Stochastic Process (ENPC)
Monday, 26th of January 2026 (3h)

Vocabulary (english/*français*): bracket=*crochet*; nonnegative=*positif*; non-dcreasing=*croissant*. $[0,1) =$ *[0,1[*; w.r.t.= with respect to.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathbb{F} = (\mathcal{F}_n, n \in \mathbb{N})$ be a filtration with $\bigvee_{n\in\mathbb{N}} \mathcal{F}_n \subset \mathcal{F}$. We say a martingale $M = (M_n, n \in \mathbb{N})$ w.r.t. the filtration $\mathbb{F}$ is square integrable if $\mathbb{E}[M_n^2] < +\infty$ for all $n \in \mathbb{N}$. Its bracket $\langle M \rangle = (\langle M \rangle_n, n \in \mathbb{N})$ is defined by $\langle M \rangle_0 = 0$ and for $n \in \mathbb{N}^*$ a.s.:

$$\langle M \rangle_n = \sum_{k=1}^{n} \mathbb{E}\left[\Delta_k^2 \,|\, \mathcal{F}_{k-1}\right] \quad \text{with} \quad \Delta_k = M_k - M_{k-1}.$$

If it exists in $[-\infty, +\infty]$, the limit of $(\langle M \rangle_n, n \in \mathbb{N})$ will be denoted by $\langle M \rangle_\infty$.

Exercise 1 is used in Exercises 2 and 3; the latter two are independent from each other.

Exercise 1 (The square of a martingale)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathbb{F} = (\mathcal{F}_n, n \in \mathbb{N})$ be a filtration with $\bigvee_{n\in\mathbb{N}} \mathcal{F}_n \subset \mathcal{F}$. By convention, we set $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$. We recall that a stochastic process $X = (X_n, n \in \mathbb{N})$ is $\mathbb{F}$-adapted if X_n is $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$; and that it is $\mathbb{F}$-predictable if X_n is $\mathcal{F}_{n-1}$-measurable for all $n \in \mathbb{N}$.

Let $M = (M_n, n \in \mathbb{N})$ be a square integrable martingale w.r.t. $\mathbb{F}$. We consider the process:

$$N = (N_n, n \in \mathbb{N}) \quad \text{defined by} \quad N_n = M_n^2 - \langle M \rangle_n. \tag{1}$$

1. Check that $\langle M \rangle$ is a nonnegative non-decreasing $\mathbb{F}$-predictable process.
2. Prove that a.s. the limit $\langle M \rangle_\infty$ of $(\langle M \rangle_n, n \in \mathbb{N})$ exists and belongs to $[0, +\infty]$.
3. Prove that N is a martingale.

△

Exercise 2 (The elephant random walk[1])**.** We present a random walk model, where each step is a step from the past chosen at random with its signed changed at random. To do this, we shall consider the Rademacher probability distribution $\mathcal{R}(r)$ with parameter $r \in (0,1)$ where the random variable Y has distribution $\mathcal{R}(r)$ (denoted by $Y \sim \mathcal{R}(r)$) if a.s. $Y \in \{-1, 1\}$ and:

$$\mathbb{P}(Y = 1) = 1 - \mathbb{P}(Y = -1) = r.$$

Let $p, q \in (0,1)$, and let X_1, $(\varepsilon_n, n \geq 2)$ and $(K_n, n \geq 2)$ be independent random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $X_1 \sim \mathcal{R}(q)$ and, for $n \geq 2$, $\varepsilon_n \sim \mathcal{R}(p)$ and K_n is uniform on $\{1, \ldots, n-1\}$. We define the *elephant random walk* $S = (S_n, n \in \mathbb{N})$ by $S_0 = 0$, $S_1 = X_1$ and for $n \geq 1$:

$$S_{n+1} = S_n + X_{n+1} \quad \text{with} \quad X_{n+1} = \varepsilon_{n+1} X_{K_{n+1}}.$$

The increments of S exhibit a long range memory, as it keeps tracks somehow of its past increments, hence its name as elephants are known to have long memory. We shall study its longtime behavior using martingales[2] and (almost) prove a phase transition at $p = 3/4$.

We set $\alpha = 2p-1$ and $\beta = 2q-1$. We consider the filtration $\mathbb{F} = (\mathcal{F}_n = \sigma(S_0, \ldots, S_n), n \in \mathbb{N})$.

Question 1 is used in the following questions, and Question 3 is also used in the Question 4 and in the optional Question 5.

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[1] G. Schütz and S. Trimper. Elephants can always remember: Exact long-range memory effects in a non-Markovian random walk. *Phys. Rev. E*, 70, 045101, 2004.

[2] B. Bercu. A martingale approach for the elephant random walk. *J. Phys. A*, 51, 015201. 2018.

1. Preliminary computations.

   (a) Check that $\mathbb{E}[X_1] = \beta$ and $\mathbb{E}[\varepsilon_n] = \alpha$ for $n \geq 2$. Check also that $|S_n| \leq n$ and $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ for $n \in \mathbb{N}^*$.

   (b) Let $n \in \mathbb{N}^*$. Compute $\mathbb{E}[X_{n+1} \,|\, \mathcal{F}_n]$.

   (c) Let $n \in \mathbb{N}^*$. Deduce that conditionally on $\mathcal{F}_n$, the random variable X_{n+1} has a Rademacher distribution with (random) parameter:

$$R_n = \frac{1}{2}\left(1 + \alpha \frac{S_n}{n}\right).$$

   (d) Is the process S a stochastic dynamical system, that is, for all $n \in \mathbb{N}$, S_{n+1} is the function of S_n and an independent innovation?

2. The particular case $p = 1/2$ (that is, $\alpha = 0$).

   (a) Prove that the random variables $(X_n, n \in \mathbb{N}^*)$ are independent and precise their distribution. Deduce that $(S_n, n \in \mathbb{N}^*)$ is a Markov chain on $\mathbb{Z}$.

   (b) Prove that a.s. $\lim_{n\to\infty} n^{-1} S_n = 0$ and that $(n^{-1/2} S_n, n \in \mathbb{N}^*)$ converges in distribution to a Gaussian random variable.

We consider the process $M = (M_n, n \in \mathbb{N})$ defined by $M_0 = \beta$, $M_1 = S_1$ and for $n \geq 2$:

$$M_n = a_n\, S_n, \quad \text{where} \quad a_n = \frac{n-1}{n-1+\alpha}\, a_{n-1} \quad \text{with } a_1 = 1.$$

We recall that $\Delta_n = M_n - M_{n-1}$ for $n \in \mathbb{N}^*$. We admit that $a_n \sim n^{-\alpha}$ as $n \to +\infty$.

3. Properties of the martingale M and its bracket.

   (a) Let $n \geq 2$. Check that:

$$\Delta_n = a_n \left(X_n - \alpha \frac{S_{n-1}}{n-1}\right).$$

   (b) Prove that M is a square integrable martingale.

   (c) Prove that $|\Delta_n| \leq 2a_n$ for $n \in \mathbb{N}^*$.

   We recall that the process N defined by (1) is a martingale (see Exercise 1 Question 3).

   (d) Deduce that $\mathbb{E}[M_n^2] \leq \beta^2 + 4\sum_{k=1}^n a_k^2$.

4. The case $p > 3/4$ (that is, $\alpha > 1/2$).

   (a) Using the question 3d, check that $\sup_{n\in\mathbb{N}} \mathbb{E}[M_n^2]$ is finite.

   (b) Deduce that $(n^{-\alpha} S_n, n \in \mathbb{N}^*)$ converges (in what sense?) to a random variable Z.

   (c) Check that $\mathbb{E}[Z] = \beta$ and that Z is non trivial, that is, $\mathrm{Var}(Z) > 0$.

   It can be proven[3] that Z is not Gaussian and that the fluctuations $(S_n - n^\alpha Z)/\sqrt{n}$ converges in distribution to a non degenerate Gaussian random variable.

[3] N. Kubota and M. Takei. Gaussian fluctuations for superdiffusive elephant random walks. *J. Stat. Phys.*, 177, pp. 1157-1171. 2019.

5. OPTIONAL. The case $p < 3/4$ (that is, $\alpha < 1/2$).

   (a) Let $n \geq 2$. Check that:

$$\mathbb{E}\left[\Delta_n^2 \,|\, \mathcal{F}_{n-1}\right] = a_n^2 \left(1 - \alpha^2 \frac{S_{n-1}^2}{(n-1)^2}\right).$$

   (b) Prove that a.s. $\liminf_{n\to\infty} n^{2\alpha-1}\langle M\rangle_n > 0$.

   (c) Deduce that a.s. $\lim_{n\to\infty}\langle M\rangle_n = +\infty$.

   We admit then that a.s. $\lim_{n\to\infty} M_n/f(\langle M\rangle_n) = 0$ for any nonnegative non-decreasing function f defined on $\mathbb{R}_+$ such that $\int_0^\infty (1+f(t))^{-2}\,\mathrm{d}t < +\infty$ (see Exercise 3 Question 2d).

   (d) Prove that, for all $\varepsilon > 0$, a.s. $\lim_{n\to\infty} n^{-(1+\varepsilon)/2} S_n = 0$.

   It can be proven[3] that $S_n/\sqrt{n}$ (resp. $S_n/\sqrt{n\log(n)}$) converges in distribution to a non degenerate Gaussian random variable if $p < 3/4$ (resp. if $p = 3/4$).

△

Exercise 3 (Martingale bracket)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathbb{F} = (\mathcal{F}_n, n \in \mathbb{N})$ be a filtration with $\bigvee_{n\in\mathbb{N}} \mathcal{F}_n \subset \mathcal{F}$. We say that a random sequence $(X_n, n \in \mathbb{N}^*)$ converges a.s. on $A \in \mathcal{F}$ if:

$$\mathbb{P}\left(A \cap \left\{\liminf_{n\to\infty} X_n = \limsup_{n\to\infty} X_n\right\}\right) = \mathbb{P}(A).$$

Let $M = (M_n, n \in \mathbb{N})$ be a square integrable martingale w.r.t. $\mathbb{F}$ such that $M_0 = 0$. We recall that the process $\langle M\rangle$ is $\mathbb{F}$-predictable and converges a.s. to $\langle M\rangle_\infty \in [0, +\infty]$ and that the process N defined by (1) is a martingale (see Exercise 1).

1. We set for $T_a = \inf\{n \in \mathbb{N} :\ \langle M\rangle_{n+1} > a^2\}$ for $a > 0$, with the convention that $\inf \emptyset = +\infty$.

   (a) Prove that T_a is a stopping time.

   (b) Check that $\mathbb{E}\left[M_{T_a\wedge n}^2\right] = \mathbb{E}\left[\langle M\rangle_{T_a\wedge n}\right]$ for $n \in \mathbb{N}$ and that $\sup_{n\in\mathbb{N}} \mathbb{E}\left[M_{T_a\wedge n}^2\right] < +\infty$.

   (c) Deduce that M converges a.s. on $\{T_a = +\infty\}$.

   (d) Prove that M converges a.s. on $\{\langle M\rangle_\infty < +\infty\}$.

2. Let f be a nonnegative non-decreasing function defined on $\mathbb{R}_+$ such that $\int_0^\infty (1+f(t))^{-2}\,\mathrm{d}t$ is finite. We set $H_n = (1 + f(\langle M\rangle_{n+1}))^{-1}$ for $n \in \mathbb{N}$ and consider the stochastic process $Q = H \cdot M = (Q_n, n \in \mathbb{N})$ defined by $Q_0 = 0$ and for $n \geq 1$:

$$Q_{n+1} = Q_n + H_n(M_{n+1} - M_n).$$

   (a) Prove that Q is a square integrable martingale.

   (b) Prove that $\langle Q\rangle_n \leq \int_0^{\langle M\rangle_n} (1 + f(t))^{-2}\,\mathrm{d}t$.

   (c) Deduce that the martingale Q converges a.s..

   We recall (Kronecker lemma) that if $(u_n, n \in \mathbb{N}^*)$ and $(\delta_n, n \in \mathbb{N}^*)$ are two sequences of real numbers such that i) $\inf_{n\in\mathbb{N}^*} u_n > 0$ and $\lim_{n\to\infty} u_n = +\infty$ and ii) $\lim_{n\to\infty} \sum_{k=1}^n \delta_k/u_k$ exists and belongs to $\mathbb{R}$, then we have $\lim_{n\to\infty} u_n^{-1} \sum_{k=1}^n \delta_k = 0$.

   (d) Prove that $\lim_{n\to\infty} M_n/f(\langle M\rangle_n) = 0$ a.s. on $\{\langle M\rangle_\infty = +\infty\}$.

△

Correction

Exercise 1

1. By monotony of the conditional expectation, we deduce that $\mathbb{E}[\Delta_k^2 \,|\, \mathcal{F}_{k-1}]$ is nonnegative for $k \geq 0$. It is also $\mathcal{F}_{k-1}$-measurable. This gives that $\langle M \rangle_n$ is nonnegative and $\mathcal{F}_{n-1}$-measurable, and thus $\langle M \rangle$ is $\mathbb{F}$-predictable. Use that $\langle M \rangle_n - \langle M \rangle_{n-1} = \mathbb{E}[\Delta_n^2 \,|\, \mathcal{F}_{n-1}] \geq 0$ to get that $\langle M \rangle$ is non-decreasing.

2. This is a consequence of the fact that the sequence $(\langle M \rangle_n, n \in \mathbb{N})$ is nonnegative and non-decreasing.

3. Clearly N_n is $\mathcal{F}_n$-measurable and integrable (use that $\mathbb{E}[\langle M \rangle_n] = \sum_{k=1}^n \mathbb{E}[(M_k - M_{k-1})^2]$ is finite as M is a square integrable process). We have, using that M_{n-1} and $\langle M \rangle_n$ are $\mathcal{F}_{n-1}$-measurable:

$$\begin{aligned}\mathbb{E}[N_n|\mathcal{F}_{n-1}] &= \mathbb{E}[(M_{n-1} + \Delta_n)^2|\mathcal{F}_{n-1}] - \langle M \rangle_n \\ &= M_{n-1}^2 - \langle M \rangle_{n-1} + 2M_{n-1}\mathbb{E}[\Delta_n|\mathcal{F}_{n-1}] \\ &= N_{n-1},\end{aligned}$$

as $\mathbb{E}[\Delta_n|\mathcal{F}_{n-1}] = \mathbb{E}[M_n - M_{n-1}|\mathcal{F}_{n-1}] = 0$. Thus the process N is a martingale.

Exercise 2 We recall that the nonnegative function Γ, defined on $\mathbb{R}_+^*$ by $\Gamma(t) = \int_{\mathbb{R}_+^*} u^{t-1} \,\mathrm{e}^{-u} \,\mathrm{d}u$, satisfies:

$$\Gamma(1) = 1, \quad \Gamma(t+1) = t\Gamma(t) \quad \text{and} \quad \lim_{t \to +\infty} t^{-r} \frac{\Gamma(t+r)}{\Gamma(t)} = 1 \quad \text{for } t, r \in \mathbb{R}_+.$$

We deduce that $a_n = \Gamma(n)\Gamma(\alpha)/\Gamma(n+\alpha)$ and thus $a_n \sim n^{-\alpha}$ as $n \to +\infty$.

1. (a) We have $\mathbb{E}[Y] = 2r - 1$ for $Y \sim \mathcal{R}(r)$. (Thus $\mathbb{E}[X_1] = \beta$ and $\mathbb{E}[\varepsilon_n] = \alpha$). Notice that $|S_{n+1}| \leq |S_n| + |X_{n+1}| = |S_n| + 1$ to get $|S_n| \leq n$. As $(S_0 = 0, S_1, \ldots, S_n)$ is in (measurable) bijection with $(X_1, \ldots, X_n)$, we deduce that $\mathcal{F}_n = \sigma(S_0, \ldots, S_n) = \sigma(X_1, \ldots, X_n)$.

   (b) Let $n \in \mathbb{N}^*$, and thus $n + 1 \geq 2$. As K_{n+1} is uniform on $\{1, \ldots, n\}$ and that $(\varepsilon_{n+1}, K_{n+1})$ is independent of $(X_1, \ldots, X_n)$, we get:

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_{n+1}|X_1, \ldots, X_n] = \mathbb{E}[\varepsilon_{n+1}] \,\frac{1}{n} \sum_{k=1}^{n} X_k = \frac{\alpha}{n} S_n. \tag{2}$$

   (c) Since X_{n+1} takes values a.s. in $\{-1, +1\}$, we deduce that conditionally on $\mathcal{F}_n$, X_{n+1} has a Rademacher distribution with parameter $(\mathbb{E}[X_{n+1}|\mathcal{F}_n] + 1)/2 = R_n$.

   (d) The process S is not a stochastic dynamical system as $X_{K_{n+1}}$ is a function of the whole path $(X_1, \ldots, X_n)$ (and the innovation $(\varepsilon_{n+1}, K_{n+1})$) and not just X_n (for $n \geq 3$).

2. (a) Conditionally on $\mathcal{F}_n$, X_{n+1} is Rademacher with parameter $1/2$. Thus, X_{n+1} is independent of $\mathcal{F}_n$. This gives that the random variables $(X_n, n \in \mathbb{N})$ are independent Rademacher random variable, with $X_1 \sim (\mathcal{R}(q)$ and $X_n \sim \mathcal{R}(1/2)$ for $n \geq 2$. As $S_{n+1} = S_n + X_{n+1}$, it is a stochastic dynamical system and thus a Markov chain.

(b) By the strong law of large number (and as X_1 is a.s. finite, and thus $\lim_{n\to\infty} X_1/n = 0$ a.s.), we get that a.s. $\lim_{n\to\infty} S_n/n = \mathbb{E}[X_2] = 0$. By the central limit theorem, we also get (using Slutsky lemma and that $\lim_{n\to\infty} X_1/\sqrt{n} = 0$ a.s.), that $n^{-1/2}S_n$ converges in distribution towards the Gaussian distribution $\mathcal{N}(0, \sigma^2)$ with $\sigma^2 = \mathrm{Var}(X_2) = 1$.

3. (a) Elementary computations.

   (b) Since $|S_n| \leq n$, we get that M_n is square integrable. By construction M_n is $\mathcal{F}_n$-measurable. Then, use that $\mathbb{E}[\Delta_n|\mathcal{F}_{n-1}] = 0$ for $n \geq 2$ (see (2)) to deduce that $\mathbb{E}[M_n|\mathcal{F}_{n-1}] = M_{n-1}$ for $n \geq 2$. For $n = 1$, we have $\mathbb{E}[M_1|\mathcal{F}_0] = \mathbb{E}[M_1] = \mathbb{E}[X_1] = \beta = M_0$.

   Thus, the process M is a square integrable martingale.

   (c) We have $|\Delta_n| \leq a_n(1 + |\alpha|) \leq 2a_n$.

   (d) As N is a martingale, with $N_0 = M_0^2 = \beta^2$, we get that $\mathbb{E}[N_n] = \mathbb{E}[N_0] = \beta$ and by linearity:

$$\mathbb{E}[M_n^2] = \mathbb{E}[N_0] + \mathbb{E}[\langle M \rangle_n] = \beta^2 + \sum_{k=1}^n \mathbb{E}[\Delta_k^2] \leq \beta^2 + 4\sum_{k=1}^n a_k^2.$$

4. (a) As $a_n \sim n^{-\alpha}$ with $\alpha > 1/2$, we deduce that the series $\sum_{k\in\mathbb{N}^*} a_k^2$ is converging and thus finite. We get that $\sup_{n\in\mathbb{N}} \mathbb{E}[M_n^2]$ is finite.

   (b) Since the martingale M is bounded in L^2, it converges a.s. and in L^2 towards a random variable, say M_∞. As $a_n \sim n^{-\alpha}$, we deduce that $\lim_{n\to\infty} n^{-\alpha}S_n = M_\infty$ a.s. and in L^2.

   (c) Since the convergence holds in L^2, it also holds in L^1 and $\mathbb{E}[M_\infty] = \lim_{n\to\infty} \mathbb{E}[M_n] = \mathbb{E}[M_0] = \beta$. We also have that $\mathbb{E}[M_\infty^2]$ is the limit of the non-decreasing sequence $(\mathbb{E}[M_n^2], n \in \mathbb{N})$, as M^2 is a sub-martingale. In particular, we have $\mathbb{E}[M_\infty^2] \geq \mathbb{E}[M_1^2] = 1 > \beta^2$. Hence $\mathrm{Var}(M_\infty) > 0$ and thus M_∞ is non trivial.

5. We shall also consider the case $\alpha = 1/2$.

   (a) For $n \geq 2$, we have:

$$\begin{aligned}
\mathbb{E}\left[\Delta_n^2 \,|\, \mathcal{F}_{n-1}\right] &= a_n^2 \left(\mathbb{E}[X_n^2|\mathcal{F}_{n-1}] - 2\alpha\frac{S_{n-1}}{n-1}\mathbb{E}[X_n|\mathcal{F}_{n-1}] + \alpha^2\frac{S_{n-1}^2}{(n-1)^2}\right) \\
&= a_n^2 \left(1 - 2\alpha\frac{S_{n-1}}{n-1}\alpha\frac{S_{n-1}}{n-1} + \alpha^2\frac{S_{n-1}^2}{(n-1)^2}\right) \\
&= a_n^2 \left(1 - \alpha^2\frac{S_{n-1}^2}{(n-1)^2}\right),
\end{aligned}$$

   where we used that $X_n^2 = 1$ and (2) for the second equality.

   (b) Since $\alpha^2 < 1$, we get that:

$$(1 - \alpha^2)\, a_n^2 \leq \mathbb{E}\left[\Delta_n^2 \,|\, \mathcal{F}_{n-1}\right] \leq a_n^2.$$

   Suppose $\alpha < 1/2$. As $a_k^2 \sim k^{-2\alpha}$, we get $\sum_{k=1}^n a_k^2 \sim n^{1-2\alpha}$ as n goes to infinity; and we deduce that a.s.:

$$(1 - \alpha^2) \leq \liminf_{n\to\infty} n^{2\alpha-1}\langle M \rangle_n \leq \limsup_{n\to\infty} n^{2\alpha-1}\langle M \rangle_n \leq 1.$$

Suppose $\alpha = 1/2$. As $a_k^2 \sim k^{-1}$, we get $\sum_{k=1}^n a_k^2 \sim \log(n)$ as n goes to infinity; and we deduce that a.s.:

$$(1-\alpha^2) \leq \liminf_{n\to\infty} \frac{\langle M\rangle_n}{\log(n)} \leq \limsup_{n\to\infty} \frac{\langle M\rangle_n}{\log(n)} \leq 1.$$

(c) The previous result implies that a.s. $\lim_{n\to\infty}\langle M\rangle_n = +\infty$.

(d) Taking $f(t) = t^{(1+\delta)/2}$ with $\delta = \varepsilon/(1-2\alpha)$ for $\alpha < 1/2$ and $\delta = \varepsilon$ for $\alpha = 1/2$ gives that a.s. $\lim_{n\to\infty} n^{-(1+\varepsilon)/2} S_n = 0$ for $\alpha < 1/2$, and that a.s. $\lim_{n\to\infty} S_n/\sqrt{n \log(n)^{1+\varepsilon}} = 0$ for $\alpha = 1/2$.

Exercise 3

1. (a) We have $\{T_a > n\} = \cup_{k=0}^n \{\langle M\rangle_{k+1} \leq a\}$, and as $\langle M\rangle$ is $\mathbb{F}$-predictable, we deduce that $\{T_a > n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$. This proves that T_a is an $\mathbb{F}$-stopping time.

   (b) Since N is a martingale and $T_a \wedge n$ a bounded stopping time, we deduce from the stopping theorem that $\mathbb{E}[N_{T_a\wedge n}] = \mathbb{E}[N_0] = \mathbb{E}[M_0] = 0$. Since M is square integrable, we also deduce that $M_{T_a\wedge n}^2 = \sum_{k=0}^n M_k^2 \mathbf{1}_{\{T_a\wedge n = k\}}$ belongs to L^1. By linearity, we deduce that $\mathbb{E}\left[M_{T_a\wedge n}^2\right] = \mathbb{E}\left[\langle M\rangle_{T_a\wedge n}\right]$ for $n \in \mathbb{N}$. By the definition of T_a, we get that $\langle M\rangle_{T_a\wedge n} \leq a^2$ for all $n \in \mathbb{N}$. This implies that $\sup_{n\in\mathbb{N}} \mathbb{E}[\langle M\rangle_{T_a\wedge n}] \leq a^2 < +\infty$. This gives the result.

   (c) Since the stopped martingale M^{T_a} is bounded in L^2 it converges a.s. and in L^2 towards a random variable, say Z_a. On $\{T_a = +\infty\}$, we have $M_n^{T_a} = M_n$, and thus M converges a.s. to Z_a on $\{T_a = +\infty\}$ (and Z_a is a.s. finite).

   (d) We deduce that M converges a.s. to a finite random variable on $\cup_{a\in\mathbb{N}^*}\{T_a = +\infty\} = \{\langle M\rangle_\infty < +\infty\}$.

2. (a) The process Q is is the discrete stochastic integral. We shall prove this is indeed a martingale. Notice that H_n is nonnegative and bounded by 1, and thus $H_n(M_{n+1} - M_n)$ is square integrable; it is $\mathcal{F}_n$ also measurable (as $\langle M\rangle$ is $\mathbb{F}$-predictable) and thus Q is square integrable and $\mathbb{F}$-adapted (as $Q_0 = 0$). We have for $n \in \mathbb{N}$, as H_n is $\mathcal{F}_n$-measurable:

$$\mathbb{E}[Q_{n+1}|\mathcal{F}_n] = Q_n + H_n \mathbb{E}[M_{n+1} - M_n|\mathcal{F}_n] = Q_n.$$

   This proves that Q is a square-integrable martingale.

   (b) We have $\langle Q\rangle_{n+1} - \langle Q\rangle_n = \mathbb{E}[(Q_{n+1} - Q_n)^2|\mathcal{F}_n] = H_n^2 \Delta_{n+1}$ and thus, as f non-decreasing and nonnegative:

$$\langle Q\rangle_{n+1} - \langle Q\rangle_n = \frac{\langle M\rangle_{n+1} - \langle M\rangle_n}{(1 + f(\langle M\rangle_{n+1}))^2} \leq \int_{\langle M\rangle_n}^{\langle M\rangle_{n+1}} \frac{\mathrm{d}t}{(1+f(t))^2}.$$

   As $\langle Q\rangle_0 = 0$, we deduce that $\langle Q\rangle_n \leq \int_0^{\langle M\rangle_n} (1 + f(t))^{-2}\,\mathrm{d}t$.

   (c) We deduce that $\langle Q\rangle_\infty$ is finite a.s., and from Question 1.d, that Q converges a.s..

   (d) Notice that $\lim_{r\to+\infty} f(r) = +\infty$ by the assumption on f. Using the Kronecker lemma with $u_k = 1 + f(\langle M\rangle_k) \geq 1$, which converges to $+\infty$ on $\{\langle M\rangle_\infty = +\infty\}$, and $\delta_k = M_k - M_{k-1}$, we deduce, as $M_0 = 0$ and $u_n^{-1}\sum_{k=1}\delta_k = M_n/(1 + f(\langle M\rangle_n)$, that $\lim_{n\to\infty} M_n/f(\langle M\rangle_n) = 0$ a.s. on $\{\langle M\rangle_\infty = +\infty\}$.