, k \geq n)$. Using the first part of Question 1e for the last equality, we get that:

$$\mathcal{G}_\infty = \bigcap_{n\in\mathbb{N}} \mathcal{G}_n = \sigma(\bar{X}_\infty) \vee \bigcap_{n\in\mathbb{N}} \mathcal{H}_n = \sigma(\bar{X}_\infty) \vee \mathcal{H}_\infty,$$

where the sets in $\mathcal{H}_\infty$ are of probability 0 or 1. This proves the result.

II Reversed martingales

3. (a) As $(n+1)\bar{X}_{n+1} = n\bar{X}_n + X_{n+1}$, that is, $\bar{X}_{n+1}$ is $\mathcal{G}'_n$-measurable, we deduce that $\mathcal{G}'_{n+1} \subset \mathcal{G}'_n$.

   (b) The random variable $\bar{X}_n$ is integrable as the X_k's are integrable; it is also clearly $\mathcal{G}'_n$-measurable. We have:

$$\mathbb{E}[\bar{X}_n \,|\, \mathcal{G}'_{n+1}] = \frac{n+1}{n}\bar{X}_{n+1} - \frac{1}{n}\,\mathbb{E}[X_{n+1} \,|\, \mathcal{G}'_{n+1}]. \tag{1}$$

   Since X is exchangeable, we get that a.s. for all $k \in \{1, \ldots, n+1\}$:

$$\mathbb{E}[X_k \,|\, \mathcal{G}'_{n+1}] = \mathbb{E}[X_{n+1} \,|\, \mathcal{G}'_{n+1}].$$

   Summing over $k \in \{1, \ldots, n+1\}$ gives that:

$$(n+1)\bar{X}_{n+1} = \sum_{k=1}^{n+1} \mathbb{E}[X_k \,|\, \mathcal{G}'_{n+1}] = (n+1)\mathbb{E}[X_{n+1} \,|\, \mathcal{G}'_{n+1}].$$

   Plugging this in (1), we deduce that:

$$\mathbb{E}[\bar{X}_n \,|\, \mathcal{G}'_{n+1}] = \frac{n+1}{n}\bar{X}_{n+1} - \frac{1}{n}\,\bar{X}_{n+1} = \bar{X}_n.$$

   In conclusion, we get that $(\bar{X}_n, n \in \mathbb{N}^*)$ is a reversed martingale with respect to $\mathbb{G}'$.

   (c) This is a direct consequence of the theorem on reverse martingales.

   (d) Since the random variable $\bar{X}_\infty$ is $\mathcal{G}_\infty$-measurable, we deduce from Question I.1e, that it is constant. The theorem on reverse martingales implies also that $\bar{X}_n$ converges also in L^1 to $\bar{X}_\infty$. This gives:

$$\lim_{n\to\infty} \mathbb{E}[\bar{X}_n] = \mathbb{E}[\bar{X}_\infty].$$

   This readily gives that a.s. $\bar{X}_\infty = \mathbb{E}[X_1]$, which proves the strong law of large numbers.

4. (a) If X is $\mathcal{H}$-measurable and Y is $\mathcal{G}$-measurable, and X, Y are square integrable, we get that $\mathbb{E}[XY] = \mathbb{E}\left[\mathbb{E}[XY \,|\mathcal{H}]\right] = \mathbb{E}\left[X\mathbb{E}[Y \,|\mathcal{H}]\right]$. This gives that:

$$\mathbb{E}\left[\mathbb{E}[V \,|\, \mathcal{H}]\, \mathbb{E}[V \,|\, \mathcal{G}]\right] = \mathbb{E}\left[(\mathbb{E}[V \,|\, \mathcal{H}])^2\right] = \mathbb{E}\left[(\mathbb{E}[V \,|\, \mathcal{G}])^2\right],$$

   where for the last equality, we used that $\mathbb{E}[V \,|\, \mathcal{H}]$ and $\mathbb{E}[V \,|\, \mathcal{G}]$ have the same distribution and thus the same expectation of their square. This readily implies that $\mathbb{E}\left[(\mathbb{E}[V \,|\, \mathcal{H}] - \mathbb{E}[V \,|\, \mathcal{G}])^2\right] = 0$ and thus a.s. $\mathbb{E}[V \,|\, \mathcal{H}] = \mathbb{E}[V \,|\, \mathcal{G}]$.

(b) We have:

$$\mathbb{E}[V'\mathbf{1}_{\{Y'\in A\}}] = \mathbb{E}[V\mathbf{1}_{\{Y\in A\}}] = \mathbb{E}\left[\mathbb{E}[V\,|\,Y]\mathbf{1}_{\{Y\in A\}}\right] = \mathbb{E}[\varphi(Y)\mathbf{1}_{\{Y\in A\}}]$$
$$= \mathbb{E}[\varphi(Y')\mathbf{1}_{\{Y'\in A\}}],$$

where for the first and last equalities, we used that (V', Y') and (V, Y) have the same distribution. By the characterization of the conditional expectation, we deduce that a.s $\mathbb{E}[V'\,|\,Y'] = \varphi(Y')$, and thus $\mathbb{E}[V\,|\,Y]$ and $\mathbb{E}[V'\,|\,Y']$ have the same distribution.

(c) Assume that for all bounded real-valued measurable function φ, a.s. we have:

$$\mathbb{E}\left[\varphi(Y)|\,\mathcal{H}\vee\mathcal{G}_\infty\right] = \mathbb{E}\left[\varphi(Y)\,|\,\mathcal{G}_\infty\right].$$

Let $B \in \mathcal{H}$. We have:

$$\begin{aligned}\mathbb{E}\left[\varphi(Y)\mathbf{1}_B\,|\,\mathcal{G}_\infty\right] &= \mathbb{E}\left[\mathbb{E}\left[\varphi(Y)\mathbf{1}_B\,|\,\mathcal{H}\vee\mathcal{G}_\infty\right]\,|\,\mathcal{G}_\infty\right]\\ &= \mathbb{E}\left[\mathbf{1}_B\mathbb{E}\left[\varphi(Y)\,|\,\mathcal{H}\vee\mathcal{G}_\infty\right]\,|\,\mathcal{G}_\infty\right]\\ &= \mathbb{E}\left[\mathbf{1}_B\mathbb{E}\left[\varphi(Y)\,|\,\mathcal{G}_\infty\right]\,|\,\mathcal{G}_\infty\right]\\ &= \mathbb{E}\left[\mathbf{1}_B\,|\,\mathcal{G}_\infty\right]\mathbb{E}\left[\varphi(Y)\,|\,\mathcal{G}_\infty\right].\end{aligned}$$

This gives that Y is independent of $\mathcal{H}$ conditionally on $\mathcal{G}_\infty$.

5. (a) Clearly the process $(M_n, n \in \mathbb{N}^*)$ is a reverse martingale with respect to $\mathbb{G}$, and thus, as $M_1 = \varphi(X_1)$, it converges a.s. and in L^1 towards:

$$M_\infty = \mathbb{E}[M_1\,|\,\mathcal{G}_\infty] = \mathbb{E}[\varphi(X_1)\,|\,\mathcal{G}_\infty].$$

(b) Let $n \geq 2$ and $n' \geq 2$. Set $V' = V = \varphi(X_1)$, $Y = (X_{n+k}, k \geq 0)$ and $Y' = (X_{n'+k}, k \geq 0)$. Since X is exchangeable, we deduce that (V, Y) and (V', Y') have the same distribution. We deduce from Question 4b, that the random variables M_n and $M_{n'}$ have the same distribution. Hence, the random variables $(M_n, 2 \leq n < \infty)$ have the same distribution. Then use the previous question to deduce that they also have the same distribution as M_∞ (because the distribution of M_∞ is the limit of the distribution of the M_n's).

(c) From Question 4a, we deduce that a.s.:

$$\mathbb{E}[\varphi(X_1)\,|\,\mathcal{G}_2] = \mathbb{E}[\varphi(X_1)\,|\,\mathcal{G}_\infty]. \tag{2}$$

From Question 4c with $Y = X_1$ and $\mathcal{H} = \mathcal{G}_2$, we get that X_1 is independent of $\mathcal{G}_2$ conditionally on $\mathcal{G}_\infty$.

(d) By iteration, using the previous question we get that X_n is independent of $\mathcal{G}_n$ conditionally on $\mathcal{G}_\infty$, for all $n \in \mathbb{N}^*$. This gives that $(X_1, \dots, X_n)$ are independent conditionally on $\mathcal{G}_\infty$. Then use the exchangeability to get that a.s. $\mathbb{E}[\varphi(X_1)\,|\,\mathcal{G}_{n+1}] = \mathbb{E}[\varphi(X_n)\,|\,\mathcal{G}_{n+1}]$ and then use (2) to conclude that a.s. $\mathbb{E}[\varphi(X_1)\,|\,\mathcal{G}_\infty] = \mathbb{E}[\varphi(X_n)\,|\,\mathcal{G}_\infty]$. This gives de Finetti's theorem.

III Pólya urn

6. (a) Since $X_n = R_n - R_{n-1}$ and $R_n = r + \sum_{k=1}^n X_k$ for $n \in \mathbb{N}^*$, we deduce that $\sigma(X_1, \dots, X_n) = \sigma(R_1, \dots, R_n)$. This gives the first equality. From the description of the process, we get that:

$$\mathbb{P}\left(X_{n+1} = 1\,|\,R_1, \dots, R_n\right) = \mathbb{P}\left(X_{n+1} = 1\,|\,R_n\right) = \frac{R_n}{r+b+n}.$$

(b) An elementary recurrence gives, with $s_n = \sum_{k=1}^n x_k$ and the convention $\prod_{k=1}^0 = 1$:

$$\mathbb{P}(X_1 = x_1, \dots, X_n = x_n) = \frac{\prod_{k=1}^{s_n}(r+k-1) \prod_{k=1}^{n-s_n}(b+k-1)}{\prod_{k=1}^{n}(r+b+k-1)}. \tag{3}$$

(c) From the previous formula, we get that the distribution of $(X_1, \dots, X_n)$ depends only on $\sum_{k=1}^n X_k$. This implies that $(X_1, \dots, X_n)$ and $(X_{\pi(1)}, \dots, X_{\pi(n)})$ have the same distribution. Thus X is exchangeable.

(d) According to de Finetti's theorem, the random variables $(X_n, n \in \mathbb{N}^*)$ are, conditionally on $\mathcal{G}_\infty$, independent with the same distribution. Since $X_n \in \{0, 1\}$, we deduce that conditionally on $\mathcal{G}_\infty$, X_n is Bernoulli, with a random parameter U taking values in $[0, 1]$.

(e) We deduce from (3) that:

$$\begin{aligned}
\mathbb{E}[U^n] = \mathbb{E}\left[\mathbb{E}[\mathbf{1}_{\{X_1=1,\dots,X_n=1\}} \,|\, U]\right] &= \mathbb{P}(X_1 = 1, \dots, X_n = 1) \\
&= \frac{\prod_{k=1}^{n}(r+k-1)}{\prod_{k=1}^{n}(r+b+k-1)} \\
&= \frac{(r+n-1)!}{(r-1)!} \, \frac{(r+b-1)!}{(r+b+n-1)!}.
\end{aligned}$$

We deduce that U has the $\beta(r, b)$ distribution.

(f) The random variable R_n is distributed as $r + S_n$, where conditionally on U, S_n is binomial with parameter (n, U). From the law of large numbers, we deduce that a.s. $\lim_{n\to\infty} S_n/n = U$. This implies that a.s. $\lim_{n\to\infty} R_n/n = U$.