Stochastic Processes and Applications

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1 Measure theory and random variables

1.1 σ–fields

Exercise 1.1 (On σ-fields and topologies). Let Ω be a state space, for example $\Omega = \mathbb{R}$.

1. Describe the σ-field generated by one subset, that is by $\mathcal{C} = \{A\}$ for some $A \subset \Omega$.

2. Describe the σ-field generated by the singletons, that is by $\mathcal{C} = \{\{x\},\, x \in \Omega\}$. Prove that if Ω is finite or countable then $\sigma(\mathcal{C}) = \mathcal{P}(\Omega)$. Is this the case if Ω is uncountable?

△

Exercise 1.2 (Generated σ-fields). Consider a real-valued function f defined on $\Omega = \mathbb{R}$ (endowed with its Borel σ-field) and $\sigma(f)$, the σ-field generated by f.

1. Determine $\sigma(f)$ for the functions f defined by: a) $f(x) = \mathbf{1}_A(x)$ for some Borel set $A \subset \mathbb{R}$; b) $f(x) = x$; c) $f(x) = \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the integer part of x, that is the only $n \in \mathbb{Z}$ such that $n \leq x < n + 1$.

2. Prove that a measurable real-valued function defined on $\Omega = \mathbb{R}$ is $\sigma(|x|)$-measurable if and only if it is an even real-valued function.

3. Prove that $\sigma(|x|) = \sigma(x^2)$ (where we write $\sigma(f(x))$ for $\sigma(f)$).

△

Exercise 1.3 (Generated fields and coin tossing). We model an infinite coin tossing game by the set $\Omega = \{0, 1\}^{\mathbb{N}} = \{\omega = (\omega_1, \omega_2, \ldots); \omega_i \in \{0, 1\}\}$, and the result of the n-th tossing is given by the function X_n: $X_n(\omega) = \omega_n$, where $\omega = (\omega_1, \omega_2, \ldots)$. We denote by $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ the σ-field generated by $X_1, \ldots, X_n$. This corresponds to the information obtained when looking at the first n coin tossing.

1. Determine $\mathcal{F}_1$ and $\mathcal{F}_2$.

2. Prove that X_2 is not $\mathcal{F}_1$-measurable but that $X_2(1 - X_2)$ is $\mathcal{F}_1$-measurable.

3. Characterize all the real-valued functions defined on Ω which are $\mathcal{F}_1$ measurable and more generally which are $\mathcal{F}_n$-measurable.

△

1.2 Independence

Exercise 1.4 (Independence). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let X_1 and X_2 be two independent random variables taking values in $\{-1, 1\}$ such that $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_2 = 1) = 1/2$.

1. Determine $\sigma(X_2)$.
2. Prove that X_1 and X_1X_2 are independent of $\sigma(X_2)$.
3. Check that (X_1, X_1X_2) is not independent of $\sigma(X_2)$.
4. Deduce an example where $A, B, C \in \mathcal{F}$ are such that A and B are independent of C but $A \cap B$ is not independent[1] of C.

$\triangle$

1.3 Convergence theorems

Exercise 1.5 (Monotone convergence). Let $f_n(x) = n^{-1} |x|$ for $n \in \mathbb{N}^*$ and $x \in \mathbb{R}$. Check that $\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\, \lambda(\mathrm{d}x) \neq \int_{\mathbb{R}} \lim_{n\to\infty} f_n(x)\, \lambda(\mathrm{d}x)$. Does this contradict the monotone convergence theorem? $\triangle$

Exercise 1.6 (Convergence of the integral). Let $f_n(x) = \mathbf{1}_{[n,n+1]}(x)$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Check that $\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\, \lambda(\mathrm{d}x)$ and $\int_{\mathbb{R}} \lim_{n\to\infty} f_n(x)\, \lambda(\mathrm{d}x)$ does not coincide. Does this contradict the monotone convergence theorem, Fatou's lemma or the dominated convergence theorem? $\triangle$

Exercise 1.7 (Characteristic function). The characteristic of X a $\mathbb{R}$-valued random variable is the ψ_X defined on $\mathbb{R}$ by $\psi_X(u) = \mathbb{E}[\mathrm{e}^{iuX}]$.

1. Prove that $|\psi_X(u)| \leq 1$.
2. Using dominated convergence prove that ψ_X is continuous on $\mathbb{R}$.
3. Using dominated convergence, prove that if a sequence $(X_n, n \in \mathbb{N})$ of $\mathbb{R}$-valued random variables converges a.s. to an $\mathbb{R}$-valued random variable, then $\lim_{n\to\infty} \psi_{X_n}(u) = \psi_X(u)$ for all $u \in \mathbb{R}$.

$\triangle$

2 Conditional expectation

Exercise 2.1 (Model for daily temperature). We consider the following auto-regressive model for the daily temperature. Let X_n denote the temperature of day $n \in \mathbb{N}$. We assume that X_0 is a given constant and $X_{n+1} = aX_n + \varepsilon_{n+1}$ for $n \in \mathbb{N}$, where the random variables $(\varepsilon_n, n \in \mathbb{N}^*)$ are independent integrable with the same mean μ. Compute $\mathbb{E}[X_{n+1} \,|\, X_n]$ (the best prevision of tomorrow daily temperature, knowing the today temperature) and $\mathbb{E}[X_{n+1} \,|\, X_0, \ldots, X_n]$ (the best prevision of tomorrow daily temperature, knowing the past and today temperatures). $\triangle$

[1] Taking $\mathcal{C} = \{A, B\}$, this gives an example where all elements of $\mathcal{C}$ are independent of a σ-field $\mathcal{G} \subset \mathcal{F}$, but the σ-field $\sigma(\mathcal{C})$ is not independent of $\mathcal{G}$. However, if all elements of $\mathcal{C}$ are independent of a σ-field $\mathcal{G} \subset \mathcal{F}$ and $\mathcal{C}$ is stable by finite intersection, then using the monotone class theorem, one can prove that $\sigma(\mathcal{C})$ is then independent of $\mathcal{G}$.

Exercise 2.2 (Bernoulli random variables). Let X_1 and X_2 be two independent Bernoulli random variables with parameter $p \in (0, 1)$.

1. Compute $\mathbb{E}[X_1 + X_2 \,|\, X_1]$.

2. Compute $\mathbb{E}[X_1 \,|\, X_1 + X_2]$.

3. Compute $\mathbb{E}[X_1X_2 \,|\, X_1 + X_2]$ and deduce that X_1 and X_2 are not independent conditionally on $X_1 + X_2$.

△

Exercise 2.3 (Geometric distribution). Let $(X_n, n \in \mathbb{N}^*)$ be independent Bernoulli random variables with parameter $p \in (0, 1)$. Let $T = \inf\{n \geq 1;\, X_n = 1\}$ with the convention that $\inf \emptyset = +\infty$. (Notice that T has geometric distribution with parameter p.) Compute $\mathbb{E}[T| \, X_1]$ and deduce $\mathbb{E}[T]$. △

Exercise 2.4 (More and more precise). Let X be an $\mathbb{R}_+$-valued random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the σ-fields $\mathcal{F}_n = \sigma(\lfloor 2^n X \rfloor)$ for $n \in \mathbb{N}$, where $\lfloor x \rfloor$ is the integer part of x, that is the only integer m such that $m \leq x < m + 1$. The σ-fields $\mathcal{F}_n$ corresponds to the observation of X with precision 2^{-n}. We set $M_n = \mathbb{E}[X \,|\, \mathcal{F}_n]$.

1. Check that $\lfloor 2^{-1} \lfloor 2^{n+1} x \rfloor \rfloor = \lfloor 2^n x \rfloor$. Deduce that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ (which roughly means that the amount of information increases with the precision). Deduce that $M_n = \mathbb{E}[M_{n+1} \,|\, \mathcal{F}_n]$.

2. Check that $\mathbf{1}_{\{2^n X \in [k,k+1)\}}$ is $\mathcal{F}_n$ measurable for every $k \in \mathbb{N}$. (More generally, you can further prove that $\mathcal{F}_n$ is generated by the events $\{2^n X \in [k, k + 1)\}$ for $k \in \mathbb{N}$.)

3. We recall that $\mathbb{E}[X \mid A] = \mathbb{E}[X\mathbf{1}_A]/\mathbb{P}(A)$ for all $A \in \mathcal{F}$, with the convention that $0/0 = 0$. Prove that:
$$M_n = \sum_{k \in \mathbb{N}} \mathbf{1}_{\{2^n X \in [k,k+1)\}} \mathbb{E}[X \mid 2^n X \in [k, k + 1)].$$

4. Check that $X - 2^{-n} \leq M_n \leq X + 2^{-n}$ and deduce that a.s. $\lim_{n\to\infty} M_n = X$ (that is when the precision is infinite, everything is known).

△

Exercise 2.5 (Bernoulli random variables). Let $(X_1, \ldots, X_n)$ be independent Bernoulli random variables with parameter $p \in (0, 1)$. Recall that $S_n = \sum_{i=1}^{n} X_i$ is binomial with parameter (n, p). .

1. Prove that conditionally on S_n, the distribution of X_1 is Bernoulli with parameter S_n/n.

2. Compute the distribution of $(X_1, \ldots, X_n)$ conditionally on S_n.

3. Check that the random variables X_1 and X_2 are not independent conditionally on S_n (for $n \geq 2$).

△

3 Markov chains

3.1 First computations

Exercise 3.1 (Examples of transition matrices). In what follows $*$ designs any positive quantity, such that the matrices of size n considered are stochastic. For the following transition matrices on $\{1,\dots,n\}$, indicate when $*$ takes the value 1, represent the corresponding transition graph, give the communicating classes and precise if they are open or closed and in the latter case their period.

$$\begin{pmatrix} * & * & 0 \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \qquad \begin{pmatrix} 0 & * & * & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ * & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} * & 0 & * & 0 \\ 0 & 0 & * & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & * \end{pmatrix} \qquad \begin{pmatrix} 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & * & 0 & * & * \\ * & 0 & 0 & 0 & 0 & * & 0 \\ * & * & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\triangle$

Exercise 3.2 (Labyrinth). A mouse is in the labyrinth depicted in figure 1 with 9 squares. We consider the three classes of squares: $A = \{1, 3, 7, 9\}$ (the corners), $B = \{5\}$ (the center) and $C = \{2, 4, 6, 8\}$ the other squares. At each step $n \in \mathbb{N}$, the mouse is in a square and we denote by X_n its number and Y_n its class.

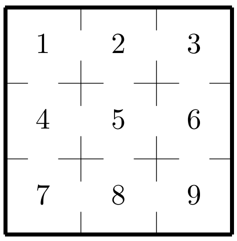

Figure 1: Labyrinth

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1. At each step, the mouse choose an adjacent square at random (and uniformly). Prove that $X = (X_n, n \in \mathbb{N})$ is a Markov chain and represent its transition graph. Classify the states of X.

2. Prove that $Y = (Y_n, n \in \mathbb{N})$ is a Markov chain and represent its transition graph. Compute the invariant probability measure of Y and deduce the one of X.

3. In fact the mouse has some memory, and at each step, the mouse choose an adjacent square at random (an uniformly), but not the one of the previous step. Prove that X is not a Markov chain. Set $Z_n = (X_{n-1}, X_n)$ and $W = (Y_{n-1}, Y_n)$. Prove that $Z = (Z_n, n \in \mathbb{N}^*)$ and $W = (W_n, n \in \mathbb{N}^*)$ are Markov chains and represent their transition graph.

4. Compute the invariant probability measure of W and deduce the one of Z. Under this invariant probability measure, what is the distribution of the first square and the second square?

$\triangle$

Exercise 3.3 (2 states Markov chain). Let $E = \{a, b\}$. The most general stochastic matrix can be written as:

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \quad \text{with} \quad \alpha, \beta \in [0,1].$$

1. Compute the invariant probability distribution(s). Give a necessary and sufficient condition for uniqueness of the invariant probability distribution.
2. Assume that $\alpha + \beta > 0$. Prove that for all $n \in \mathbb{N}$:

$$P^n = \begin{pmatrix} 1-p & p \\ 1-p & p \end{pmatrix} + \gamma^n \begin{pmatrix} p & -p \\ -1+p & 1-p \end{pmatrix},$$

   for some p and γ which shall be computed.
3. Assume that $\alpha + \beta \in (0, 2)$. Prove that for any probability measure μ_0 on E, the limit $\lim_{n\to+\infty} \mu_0 P^n$ exists and does not depend on μ_0.
4. When is P irreducible? When is P periodic? When does P have only one closed communicating class? When does P have an open communicating class?

$\triangle$

Exercise 3.4 (Examples). Let $(X_n, n \in \mathbb{N})$ be a Markov chain on $E = \{-1, 0, 1\}$ with transition matrix P. Set $Y_n = |X_n|$ for $n \in \mathbb{N}$.

1. Check that conditionally on $\{Y_n = 0\}$ the σ-fields $\sigma(Y_{n+k}, k \in \mathbb{N})$ and $\mathcal{F}_n = \sigma(X_k, 0 \leq k \leq n)$ are independent (that is for all $A \in \sigma(Y_{n+k}, k \in \mathbb{N})$ and $B \in \mathcal{F}_n$, we have $\mathbb{P}(A \bigcap B \,|\, Y_n = 0) = \mathbb{P}(A \,|\, Y_n = 0)\mathbb{P}(B \,|\, Y_n = 0)$).
2. Assume that $P(0,1) = P(-1,-1) = 0$, $P(-1,1)P(0,-1) > 0$ and $P(-1,0) \neq P(1,0)$. Compute $\mathbb{P}(Y_{n+1} = 0 \,|\, Y_n = 1, Y_{n-1} = 0)$ and $\mathbb{P}(Y_{n+1} = 0 \,|\, Y_n = 1, Y_{n-1} = 1, Y_{n-2} = 0)$ for $n \geq 2$ and deduce that $(Y_n, n \in \mathbb{N})$ is not a Markov chain.

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Exercise 3.5 (How many items will you buy?). You have an amount of $X_0 \in \mathbb{N}$ euros to spent in your favorite mall. Denote by X_k your wealth at step k. At step $k+1$, if $X_k > 0$, you buy the most fancy item you just found whose cost is uniformly distributed on $\{1, \ldots, X_k\}$; and if $X_k = 0$ then you can buy nothing. Prove that $(X_k, k \in \mathbb{N})$ is a Markov chain on $\mathbb{N}$ and classify all its states. Give its transition probability and its transition graph. Let N be the number of items you bought. Denoting $\mathbb{E}_n[\bullet] = \mathbb{E}[\bullet \,|\, X_0 = n]$, prove that for $n \in \mathbb{N}$:

$$\mathbb{E}_n[N] = \sum_{\ell=1}^{n} \frac{1}{\ell} \cdot$$

$\triangle$

3.2 Asymptotic behavior

Exercise 3.6 (Labyrinth (end)). We continue Exercise 3.2. Compute $\lim_{n\to\infty} n^{-1} \sum_{k=1} f(X_k)$ for the model given in Questions 1 and 3 and a function f defined on $\{1, \ldots, 9\}$. $\triangle$

Exercise 3.7 (Mean return time). Let $X = (X_n, n \in \mathbb{N})$ be an irreducible Markov chain on a space with a finite number N of elements.

1. Explain why there exits a unique invariant probability measure π.

2. Compute, as an explicit function of N, the mean return time under π, that is $\mathbb{E}[T]$, where X_0 is distributed as π and $T = \inf\{n \in \mathbb{N}^*,\ X_n = X_0\}$.

$\triangle$

Exercise 3.8 (Ferryman). Once upon a time, there was a dangerous river and only one brave ferryman to cross it. The ferryman can take only one passenger from one bank to the other. At each step $n \in \mathbb{N}$, one customer may appear independently on the left (resp. right) bank with probability p (resp. q) with $1 \geq p > q > 0$; during the same step the customer who just arrived either embark on the ferryboat if the ferryboat just arrived or was waiting there, or leaves if the ferryboat is not there. The embarkation and disembarkation of customers take one step (and can happens one the same step), and the crossing of the river takes one step. So, the possible states of the ferryman are: waiting or embarking on the left bank state $\mathcal{L}$; waiting or embarking on the right bank state $\mathcal{R}$; crossing the river from the left bank to the right state $\mathcal{LR}$; crossing from the right bank to the left state $\mathcal{RL}$. Thus, a very busy period for the ferryman correspond to the sequence $..., \mathcal{LR}, \mathcal{R}, \mathcal{RL}, \mathcal{L}, \mathcal{LR}, ...$, corresponding to ..., crossing, disembarking a customer and embarking an other one, crossing, disembarking a customer and embarking an other one, crossing, ... For the ferryman, waiting cost nothing but crossing has a cost c, and the price for the customer to cross is $C > c$. The ferryman consider the strategy $S^{[1]}$: waiting on the bank until there is a customer and then embarking him and crossing; and strategy $S^{[2]}$: waiting on the left bank until there is a customer and then embarking him and crossing from the left bank to the right, and staying only one step on the right bank for disembarking the customer on the ferryboat and also embarking a new customer if there is any. So a typical sequence for $S^{[1]}$ would be:

$$\mathcal{L}, \mathcal{L} \text{ (and embarking a customer)}, \mathcal{LR}, \mathcal{R}, \mathcal{R}, \mathcal{R} \text{ (and embarking a customer)}, \mathcal{RL}, ...,$$

while a typical sequence for $S^{[2]}$ would be:

$$\mathcal{L}, \mathcal{L} \text{ (and embarking a customer)}, \mathcal{LR}, \mathcal{R} \text{ (and possibly embarking nobody)}, \mathcal{RL}, ..., .$$

We denote $X_n^{[i]}$ the state of the ferryman at step $n \in \mathbb{N}$ with the strategy i.

1. Prove that $X^{[i]} = (X_n^{[i]}, n \in \mathbb{N})$ is Markov chain on $E = \{\mathcal{L}, \mathcal{R}, \mathcal{LR}, \mathcal{RL}\}$ and represent its transition graph.

2. Compute $\pi^{[i]}$ the invariant probability measure of $X^{[i]}$.

3. Denote by $G_n^{[i]}$ the gain of the ferryman at step n with the strategy i. Check that $((X_n^{[i]}, G_n^{[i]}), n \in \mathbb{N})$ is a Markov chain (notice there is nothing to do for $i = 1$, but one needs to replace the state $\mathcal{RL}$ by two states for $i = 2$).

4. Deduce $s^{[i]} = \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} G_k^{[i]}$ the asymptotic gain averaged in time. Compare the two strategies according to the values of $C > c$ and $p > q > 0$ (consider in particular the case $C = 2c$).

$\triangle$

Exercise 3.9 (Emptying the urn). We consider an urn and the following mechanism. At step $n \in \mathbb{N}$, with probability $p > 0$ a ball is added to the urn; with probability $1 - p > p$ a ball is removed from the urn (and nothing happens if the urn is empty). Let $X_n \in \mathbb{N}$ be the number of balls in the urn at step n.

1. Check that $X = (X_n, n \in \mathbb{N})$ is an irreducible aperiodic Markov chain.

2. Check that X is reversible with respect to the probability measure π on $\mathbb{N}$ defined by

$$\pi(k) = \frac{1-2p}{1-p}\left(\frac{p}{1-p}\right)^k.$$

3. Compute the asymptotic proportion of time the urn is empty.

4. Compute the mean time to empty the urn starting from an empty urn and then from an urn with only one ball.

$\triangle$

Exercise 3.10 (House of cards). Consider the house of cards Markov chain[2]: the state space is $E = \mathbb{N}$ and the transition matrix P is defined by $P(k, k+1) = p_k$, $P(k, 0) = 1 - p_k$ and $P(k, j) = 0$ if $j \notin \{0, k+1\}$, where $(p_k, k \in \mathbb{N})$ is a sequence of elements of $(0, 1)$. Check the corresponding Markov chain is irreducible. Give an interpretation of $\prod_{k=0}^n p_k$ using the Markov chain. Deduce that if $\prod_{k\in\mathbb{N}} p_k > 0$ the Markov chain is transient. $\triangle$

4 Martingales

4.1 First computations

Exercise 4.1 (Sum of squares). Let $(X_n, n \in \mathbb{N})$ be independent square integrable random variables, such that $\mathbb{E}[X_n] = 0$ for all $n \in \mathbb{N}$. Set $M_n = (\sum_{k=0}^n X_k)^2 - \sum_{k=0}^n \mathbb{E}[X_k^2]$ for $n \in \mathbb{N}$. Prove that $(M_n, n \in \mathbb{N})$ is a martingale (precise also the filtration). $\triangle$

Exercise 4.2 (Wald formula). Let $\mathbb{F} = (\mathcal{F}_n,\, n \in \mathbb{N})$ be filtration and $X = (X_n, n \in \mathbb{N}^*)$ a real-valued adapted integrable process with constant mean μ (that is $\mathbb{E}[X_n] = \mu$ for all $n \in \mathbb{N}$) and such that X_{n+1} is independent of $\mathcal{F}_n$ for all $n \in \mathbb{N}$. Set $S_0 = 0$ and $S_n = \sum_{k=1}^n X_k$ for $n \in \mathbb{N}$. Let τ be a stopping time with respect to $\mathbb{F}$. We say the Wald formula hold if

$$\mathbb{E}[S_\tau] = \mu\mathbb{E}[\tau].$$

1. Check that $M = (M_n = S_n - n\mu, n \in \mathbb{N})$ is a martingale.

2. Using the stopping time theorem, prove that $\mathbb{E}[S_{n\wedge\tau}] = \mu\mathbb{E}[n \wedge \tau]$ for all $n \in \mathbb{N}$.

3. Assume that X_n is non-negative for all $n \in \mathbb{N}$. Deduce the Wald formula holds.

4. Assume that X is bounded in L^1 (that is $\mu_0 = \sup_{n\in\mathbb{N}^*} \mathbb{E}[|X_n|]$ is finite) and $\mathbb{E}[\tau] < +\infty$. Deduce the Wald formula holds.

5. We consider the Jacob Bernoulli's game: a fair die is rolled, and if the result is $N \in \{1, \dots, 6\}$, then N dice are rolled. Give the expected total of the N dice.

$\triangle$

[2] W. Feller. *An introduction to probability theory and its applications. Vol. I.* Third edition. John Wiley & Sons, 1968. (See pages 381-382 (Examples 2.(l)), 390, 398, 403 and 408.)

Exercise 4.3 (Model for daily temperature (2)). We consider the following auto-regressive model for the daily temperature. Let X_n denote the temperature of day $n \in \mathbb{N}$. We assume that X_0 is a given constant and $X_{n+1} = aX_n + \varepsilon_{n+1}$ for $n \in \mathbb{N}$, where the random variables $(\varepsilon_n, n \in \mathbb{N}^*)$ are independent integrable, defined on Ω, with the same mean μ. Let $\mathbb{F} = (\mathcal{F}_n, n \in \mathbb{N})$ be the filtration defined by $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n$ be the σ-field generated by $(\varepsilon_1, \dots, \varepsilon_n)$. On what condition on (a, μ) do we have that $X = (X_n, n \in \mathbb{N})$ is a martingale. Give a sufficient condition for X to be a non-negative sub-martingale. △

Exercise 4.4 (A game). We consider the game: bet every thing as long as you win in a coin tossing game. Let $(X_n, n \in \mathbb{N}^*)$ be independent Bernoulli random variables such that $\mathbb{P}(X_k = 1) = \mathbb{P}(X_k = 0) = 1/2$. We consider the corresponding gain process $G = (G_n, n \in \mathbb{N})$ given by $G_0 = 1$, $G_n = 2^n \prod_{k=1}^n X_k = 2X_n G_{n-1}$ for $n \in \mathbb{N}^*$. Let $\mathbb{F}$ be the natural filtration of the process G.

1. Compute $\mathbb{E}[G_n]$ and prove that a.s. $\lim_{n\to\infty} G_n = 0$.

2. Check that G is a non-negative martingale.

3. Using the stopping time theorem, prove that $\mathbb{E}[G_\tau] = 1$ for any bounded stopping time.

4. Using Fatou's lemma deduce that $\mathbb{E}[G_\tau] \leq 1$ for any stopping time. (There is no winning strategy. And someone who is winning on average is thus cheating!)

5. Check that $\tau = \inf\{n \geq 1;\, G_n = 0\}$ is a (finite) stopping time and that it is distributed as a geometric random variable.

6. Check that $G_{\tau-1} = \sup_{n\in\mathbb{N}} G_n = 2^{\tau-1}$ and $\mathbb{E}[\sup_{n\in\mathbb{N}} G_n] = +\infty$. (The mean gain for someone who can foresee one step ahead is infinite!) Deduce that $\tau - 1$ is not a stopping time.

△

4.2 Convergence theorems

Exercise 4.5 (0-1 Kolmogorov's law). Let $(X_n, n \in \mathbb{N})$ be independent random variables. For $n \in \mathbb{N}$, set $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$ the σ-field of the past before n and $\mathcal{G}_n = \sigma(X_{n+k}, k \in \mathbb{N})$ the σ-field from the future. We set $\mathcal{F}_\infty = \bigvee_{n\in\mathbb{N}} \mathcal{F}_n$ and $\mathcal{G}_\infty = \bigcap_{n\in\mathbb{N}} \mathcal{G}_n$ the corresponding asymptotic σ-fields. Let $A \in \mathcal{G}_\infty$.

1. Check that $\mathcal{G}_\infty$ is indeed a σ-field and that $\mathcal{G}_\infty \subset \mathcal{F}_\infty$.

2. Check that $(M_n = \mathbb{E}[\mathbf{1}_A \,|\, \mathcal{F}_n], n \in \mathbb{N})$ is a martingale and that it converges a.s. to $\mathbf{1}_A$.

3. Check that M_n is a.s. constant and deduce that $\mathbb{P}(A)$ is either 1 or 0.

4. Deduce that $\{\liminf_{n\to\infty} n^{-1} \sum_{k=1}^n X_k \leq a\}$ is of probability either 0 or 1 for all $a \in [-\infty, +\infty]$. (In particular, averaging independent random variables either a.s. converges to a constant in $[-\infty, +\infty]$ or a.s. oscillates between two distinct deterministic bounds.)

△