Stochastic Process (ENPC)
Monday, 22rd of January 2024 (2h30)

Vocabulary (english/*français*): positive = *strictement positif*; cumulative distribution functio =*fonction de répartition*.

We write $x_+ = \max(x, 0)$ the nonnegative part of $x \in \mathbb{R}$ and use the convention $\inf \emptyset = +\infty$.

Exercise 1 (Lindley processes)**.** We shall study the stability of queueing model where services take place at discrete times $n \in \mathbb{N}$. We denote by X_n the queue length at time n, A_{n+1} the number of new customers arriving between time n and $n+1$, and D_{n+1} the number of maximal customers that can be served between time n and $n+1$; so that, with $V_{n+1} = A_{n+1} - D_{n+1}$, we have the Lindley equation[1] for the process $\mathbf{X} = (X_n,\, n \in \mathbb{N})$:

$$X_{n+1} = (X_n + V_{n+1})_+ \quad \text{for } n \in \mathbb{N}. \tag{1}$$

We assume that X_0 is a random variable taking values in $\mathbb{N}$, that the random variables $(V_n, n \in \mathbb{N}^*)$ are independent with the same distribution as a random variable V taking values in $\mathbb{Z}$, and are independent of X_0.

1. Check that $\mathbf{X}$ is a Markov process on $\mathbb{N}$ with respect to a filtration which has to be precised.

We also consider the random walk $\mathbf{S} = (S_n, n \in \mathbb{N})$ on $\mathbb{Z}$ defined by $S_0 = 0$ and:

$$S_{n+1} = S_n + V_{n+1} \quad \text{for } n \in \mathbb{N}.$$

2. We set $M_n = \max_{n \geq k \geq 0} S_k$ for $k \in \mathbb{N}$.
   (a) Check that $X_n - X_{n-k} \geq S_n - S_{n-k}$ for $n \geq k \geq 0$.
   (b) Deduce that for all $n \in \mathbb{N}$:
   $$X_n = \max\left(X_0 + S_n, S_n - S_1, \ldots, S_n - S_{n-1}, 0\right).$$
   (c) Let $n \in \mathbb{N}$. Prove that X_n and $\max(X_0 + S_n, M_{n-1})$ have the same distribution.

We assume that V is integrable and set $\mu = \mathbb{E}[V]$.

3. Assume that $\mu > 0$. Prove that a.s. $\lim_{n\to\infty} X_n = +\infty$, and then that $\mathbf{X}$ is transient.
4. Assume that $\mu < 0$ and that $\mathbf{X}$ is irreducible. We set $M = \sup_{k\in\mathbb{N}} S_k$.
   (a) Prove that a.s. M is finite.
   (b) Using Question 2c, prove that $\mathbf{X}$ converges in distribution to M.
   (c) Deduce that $\mathbf{X}$ is positive recurrent and the invariant distribution is the law of M.

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Exercise 2 (Lyapounov functions)**.** Let $\mathbf{X} = (X_n, n \in \mathbb{N})$ be an irreducible homogeneous Markov chain on a discrete countable space E with transition matrix P, and $(\mathcal{F}_n, n \in \mathbb{N})$ its natural filtration. We recall that $\mathbb{P}_x$-a.s. we have $X_0 = x$. Let $\mathcal{B}_+$ denote the set of finite nonnegative functions defined on E. For $h \in \mathcal{B}_+$, we write $\lim_{x\to\infty} h(x) = +\infty$ if for all $n \in \mathbb{N}$, the set $\{x \in E :\ h(x) \leq n\}$ is finite.

We say that a function $h \in \mathcal{B}_+$ is *harmonic* (resp. *super-harmonic*) if $Ph = h$ (resp. $Ph \leq h$).

[1] S. Asmussen. *Applied Probability and Queues.* Springer-Verlag, 2003.

1. Assume $h \in \mathcal{B}_+$ is harmonic (resp. super-harmonic). Prove that $(h(X_n), n \in \mathbb{N})$ is a martingale (resp. super-martingale) provided that $\mathbb{E}[h(X_0)] < \infty$.

2. Assume that $\mathbf{X}$ is recurrent. Recall the cardinal of $\{n \in \mathbb{N} : X_n = x\}$ for $x \in E$. Prove that any function in $\mathcal{B}_+$ which is super-harmonic is then constant.

3. Let $x_0 \in E$. For $h \in \mathcal{B}_+$, we consider the condition:

$$Ph(x) = h(x) \quad \text{for } x \neq x_0. \tag{2}$$

   (a) Assume that $h \in \mathcal{B}_+$ is bounded non constant and satisfies (2). Using Question 2 prove that $\mathbf{X}$ is transient.

   (b) Assume that $\mathbf{X}$ is transient. Set $\tau_0 = \inf\{n \in \mathbb{N} : X_n = x_0\}$. Prove that the function $h \in \mathcal{B}_+$, defined by $h(x) = \mathbb{P}_x(\tau_0 = \infty)$ for $x \in E$, satisfies (2) and is non constant.

Let $E_0 \subset E$ be finite and non-empty and set $\tau = \inf\{n \in \mathbb{N} : X_n \in E_0\}$. Let $h \in \mathcal{B}_+$ and set $\mathbf{Y} = (Y_n, n \in \mathbb{N})$ with $Y_n = h(X_n)\mathbf{1}_{\{\tau > n\}}$.

4. Assume that $h \in \mathcal{B}_+$ is such that:

$$\lim_{x \to \infty} h(x) = +\infty \quad \text{and} \quad Ph(x) \leq h(x) \quad \text{for } x \notin E_0. \tag{3}$$

   (a) Check that τ is a stopping time.

   (b) Let $x \notin E_0$. Prove that $\mathbf{Y}$ is a super-martingale under $\mathbb{P}_x$.

   (c) Prove by contradiction that $\mathbf{X}$ is recurrent.

5. (Facultatif) Let $\delta > 0$. Assume that $h \in \mathcal{B}_+$ is such that:

$$Ph(x) < +\infty \quad \text{for } x \in E_0 \quad \text{and} \quad Ph(x) \leq h(x) - \delta \quad \text{for } x \notin E_0. \tag{4}$$

   (a) Let $x \notin E_0$. Prove that $\mathbb{E}_x[Y_{n+1}|\mathcal{F}_n] \leq Y_n - \delta \mathbf{1}_{\{\tau > n\}}$ for $n \in \mathbb{N}$.

   (b) Let $x \in E$. Prove that $\mathbb{E}_x[\tau] = \sum_{n \in \mathbb{N}} \mathbb{P}_x(\tau > n)$ and deduce that $\mathbb{E}_x[\tau] \leq \delta^{-1} h(x)$.

   We consider the successive visiting times of E_0 by $T_0 = \tau$ and for $k \in \mathbb{N}$, $T_{k+1} = \inf\{n > T_k : X_n \in E_0\}$. Define the restriction of the Markov chain $\mathbf{X}$ to E_0 by $\mathbf{X}^0 = (X^0_n = X_{T_n}, n \in \mathbb{N})$.

   (c) Prove that $\mathbf{X}^0$ is a Markov chain with respect to a filtration which shall be defined.

   (d) Prove that $\mathbf{X}^0$ is irreducible and positive recurrent.

   (e) Prove that for all $y \in E_0$, $\mathbb{E}_y[T_1] \leq 1 + \mathbb{E}_y[\mathbb{E}_{X_1}[\tau]] \leq 1 + \delta^{-1} Ph(y)$.

   (f) Prove that $\mathbf{X}$ is positive recurrent.

6. Consider the queueing model of Exercise 1 with $E = \mathbb{N}$ and $\mathbf{X}$ given by (1). Assume that $\mathbf{X}$ is irreducible and V is integrable with $\mathbb{E}[V] < 0$.

   (a) Prove there exists $n_0 \geq 2$ such that $\mathbb{E}[V \mathbf{1}_{\{V > -n_0\}}] < 0$.

   (b) Check that the identity function satisfies (4) with $E_0 = \{0, \ldots, n_0\}$ and, using Question 5f, deduce that $\mathbf{X}$ is positive recurrent.

△

Correction

Exercise 1

1. The process $\mathbf{X}$ is a stochastic dynamical system and thus a Markov chain with respect to the filtration $(\mathcal{F}_n, n \in \mathbb{N})$ with $\mathcal{F}_0 = \sigma(X_0)$ and $\mathcal{F}_n = \sigma(X_0, V_1, \ldots, V_n)$ for $n \geq 1$.

2. (a) We have $X_{n+1} \geq X_n + V_{n+1}$ for all $n \in \mathbb{N}$ and thus $X_n \geq X_{n-k} + V_n + \ldots + V_{n-k+1} = X_{n-k} + S_n - S_k$.

   (b) We deduce from the previous question that:

   $$X_n \geq \max\left(X_0 + S_n, S_n - S_1, \ldots, S_n - S_{n-1}, 0\right).$$

   To get the equality, we consider the last zero of the sequence $X_0, \ldots, X_n$: $L = \sup\{k \in \{0, \ldots, n\} : X_k = 0\}$ with the convention that $\sup \emptyset = 0$. Notice that L is not a stopping time. By construction of L, we have that if $L = n$ then $X_n = 0$, if $L = 0$, then $X_n = X_0 + S_n$ and otherwise $X_n = V_{L+1} + \cdots V_n = S_n - S_L$. This gives the equality.

   (c) Notice that $(S_n, S_n - S_1, \ldots, S_n - S_{n-1}, 0)$ is distributed as $(S_n, S_{n-1}, \ldots, S_1, 0)$ as $(V_1, \ldots, V_n)$ is distributed as $(V_n, \ldots, V_1)$. This gives the result.

3. We deduce from the strong law of large number that a.s. $\lim_{n\to\infty} S_n/n = \mu > 0$ and thus that a.s. $\lim_{n\to\infty} S_n = \infty$. Since $X_n \geq X_0 + S_n \geq S_n$, we also deduce that a.s. $\lim_{n\to\infty} X_n = \infty$ and thus $\mathbf{X}$ is transient.

4. (a) Arguing as in the previous question, we deduce that a.s. $\lim_{n\to\infty} S_n = -\infty$ and thus a.s. M is finite.

   (b) From the previous question, we deduce that a.s. $\lim_{n\to\infty} \max(X_0 + S_n, M_n) = M$. This and Question 2c readily implies that $\mathbf{X}$ converges in distribution to M (for details notice that if h is continuous bounded, we have: $\mathbb{E}[h(X_n)] = \mathbb{E}[h(\max(X_0 + S_n, M_n))] \to \mathbb{E}[h(M)]$ as n goes to infinity).

   (c) Since $\mathbf{X}$ converges in distribution, we deduce that $\lim_{k\to\infty} \nu_0 P^k \mathbf{1}_{\{x\}} = \mathbb{P}(M = x)$ for any initial probability distribution ν_0 and any $x \in \mathbb{N}$. Thus we get that $\lim_{n\to\infty} n^{-1} \sum_{k=1}^n \nu_0 P^k \mathbf{1}_{\{x\}} = \mathbb{P}(M = x)$. From the ergodic theorem and dominated convergence we also get that $\lim_{n\to\infty} n^{-1} \sum_{k=1}^n \nu_0 P^k \mathbf{1}_{\{x\}} = \pi(x)$, with $\pi(x) = 1/\mathbb{E}[T_x]$ and T_x the first return time at x. This implies that π is a probability distribution and thus that $\mathbf{X}$ is recurrent positive. Since $\mathbf{X}$ is recurrent positive, we also get that π and thus the probability distribution of M is invariant.

Exercise 2

1. The process $(h(X_n), n \in \mathbb{N})$ is adapted to the natural filtration of $\mathbf{X}$. Since $h(X_{n+1})$ is non-negative, we can compute, using the Markov property for the second equality:

   $$\mathbb{E}[h(X_{n+1}) \,|\, \mathcal{F}_n] = \mathbb{E}[h(X_{n+1}) \,|\, X_n] = Ph(X_n) \leq h(X_n).$$

   Taking the expectation we get $\mathbb{E}[h(X_{n+1})] \leq \mathbb{E}[h(X_n)]$ and by iteration that $\mathbb{E}[h(X_n)] \leq \mathbb{E}[h(X_0)]$. Thus if $\mathbb{E}[h(X_0)] < \infty$, we deduce that $h(X_n)$ is integrable for all $n \in \mathbb{N}$. This gives that $(h(X_n), n \in \mathbb{N})$ is a super-martingale.

2. Recall that $X_0 = x$ under $\mathbb{P}_x$. Since $\mathbb{E}_x[h(X_0)] = h(x) < \infty$, we deduce that $(h(X_n), n \in \mathbb{N})$ is a super-martingale under $\mathbb{P}_x$. Since it is non-negative it converges a.s. to a limit say Z which is integrable. Since $\mathbf{X}$ is recurrent, it visits a.s. $y \in E$ infinitely often. This implies that a.s. $h(y) = Z$. Since this holds for any $y \in E$, we deduce that h is constant.

3. (a) Since h is bounded, we get that $Ph(x_0)$ is finite. If $Ph(x_0) \leq h(x_0)$ we get that h is super-harmonic. By contraposition of Question 2, we deduce that $\mathbf{X}$ is not recurrent, and, as it is irreducible, it has to be transient. If $Ph(x_0) \geq h(x_0)$ then the function $g = -h + \max h$ belongs to $\mathcal{B}_+$, is bounded satisfies (2) as well as $Pg(x_0) \leq g(x_0)$, since $P\mathbf{1} = \mathbf{1}$. Thus the function g is super-harmonic. Similarly, we deduce that $\mathbf{X}$ is transient. In conclusion $\mathbf{X}$ is transient.

   (b) We have $h(x_0) = 0$, while $h(x) > 0$ for all $x \neq x_0$ as $\mathbf{X}$ is irreducible and transient. Furthermore, we have for $x \neq x_0$ that $\tau_0 > 0$ and thus:

$$\mathbb{E}_x[\mathbf{1}_{\{\tau_0=\infty\}} \,|\, \mathcal{F}_1] = \mathbb{E}_x[\mathbf{1}_{\{\tau_0=\infty\}} \,|\, X_1] = h(X_1),$$

   where we used the Markov property for each equality. Taking the expectation in the last equalities, we get for $x \neq x_0$ that:

$$h(x) = \mathbb{E}_x[\mathbb{E}_x[\mathbf{1}_{\{\tau_0=\infty\}} \,|\, \mathcal{F}_1]] = \mathbb{E}_x[h(X_1)] = Ph(x).$$

4. (a) Since $\{\tau \leq n\} = \cup_{k=0}^n \{X_k \in E_0\}$ belongs to $\mathcal{F}_n$, we deduce that τ is a stopping time.

   (b) From the previous question, we get that Y_n is $\mathcal{F}_n$ measurable. Let $x \notin E_0$. Since Y_{n+1} is nonnegative, we can compute:

$$\begin{aligned}
\mathbb{E}_x[Y_{n+1} \,|\, \mathcal{F}_n] &= \mathbb{E}_x[h(X_{n+1})\mathbf{1}_{\{\tau>n+1\}} \,|\, \mathcal{F}_n] \\
&\leq \mathbb{E}_x[h(X_{n+1})\mathbf{1}_{\{\tau>n\}} \,|\, \mathcal{F}_n] \\
&= \mathbb{E}_x[h(X_{n+1}) \,|\, \mathcal{F}_n]\mathbf{1}_{\{\tau>n\}} \\
&= \mathbb{E}_x[h(X_{n+1}) \,|\, X_n]\mathbf{1}_{\{\tau>n\}} \\
&= Ph(X_n)\mathbf{1}_{\{\tau>n\}} \qquad (5)\\
&= Y_n,
\end{aligned}$$

   where we used that h is nonnegative for the inequality, that τ is a stopping time for the second equality, the Markov property for the third and fourth equalities.

   Taking the expectation we get $\mathbb{E}_x[Y_{n+1}] \leq \mathbb{E}_x[Y_n]$ and by iteration that $\mathbb{E}_x[Y_n] \leq \mathbb{E}_x[Y_0] = h(x) < \infty$, as, since $x \notin E_0$, we get that $\mathbb{P}_x$-a.s. $\tau > 0$. Thus, we deduce that Y_n is integrable for all $n \in \mathbb{N}$. This gives that $\mathbf{Y}$ is a super-martingale under $\mathbb{P}_x$.

   (c) Recall $x \notin E_0$. Since $\mathbf{Y}$ is a nonnegative super-martingale under $\mathbb{P}_x$, we get that its converges $\mathbb{P}_x$-a.s. towards an integrable limit, say Z. If $\mathbf{X}$ is transient, in this case, a.s. $\mathbf{X}$ does not belong in any finite set. This implies that $\mathbb{P}_x$-a.s. we have $\lim_{n\to\infty} h(X_n) = +\infty$ for all $x \notin E_0$. We deduce that $\mathbb{P}_x$-a.s. τ is finite for all $x \notin E_0$. Using the strong Markov properties at the exit times of E_0, we deduce that the number of visits of E_0 is a.s. infinite. Since E_0 is finite and $\mathbf{X}$ is transient, we get a contradiction, and thus deduce that $\mathbf{X}$ is recurrent.

5. (a) Notice that (5) still holds, so that, thanks to (4) for the second inequality, we get:

$$\mathbb{E}_x[Y_{n+1} \,|\, \mathcal{F}_n] \leq Ph(X_n)\mathbf{1}_{\{\tau>n\}} \leq Y_n - \delta\mathbf{1}_{\{\tau>n\}}.$$

(b) Since $\mathbb{P}_x$-a.s. $\tau = 0$ if $x \in E_0$, we therefore only need to consider the case $x \notin E_0$. From the previous question, we get that $\delta \mathbf{1}_{\{\tau > n\}} + \mathbb{E}_x[Y_{n+1} \,|\, \mathcal{F}_n] \leq Y_n$. Taking the expectation, we get by recurrence that $\delta \sum_{k=0}^n \mathbb{P}(\tau > k) + \mathbb{E}_x[Y_{n+1}] \leq \mathbb{E}_x[Y_0] = h(x)$ for all $n \in \mathbb{N}$. This gives $\sum_{k=0}^n \mathbb{P}_x(\tau > k) \leq \delta^{-1} h(x)$ for all $n \in \mathbb{N}$, and letting n goes to infinity, we obtain that $\sum_{k \in \mathbb{N}} \mathbb{P}_x(\tau > k) \leq \delta^{-1} h(x) < \infty$. By Fubini, we have:

$$\sum_{k \in \mathbb{N}} \mathbb{P}_x(\tau > k) = \sum_{k \in \mathbb{N}} \mathbb{E}_x[\mathbf{1}_{\{\tau > k\}}] = \mathbb{E}_x[\sum_{k \in \mathbb{N}} \mathbf{1}_{\{\tau > k\}}] = \mathbb{E}_x[\tau].$$

In conclusion, we have obtained that $\mathbb{E}_x[\tau] \leq \delta^{-1} h(x)$.

(c) First notice that all the random times are stopping times. From the previous question we get that $\mathbb{P}_x$-a.s. T_0 is finite. By construction $T_0 = 0$ $\mathbb{P}_x$-a.s. for $x \in E_0$. So we deduce that T_0 is a.s. finite. By the strong Markov property, we deduce that the process $(X_{T_0+n}, n \in \mathbb{N})$ is conditionally on X_{T_0} independent of $\mathcal{F}_{T_0}$. Then notice that conditionally on X_{T_0} we have $T_1 - T_0$ is distributed as τ if $X_{T_0} \notin E_0$ and equal to 1 if $X_{T_0} \in E_0$. This implies that $T_1 - T_0$, as well as T_1, are a.s. finite. By recurrence, we get that the stopping times T_n are a.s. finite for all $n \in \mathbb{N}$. Using again the strong Markov property, we deduce that $\mathbf{X}^0$ is a Markov chain with respect to the filtration $(\mathcal{G}_n, n \in \mathbb{N})$ where $\mathcal{G}_n = \mathcal{F}_{T_n}$.

(d) For all $x, y \in E_0$ there is a path for $\mathbf{X}$ from x to y with positive probability. The trace on E_0 of this path is thus a path for $\mathbf{X}^0$ from x to y with positive probability. This gives that $\mathbf{X}^0$ is irreducible on E_0. Since E_0 is finite, we get that $\mathbf{X}^0$ is positive recurrent.

(e) For $y \in E_0$, we have:

$$\mathbb{E}_y[T_1] = 1 + \mathbb{E}_y[\mathbb{E}_{X_1}[\tau] \mathbf{1}_{\{X_1 \notin E_0\}}] \leq 1 + \delta^{-1} \mathbb{E}_y[h(X_1)] = 1 + \delta^{-1} Ph(y),$$

where we used the Markov property at time $n = 1$ for $\mathbf{X}$ and a decomposition according to $X_1 \in E_0$ and $X_1 \notin E_0$ for the first equality, and Question 5b for the inequality.

(f) Let $x \in E_0$ and $\sigma = \inf\{n \geq 1 : X_n^0 = x\}$ be the first return time to x for $\mathbf{X}^0$. Notice that σ is a stopping time with respect to the filtration $(\mathcal{G}_n, n \in \mathbb{N})$. Since $\mathbf{X}^0$ is positive recurrent, we have $\mathbb{E}_x[\sigma] < \infty$. Let $T^x = \inf\{n \geq 1 : X_n = x\}$ be the first return time to x for $\mathbf{X}$. We shall prove that $\mathbb{E}_x[T^x] < \infty$ to deduce that $\mathbf{X}$ is positive recurrent. Under $\mathbb{P}_x$, we have that $T^x = T_\sigma = \sum_{k=1}^{\sigma} (T_k - T_{k-1})$. This gives:

$$\mathbb{E}_x[T^x] = \sum_{k=1}^{\infty} \mathbb{E}[\mathbf{1}_{\{\sigma \geq k\}} (T_k - T_{k-1})] = \sum_{k=1}^{\infty} \mathbb{E}_x \left[\mathbb{E}_{X_{T_{k-1}}}[T_1] \, \mathbf{1}_{\{\sigma \geq k\}} \right],$$

where we conditioned by $\mathcal{G}_{k-1} = \mathcal{F}_{T_{k-1}}$ with $\{\sigma \geq k\} \in \mathcal{G}_{k-1}$ and use the strong Markov property for $\mathbf{X}$ at time T_{k-1}.

We set $M = 1 + \max_{y \in E_0} Ph(y)$ which is finite by the first part of (4) and since E_0 is finite. Using the previous question, we get:

$$\mathbb{E}_x[T^x] \leq \sum_{k=1}^{\infty} M \mathbb{E}_x \left[\mathbf{1}_{\{\sigma \geq k\}} \right] = M \mathbb{E}_x[\sigma] < \infty.$$

This finishes the proof.

6. (a) Since V is integrable, we deduce by dominated convergence that $\lim_{n\to\infty} \mathbb{E}[V\mathbf{1}_{\{V>-n\}}] = \mathbb{E}[V]$. Use that $\mathbb{E}[V] < 0$ to conclude at the existence of such n_0.

   (b) Let h be the identity function on $\mathbb{N}$. Set $\delta = -\mathbb{E}[V\mathbf{1}_{\{V>-n_0\}}] > 0$. We have for $n > n_0$ that:

$$Ph(n) = \mathbb{E}[(n+V)_+] = \mathbb{E}[(n+V)\mathbf{1}_{\{V>-n\}}] \leq n - \delta = h(n) - \delta.$$

   For $n \in E_0$, we also have $Ph(n) = \mathbb{E}[(n+V)_+] \leq \mathbb{E}[(n+V)] = n + \mu \leq n < \infty$. Hence, the function h belongs to $\mathcal{B}_+$ and satisfies (4). We deduce from Question 5f that $\mathbf{X}$ is positive recurrent.