EFFECTIVE REPRODUCTION NUMBER: CONVEXITY, INVARIANCE AND CORDONS SANITAIRES

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Abstract. We consider the problem of optimal allocation strategies for a (perfect) vaccine in an infinite-metapopulation model (including SIS, SIR, SEIR, ...), when the loss function is given by the effective reproduction number R_e, which is defined as the spectral radius of the effective next generation matrix (in finite dimension) or more generally of the effective next generation operator (in infinite dimension). We give sufficient conditions for R_e to be a convex or a concave function of the vaccination strategy. Then, following a previous work, we consider the bi-objective problem of minimizing simultaneously the cost and the loss of the vaccination strategies. In particular, we prove that a cordon sanitaire might not be optimal, but it is still better than the "worst" vaccination strategies. Inspired by the graph theory, we compute the minimal cost which ensures that no infection occurs using independent sets. Using Frobenius decomposition of the whole population into irreducible sub-populations, we give some explicit formulae for optimal ("best" and "worst") vaccinations strategies. Eventually, we provide equivalence properties on models which ensure that the function R_e is unchanged.

1. Introduction

1.1. **Vaccination in metapopulation models.** The study of vaccination strategies for metapopulation models with $N \geq 2$ sub-populations, naturally leads to an easily stated linear algebra problem: given a matrix K, of size $N \times N$, with non-negative entries, what can be said about the function

$$R_e : \begin{cases} \Delta & \to \mathbb{R}, \\ \eta & \mapsto \text{spectral radius of } K \cdot \mathrm{Diag}(\eta), \end{cases} \tag{1}$$

where $\Delta = [0,1]^N$, $\mathrm{Diag}(\eta)$ denotes the $N \times N$ matrix with diagonal elements $\eta = (\eta_1, \ldots, \eta_N)$, and the spectral radius is the largest modulus of the eigenvalues. In this form, the problem appears for instance, with a mathematical point of view, in Elsner and Hadeler [15], see also Friedland [18] and Nussbaum [30].

In metapopulation epidemiological models, the indices $i = 1, \ldots, N$ correspond to various sub-populations with respective proportional size $\mu_1, \ldots, \mu_N$. Following [21], the entry K_{ij} of the so-called next-generation matrix K is equal to the expected number of secondary infections for people in subgroup i resulting from a single randomly selected non-vaccinated infectious person in subgroup j. Finally, η represents a vaccination strategy, that is, η_i is the **fraction of non-vaccinated** individuals in the i^{th} sub-population; **thus $\eta_i = 0$ when the i^{th} sub-population is fully vaccinated, and 1 when it is not vaccinated at all**. (This seemingly unnatural convention is in particular motivated by the simple form of Equation (1)). So, the strategy $\mathbb{1} \in \Delta$, with all its entries equal to 1, corresponds to an entirely non-vaccinated

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population. The quantity R_e, referred to as the *effective reproduction number*, may then be interpreted as the mean number of infections coming from a typical case. In particular, we denote by $R_0 = R_e(\mathbb{1})$ the so-called *basic reproduction number* associated to the metapopulation epidemiological model. With the interpretation of the function R_e in mind, it is then very natural to minimize it under a constraint on the cost $C(\eta)$ of the vaccination strategies η. A natural choice for the cost function is given by the uniform cost $C(\eta) = 1 - \sum_i \eta_i \mu_i$, which corresponds to the fraction of vaccinated individuals in the population. This constrained optimization problem appears in most of the literature for designing efficient vaccination strategies for multiple epidemic situation (SIR/SEIR), see [6, 14, 16, 21, 29, 32, 38]. Note that in some of these references, the effective reproduction number is defined as the spectral radius of the matrix $\mathrm{Diag}(\eta) \cdot K$. Since the eigenvalues of $\mathrm{Diag}(\eta) \cdot K$ are exactly the eigenvalues of the matrix $K \cdot \mathrm{Diag}(\eta)$, this actually defines the same function R_e. In Section 2, we discuss the generalization of the effective reproduction number to the kernel model that offers a finer description of the contacts within the population.

The goal of this paper is to prove a number of properties of R_e, that shed a light on how to vaccinate in the best possible way. In previous works [7, 10], we introduced a general infinite-dimensional kernel framework in which the matrix formulation appears as a special finite-dimensional case. We state our results in this general framework, but for ease of the presentation, we shall stick to the matrix formulation in this introduction. Finally, the results of this paper are applied and illustrated in detail on various examples in the companion papers [8, 9, 12].

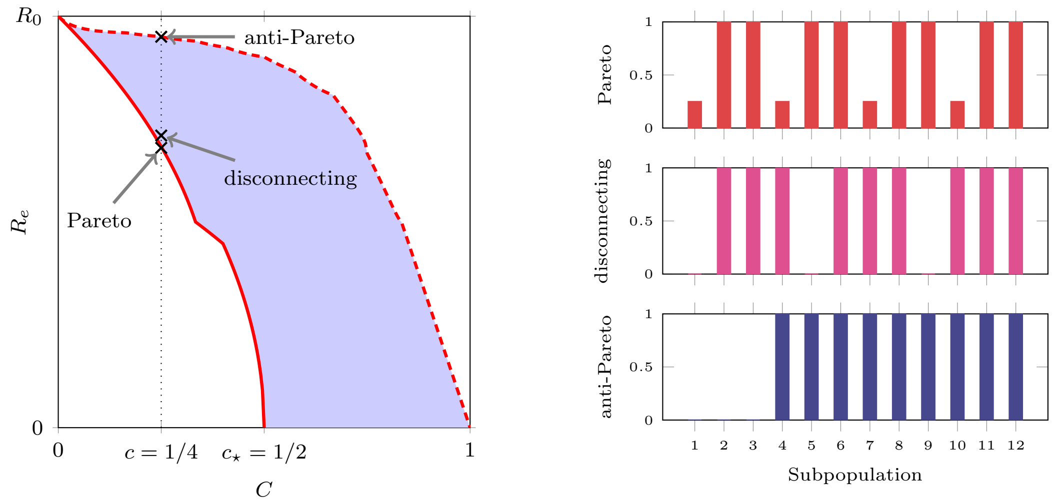
1.2. **Convexity properties of the effective reproduction number.** Given the importance of convexity to solve optimization problems efficiently, it is natural to look for conditions on the matrix K that imply convexity or concavity for the map R_e defined by (1). In their investigation of the behavior of this map in the finite dimensional matrix setting, Hill and Longini conjecture in [21] sufficient spectral conditions to get either concavity or convexity. More precisely, guided by explicit examples, they state that R_e should be convex if all the eigenvalues of K are non negative real numbers, and that it should be concave if all eigenvalues are real, with only one positive eigenvalue.

Our first series of results show that, while this conjecture cannot hold in full generality, see Section 5.1, it is true under an additional symmetry hypothesis. Recall that a matrix K is called diagonally symmetrizable if there exist positive numbers $(d_1, \dots d_N)$ such that for all i, j, $d_i K_{ij} = d_j K_{ji}$. Such a matrix is necessarily diagonalizable with real eigenvalues. The following result, which appears below in the text as Theorem 5.1, settles the conjecture for diagonally symmetrizable matrices. It is a special case of the more general Theorem 5.5, which holds in the infinite dimensional kernel setting, and for which the symmetry assumption has to be carefully worded. Let us mention that the eigenvalue λ_1 in the theorem below is non-negative and is equal to the spectral radius of K, that is, $\lambda_1 = R_e(\mathbb{1}) = R_0$, thanks to the Perron-Frobenius theory.

Theorem 1.1. *Let K be an $N \times N$ matrix with non-negative entries. Suppose that K is diagonally symmetrizable with eigenvalues $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_N$.*

(i) If $\lambda_N \geq 0$, then the function R_e is convex.
(ii) If $\lambda_2 \leq 0$, then the function R_e is concave.

Note that the case (i) appears already in Cairns [6]; see also [17, 18] and Section 5.1 below for a detailed comparison with existing results.



(A) Thick red line: Pareto frontier; Dashed line: anti-Pareto frontier; x marker: outcomes of various strategies; light blue: all possible outcomes $(C(\eta), R_e(\eta))$ for $\eta \in \Delta$.

(B) Profile of various strategies.

FIGURE 1. Performance of the disconnecting vaccination strategy "one in 4" for the non-oriented cycle graph with 12 nodes and uniform cost 1/4.

It is easy to see that if K and K' are diagonally similar up to transposition, they define the same function R_e (see [11] for more results in this direction). We check in Section 4 that this is essentially still true in the generalized kernel setting.

1.3. **Properties of Pareto and anti-Pareto optima, cordons sanitaires.** Let us now come back to the problem of finding optimal vaccination strategies. In contrast with our previous work [10], where we put minimal assumptions on the loss function which measures the efficiency of the vaccination strategies, we consider here that the loss of a strategy η is given by its effective reproduction number $R_e(\eta)$. This focus and the fact that we consider strictly decreasing cost functions (because vaccinating more costs more, see Section 6.1), allow us to simplify some of the statements of [10] and to give additional specific results.

The problem of minimizing the effective reproduction number while keeping the cost of the vaccination low leads to a bi-objective optimization problem. We recall in Section 6.1 the setting introduced in detail in [10] for a general framework. One can identify Pareto optimal and anti-Pareto optimal vaccinations strategies, informally "best" and "worst" vaccination strategies, and consider the Pareto frontier $\mathcal{F}$ (resp. anti-Pareto frontier $\mathcal{F}^{\text{Anti}}$) as the outcomes $(C(\eta), R_e(\eta))$ of the Pareto (resp. anti-Pareto) optimal strategies η.

In Figure 1(A), we have plotted in red the Pareto frontier and in a dashed red line the anti-Pareto frontier when the next-generation matrix is the adjacency matrix of the non-oriented cycle graph with $N = 12$ nodes from Figure 2(A) and Example 1.2, see also Example 2.1.

1.3.1. *A cordon sanitaire is not the worst vaccination strategy.* Recall that a matrix K is reducible if there exists a permutation σ such that $(K_{\sigma(i)\sigma(j)})_{i,j}$ is block upper triangular, and irreducible otherwise. A *cordon sanitaire* is a vaccination strategy η such that the infection matrix between non-vaccinated people, $K \cdot \text{Diag}(\eta)$, is reducible: informally, such a vaccination cuts the effective population in two or more groups that do not infect one another.

Disconnecting the population by creating a cordon sanitaire is not always the "best" choice, that is, it may not be Pareto optimal. However, we prove in Proposition 6.6 that a cordon sanitaire can never be anti-Pareto optimal; this result still holds in the general kernel framework, provided that the definition of cordon sanitaires is generalized in an appropriate way.

Example 1.2 (Non-oriented cycle graph). Suppose that the matrix K is given by the adjacency matrix (see Figure 2(B) for a grayplot representation) of the non-oriented cycle graph with $N = 12$ nodes; see Figure 2(A). For a cost $C_{\text{uni}} = 1/4$, there is a disconnecting strategy η that consists in vaccinating one sub-population in four; see Figure 2(C) (and Figure 2(D) for a grayplot representation of the corresponding adjacency matrix). The effective reproduction number associated is equal to $\sqrt{2}$. This strategies performs better than the anti-Pareto optimal strategy and is out-performed by the Pareto optimal one as we can see in Figure 1. This example is discussed in detail in [9, Section 2.4].

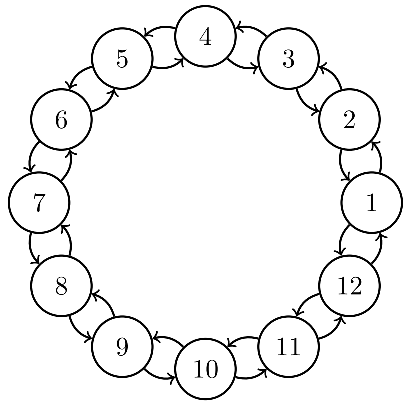
1.3.2. *Minimal cost required to completely stop the transmission of the disease.* A vaccination strategy η such that $R_e(\eta) = 0$ completely eradicates the epidemic. Section 6.4 is devoted to the characterization of the minimal cost of such vaccinations, which is denoted by $c_\star$. This quantity is introduced and discussed in [10] under general assumption for the loss function. Since we consider here the special case of measuring the loss by the effective reproduction number R_e, we are able to give in Proposition 6.11 an explicit expression of this quantity in the kernel model. In the symmetric matrix case, when the cost is uniform (the cost is proportional to the number of vaccinated individuals), this expression is proportional to the size of maximal independent sets of the non-oriented graph with vertices $\{1, \ldots, N\}$, where there is an edge between i and j if and only if $K_{ij} > 0$.

We can observe this property in Figure 1(A) as the size of the maximal independent set of the non-oriented cycle graph of size N from Example 1.2 is equal to $\lfloor N/2 \rfloor$.

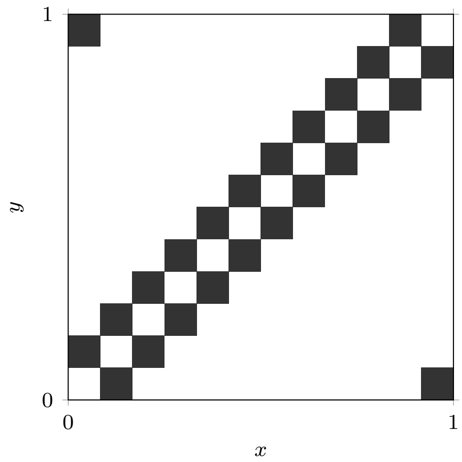
1.3.3. *Reducible case.* When the matrix K happens to be reducible, up to a relabeling, we may assume that it is block upper triangular. Denoting by m the number of blocks and $I_1, \ldots, I_m$ the sets of indices describing the blocks, this means that for all $\ell > k$ and $(i, j) \in I_\ell \times I_k$, we have $K_{ij} = 0$. In the epidemiological interpretation, this means that the populations with indices in I_k never infect the ones with indices in I_ℓ. One may then hope that the study of R_e can be effectively reduced to the study of the effective radius of the square sub-matrices $(K_{ij})_{i,j\in I_k}$ describing the infections within block I_k. This is indeed the case, and we give in Section 7 a complete picture of the Pareto and anti-Pareto frontiers of R_e, in terms of the effective reproduction numbers restricted to each irreducible component of the infection kernel or matrix. In particular, this allows a better understanding of the possible disconnection of the anti-Pareto frontier, whereas the Pareto frontier is always connected. Once more, special care has to be taken with the definitions when handling the infinite dimensional kernel case.

1.3.4. *Optimal ray.* It is observed by Poghotanyan, Feng, Glasser and Hill in [32], that if there exists a Pareto optimal strategy η with all its entries strictly less than 1, then all the strategies $\lambda\eta$, with $\lambda \geq 0$ such that $\lambda\eta \in \Delta$, are Pareto optimal. We give a short proof on the existence of such optimal rays in Section 6.2, when one assumes that the cost function C is affine on Δ.

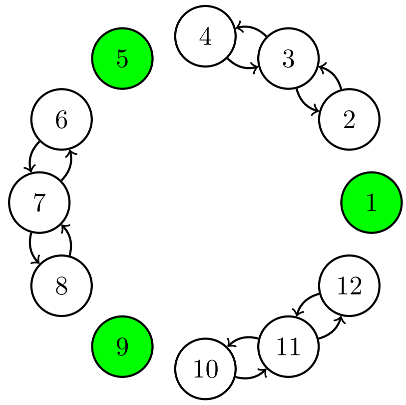
1.4. **Structure of the paper.** We discuss in Section 2 the generality of the setting, showing that studying vaccination strategies in many different epidemic models gives rise to the same optimization problem. After recalling formally our infinite dimensional kernel setting in Section 3, we discuss invariance properties of R_e in Section 4. The convexity properties of R_e and the related conjecture of Hill and Longini are discussed in Section 5. Various properties of the



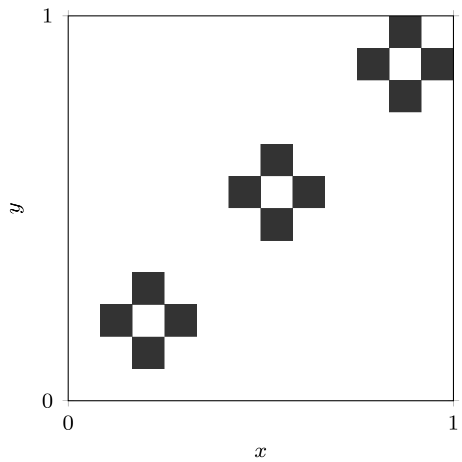
(A) The non-oriented cycle graph.



(B) Grayplot of the corresponding kernel.



(C) Cordon sanitaire corresponding to the “one in 4” vaccination strategy (in green the vaccinated groups).



(D) Grayplot of the corresponding kernel.

Figure 2. Example of disconnecting vaccination strategy on the non-oriented cycle graph with $N = 12$ nodes.

Pareto and anti-Pareto frontiers, and in particular the fact that establishing a *cordon sanitaire* by disconnecting the population is never the worst solution, are discussed in Section 6. Finally, the case of reducible kernels is treated in Section 7.

2. Discussion on the next-generation operator

In [7, 10], we developed a framework that we call the kernel model where the population is represented as an abstract probability space $(\Omega, \mathscr{F}, \mu)$. Individuals are characterized by a feature $x \in \Omega$, and the relative size of the sub-population with feature x is given by $\mu(\mathrm{d}x)$. The underlying structure described by this feature can be very varied, typical examples being one or several of the following characteristics: spatial position, social contacts, susceptibility, infectiousness, characteristics of the immunological response, ... The analogue of the next-generation matrix K is then the kernel operator defined formally by:

$$T_{\mathrm{k}}(g)(x) = \int_\Omega \mathrm{k}(x,y)\, g(y)\, \mathrm{d}\mu(y);$$

where the non-negative kernel k is defined on $\Omega \times \Omega$ and $\mathrm{k}(x,y)$ still represents a strength of infection from y to x. Vaccination strategies $\eta : \Omega \to [0,1]$ encode the density of non-vaccinated individuals with respect to the measure μ. The (sub-probability) measure $\eta(y)\,\mu(\mathrm{d}y)$ may then be understood as an effective population, giving rise to an effective next-generation operator:

$$T_{\mathrm{k}\eta}(g)(x) = \int_\Omega \mathrm{k}(x,y)\, g(y)\, \eta(y)\, \mu(\mathrm{d}y).$$

The effective reproduction number is then defined by $R_e(\eta) = \rho(T_{\mathrm{k}\eta})$, where ρ stands for the spectral radius of the operator and $\mathrm{k}\eta$ for the kernel $(\mathrm{k}\eta)(x,y) = \mathrm{k}(x,y)\eta(y)$.

Most of the results mentioned in the introduction will be given in this general framework as we argue that the latter is sufficiently flexible to describe a wide range of epidemic models from the literature including the metapopulation models. We give in the following a few examples to support this claim: in each of them, the spectral radius of a particular, explicit kernel operator appears as a threshold parameter, and the epidemic either "invades/survives" or "dies out" depending on the value of this parameter. Classical notations are used: S denotes the proportion of susceptible individuals, E the proportion of those who have been exposed to the disease, I the proportion of infected individuals, R the proportion of removed individuals in the population.

Example 2.1 (Meta-population models). Recall that in metapopulation models, the population is divided into $N \geq 2$ different sub-populations of respective proportional size $\mu_1, \ldots, \mu_N$, and the reproduction number is given by $R_e(\eta) = \rho(K \cdot \mathrm{Diag}(\eta))$, where K is the next generation matrix and η belongs to $[0,1]^N$ and gives the proportion of non-vaccinated individuals in each sub-population. To express the function R_e as the effective reproduction number of a kernel model, consider the discrete state space $\Omega_{\mathrm{d}} = \{1, \ldots, N\}$ equipped with the probability measure μ_{d} defined by $\mu_{\mathrm{d}}(\{i\}) = \mu_i$, and let k_{d} denote the discrete kernel on Ω_{d} defined by:

$$\mathrm{k}_{\mathrm{d}}(i,j) = K_{ij}/\mu_j. \tag{2}$$

For all $\eta \in \Delta = [0,1]^N$, the matrix $K \cdot \mathrm{Diag}(\eta)$ is the matrix representation of the endomorphism $T_{\mathrm{k_d}\eta}$ in the canonical basis of $\mathbb{R}^N$. In particular, we have: $R_e(\eta) = \rho(T_{\mathrm{k}\eta}) = \rho(K \cdot \mathrm{Diag}(\eta))$.

In Figure 2(B), we have plotted the kernel on $[0,1]$ associated to $\mathrm{k_d}$ for the non-oriented cycle graph when the sub-populations have the same size.

Example 2.2 (An SIR model with nonlinear incidence rate and vital dynamics). In [35], Thieme proposed an SIR model in an infinite-dimensional population structure with a nonlinear incidence rate. The structure space is given by Ω a compact subset of $\mathbb{R}^N$ equipped with the normalized Lebesgue measure. We restrict slightly his assumption so that the incidence rate is a linear function of the number of susceptible. The dynamic of the epidemic then writes:

$$\text{For } t \geq 0,\ x \in \Omega, \qquad \begin{cases} \partial_t S(t,x) = \Lambda(x) - \nu(x) S(t,x) - S(t,x) \int_\Omega f(I(t,y),x,y)\,\mathrm{d}y, \\[1em] \partial_t I(t,x) = S(t,x) \int_\Omega f(I(t,y),x,y)\,\mathrm{d}y - (\gamma(x) + \nu(x)) I(t,x), \\[1em] \partial_t R(t,x) = \gamma(x) I(t,x). \end{cases} \tag{3}$$

Here $\Lambda(x)$ is the rate at which fresh susceptible individuals are recruited into the population at location x, $\nu(x)$ is the *per capita* death rate of the individuals, and $\gamma(x)$ is the *per capita* recovery rate of infectious individuals The integral term describes the incidence at x at time t, *i.e.*, the rate of new infections. Thieme identified a threshold parameter that plays the role of the reproduction number, and is given by the spectral radius of the operator T_{k} with the kernel given by:

$$\mathrm{k}(x,y) = \frac{\Lambda(x)}{\gamma(x)+\nu(x)}\partial_I f(0,x,y), \quad x,y \in \Omega, \tag{4}$$

where $\partial_I f(0,x,y)$, the derivative of f with respect to I, is supposed to be non-negative.

Suppose that individuals at location x are vaccinated with probability $1-\eta(x)$ at birth so that the susceptible individuals with feature x are recruited at rate $\eta(x)\Lambda(x)$ and recovered/immunized individuals are also recruited at rate $(1-\eta(x))\Lambda(x)$ at location x. The threshold parameter $R_e(\eta)$ is then given by the spectral radius of the integral operator $T_{\eta\mathrm{k}}$ with kernel $\eta\mathrm{k}$ given by $(\eta\mathrm{k})(x,y) = \eta(x)\mathrm{k}(x,y)$. According to Lemma 3.1 (ii), we have $\rho(T_{\eta\mathrm{k}}) = \rho(T_{\mathrm{k}\eta})$, and our framework can be used for this model.

Under regularity assumptions on the parameters of the model, Thieme proved that if $R_e(\eta)$ is greater than 1, then there exists an endemic equilibrium that attracts all the solutions while if $R_e(\eta)$ is smaller than 1, then $I(t,x)$ converges to 0 for all $x \in \Omega$ as t goes to infinity.

Example 2.3 (An SEIR model without vital dynamics). In [1], Almeida, Bliman, Nadin and Perthame studied an heterogeneous SEIR model where the population is again structured with a bounded subset $\Omega \subset \mathbb{R}^N$ equipped with the normalized Lebesgue measure. The dynamic of the susceptible, exposed, infected and recovered individuals writes:

$$\text{For } t \geq 0,\ x \in \Omega, \quad \begin{cases} \partial_t S(t,x) = -S(t,x) \int_\Omega k(x,y) I(t,y)\,\mathrm{d}y, \\[1ex] \partial_t E(t,x) = S(t,x) \int_\Omega k(x,y) I(t,y)\,\mathrm{d}y - \alpha(x)E(t,x), \\[1ex] \partial_t I(t,x) = \alpha(x)E(t,x) - \gamma(x)I(t,x), \\[1ex] \partial_t R(t,x) = \gamma(x)I(t,x). \end{cases} \tag{5}$$

Here, the average incubation rate is denoted by $\alpha(x)$ and the average recovery rate by $\gamma(x)$; both quantities may depend upon the trait x. The function k is the transmission kernel of the disease. In this model, the basic reproduction number is given by the spectral radius of the integral operator T_{k} with kernel $\mathrm{k} = k/\gamma$:

$$\mathrm{k}(x,y) = k(x,y)/\gamma(y). \tag{6}$$

Suppose that, prior to the beginning of the epidemic, the decision maker immunizes a density $1-\eta$ of individuals. According to [1, Section 3.2], the effective reproduction number is given by $\rho(T_{\eta\mathrm{k}})$ which is also equal to $\rho(T_{\mathrm{k}\eta})$, see Lemma 3.1 (ii) below, and our model is indeed suitable for studying the vaccination strategies in this context.

Example 2.4 (An SIS model without vital dynamic). In [7], generalizing the discrete model of Lajmanovich and Yorke [27], we introduced the following heterogeneous SIS model where the population is structured with an abstract probability space $(\Omega, \mathscr{F}, \mu)$:

$$\text{For } t \geq 0,\ x \in \Omega, \quad \begin{cases} \partial_t S(t,x) = -S(t,x) \int_\Omega k(x,y) I(t,y)\,\mathrm{d}y + \gamma(x)I(t,x), \\[1ex] \partial_t I(t,x) = S(t,x) \int_\Omega k(x,y) I(t,y)\,\mathrm{d}y - \gamma(x)I(t,x). \end{cases} \tag{7}$$

The function γ is the *per-capita* recovery rate and k is the transmission kernel. For this model, $R_e(\eta) = \rho(T_{\mathrm{k}\eta})$ where $\mathrm{k} = k/\gamma$ is defined by (6).

Suppose that, prior to the beginning of the epidemic, a density $1-\eta$ of individuals is vaccinated with a perfect vaccine. In the same way as for the SEIR model, we proved, as t goes to infinity, that if $R_e(\eta)$ is smaller than or equal to 1, then $I(t,\cdot)$ converges to 0, and, under a connectivity assumption on the kernel k, that if $R_e(\eta)$ is greater than 1, then $I(t,\cdot)$ converges to an endemic equilibrium. This highlights the importance of R_e in the design of vaccination strategies.

3. Setting, notations and previous results

3.1. **Spaces, operators, spectra.** All metric spaces (S,d) are endowed with their Borel σ-field denoted by $\mathscr{B}(S)$. The set $\mathscr{K}$ of compact subsets of $\mathbb{C}$ endowed with the Hausdorff distance d_{H} is a metric space, and the function rad from $\mathscr{K}$ to $\mathbb{R}_+$ defined by $\mathrm{rad}(K)=\max\{|\lambda|\,,\,\lambda\in K\}$ is Lipschitz continuous from $(\mathscr{K},d_{\mathrm{H}})$ to $\mathbb{R}$ endowed with its usual Euclidean distance.

Let $(\Omega,\mathscr{F},\mu)$ be a probability space. We denote by Δ the set of $[0,1]$-valued measurable functions defined on Ω. For f and g real-valued functions defined on Ω, we may write $\langle f,g\rangle$ or $\int_\Omega fg\,\mathrm{d}\mu$ for $\int_\Omega f(x)g(x)\,\mu(\mathrm{d}x)$ whenever the latter is meaningful. For $p\in[1,+\infty]$, we denote by $L^p=L^p(\mu)=L^p(\Omega,\mu)$ the space of real-valued measurable functions g defined Ω such that $\|g\|_p=\left(\int|g|^p\,\mathrm{d}\mu\right)^{1/p}$ (with the convention that $\|g\|_\infty$ is the μ-essential supremum of $|g|$) is finite, where functions which agree μ-a.s. are identified. We denote by L^p_+ the subset of L^p of non-negative functions.

Let $(E,\|\cdot\|)$ be a Banach space. We denote by $\|\cdot\|_E$ the operator norm on $\mathcal{L}(E)$ the Banach algebra of bounded operators. The spectrum $\mathrm{Spec}(T)$ of $T\in\mathcal{L}(E)$ is the set of $\lambda\in\mathbb{C}$ such that $T-\lambda\mathrm{Id}$ does not have a bounded inverse operator, where Id is the identity operator on E. Recall that $\mathrm{Spec}(T)$ is a compact subset of $\mathbb{C}$, and that the spectral radius of T is given by:

$$\rho(T)=\mathrm{rad}(\mathrm{Spec}(T))=\lim_{n\to\infty}\|T^n\|_E^{1/n}. \tag{8}$$

The element $\lambda\in\mathrm{Spec}(T)$ is an eigenvalue if there exists $x\in E$ such that $Tx=\lambda x$ and $x\neq 0$. Following [26], we define the (algebraic) multiplicity of $\lambda\in\mathbb{C}$ by:

$$\mathrm{m}(\lambda,T)=\dim\left(\bigcup_{k\in\mathbb{N}^*}\ker(T-\lambda\mathrm{Id})^k\right),$$

so that λ is an eigenvalue if $\mathrm{m}(\lambda,T)\geq 1$. We say the eigenvalue λ of T is *simple* if $\mathrm{m}(\lambda,T)=1$.

If E is also an algebra, for $g\in E$, we denote by M_g the multiplication (possibly unbounded) operator defined by $M_g(h)=gh$ for all $h\in E$.

3.2. **Invariance and continuity of the spectrum for compact operators.** We collect some known results on the spectrum and multiplicity of eigenvalues related to compact operators. Let $(E,\|\cdot\|)$ be a Banach space. Let $A\in\mathcal{L}(E)$. We denote by $A^\top$ the adjoint of A. A sequence $(A_n,n\in\mathbb{N})$ of elements of $\mathcal{L}(E)$ converges strongly to $A\in\mathcal{L}(E)$ if $\lim_{n\to\infty}\|A_nx-Ax\|=0$ for all $x\in E$. Following [2], a set of operators $\mathscr{A}\subset\mathcal{L}(E)$ is *collectively compact* if the set $\{Ax:\ A\in\mathscr{A},\ \|x\|\leq 1\}$ is relatively compact. Recall that the spectrum of a compact operator is finite or countable and has at most one accumulation point, which is 0. Furthermore, 0 belongs to the spectrum of compact operators in infinite dimension. We refer to [33] for an introduction to Banach lattices and positive operators; we shall only consider the Banach lattices $L^p(\Omega,\mu)$ for $p\geq 1$ on a probability space $(\Omega,\mathscr{F},\mu)$ and a bounded operator A is positive if $A(L^p_+)\subset L^p_+$.

Lemma 3.1. *Let A,B be elements of $\mathcal{L}(E)$.*

(i) If E is a Banach lattice, and if A, B and $A-B$ are positive operators, then we have:

$$\rho(A) \geq \rho(B). \tag{9}$$

(ii) If A is compact, then we have AB and BA compact and:

$$\mathrm{Spec}(A) = \mathrm{Spec}(A^\top) \quad \textit{and} \quad \mathrm{m}(\lambda, A) = \mathrm{m}(\lambda, A^\top) \quad \textit{for } \lambda \in \mathbb{C}^*, \tag{10}$$

$$\mathrm{Spec}(AB) = \mathrm{Spec}(BA \quad \textit{and} \quad \mathrm{m}(\lambda, AB) = \mathrm{m}(\lambda, BA) \quad \textit{for } \lambda \in \mathbb{C}^*, \tag{11}$$

and in particular:

$$\rho(AB) = \rho(BA). \tag{12}$$

(iii) Let $(E', \|\cdot\|')$ be a Banach space such that E' is continuously and densely embedded in E. Assume that $A(E') \subset E'$, and denote by A' the restriction of A to E' seen as an operator on E'. If A and A' are compact, then we have:

$$\mathrm{Spec}(A) = \mathrm{Spec}(A') \quad \textit{and} \quad \mathrm{m}(\lambda, A) = \mathrm{m}(\lambda, A') \quad \textit{for } \lambda \in \mathbb{C}^*. \tag{13}$$

(iv) Let $(A_n, n \in \mathbb{N})$ be a collectively compact sequence which converges strongly to A. Then, we have $\lim_{n\to\infty} \mathrm{Spec}(A_n) = \mathrm{Spec}(A)$ in $(\mathscr{K}, d_{\mathrm{H}})$, $\lim_{n\to} \rho(T_n) = \rho(T)$ and for $\lambda \in \mathrm{Spec}(A) \cap \mathbb{C}^*$, $r > 0$ such that $\lambda' \in \mathrm{Spec}(A)$ and $|\lambda - \lambda'| \leq r$ implies $\lambda = \lambda'$, and all n large enough:

$$\mathrm{m}(\lambda, A) = \sum_{\lambda' \in \mathrm{Spec}(A_n),\, |\lambda - \lambda'| \leq r} \mathrm{m}(\lambda', A_n). \tag{14}$$

Proof. Property (i) can be found in [28, Theorem 4.2]. Equation (10) from Property (ii) can be deduced from from [26, Theorem p. 20]. Using [26, Proposition p. 25], we get the second part of (11) and $\mathrm{Spec}(AB) \cap \mathbb{C}^* = \mathrm{Spec}(BA) \cap \mathbb{C}^*$, and thus (12) holds. To get the first part of (11), see [10, Lemma 3.2].

We now provide a short proof for Property (iii). According to [20, Corollary 1 and Section 6], we have $\mathrm{Spec}(A) = \mathrm{Spec}(A')$. Let $\lambda \in \mathrm{Spec}(A) \cap \mathbb{C}^*$. Since the multiplicity of λ for A is finite, we get that $\mathrm{m}(\lambda, A) = \dim(\ker(A - \lambda \mathrm{Id})^n)$ for n large enough, and similarly for $\mathrm{m}(\lambda, A')$. Clearly, we have $\ker(A' - \lambda \mathrm{Id})^n \subset \ker(A - \lambda \mathrm{Id})^n$. Let us prove that $\ker(A - \lambda \mathrm{Id})^n \subset \ker(A' - \lambda \mathrm{Id})^n$. Let $x \in \ker(A - \lambda \mathrm{Id})^n$ and $(x_\ell, \ell \in \mathbb{N})$ be a sequence of elements of E' which converges (in E) towards x. Up to taking a sub-sequence, since A' is compact, we can assume that $A' x_\ell$ converges in E', say towards $y \in E'$. We deduce that:

$$\begin{aligned}
\lambda^n x &= \sum_{k=1}^n \binom{n}{k} (-\lambda)^{n-k+1} A^k x \\
&= \lim_{\ell\to\infty} \sum_{k=1}^n \binom{n}{k} (-\lambda)^{n-k+1} A^k x_\ell \\
&= \lim_{\ell\to\infty} \sum_{k=1}^n \binom{n}{k} (-\lambda)^{n-k+1} (A')^{k-1} (A' x_\ell) \\
&= \sum_{k=1}^n \binom{n}{k} (-\lambda)^{n-k+1} (A')^{k-1} y.
\end{aligned}$$

Since $\lambda \neq 0$, we get that x belongs to E' and thus $(A' - \lambda \mathrm{Id})^n x = (A - \lambda \mathrm{Id})^n x = 0$, that is $\ker(A - \lambda \mathrm{Id})^n \subset \ker(A' - \lambda \mathrm{Id})^n$. Then use the definition of the multiplicity to conclude.

We eventually check Point (iv). We deduce from [2, Theorems 4.8 and 4.16] (see also (d), (g) [take care that $d(\lambda, K)$ therein is the algebraic multiplicity of λ for the compact operator K and not the geometric multiplicity] and (e) in [3, Section 3]) that $\lim_{n\to\infty} \mathrm{Spec}(T_n) = \mathrm{Spec}(T)$ and (14). Then use that the function rad is continuous to deduce the convergence of the spectral radius from the convergence of the spectra. □

3.3. **Kernel operators.** We define a *kernel* (resp. *signed kernel*) on Ω as a $\mathbb{R}_+$-valued (resp. $\mathbb{R}$-valued) measurable function defined on $(\Omega^2, \mathscr{F}^{\otimes 2})$. For f, g two non-negative measurable functions defined on Ω and k a kernel on Ω, we denote by $f\mathrm{k}g$ the kernel defined by:

$$f\mathrm{k}g : (x, y) \mapsto f(x)\, \mathrm{k}(x, y) g(y). \tag{15}$$

For $p \in (1, +\infty)$, we define the double norm of a signed kernel k on L^p by:

$$\|\mathrm{k}\|_{p,q} = \left(\int_\Omega \left(\int_\Omega |\mathrm{k}(x, y)|^q \, \mu(\mathrm{d}y) \right)^{p/q} \mu(\mathrm{d}x) \right)^{1/p} \quad \text{with } q \text{ given by } \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{16}$$

We say that k has a finite double norm, if there exists $p \in (1, +\infty)$ such that $\|\mathrm{k}\|_{p,q} < +\infty$. To such a kernel k, we then associate the positive integral operator T_k on L^p defined by:

$$T_\mathrm{k}(g)(x) = \int_\Omega \mathrm{k}(x, y)\, g(y)\, \mu(\mathrm{d}y) \quad \text{for } g \in L^p \text{ and } x \in \Omega. \tag{17}$$

According to [19, p. 293], T_k is compact. It is well known and easy to check that:

$$\|T_\mathrm{k}\|_{L^p} \leq \|\mathrm{k}\|_{p,q}. \tag{18}$$

We define the *reproduction number* associated to the operator T_k as:

$$R_0[\mathrm{k}] = \rho(T_\mathrm{k}). \tag{19}$$

The proof of the next stability result appears already in [10] (but for (20) whose proof relies on (14) and is left to the reader).

Corollary 3.2. *Let $p \in (1, +\infty)$. Let $(\mathrm{k}_n, n \in \mathbb{N})$ and k be kernels on Ω with finite double norms on L^p such that $\lim_{n\to\infty} \|\mathrm{k}_n - \mathrm{k}\|_{p,q} = 0$. Then, we have $\lim_{n\to\infty} \mathrm{Spec}(T_{\mathrm{k}_n}) = \mathrm{Spec}(T_\mathrm{k})$ in $(\mathscr{K}, d_\mathrm{H})$, $\lim_{n\to} \rho(T_{\mathrm{k}_n}) = \rho(T_\mathrm{k})$ and for $\lambda \in \mathrm{Spec}(T_\mathrm{k}) \cap \mathbb{C}^*$, $r > 0$ such that $\lambda' \in \mathrm{Spec}(T_\mathrm{k})$ and $|\lambda - \lambda'| \leq r$ implies $\lambda = \lambda'$, and all n large enough:*

$$\mathrm{m}(\lambda, T_\mathrm{k}) = \sum_{\lambda' \in \mathrm{Spec}(T_{\mathrm{k}_n}),\, |\lambda - \lambda'| \leq r} \mathrm{m}(\lambda', T_{\mathrm{k}_n}). \tag{20}$$

3.4. **Irreducibility, quasi-irreducibility and monatomic kernel.** We first define *irreducible* and *monatomic* kernels. For $A, B \in \mathscr{F}$, we write $A \subset B$ a.s. if $\mu(B^c \cap A) = 0$ and $A = B$ a.s. if $A \subset B$ a.s. and $B \subset A$ a.s. For $A, B \in \mathscr{F}$, $x \in \Omega$ and an integrable kernel k, we simply write $\mathrm{k}(x, A) = \int_A \mathrm{k}(x, y)\, \mu(\mathrm{d}y)$, $\mathrm{k}(B, x) = \int_B \mathrm{k}(z, x)\, \mu(\mathrm{d}z)$ and:

$$\mathrm{k}(B, A) = \int_{B \times A} \mathrm{k}(z, y)\, \mu(\mathrm{d}z)\mu(\mathrm{d}y).$$

A set $A \subset \mathscr{F}$ is k-*invariant*, or simply *invariant* when there is no ambiguity on the kernel k, if $\mathrm{k}(A^c, A) = 0$. In the epidemiological setting, the set A is invariant if the sub-population A does not infect the sub-population A^c. If k is symmetric, then A is invariant if and only if A^c is invariant.

A kernel k is *irreducible* or *connected* if any k-invariant set A is such that a.s. $A = \emptyset$ or a.s. $A = \Omega$. According to [33, Theorem V.6.6], if k is an irreducible kernel with finite double norm, then we have $R_0[\mathrm{k}] > 0$. If the kernel is positive a.s., then it is irreducible. Following [5, Definition 2.11], we say that a kernel is *quasi-irreducible* if k restricted to $\{\mathrm{k} \equiv 0\}^c$, with $\{\mathrm{k} \equiv 0\} = \{x \in \Omega : \mathrm{k}(x, \Omega) + \mathrm{k}(\Omega, x) = 0\}$, is irreducible. The quasi-irreducible property was introduced for symmetric kernel; for general kernel one can consider the following weaker property. A kernel k is *monatomic* if the operator T_{k} has a unique (up to a multiplicative constant) non-negative eigenfunction. Intuitively, this corresponds to have only one irreducible component. Formally, this is also equivalent to the following two properties:

(i) There exists a measurable subset $\Omega_{\mathrm{a}} \subset \Omega$, the irreducible component or *atom* such that:
  - $\mu(\Omega_{\mathrm{a}}) > 0$ and the kernel k restricted to Ω_a is irreducible.
  - If a.s. $\Omega_{\mathrm{a}}^c \neq \emptyset$ then the restriction of T_{k} to Ω_{a}^c is quasi-nilpotent, that is, $R_e[\mathrm{k}](\mathbb{1}_{\Omega_{\mathrm{a}}^c}) = 0$.

(ii) There exists a measurable subset $\Omega_{\mathrm{i}} \subset \Omega_{\mathrm{a}}^c$, "the sub-population infected by" Ω_{a}, such that:
  - The sets $\Omega_{\mathrm{a}} \cup \Omega_{\mathrm{i}}$ and Ω_{i} are invariant.
  - The set Ω_{i} is the minimal set such that $\Omega_{\mathrm{a}} \cup \Omega_{\mathrm{i}}$ is invariant: if A is invariant and $\Omega_{\mathrm{a}} \subset A$ then a.s. $\Omega_{\mathrm{i}} \subset A$.

In the epidemiological setting, the sub-population Ω_{i} can only infect itself, and the sub-population Ω_{a} infects only itself and Ω_{i}; the set $\Omega_{\mathrm{a}} \cup \Omega_{\mathrm{i}}$ corresponds to the support of the endemic equilibrium in the supercritical regime, see [10, Lemma 5.12]. We refer to [34] for further details on the decomposition of a kernel on its irreducible components; in particular the sets Ω_{a} and Ω_{i} are unique up to the a.s. equivalence. We represented in Figure 3(A) a monatomic kernel and in Figure 3(B) a quasi-irreducible kernel; the set Ω being "nicely ordered" so that the representation of the kernels are upper triangular.

Remark 3.3. Irreducible and quasi-irreducible kernels are also monatomic (take $\Omega_{\mathrm{a}} = \{\mathrm{k} \equiv 0\}^c$ and $\Omega_{\mathrm{i}} = \emptyset$). If the kernel k is monatomic and symmetric, then we get $\mathrm{k} = \mathbb{1}_{\Omega_{\mathrm{a}}} \mathrm{k} \mathbb{1}_{\Omega_{\mathrm{a}}}$ and thus the kernel k is quasi-irreducible.

The notion of irreducibility of a kernel depends only on its support: the kernel k is irreducible (resp. quasi-irreducible, resp. monatomic) if and only if the kernel $\mathbb{1}_{\{\mathrm{k}>0\}}$ is irreducible (resp. quasi-irreducible, resp. monatomic). Furthermore, if k is monatomic, then the kernels k and $\mathbb{1}_{\{\mathrm{k}>0\}}$ have the same atom Ω_{a} and the same set Ω_{i} infected by Ω_{a}.

The introduction of monatomic kernel is also motivated by the following result which can be deduced from [33, Theorem V.6.6] and [34, Theorem 8], see also Section 7.

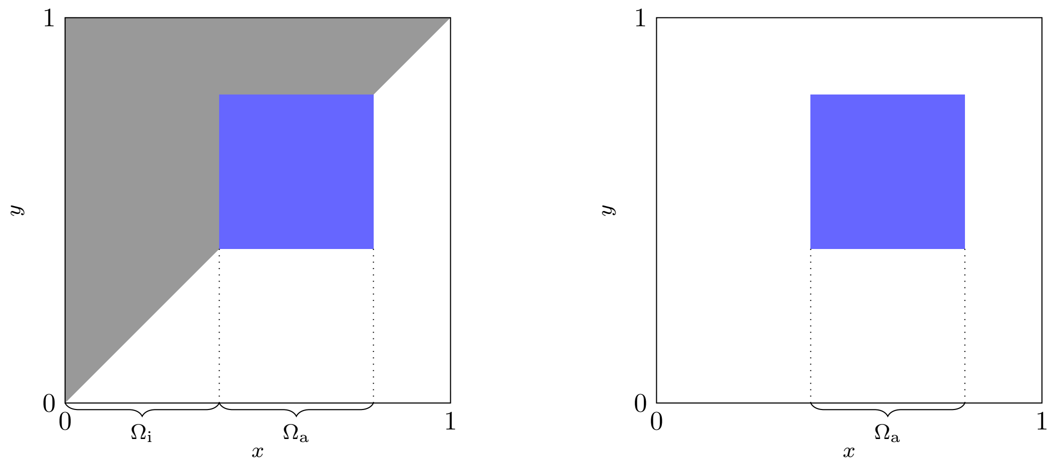
Lemma 3.4. *Let* k *be a kernel with finite double norm and set* $R_0 = R_0[\mathrm{k}]$. *If the kernel* k *is monatomic then* $R_0 > 0$ *and* R_0 *is simple (*i.e. $\mathrm{m}(R_0, T_{\mathrm{k}}) = 1$*). If* R_0 *is simple and the only eigenvalue in* $(0, +\infty)$*, then the kernel* k *is monatomic.*

3.5. **The effective reproduction number R_e.** A *vaccination strategy* η of a vaccine with perfect efficiency is an element of Δ, where $\eta(x)$ represents the proportion of ***non-vaccinated*** individuals with feature x. In particular $\eta = \mathbb{1}$ (the constant function equal to 1) corresponds to no vaccination and $\eta = \mathbb{0}$ (the constant function equal to 0) corresponds to the whole population vaccinated. Notice that $\eta \, \mathrm{d}\mu$ corresponds in a sense to the effective population. Let k be a kernel on Ω with finite double norm on L^p. For $\eta \in \Delta$, the kernel $\mathrm{k}\eta$ has also a finite double norm on L^p and the operator M_η is bounded, so that the operator $T_{\mathrm{k}\eta} = T_{\mathrm{k}} M_\eta$ is compact. We can define the *effective spectrum* function Spec[k] from Δ to $\mathscr{K}$ by:

$$\mathrm{Spec}[\mathrm{k}](\eta) = \mathrm{Spec}(T_{\mathrm{k}\eta}), \tag{21}$$

the *effective reproduction number* function $R_e[\mathrm{k}] = \mathrm{rad} \circ \mathrm{Spec}[\mathrm{k}]$ from Δ to $\mathbb{R}_+$ by:

$$R_e[\mathrm{k}](\eta) = \mathrm{rad}(\mathrm{Spec}(T_{\mathrm{k}\eta})) = \rho(T_{\mathrm{k}\eta}), \tag{22}$$



(A) A representation of a monatomic kernel.

(B) A representation of a quasi-irreducible kernel.

FIGURE 3. Example of a monatomic and quasi-irreducible kernels $(x, y) \mapsto \mathrm{k}(x, y)$, where $\mathrm{k}(x, y) = 0$ on the white zone and k reduced to the blue zone is irreducible.

and the corresponding reproduction number is then given by $R_0[\mathrm{k}] = R_e[\mathrm{k}](\mathbb{1})$. When there is no risk of confusion on the kernel k, we simply write R_e and R_0 for the function $R_e[\mathrm{k}]$ and the number $R_0[\mathrm{k}]$.

We can see Δ as a subset of L^1, and consider the corresponding *weak topology*: a sequence $(g_n, n \in \mathbb{N})$ of elements of Δ converges weakly to g if for all $h \in L^\infty$ we have:

$$\lim_{n\to\infty} \int_\Omega h g_n \,\mathrm{d}\mu = \int_\Omega h g \,\mathrm{d}\mu. \tag{23}$$

Notice that (23) can easily be extended to any function $h \in L^q$ for any $q \in (1, +\infty)$; so that the weak-topology on Δ, seen as a subset of L^p with $1/p + 1/q = 1$, can be seen as the trace on Δ of the weak topology on L^p. From the Banach-Alaoglu theorem, we get that the set Δ endowed with the weak topology is compact and sequentially compact, see [10, Lemma 3.1].

We also recall the properties of the effective reproduction number given in [10, Proposition 4.1 and Theorem 4.2].

Proposition 3.5. *Let* k *be a kernel on a probability space* $(\Omega, \mathscr{F} \mu)$ *with finite double norm. Then, the functions* $\mathrm{Spec}[\mathrm{k}]$ *and* $R_e = R_e[\mathrm{k}]$ *are continuous functions from* Δ *respectively to* $\mathscr{K}$ *(endowed with the Hausdorff distance) and to* $\mathbb{R}_+$. *Furthermore, the function* $R_e = R_e[\mathrm{k}]$ *satisfies the following properties:*

(i) $R_e(\eta_1) = R_e(\eta_2)$ *if* $\eta_1 = \eta_2$, μ *a.s., and* $\eta_1, \eta_2 \in \Delta$,
(ii) $R_e(\mathbb{0}) = 0$ *and* $R_e(\mathbb{1}) = R_0$,
(iii) $R_e(\eta_1) \leq R_e(\eta_2)$ *for all* $\eta_1, \eta_2 \in \Delta$ *such that* $\eta_1 \leq \eta_2$,
(iv) $R_e(\lambda\eta) = \lambda R_e(\eta)$, *for all* $\eta \in \Delta$ *and* $\lambda \in [0, 1]$.

We complete Corollary 3.2 on the stability property of the spectrum and spectral radius with respect to the kernel k, see [10, Proposition 4.3].

Proposition 3.6 (Stability of $R_e[\mathrm{k}]$ and $\mathrm{Spec}[\mathrm{k}]$)**.** *Let $p \in (1, +\infty)$. Let $(\mathrm{k}_n, n \in \mathbb{N})$ and k be kernels on Ω with finite double norms on L^p. If $\lim_{n\to\infty} \|\mathrm{k}_n - \mathrm{k}\|_{p,q} = 0$, then we have:*

$$(24)\quad \lim_{n\to\infty} \sup_{\eta\in\Delta} \Big| R_e[\mathrm{k}_n](\eta) - R_e[\mathrm{k}](\eta) \Big| = 0 \quad \text{and} \quad \lim_{n\to\infty} \sup_{\eta\in\Delta} d_{\mathrm{H}}\Big(\mathrm{Spec}[\mathrm{k}_n](\eta), \mathrm{Spec}[\mathrm{k}](\eta) \Big) = 0.$$

4. Spectrum-preserving transformations

In this section, we consider a given probability state space $(\Omega, \mathscr{F}, \mu)$, and we discuss two operations on the kernel k that leave the functions $\mathrm{Spec}[\mathrm{k}]$ and $R_e[\mathrm{k}]$ defined on Δ. Recall the convention (15) for the kernel $f\mathrm{k}g$ defined from the kernel k and the non-negative functions f and g.

Lemma 4.1. *Let k be a kernel on Ω and h be a non-negative measurable function on Ω.*

(i) If $h\mathrm{k}$ and $\mathrm{k}h$ have finite double norms (with possibly different p), then we have:

$$\mathrm{Spec}[h\mathrm{k}] = \mathrm{Spec}[h\mathrm{k}\mathbb{1}_{\{h>0\}}] = \mathrm{Spec}[\mathbb{1}_{\{h>0\}}\mathrm{k}h] = \mathrm{Spec}[\mathrm{k}h],$$
$$R_e[h\mathrm{k}] = R_e[h\mathrm{k}\mathbb{1}_{\{h>0\}}] = R_e[\mathbb{1}_{\{h>0\}}\mathrm{k}h] = R_e[\mathrm{k}h].$$

(ii) If h is positive and if k and $h\mathrm{k}/h$ have finite double norms (with possibly different p), then we have:

$$\mathrm{Spec}[\mathrm{k}] = \mathrm{Spec}[h\mathrm{k}/h] \quad \text{and} \quad R_e[\mathrm{k}] = R_e[h\mathrm{k}/h].$$

(iii) If k and its transpose $\mathrm{k}^\top : (x, y) \mapsto \mathrm{k}(y, x)$ have finite double norms (with possibly different p), then we have:

$$\mathrm{Spec}[\mathrm{k}] = \mathrm{Spec}[\mathrm{k}^\top] \quad \text{and} \quad R_e[\mathrm{k}] = R_e[\mathrm{k}^\top].$$

Even if (ii) is a consequence of (i), we state it separately since (ii) and (iii) describe two modifications of k that leave the functions R_e and Spec invariant. See Equation (48) for an other transformation on the kernels which leaves the functions R_e and Spec invariant. See also [11] for further results in the finite dimensional case.

Proof. Since $R_e = \mathrm{rad} \circ \mathrm{Spec}$, we only need to prove (i)-(iii) for the function Spec. We give the detailed proof of (ii) and leave the proof of (i), which is very similar, to the reader. We first assume that k, h and $1/h$ are bounded. The operators $T_{\mathrm{k}\eta}$ and $T_{h\mathrm{k}\eta/h}$ and the multiplication operators M_h and $M_{1/h}$ are bounded operators on L^p for $p \in (1, +\infty)$. We have, using that $T_{\mathrm{k}\eta/h} = T_\mathrm{k}\, M_{\eta/h}$ is compact and (11) for the second equality:

$$\mathrm{Spec}(T_{\mathrm{k}\eta}) = \mathrm{Spec}(T_{\mathrm{k}\eta/h}\, M_h) = \mathrm{Spec}(M_h T_{\mathrm{k}\eta/h}) = \mathrm{Spec}(T_{h\mathrm{k}\eta/h}).$$

Since $\eta \in \Delta$ is arbitrary, this gives that $\mathrm{Spec}[\mathrm{k}] = \mathrm{Spec}[h\mathrm{k}/h]$.

In the general case, we use an approximation scheme. Define the kernel $\mathrm{k}_n = (v_n \mathrm{k} v_n) \wedge n$ with $v_n = \mathbb{1}_{\{n\geq h\geq 1/n\}}$ and the function $h_n = n^{-1} \vee (h \wedge n)$ for $n \in \mathbb{N}^*$. From the first part of the proof, we get $\mathrm{Spec}[\mathrm{k}_n] = \mathrm{Spec}[\mathrm{k}'_n]$, with $\mathrm{k}'_n = h_n \mathrm{k}_n / h_n$. Since $\|\mathrm{k}\|_{p,q}$ is finite for some $p \in (1, +\infty)$, we get by dominated convergence that $\lim_{n\to\infty} \|\mathrm{k} - \mathrm{k}_n\|_{p,q} = 0$, and we deduce from Proposition 3.6 that $\lim_{n\to\infty} \mathrm{Spec}[\mathrm{k}_n] = \mathrm{Spec}[\mathrm{k}]$. Similarly, setting $\mathrm{k}' = h\mathrm{k}/h$, the norm $\|\mathrm{k}'\|_{p',q'}$ is finite for some $p' \in (1, +\infty)$, and thus $\lim_{n\to\infty} \|\mathrm{k}' - \mathrm{k}'_n\|_{p',q'} = 0$, so that $\lim_{n\to\infty} \mathrm{Spec}[\mathrm{k}'_n] = \mathrm{Spec}[\mathrm{k}']$. This proves that $\mathrm{Spec}[\mathrm{k}] = \mathrm{Spec}[\mathrm{k}']$, and thus (ii).

We now prove (iii). For any $\eta \in \Delta$, the kernel $\mathrm{k}^\top\eta$ defines a bounded integral operator in L^q, whose adjoint is $T_{\eta\mathrm{k}}$. Since the spectrum of an operator and its adjoint are the same, we get $\mathrm{Spec}[\mathrm{k}^\top](\eta) = \mathrm{Spec}(T_{\mathrm{k}^\top\eta}) = \mathrm{Spec}(T_{\eta\mathrm{k}}) = \mathrm{Spec}(M_\eta T_\mathrm{k}) = \mathrm{Spec}(T_\mathrm{k} M_\eta) = \mathrm{Spec}[\mathrm{k}](\eta)$, where the fourth equality follows once more from (11). Since this is true for any $\eta \in \Delta$, this gives $\mathrm{Spec}[\mathrm{k}^\top] = \mathrm{Spec}[\mathrm{k}]$. □

Remark 4.2. In the infinite dimensional SIS model developed in [7], the next generation operator is given by the integral operator T_{k}, where the kernel $\mathrm{k} = k/\gamma$ is defined in terms of a transmission rate kernel $k(x,y)$ and a recovery rate function γ by the product $\mathrm{k}(x,y) = k(x,y)/\gamma(y)$; and the reproduction number R_0 is then the spectral radius $\rho(T_{\mathrm{k}})$ of T_{k}. Furthermore the operator $T_{\gamma^{-1}k}$ appears very naturally in the definition of the maximal equilibrium $\mathfrak{g}$ which is solution to [7, Equation (24)], that is $T_{\gamma^{-1}k}(\mathfrak{g}) = \mathfrak{g}/(1-\mathfrak{g})$. According to Lemma 4.1 (i), provided that k/γ and $\gamma^{-1}k$ have finite double norms, the next generation operator $T_{k/\gamma}$ and $T_{\gamma^{-1}k}$ have the same effective spectrum function.

We shall use the following extension in the proof of Lemma 5.12.

Remark 4.3. Following closely the proof of Lemma 4.1 (ii) and using Corollary 3.2, we also get that if h is positive and if k and $h\mathrm{k}/h$ have finite double norms (with possibly different p), then we have:

$$\mathrm{m}(\lambda, T_{\mathrm{k}}) = \mathrm{m}(\lambda, T_{h\mathrm{k}/h}) \quad \text{for all } \lambda \in \mathbb{C}^*. \tag{25}$$

5. Sufficient conditions for convexity or concavity of R_e

5.1. A conjecture from Hill and Longini.

Recall that, in the metapopulation framework, the effective reproduction number is equal to the spectral radius of the matrix $K \cdot \mathrm{Diag}(\eta)$, where K has non-negative entries and is the next-generation matrix and η is the vaccination strategy giving the proportion of non-vaccinated people in each groups. The Hill-Longini conjecture appears in [21] and gives conditions on the spectrum of the next-generation matrix that implies the convexity or the concavity of the effective reproduction number. It states that the function $R_e[K]$ is:

(i) convex when $\mathrm{Spec}(K) \subset \mathbb{R}_+$,
(ii) concave when $\mathrm{Spec}(K)\backslash\{R_0\} \subset \mathbb{R}_-$.

It turns out that the conjecture cannot be true without additional assumption on the matrix K. Indeed, consider the following next-generation matrix:

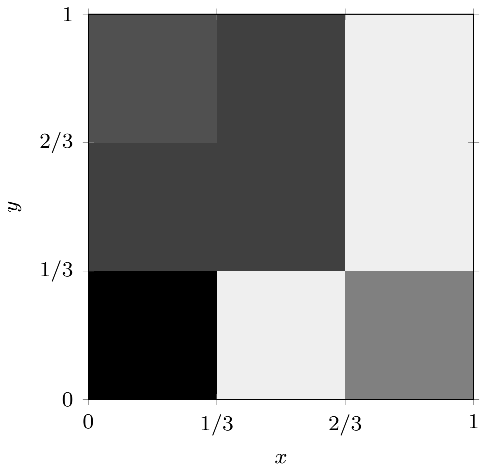
$$K = \begin{pmatrix} 16 & 12 & 11 \\ 1 & 12 & 12 \\ 8 & 1 & 1 \end{pmatrix} \tag{26}$$

Its eigenvalues are approximately equal to 24.8, 2.9 and 1.3. Since R_e is homogeneous, the function is entirely determined by the value it takes on the plane $\{\eta : \eta_1 + \eta_2 + \eta_3 = 1/3\}$. The graph of the function R_e restricted to this set has been represented in Figure 4(B). The view clearly shows the saddle nature of the surface. Hence, the Hill-Longini conjecture (i) is contradicted in its original formulation. In Figure 4(A), we have represented the corresponding kernel model when the population is split equally into three groups, *i.e.*, $\mu_1 = \mu_2 = \mu_3 = 1/3$.

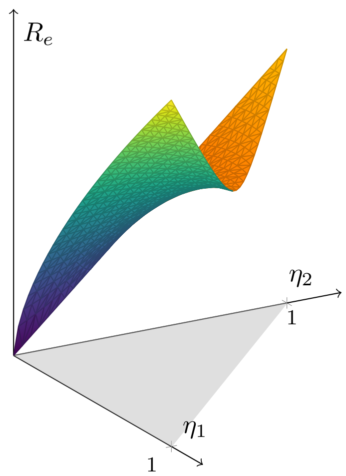
In the same manner, the eigenvalues of the following next-generation matrix:

$$K = \begin{pmatrix} 9 & 13 & 14 \\ 18 & 6 & 5 \\ 1 & 6 & 6 \end{pmatrix} \tag{27}$$

are approximately equal to 26.3, -1.4 and -3.9. Thus, K satisfies the condition that should imply the concavity of the effective reproduction number in the Hill-Longini conjecture (ii). However, as we can see in Figure 5(B), the function R_e is neither convex nor concave. In Figure 5(A), we have represented the corresponding kernel model when the population is splitted equally into three groups, *i.e.*, $\mu_1 = \mu_2 = \mu_3 = 1/3$.



(A) Grayplot of the kernel corresponding to the next-generation matrix (26) when the sub-populations have same size.



(B) The plan of strategies η with cost $C_{\mathrm{uni}}(\eta) = 2/3$ is represented as a gray surface. The triangulated surface corresponds to the graph of R_e restricted to this set.

Figure 4. Counter-example of the Hill-Longini conjecture (convex case).

Despite these counter-examples, the Hill-Longini conjecture is indeed true when making further assumption on the next-generation matrix. Let M be a square real matrix. The matrix M is *diagonally similar* to a matrix M' if there exists a non singular real diagonal matrix D such that $M = D \cdot M' \cdot D^{-1}$. The matrix M is said to be *diagonally symmetrizable* or simply *symmetrizable* if it is diagonally similar to a symmetric matrix, or, equivalently, if M admits a decomposition $M = D \cdot S$ (or $M = S \cdot D$), where D is a diagonal matrix with positive diagonal entries and S is a symmetric matrix. If a matrix M is diagonally symmetrizable, then its eigenvalues are real since similar matrices share the same spectrum. We obtained the following result when the next-generation matrix is symmetrizable.

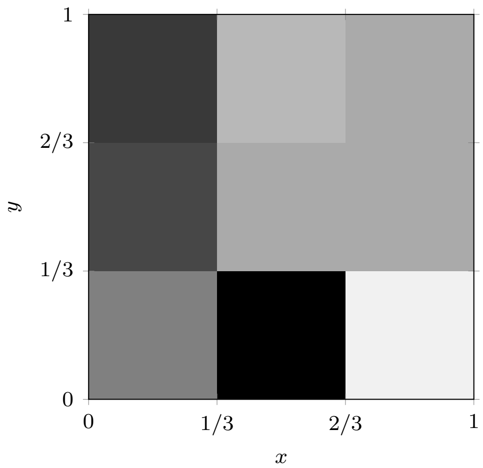
Theorem 5.1. *Suppose the non-negative matrix K is diagonally symmetrizable.*

(i) If $\mathrm{Spec}(K) \subset \mathbb{R}_+$, then the function $R_e[K]$ is convex.
(ii) If R_0 is a simple eigenvalue of K and $\mathrm{Spec}(K) \subset \mathbb{R}_- \cup \{R_0\}$, then the function $R_e[K]$ is concave.

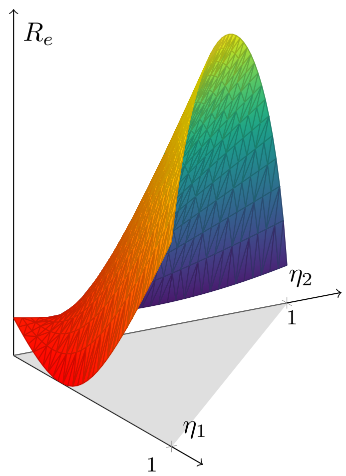
This result is a particular case of Theorem 5.5 below. The first point (i) has been proved by Cairns in [6]. In [18], Friedland obtained that, if the next-generation matrix K is not singular and if its inverse is an M-matrix (*i.e.*, its non-diagonal coefficients are non-positive), then R_e is convex. Friedland's condition does not imply that K is symmetrizable nor that $\mathrm{Spec}(K) \subset \mathbb{R}_+$. On the other hand, the following matrix is symmetric definite positive (and thus R_e is convex) but its inverse is not an M-matrix.

$$K = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix} \qquad \text{with inverse} \quad K^{-1} = \begin{pmatrix} 1.4 & -1.6 & 0.4 \\ -1.6 & 2.4 & -0.6 \\ 0.4 & -0.6 & 0.4 \end{pmatrix}.$$

Thus Friedland's condition and Property (i) in Theorem 5.1 are not comparable. Note that if K is symmetrizable and its inverse is an M-matrix, then the eigenvalues of K are actually non-negative thanks to [4, Chapter 6 Theorem 2.3] and one can apply Theorem 5.1 (i).



(A) Grayplot of the kernel corresponding to the next-generation matrix (27) when the sub-populations have same size.



(B) The plan of strategies η with cost $C_{\text{uni}}(\eta) = 2/3$ is represented as a gray surface. The triangulated surface corresponds to the graph of R_e restricted to this set.

FIGURE 5. Counter-example of the Hill-Longini conjecture (concave case).

5.2. **Generalization for the kernel model.** In this section, we give the analogue of Theorem 5.1 for kernels instead of matrices. First, we proceed with some definitions.

We say that a kernel k' is an Hilbert-Schmidt non-negative symmetric kernel if $\mathrm{k}' \geq 0$, $\|\mathrm{k}'\|_{2,2} < +\infty$ and $\mu(\mathrm{d}x) \otimes \mu(\mathrm{d}y)$-a.e. $\mathrm{k}'(x, y) = \mathrm{k}'(y, x)$. By analogy with the matrix case and following [37, Example A, p252], we introduce the notion of symmetrizability in the context of kernels.

Definition 5.2 (Diagonally HS kernel)**.** *A kernel* k *on* Ω *is diagonally HS if there exists an Hilbert-Schmidt symmetric non-negative kernel* k' *on* Ω *and two positive measurable functions* f, g *defined on* Ω *such that* $\mathrm{k} = f\mathrm{k}'g$ *a.s., that is* $\mu(\mathrm{d}x) \otimes \mu(\mathrm{d}y)$*-a.s.:*

$$\mathrm{k}(x, y) = f(x)\,\mathrm{k}'(x, y)\,g(y). \tag{28}$$

If furthermore f *and* g *are bounded and bounded away from* 0*, then we say that the kernel* k *is strongly diagonally HS.*

The notion of diagonally HS kernel appears naturally when considering the SIS model on graphons; see [7, Example 1.3], where the kernel k is written as $\mathrm{k} = \beta W \theta$, where $\beta(x)$ represents the susceptibility and $\theta(x)$ the infectiousness of the individuals with feature x, and W models the graph of the contacts within the population with the quantity $W(x, y) = W(y, x) \in [0, 1]$ representing the density of contacts between individuals with features x and y.

Remark 5.3. We complete the notion of diagonally HS kernel with three comments.

(i) In finite dimension (*i.e.* Ω finite), a diagonally HS kernel is strongly diagonally HS.
(ii) Notice that a strongly diagonally HS kernel has finite double norm in L^2.
(iii) Consider the decomposition (28), where f and g are assumed to be non-negative instead of positive, with the other assumptions unchanged. Then using Lemma 4.1 (i) and assuming that k in (28) has a finite double norm, we get that $R_e[\mathrm{k}] = R_e[\mathrm{k}_0]$ coincide on Δ, where $\mathrm{k}_0 = \mathbb{1}_{\{fg>0\}}\,\mathrm{k}\,\mathbb{1}_{\{fg>0\}}$. As $\mathrm{k}_0 = f'\,\mathrm{k}'_0\,g'$ with $\mathrm{k}'_0 = \mathbb{1}_{\{fg>0\}}\,\mathrm{k}'\,\mathbb{1}_{\{fg>0\}}$ and $h' = h + \mathbb{1}_{\{fg=0\}}$ for $h \in \{f, g\}$, we get that the kernel k_0 is diagonally HS (indeed f'

and g' are positive, and the other assumptions hold). So, as far as the study of $R_e[\mathrm{k}]$ is concerned, without loss of generality one can indeed assume that the functions f and g which appear in the decomposition of a diagonally HS kernel are positive instead of non-negative.

The following elementary lemma states that the integral operator of a diagonally HS kernel has real eigenvalues.

Lemma 5.4. *Let* k *be a diagonally HS kernel with finite double norm. The spectrum of* T_k *is real, that is,* $\mathrm{Spec}(T_\mathrm{k}) \subset \mathbb{R}$.

Proof. Let k', f and g as in (28) and for $n \in \mathbb{N}^*$ set:

$$v_n = \mathbb{1}_{\{n \geq f \geq 1/n \text{ and } n \geq g \geq 1/n\}}. \tag{29}$$

Let $p \in (1, +\infty)$ be such that $\|\mathrm{k}\|_{p,q}$ is finite. By monotone convergence, we have $\lim_{n\to\infty} \|\mathrm{k} - f v_n \mathrm{k}' v_n g\|_{p,q} = 0$. We deduce that:

$$\mathrm{Spec}(T_\mathrm{k}) = \mathrm{Spec}(T_{f\mathrm{k}'g}) = \lim_{n\to\infty} \mathrm{Spec}(T_{f v_n \mathrm{k}' v_n g}) = \lim_{n\to\infty} \mathrm{Spec}(T_{\sqrt{fg}\, v_n \mathrm{k}' v_n \sqrt{fg}}),$$

where we used (28) for the first equality, Corollary 3.2 for the second, Lemma 4.1 (ii) with $h = v_n \sqrt{g/f} + (1 - v_n)$ for the last. Since the kernel $\sqrt{fg}\, v_n \mathrm{k}' v_n \sqrt{fg}$ is symmetric with finite double norm in L^2, we deduce that the associated compact integral operator is self-adjoint, and thus $\mathrm{Spec}(T_{\sqrt{fg}\, v_n \mathrm{k}' v_n \sqrt{fg}}) \subset \mathbb{R}$. Then, use that $\mathbb{R}$ is closed for the Hausdorff distance to deduce that $\mathrm{Spec}(T_\mathrm{k}) \subset \mathbb{R}$. $\square$

For a compact operator T, we denote by $\mathrm{p}(T)$ and $\mathrm{n}(T)$ the number of its positive and negative eigenvalues with their multiplicity:

$$\mathrm{p}(T) = \sum_{\lambda > 0} \mathrm{m}(\lambda, T) \quad \text{and} \quad \mathrm{n}(T) = \sum_{\lambda < 0} \mathrm{m}(\lambda, T).$$

Note that $R_0[\mathrm{k}] > 0$ implies that $\mathrm{p}(T_\mathrm{k}) \geq 1$.

The following result is the analogue of Theorem 5.1 for the kernel model.

Theorem 5.5 (Convexity/Concavity of R_e)**.** *Let* k *be a strongly diagonally HS kernel. We consider the function* $R_e = R_e[\mathrm{k}]$ *defined on* Δ.

(i) If $\mathrm{n}(T_\mathrm{k}) = 0$*, then the function* R_e *is convex.*
(ii) If $\mathrm{p}(T_\mathrm{k}) = 1$*, then the function* R_e *is concave.*

In the case of diagonally HS kernels, we have the following partial result.

Proposition 5.6. *Let* k *be a diagonally HS kernel of finite double norm, with the HS kernel* k' *from* (28)*. We consider the function* $R_e = R_e[\mathrm{k}]$ *defined on* Δ.

(i) If $\mathrm{n}(T_{\mathrm{k}'}) = 0$*, then* $\mathrm{n}(T_\mathrm{k}) = 0$ *and the function* R_e *is convex.*
(ii) If $\mathrm{p}(T_{\mathrm{k}'}) = 1$*, then* $\mathrm{p}(T_\mathrm{k}) = 1$ *and the function* R_e *is concave.*

The proof for HS kernels is given in Section 5.4.1 for the convex case and in Section 5.4.2 for the concave case; the latter relies on the Sylvester's inertia theorem which is presented in Section 5.3. The extension to (strongly) diagonally HS kernel follows from Sections 5.5.

Remark 5.7 (Concavity and monatomicity). We assume $R_0 > 0$ with $R_0 = R_0[\mathrm{k}]$.

(i) If $R_e[\mathrm{k}]$ is concave, then k is monatomic, see Lemma 7.3.
(ii) If k is a strongly diagonally HS kernel, then the condition $\mathrm{p}(T_\mathrm{k}) = 1$ in Theorem 5.5 (ii) implies that R_0 is simple and $\mathrm{Spec}(T_\mathrm{k}) \subset \mathbb{R}_- \cup \{R_0\}$, and thus k is monatomic, see Lemma 3.4.

(iii) More generally, using the decomposition of a reducible kernel from Lemma 7.2, we get that if $\mathrm{Spec}(T_{\mathrm{k}}) \subset \mathbb{R}_- \cup \{R_0\}$ and k is a strongly diagonally HS kernel, then the function R_e is the maximum of $m = \mathrm{m}(R_0, T_{\mathrm{k}})$ concave functions which are non-zero on m pairwise disjoint subsets of Δ.

Remark 5.8. It is unclear whether or not $\mathrm{p}(T_{\mathrm{k}}) = 1$ (resp. $\mathrm{n}(T_{\mathrm{k}}) = 0$) in Proposition 5.6 implies that $\mathrm{p}(T_{\mathrm{k}'}) = 1$ (resp. $\mathrm{n}(T_{\mathrm{k}'}) = 0$).

Remark 5.9. A *configuration model* corresponds in finite dimension to the next generation matrix having rank one, this is the so-called *proportionate mixing* model in the metapopulation literature; see Cairns [6] for optimal vaccinations strategies in this setting.

Motivated by the finite dimensional case, we say that a kernel k is a *configuration* kernel if there exist $p \in (1, +\infty)$, $f \in L^p$ and $g \in L^q$ where $q = p/(p-1)$ such that $\mathrm{k}(x, y) = f(x)g(y)$, $\mu \otimes \mu$-almost surely. We also suppose that $\mu(fg > 0) > 0$. Such a kernel has finite double norm, as $\|\mathrm{k}\|_{p,q} = \|f\|_p \|g\|_q$. Following Remark 5.3 (iii), we have $R_e[\mathrm{k}] = R_e[\mathbb{1}_{\{fg>0\}}\, \mathrm{k}\, \mathbb{1}_{\{fg>0\}}]$ with $\mathbb{1}_{\{fg>0\}}\, \mathrm{k}\, \mathbb{1}_{\{fg>0\}}$ diagonally HS as $\mathbb{1}_{\{fg>0\}}\, \mathrm{k}\, \mathbb{1}_{\{fg>0\}} = (f + \mathbb{1}_{\{f=0\}})\mathbb{1}_{\{fg>0\}}\mathbb{1}_{\{fg>0\}}(g + \mathbb{1}_{\{g=0\}})$. Besides, the only eigenvalue of the kernel $\mathbb{1}_{\{fg>0\}}(x)\mathbb{1}_{\{fg>0\}}(y)$ different from 0 is its spectral radius equal to $\mu(fg > 0)$ and it has multiplicity 1. Applying Proposition 5.6, we obtain that R_e is convex and concave and thus linear. This can be checked directly as:

$$R_e[\mathrm{k}](\eta) = \int_\Omega fg\, \eta \, \mathrm{d}\mu. \tag{30}$$

We shall provide in [12] a deeper study of configuration kernels in the context of epidemiology.

5.3. **Sylvester's inertia theorem.** Following [31, Section 4.1.2], we state and provide a short proof for the Sylvester's inertia theorem in our context; see also [23, Theorem 4.5.8] in finite dimension. This result will be used to prove the concavity of R_e.

Theorem 5.10 (Sylvester's inertia theorem). *Let $(\Omega, \mathscr{F}, \mu)$ be a probability space. Let T' be a self-adjoint compact operator on $L^2(\mu)$, and two non-negative measurable functions f, g defined on Ω which are bounded and bounded away from* 0*. Set $T = M_f T' M_g$. Then, we have* $\mathrm{Spec}(T) \subset \mathbb{R}$ *and:*

$$\mathrm{p}(T) = \mathrm{p}(T') \quad \textit{as well as} \quad \mathrm{n}(T) = \mathrm{n}(T'). \tag{31}$$

Proof. Set $h = \sqrt{f/g}$, $M = M_{\sqrt{fg}}$ and

$$T'' = MT'M,$$

so that $T = M_h T'' M_{1/h}$. Thanks to (11), we get that $\mathrm{m}(\lambda, T) = \mathrm{m}(\lambda, T'')$ for all $\lambda \in \mathbb{C}^*$. So, we need to prove (31) with T replaced by T''. We only consider the number of positive eigenvalues as the number of negative eigenvalues can be handled similarly.

We introduce some general notations. For a self-adjoint compact operator S on $L^2(\mu)$, let $(u_i, i \in I)$, with I at most countable and $\sharp I = \mathrm{p}(S)$, be a sequence of orthogonal eigenvectors associated to the positive eigenvalues $(\lambda_i, i \in I)$ of S. Let $U \subset L^2(\mu)$ be the (closed) vector subspace spanned by $(u_i, i \in I)$. The orthogonal complement of U, say $U^\top$ is the (closed) vector space spanned by the kernel of S and the eigenvectors associated to the negative eigenvalues. We consider the quadratic form Q_S on $L^2(\mu)$ defined by:

$$Q_S(u) = \langle u, Su \rangle.$$

Let P_S be the orthogonal projection on $U^\top$. By decomposing u on $U \oplus U^\top$, we get:

$$Q_S(u) = \sum_{i \in I} \lambda_i \langle u, u_i \rangle^2 + Q_S(P_S(u)),$$

and the quadratic form $Q_S \circ P_S$ is negative semi-definite.

We shall now prove that $\mathrm{p}(T'') = \mathrm{p}(T')$ by contradiction. First assume that $\mathrm{p}(T') < \mathrm{p}(T'')$, so in particular $\mathrm{p}(T')$ is finite. Let $(u''_i, i \in I'')$ be a sequence of orthogonal eigenvectors associated to the positive eigenvalues $(\lambda''_i, i \in I'')$ of T''. Set $v_i = M u''_i$ for $i \in I''$. In particular, the dimension of the space spanned by $(v_i, i \in I'')$, which is equal to $\mathrm{p}(T'')$, is larger than the finite dimension of the space U spanned by the orthogonal eigenvectors $(u'_i, i \in I')$ associated to the positive eigenvalues of T'. Thus, solving a linear system, we get there exists $(c_i, i \in I'')$ such that $c_i \neq 0$ for at most $\mathrm{p}(T') + 1$ indices, $u = \sum_{i \in I''} c_i v_i \neq 0$, and $u \in U^\top$. On one hand, since $Q_{T'}$ is negative semi-definite on $U^\top$, we get $Q_{T'}(u) \leq 0$. On the other hand, we have:

$$Q_{T'}(u) = \langle u, T'u \rangle = \sum_{i,j \in I''} c_i c_j \langle v_i, T' v_j \rangle = \sum_{i,j \in I''} c_i c_j \langle u''_i, T'' u''_j \rangle = \sum_{i \in I''} c_i^2 \lambda''_i > 0.$$

By contradiction, we deduce that $\mathrm{p}(T') \geq \mathrm{p}(T'')$, and by symmetry $\mathrm{p}(T') = \mathrm{p}(T'')$. □

5.4. **The symmetric case.** Let k be an Hilbert-Schmidt non-negative symmetric kernel. As $R_0[\mathrm{k}] = 0$ implies $R_e[\mathrm{k}] = 0$ by (9), we shall only consider the case $R_0[\mathrm{k}] > 0$. We now prove Theorem 5.5 when k is symmetric with finite double norm in $L^2(\mu)$ and $R_0[\mathrm{k}] > 0$.

5.4.1. *The convex case.* The proof relies on an idea from [18] (see therein just before Theorem 4.3). Let k be an Hilbert-Schmidt non-negative symmetric kernel such that $\mathrm{Spec}(T_\mathrm{k}) \subset \mathbb{R}_+$, where T_k is the corresponding integral operator on $L^2(\mu)$. Since T_k is a self-adjoint positive semi-definite operator on $L^2(\mu)$, there exists a self-adjoint positive semi-definite operator Q on $L^2(\mu)$ such that $Q^2 = T$. Recall that for a real-valued function u defined on Ω, M_u denotes the multiplication by u operator. Thanks to (12), we have for $\eta \in \Delta$:

$$R_e[\mathrm{k}](\eta) = \rho(T_\mathrm{k}\, M_\eta) = \rho(Q^2\, M_\eta) = \rho(Q\, M_\eta\, Q).$$

Since the self-adjoint operator $Q\, M_\eta\, Q$ (on $L^2(\mu)$) is also positive semi-definite, we deduce from the Courant-Fischer-Weyl min-max principle that:

$$R_e[\mathrm{k}](\eta) = \rho\,(Q\, M_\eta\, Q) = \sup_{u \in L^2(\mu)\setminus\{0\}} \frac{\langle u, Q\, M_\eta\, Qu \rangle}{\langle u, u \rangle}.$$

Since the map $\eta \mapsto \langle u, Q\, M_\eta\, Qu \rangle$ defined on Δ is linear, we deduce that $\eta \mapsto R_e[\mathrm{k}](\eta)$ is convex as a supremum of linear functions.

5.4.2. *The concave case.* Let k be an Hilbert-Schmidt non-negative symmetric kernel such that $\mathrm{p}(T_\mathrm{k}) = 1$. In particular k is monatomic, see Lemma 3.4. Let Δ^* be the subset of Δ of the functions which are bounded away from 0. The set Δ^* is a dense convex subset of Δ. So its suffice to prove that $R_e = R_e[\mathrm{k}]$ is concave on Δ^*. Let η_0, η_1 be elements of Δ^*, and set $\eta_\alpha = (1-\alpha)\eta_0 + \alpha\eta_1$ for $\alpha \in [0,1]$ (which is also an element of Δ^*). We write $T_\alpha = T_{\mathrm{k}\eta_\alpha}$, so that $T_\alpha = T_0 + \alpha T_\mathrm{k} M$, where M is the multiplication by $(\eta_1 - \eta_0)$ operator, and:

$$R(\alpha) = R_e(\eta_\alpha) = \rho(T_\alpha) = \rho(T_0 + \alpha T_\mathrm{k} M).$$

So, to prove that R_e is concave on Δ^* (and thus on Δ), it is enough to prove that $\alpha \mapsto R(\alpha)$ is concave on $(0,1)$. As η_α is also bounded away from 0, we get that $\mathrm{k}\eta_\alpha$ is monatomic and its spectral radius $R(\alpha)$ is positive and a simple eigenvalue, thanks to Lemma 3.4. Thanks to Sylvester's inertia theorem, see Theorem 5.10 (with $f = 1$ and $g = \eta_\alpha$), we also get that $\mathrm{p}(T_\alpha) = 1$.

We consider the following scalar product on $L^2(\mu)$ defined by $\langle u, v \rangle_\alpha = \langle u, \eta_\alpha v \rangle$. The operator T_α is self-adjoint and compact on $L^2(\eta_\alpha \mathrm{d}\mu)$ with spectrum $\mathrm{Spec}(T_\alpha)$ thanks to Lemma 3.1 (iii). Let $(\lambda_n, n \in I = [\![0, N[\![)$, with $N \in \mathbb{N} \cup \{\infty\}$ be an enumeration of the non-zero eigenvalues of T_α with their multiplicity so that $\lambda_0 = R(\alpha) > 0$ and thus $\lambda_n < 0$ for $n \in I^* = I \setminus \{0\}$; and denote by $(u_n, n \in I)$ a corresponding sequence of orthogonal eigenvectors. The functions $v_\alpha = u_0$ and $\phi_\alpha = \eta_\alpha u_0$ are the right and left-eigenvectors for T_α (seen as an operator on $L^2(\mu)$) associated to $R(\alpha)$.

We now follow [25] to get that $\alpha \mapsto R(\alpha) = \rho(T_0 + \alpha T_{\mathrm{k}} M)$ is analytic and compute its second derivative. Let π_α be the projection on the $(\langle \cdot, \cdot \rangle_\alpha)$-orthogonal of v_α, and define:

$$S_\alpha = (T_\alpha - R(\alpha))^{-1} \pi_\alpha.$$

In other words, S_α maps u_0 to 0 and u_i to $(\lambda_i - R(\alpha))^{-1} u_i$. Let $\alpha \in (0,1)$ and ε small enough so that $\alpha + \varepsilon \in [0,1]$. We have:

$$T_{\alpha+\varepsilon} = T_\alpha + \varepsilon T_{\mathrm{k}} M,$$

and thus $\|T_{\alpha+\varepsilon} - T_\alpha\|_{L^2(\eta_\alpha \mathrm{d}\mu)} = O(\varepsilon)$. Using [25, Theorem 2.6] on the Banach space $L^2(\eta_\alpha \, \mathrm{d}\mu)$, we get that:

$$R(\alpha + \varepsilon) = R(\alpha) + \varepsilon \langle v_\alpha, T_{\mathrm{k}} M v_\alpha \rangle_\alpha - \varepsilon^2 \langle v_\alpha, T_{\mathrm{k}} M S_\alpha T_{\mathrm{k}} M v_\alpha \rangle_\alpha + O(\varepsilon^3).$$

Let $N_\alpha = M_{1/\eta_\alpha} M = M M_{1/\eta_\alpha}$ be the multiplication by $(\eta_1 - \eta_0)/\eta_\alpha$ bounded operator. Since $\alpha \mapsto R(\alpha)$ is analytic and T_{k} self-adjoint (with respect to $\langle \cdot, \cdot \rangle$), we get that:

$$\begin{aligned} R''(\alpha) &= -2 \langle v_\alpha, T_{\mathrm{k}} M S_\alpha T_{\mathrm{k}} M v_\alpha \rangle_\alpha \\ &= -2 \langle M T_\alpha v_\alpha, S_\alpha T_{\mathrm{k}} M v_\alpha \rangle \\ &= -2 R(\alpha) \langle M v_\alpha, S_\alpha T_{\mathrm{k}} M v_\alpha \rangle \\ &= -2 R(\alpha) \langle N_\alpha v_\alpha, S_\alpha T_\alpha N_\alpha v_\alpha \rangle_\alpha . \end{aligned}$$

Since the kernel and the image of T_α are orthogonal (in $L^2(\eta_\alpha \mathrm{d}\mu)$), and the latter is generated by $(u_n, n \in I)$, we have the decomposition $N_\alpha v_\alpha = g + \sum_{n \in I} a_n u_n$ with $g \in \mathrm{Ker}(T_\alpha)$ and $a_n = \langle N_\alpha v_\alpha, u_n \rangle_\alpha$. This gives, with $I^* = I \setminus \{0\}$:

$$R''(\alpha) = 2R(\alpha) \sum_{n \in I^*} \frac{\lambda_n}{R(\alpha) - \lambda_n} \, a_n^2 \, \langle u_n, u_n \rangle_\alpha . \tag{32}$$

Since $\lambda_n < 0$ for all $n \in I^*$, we deduce that $R''(\alpha) \leq 0$ and thus $\alpha \mapsto R(\alpha)$ is concave on $[0,1]$. This implies that $R_e[\mathrm{k}]$ is concave.

Remark 5.11. The same proof with obvious changes gives that if k is an Hilbert-Schmidt non-negative symmetric monatomic (and thus quasi-irreducible) kernel such that $\mathrm{n}(T_{\mathrm{k}}) = 0$, then $R_e[\mathrm{k}]$ is convex on Δ. This result is however less general than the one obtained in Section 5.4.1.

5.5. **Proof of Theorem 5.5 and Proposition 5.6.** We first consider the following technical Lemma.

Lemma 5.12. *Let* k *be a diagonally HS kernel, with the HS kernel* k′ *from* (28). *We have:*

$$\mathrm{p}(T_{\mathrm{k}}) \leq \mathrm{p}(T_{\mathrm{k}'}) \quad \textit{and} \quad \mathrm{n}(T_{\mathrm{k}}) \leq \mathrm{n}(T_{\mathrm{k}'}).$$

If furthermore k *is strongly diagonally HS, then the previous inequalities are in fact equalities.*

Proof. We only consider the number of positive eigenvalues as the number of negative eigenvalues can be handled similarly. Let f, g be the functions from (28) and v_n defined in (29) for $n \in \mathbb{N}^*$. For simplicity, we write $\mathrm{p}(\mathrm{k}'')$ for $\mathrm{p}(T_{\mathrm{k}''})$ when k″ is a kernel with finite double norm. Let $m \in \mathbb{N}^*$. As the function $w_{n,m} = \sqrt{fg}\, v_n + m^{-1}(1 - v_n)$ is bounded and bounded away from 0, we deduce from the Sylvester's inertia Theorem 5.10 that:

$$\mathrm{p}(\mathrm{k}') = \mathrm{p}\left(w_{n,m} \, \mathrm{k}' \, w_{n,m} \right). \tag{33}$$

Notice that $\lim_{m\to\infty}\|\sqrt{fg}\, v_n\, \mathrm{k}'\, v_n\sqrt{fg} - w_{n,m}\, \mathrm{k}'\, w_{n,m}\|_{2,2} = 0$. Letting m goes to infinity, we deduce from (20) in Corollary 3.2 and the fact that the spectrum is real that:

$$\mathrm{p}(\mathrm{k}') \geq \mathrm{p}\left(\sqrt{fg}\, v_n\, \mathrm{k}'\, v_n\sqrt{fg}\right). \tag{34}$$

We also deduce from Remark 4.3, with $h = \sqrt{f/g}\, v_n + (1 - v_n)$ that:

$$\mathrm{p}\left(\sqrt{fg}\, v_n\, \mathrm{k}'\, v_n\sqrt{fg}\right) = \mathrm{p}\left(f v_n\, \mathrm{k}'\, v_n g\right).$$

Recall k has a finite double norm in some L^p. By monotone convergence, we get that $\lim_{m\to\infty}\|f\, \mathrm{k}'\, g - f\, v_n\, \mathrm{k}'\, v_n g\|_{p,q} = 0$. Letting n goes to infinity, we also deduce from (20) in Corollary 3.2 and the fact that the spectra of $T_{f v_n\, \mathrm{k}'\, v_n g}$ and $T_{f\, \mathrm{k}'\, g}$ are real according to Lemma 5.4, that:

$$\liminf_{n\to\infty} p\left(f v_n\, \mathrm{k}'\, v_n g\right) \geq p\left(f\, \mathrm{k}'\, g\right). \tag{35}$$

Thus, we have $\mathrm{p}(\mathrm{k}') \geq \mathrm{p}(\mathrm{k})$.

Notice that if k is strongly diagonally HS, then $v_n = 1$ for n large enough, so that inequalities (34) and (35) are in fact equalities and thus $\mathrm{p}(\mathrm{k}') = \mathrm{p}(\mathrm{k})$. □

Proof of Proposition 5.6. We only prove (ii) as the proof of (i) is similar and easier for the last part. We keep notations from the proof of Lemma 5.12. Assume that $\mathrm{p}(\mathrm{k}') = 1$. We deduce from (33) and from Section 5.4.2 that $R_e[w_{n,m}\, \mathrm{k}'\, w_{n,m}]$ is concave. We deduce from Corollary 3.2, letting m goes to infinity, that $R_e[\sqrt{fg}\, v_n\, \mathrm{k}'\, v_n\sqrt{fg}]$ is concave. Use Lemma 4.1 (ii) with $h = \sqrt{f/g}\, v_n + (1 - v_n)$ to obtain that $R_e[f v_n\, \mathrm{k}'\, v_n g]$ is concave. Then, letting n goes to infinity and using again Corollary 3.2, we deduce that $R_e[f\mathrm{k}'g] = R_e[\mathrm{k}]$ is concave.

Use also Lemma 5.12 to get $\mathrm{p}(\mathrm{k}) \leq \mathrm{p}(\mathrm{k}')$. Now if $\mathrm{p}(\mathrm{k}) = 0$, then we have that $R_0[\mathrm{k}] = 0$ which is equivalent to $R_0[\mathbb{1}_{\{\mathrm{k}>0\}}] = 0$. Since $\{\mathrm{k} > 0\} = \{\mathrm{k}' > 0\}$, this is also equivalent to $R_0[\mathrm{k}'] = 0$. As this is ruled out because $\mathrm{p}(\mathrm{k}') = 1$, we deduce that $\mathrm{p}(\mathrm{k}) = 1$. □

Proof of Theorem 5.5. The result is an immediate consequence of Proposition 5.6 and the second part of Lemma 5.12. □

6. Three properties of the Pareto and anti-Pareto frontiers

We introduce in Section 6.1 the bi-objective minimization problem, where one tries to minimize simultaneously the cost of the vaccination and the effective reproduction number, and recall results from [10] on the Pareto and anti-Pareto optimal strategies and frontiers. Then, we derive in Section 6.2 the existence of Pareto optimal rays as soon as there exists a Pareto optimal strategy uniformly strictly bounded from above by 1. We prove in Section 6.3 that creating a cordon sanitaire is not the worst idea in the sense that it is not anti-Pareto optimal (and it can be Pareto optimal or not). Eventually, in Section 6.4 we give a characterization of $c_\star = C_\star(0)$ using the notion of independent set from graph theory.

6.1. **Pareto and anti-Pareto frontiers.** We quantify the cost of the vaccination strategy $\eta \in \Delta$ by a function $C : \Delta \to \mathbb{R}^+$, and we assume that $C(\mathbb{1}) = 0$ (doing nothing costs nothing), C is non-increasing (doing more costs more) and continuous for the weak topology on Δ defined in Section 3.5. Recall that $1 - \eta$ represents the proportion of the population which has been vaccinated when using the strategy η. One natural choice is the uniform cost function $C = C_{\mathrm{uni}}$ defined for $\eta \in \Delta$ by:

$$C_{\mathrm{uni}}(\eta) = \int_\Omega (1 - \eta)\, \mathrm{d}\mu. \tag{36}$$

In [10], we formalized and study the problem of optimal allocation strategies for a perfect vaccine. This question may be viewed as a bi-objective minimization problem, where one tries to minimize simultaneously the cost of the vaccination and the effective reproduction number:

$$\min_{\Delta}(C, R_e). \tag{37}$$

We briefly summarize the results from [10]. We shall assume that the kernel k has a finite double norm, the loss function is given by the effective reproduction function $R_e[\mathrm{k}]$, and the cost function C is furthermore *decreasing* (this is the case of the uniform cost), that is, for any $\eta_1, \eta_2 \in \Delta$:

$$\eta_1 \leq \eta_2 \quad \text{and} \quad \int_\Omega \eta_1 \,\mathrm{d}\mu < \int_\Omega \eta_2 \,\mathrm{d}\mu \implies C(\eta_1) > C(\eta_2).$$

To be precise, the next results can be found in [10, Propositions 5.4 and 5.5] (notice in particular, that Assumptions 4 and 5 holds thanks to Lemma 5.13 therein). By definition, we have $R_0 = \max_\Delta R_e$ and we set $c_{\max} = \max_\Delta C$ which is positive as C is decreasing. Related to the minimization problem (37), we shall consider $R_{e\star}$ the *optimal loss* function and $C_\star$ the *optimal cost* function defined by:

$$\begin{aligned} R_{e\star}(c) &= \min\,\{\, R_e(\eta) :\ \eta \in \Delta,\, C(\eta) \leq c \,\} \quad \text{for } c \in [0, c_{\max}], \\ C_\star(\ell) &= \min\,\{\, C(\eta) :\ \eta \in \Delta,\, R_e(\eta) \leq \ell \,\} \quad \text{for } \ell \in [0, R_0]. \end{aligned}$$

We have $C_\star(R_0) = 0$ and $R_{e\star}(0) = R_0$ since C is decreasing. For convenience, we write $c_\star$ for the minimal cost required to completely stop the transmission of the disease:

$$c_\star = C_\star(0) = \inf\{c \in [0, c_{\max}] :\ R_{e\star}(c) = 0\}. \tag{38}$$

The function $R_{e\star}$ is continuous, decreasing on $[0, c_\star]$ and zero on $[c_\star, 1]$; the function $C_\star$ is continuous and decreasing on $[0, R_0]$; and the functions $R_{e\star}$ and $C_\star$ are the inverse of each other, that is, $R_{e\star} \circ C_\star(\ell) = \ell$ for $\ell \in [0, R_0]$ and $C_\star \circ R_{e\star}(c) = c$ for $c \in [0, c_\star]$.

We define the Pareto optimal strategies $\mathcal{P}$ as the "best" solutions of the minimization problem (37) (we refer to [10] for a precise justification of this terminology):

$$\mathcal{P} = \{\eta \in \Delta :\ C(\eta) = C_\star(R_e(\eta)) \quad \text{and} \quad R_e(\eta) = R_{e\star}(C(\eta))\},$$

and the Pareto frontier as their outcomes:

$$\mathcal{F} = \{(C(\eta), R_e(\eta)) :\ \eta \in \mathcal{P}\}.$$

The set $\mathcal{P}$ is a non empty compact (for the weak topology) in Δ and furthermore the Pareto frontier can be easily represented using the graph of the optimal loss function or cost function:

$$\mathcal{F} = \{(C_\star(\ell), \ell) :\ \ell \in [0, R_0]\} = \{(c, R_{e\star}(c)) :\ c \in [0, c_\star]\}.$$

It is also of interest to consider the "worst" strategies which can be viewed as solutions to the bi-objective maximization problem:

$$\max_{\Delta}(C, R_e). \tag{39}$$

To be precise, the next results can be found in [10, Propositions 5.8 and 5.9] (notice in particular that Assumption 6 holds in general but that Assumption 7 holds under the stronger condition that the kernel k is monatomic, see Section 5.4.2 therein). Related to the maximization problem (39), we shall consider $R_e^\star$ the *optimal loss* function and $C^\star$ the *optimal cost* function defined by:

$$R_e^\star(c) = \max\,\{\, R_e(\eta) : \,\eta \in \Delta,\, C(\eta) \geq c \,\} \quad \text{for } c \in [0, c_{\max}],$$
$$C^\star(\ell) = \max\,\{\, C(\eta) : \,\eta \in \Delta,\, R_e(\eta) \geq \ell \,\} \quad \text{for } \ell \in [0, R_0].$$

We have $C^\star(0) = c_{\max}$ and $R_e^\star(c_{\max}) = 0$ since C is decreasing and $C(\mathbb{0}) = c_{\max}$. Since, for $\varepsilon \in (0,1)$ we have $C(\varepsilon\mathbb{1}) < c_{\max}$ as C is decreasing and $R_e(\varepsilon\mathbb{1}) = \varepsilon R_0 > 0$, we deduce that $C^\star(0+) = c_{\max}$. For convenience, we write $c^\star$ for the maximal cost of totally inefficient strategies:

$$c^\star = C^\star(R_0) = \max\{c \in [0, c_{\max}] : \, R_e^\star(c) = R_0\}. \tag{40}$$

The function $C^\star$ is decreasing on $[0, R_0]$; the function $R_e^\star$ is constant equal to R_0 on $[0, c^\star]$; we have $R_e^\star \circ C^\star(\ell) = \ell$ for $\ell \in [0, R_0]$. This latter property implies that the function $R_e^\star$ is continuous.

We define the anti-Pareto optimal strategies $\mathcal{P}^{\text{Anti}}$ as the "worst" strategies, that is solutions of the maximization problem (39):

$$\mathcal{P}^{\text{Anti}} = \{\eta \in \Delta : \, C(\eta) = C^\star(R_e(\eta)) \quad \text{and} \quad R_e(\eta) = R_e^\star(C(\eta))\}\,,$$

and the anti-Pareto frontier as their outcomes:

$$\mathcal{F}^{\text{Anti}} = \left\{(C(\eta), R_e(\eta)) : \, \eta \in \mathcal{P}^{\text{Anti}}\right\}.$$

The set $\mathcal{P}$ is non empty and furthermore the Pareto frontier can be easily represented using the graph of the optimal cost function:

$$\mathcal{F}^{\text{Anti}} = \{(C^\star(\ell), \ell) : \, \ell \in [0, R_0]\}. \tag{41}$$

We also have that the feasible region or set of possible outcomes for (C, R_e):

$$\mathbf{F} = \{(C(\eta), R_e(\eta)) : \, \eta \in \Delta\}$$

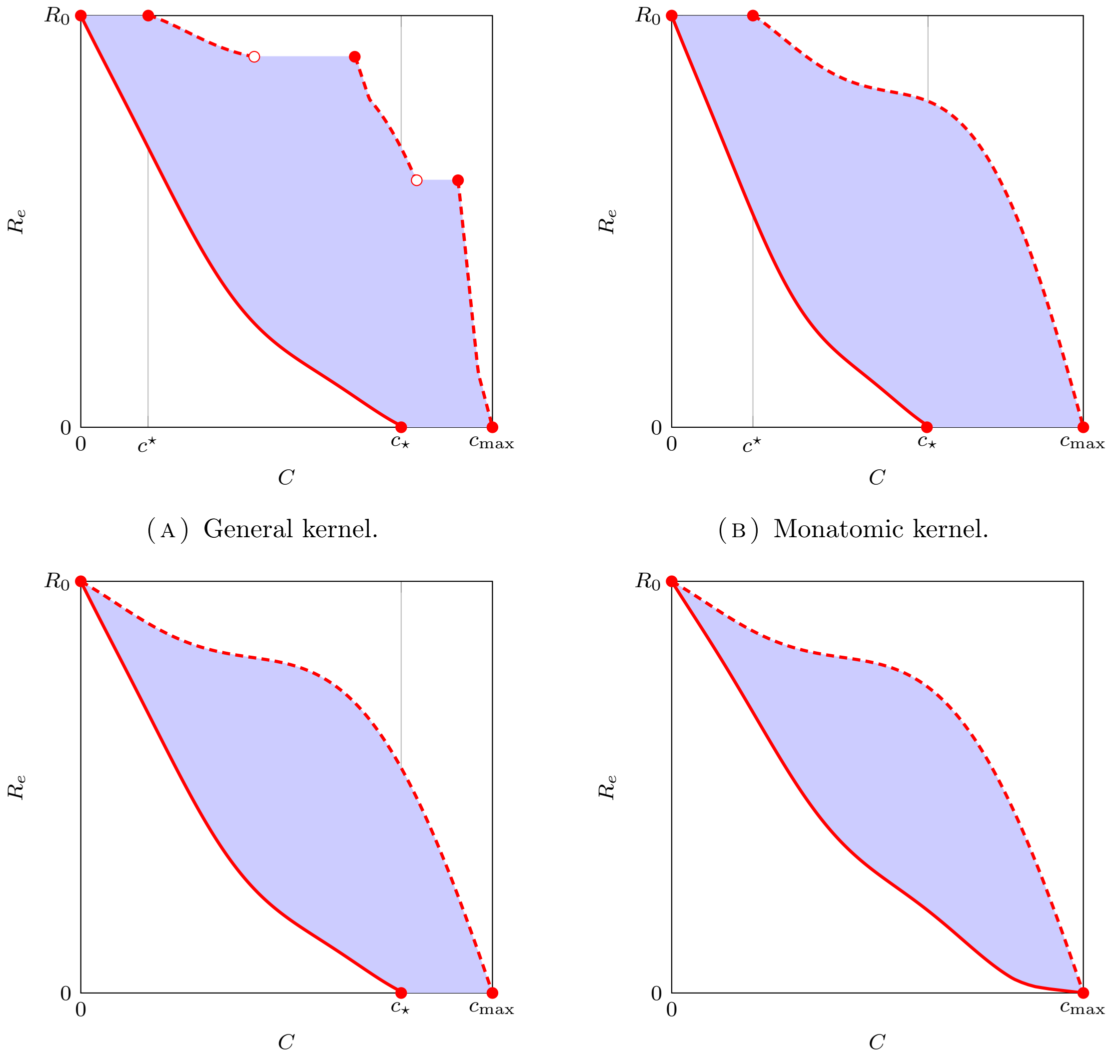
is compact, path connected, and its complement is connected in $\mathbb{R}^2$. It is the whole region between the graphs of the one-dimensional value functions:

$$\begin{aligned}\mathbf{F} &= \{(c, \ell) \in [0, c_{\max}] \times [0, R_0] : \, R_{e\star}(c) \leq \ell \leq R_e^\star(c)\}\\ &= \{(c, \ell) \in [0, c_{\max}] \times [0, R_0] : \, C_\star(\ell) \leq c \leq C^\star(\ell)\}.\end{aligned}$$

If furthermore k is monatomic with atom Ω_{a}, then thanks to [10, Lemma 5.13], we have $c^\star = C(\mathbb{1}_{\Omega_{\mathrm{a}}})$ (which is 0 if k is irreducible); the function $R_e^\star$ is continuous, decreasing on $[c^\star, c_{\max}]$; the function $C^\star$ is continuous and decreasing on $[0, R_0]$; the functions $R_e^\star$ and $C^\star$ are the inverse of each other, that is, $R_e^\star \circ C^\star(\ell) = \ell$ for $\ell \in [0, R_0]$ and $C^\star \circ R_e^\star(c) = c$ for $c \in [c^\star, c_{\max}]$; and the set $\mathcal{P}^{\text{Anti}}$ is compact and $\mathcal{F}^{\text{Anti}} = \{(c, R_e^\star(c)) : \, c \in [c^\star, c_{\max}]\}$.

We plotted in Figure 6 the typical Pareto and anti-Pareto frontiers for a general kernel (notice the anti-Pareto frontier is not connected *a priori*), a monatomic kernel (notice the anti-Pareto frontier is connected), and a positive kernel. In the latter case, the properties of the frontiers are stated in the next lemma.

Lemma 6.1. *Suppose that the cost function C is continuous decreasing with $C(\mathbb{1}) = 0$ and consider the loss function $R_e[\mathrm{k}]$, with* k *a finite double norm kernel such that a.s.* $\mathrm{k} > 0$. *Then, we have $R_0[\mathrm{k}] > 0$, $c^\star = 0$, $c_\star = c_{\max}$ and the strategy $\mathbb{1}$ (resp. $\mathbb{0}$) is the only Pareto optimal as well as the only anti-Pareto optimal strategy with cost $c = 0$ (resp. $c = 1$).*



(A) General kernel.

(B) Monatomic kernel.

(C) Irreducible kernel.

(D) Kernel strictly positive almost surely.

FIGURE 6. Generic aspect of the feasible region (light blue), the Pareto frontier (thick red line) and the anti Pareto frontiers (dashed red line) for the cost function $R_e[\mathrm{k}]$, with kernel k, and a continuous decreasing cost function C.

Proof. Since $\mathrm{k} > 0$, we get that k is irreducible (and thus monatomic) and $R_0 > 0$, thanks to Lemma 3.4. We get that $c^\star = 0$. This implies that the strategy $\mathbb{1}$ is anti-Pareto optimal. As C is decreasing, we also get that the strategy $\mathbb{1}$ is Pareto optimal.

Let $\eta \in \Delta$ be different from $\mathbb{0}$. We get that the kernel $\mathrm{k}\eta$ restricted to the set of positive μ-measure $\{\eta > 0\}$ is positive, thus $\mathrm{k}\eta$ is monatomic (with $\Omega_\mathrm{a} = \{\eta > 0\}$ and $\Omega_\mathrm{i} = \Omega_\mathrm{a}^c$). Thanks Lemma 3.4, we get that $R_e(\eta) > 0$. This readily implies that $c_\star = c_{\max}$ and that the strategy $\mathbb{0}$ is Pareto optimal. As C is decreasing, we also get that the strategy $\mathbb{0}$ is anti-Pareto optimal. $\square$

6.2. **Optimal ray.** As the loss function R_e is convex and homogeneous, and if the cost function is affine, then the set $\mathcal{P}$ of Pareto optimal strategies may contains a non-trivial optimal ray $\{\lambda\eta : \lambda \in [0,1]\}$. This optimal ray has already been observed in finite dimension, see [32].

Proposition 6.2 (Optimal ray)**.** *Suppose that the cost function C is continuous decreasing and affine and that the loss function $R_e[\mathrm{k}]$, with k a finite double norm kernel, is convex. If there exists a Pareto optimal strategy $\eta_\star \in \mathcal{P}$ such that $\sup_\Omega \eta_\star \in (0,1)$, then the strategies $\lambda\eta_\star$ are Pareto optimal for all $\lambda \in [0, 1/\sup_\Omega \eta_\star]$.*

Remark 6.3. Suppose assumptions of Proposition 6.2 hold so that there is an optimal ray $\{\lambda\eta_\star : \lambda \in [0,1]\} \subset \mathcal{P}$, where $\sup_\Omega \eta_\star = 1$. Then, by homogeneity of the loss function, the Pareto frontier has a linear part (from $(C(\eta_\star), R_e(\eta_\star))$ to $(c_{\max}, 0)$).

Remark 6.4. Suppose that C is continuous decreasing and affine and that R_e is concave. With a similar proof (but for the last part which has to be replaced by the fact that $C^\star(0+) = c_{\max}$), we can show that if $\eta^\star$ is anti-Pareto optimal such that $\sup_\Omega \eta^\star \in (0,1)$, then $\lambda\eta^\star$ is also anti-Pareto optimal for all $\lambda \in [0, 1/\sup_\Omega \eta^\star]$.

Proof of Proposition 6.2. Assume there exists $\eta_\star \in \mathcal{P}$ such that $\sup_\Omega \eta_\star \in (0,1)$. Let $\lambda \in (0, 1/\sup_\Omega \eta_\star]$, so that $\lambda\eta_\star \in \Delta$, and let $\eta \in \Delta$ such that $R_e(\eta) \leq R_e(\lambda\eta_\star)$, and thus $R_e(\eta) \leq \lambda R_e(\eta_\star)$. Since $\sup_\Omega \eta_\star < 1$, there exists $s \in (0,1]$ such that $(1-s)\eta_\star + s\eta/\lambda \in \Delta$. Using the homogeneity and the convexity of R_e, we get:

$$\begin{aligned}
R_e((1-s)\eta_\star + s\eta/\lambda) &= \frac{1}{\lambda} R_e((1-s)\lambda\eta_\star + s\eta) \\
&\leq (1-s)R_e(\lambda\eta_\star)/\lambda + sR_e(\eta)/\lambda \\
&\leq R_e(\eta_\star).
\end{aligned}$$

Since $\eta_\star$ is Pareto optimal, we deduce that $C((1-s)\eta_\star + s\eta/\lambda) \geq C(\eta_\star)$. Since C is affine, we get that $C(\eta) \geq C(\lambda\eta_\star)$. Hence, $\lambda\eta_\star$ is solution of the problem $\min C(\eta)$ for $\eta \in \Delta$ such that $R_e(\eta) \leq \ell$ with $\ell = R_e(\lambda\eta_\star)$. We conclude that $\lambda\eta_\star$ is Pareto optimal using [10, Proposition 5.5 (ii)]. Use that the Pareto optimal set is closed, see [10, Corollary 5.7] to get that $\lambda\eta_\star$ is Pareto optimal for $\lambda = 0$. □

6.3. **Creating a cordon sanitaire is not the worst idea.** We say a strategy $\eta \in \Delta$ is a *cordon sanitaire* or *disconnecting* (for the kernel k) if $\eta \neq \mathbb{0}$ and the kernel k restricted to the set $\{\eta > 0\}$ is not connected (or equivalently not irreducible). We make some elementary comments on disconnecting strategies.

Remark 6.5. Let k be a kernel on Ω.

(i) The strategy $\eta = \mathbb{1}$ is disconnecting if and only if k is not connected.
(ii) A strategy η is disconnecting if and only if the strategy $\mathbb{1}_{\{\eta>0\}}$ is disconnecting.
(iii) If $\mathrm{k} > 0$, then there is no disconnecting strategy.

The next proposition states that if the strategy η is anti-Pareto optimal for a kernel k and non zero, then the kernel k restricted to $\{\eta > 0\}$ is irreducible and thus the kernel $\mathbb{1}_{\{\eta>0\}}\mathrm{k}\mathbb{1}_{\{\eta>0\}}$ is quasi-irreducible. Let us remark that in general none of those implications are equivalences.

Proposition 6.6 (A cordon sanitaire is never the worst idea)**.** *Suppose that the cost function C is continuous decreasing and consider the loss function $R_e[\mathrm{k}]$, with k a finite double norm kernel on Ω such that $R_0[\mathrm{k}] > 0$. Then, a disconnecting strategy is not anti-Pareto optimal.*

In the non-oriented cycle graph from Example 1.2, this property is illustrated in Figure 1 as the disconnecting strategy "one in 4", see Figure 2, is not anti-Pareto.

The proof of the proposition relies on the next lemma which is a direct application of [34, Lemma 11] to our setting. For $A \in \mathscr{F}$, let $\mathrm{m}(\lambda, \mathrm{k}, A)$ be the multiplicity (possibly equal to 0) of the eigenvalue $\lambda \in \mathbb{C}^*$ for the integral operator $T_{\mathrm{k}\mathbb{1}_A}$ associated to the kernel $\mathrm{k}\mathbb{1}_A$.

Lemma 6.7. *Let* k *be kernel with finite double norm. Let $A, B \in \mathscr{F}$ be such that $A \cap B = \emptyset$ a.s. and $\mathrm{k}(B, A) = 0$. For all $\lambda \in \mathbb{C}^*$, we have:*

$$\mathrm{m}(\lambda, \mathrm{k}, A \cup B) = \mathrm{m}(\lambda, \mathrm{k}, A) + \mathrm{m}(\lambda, \mathrm{k}, B),$$

and thus

$$R_e[\mathrm{k}](\mathbb{1}_A + \mathbb{1}_B) = \max\big(R_e[\mathrm{k}](\mathbb{1}_A), R_e[\mathrm{k}](\mathbb{1}_B)\big). \tag{42}$$

We are now in a position to prove Proposition 6.6.

Proof of Proposition 6.6. Let η be a disconnecting strategy, and thus $\eta \neq \mathbb{0}$. Since η is disconnecting, that is, k restricted to $\{\eta > 0\}$ is not irreducible, we deduce there exists $A, B \in \mathscr{F}$ such that $\mu(A) > 0$, $\mu(B) > 0$, $(\mathrm{k}\eta)(B, A) = 0$ and a.s. $A \cup B = \{\eta > 0\}$ and $A \cap B = \emptyset$. In particular (42) holds with k replaced by $\mathrm{k}\eta$. First assume that $R_e[\mathrm{k}\eta](\mathbb{1}_A) \geq R_e[\mathrm{k}\eta](\mathbb{1}_B)$, so that (42) yields:

$$R_e[\mathrm{k}](\eta) = R_e[\mathrm{k}\eta](\mathbb{1}_A + \mathbb{1}_B) = R_e[\mathrm{k}\eta](\mathbb{1}_A).$$

For $\theta \in [0, 1]$, define the strategy $\eta_\theta = \eta\mathbb{1}_A + \theta\eta\mathbb{1}_B$. We deduce that:

$$\begin{aligned} R_e[\mathrm{k}](\eta_\theta) = R_e[\mathrm{k}\eta_\theta](\mathbb{1}_A + \mathbb{1}_B) &= \max(R_e[\mathrm{k}\eta_\theta](\mathbb{1}_A), R_e[\mathrm{k}\eta_\theta](\mathbb{1}_B)) \\ &= \max(R_e[\mathrm{k}\eta](\mathbb{1}_A), \theta R_e[\mathrm{k}\eta](\mathbb{1}_B)) \\ &= R_e[\mathrm{k}\eta](\mathbb{1}_A) \\ &= R_e[\mathrm{k}](\eta), \end{aligned}$$

where we used (42) with k replaced by $\mathrm{k}\eta_\theta$ for the second equality as $(\mathrm{k}\eta_\theta)(B, A) = 0$, and the homogeneity of the spectral radius in the third. Thus, the map $\theta \mapsto R_e[\mathrm{k}](\eta_\theta)$ is constant on $[0, 1]$. Since $\mu(B) > 0$ and C is decreasing, we get that $\theta \mapsto C(\eta_\theta)$ is decreasing. This implies that η_θ is worse than η for any $\theta \in [0, 1)$, and thus η is not anti-Pareto optimal.

The case $R_e[\mathrm{k}\eta](\mathbb{1}_B) \geq R_e[\mathrm{k}\eta](\mathbb{1}_A)$ is handled similarly. □

Remark 6.8. If the kernel k is irreducible, then the upper boundary of the set of outcomes **F** is the anti-Pareto frontier, see Figure 6(C) for an instance. We deduce from Proposition 6.6 that if η_0 is a disconnecting strategy, then we have $R_e[\mathrm{k}](\eta_0) < \sup\{R_e[\mathrm{k}](\eta) : C(\eta) = C(\eta_0)\}$.

However, if the kernel k is not irreducible, then the trivial strategy $\mathbb{1}$ is disconnecting. Furthermore, the upper boundary of the set of outcomes **F** is not reduced to the anti-Pareto frontier, see Figure 6(A) for instance. In fact, there exists disconnecting strategies that are not anti-Pareto optimal, but whose outcomes lie on the flat parts of the upper boundary of **F**. In particular, such strategies have the worst loss given their cost. However, it is not difficult to check that they do not disconnect further than the trivial strategy $\mathbb{1}$.

6.4. **A characterization of $c_\star = C_\star(0)$ when the support of k is symmetric.** We characterize the Pareto optimal strategies which minimize R_e when the kernel k has a symmetric support; and we get a very simple representation of $C_\star(0)$ when the cost is uniform $C = C_{\mathrm{uni}}$.

Let us first recall a notion from graph theory. If $G = (V, E)$ is an non-oriented graph with vertices set V and edge set E, an *independent* set of G is a subset $A \subset V$ of vertices which are pairwise not adjacent, that is, $i, j \in A$ implies $ij \notin E$. The *independence number* of a graph G, denoted by $\alpha(G)$, is the maximum of $\sharp A/\sharp G$, over all the independent sets A of G. Following [22], we generalize this definition to kernels.

Definition 6.9 (Independent sets for kernels)**.** *Let* k *be a kernel on* Ω*. A measurable set* $A \in \mathscr{F}$ *is an independent set of* k *if* $\mathrm{k} = 0$ $\mu^{\otimes 2}$*-a.s. on* $A \times A$*. The independence number* $\alpha(\mathrm{k})$ *of the kernel* k *is:*

$$\alpha(\mathrm{k}) = \sup\{\mu(A) : \, A \textit{ is an independent set of } \mathrm{k}\}.$$

A compactness argument will show that the supremum defining α is reached.

Proposition 6.10 (Existence of a maximal independent set)**.** *For any kernel* k *on* Ω*, there exists an independent set* A *of* k *that is* maximal*, in the sense that* $\mu(A) = \alpha(\mathrm{k})$*.*

Proof. First, notice that the independent sets and maximal independent sets of a kernel k depends only on the support $\{ \mathrm{k} > 0 \}$ of k. Therefore, the maximal independent sets of the kernel k and of the kernel $\mathbb{1}_{\{ \mathrm{k} > 0 \}}$ are the same. In particular, we can assume without loss of generality that the kernel k is bounded.

Let $(A_n, \, n \in \mathbb{N})$ be a sequence of independent sets for k such that:

$$\lim_{n \to \infty} \mu(A_n) = \alpha(\mathrm{k}).$$

Since Δ is sequentially compact for the weak topology, up to taking a sub-sequence, we may assume that the sequence $(\mathbb{1}_{A_n}, \, n \in \mathbb{N})$ converges weakly to some function $g \in \Delta$. Since k is bounded, the integral operator T_k is well defined. We deduce that $T_\mathrm{k}(\mathbb{1}_{A_n})$ belongs to Δ and converges a.s. towards $T_\mathrm{k}(g)$. This implies that $\mathbb{1}_{A_n} T_\mathrm{k}(\mathbb{1}_{A_n})$ converges weakly towards $g T_\mathrm{k}(g)$. We deduce that:

$$\int_\Omega g T_\mathrm{k}(g) \, \mathrm{d}\mu = \lim_{n \to \infty} \int_\Omega \mathbb{1}_{A_n} T_\mathrm{k}(\mathbb{1}_{A_n}) \, \mathrm{d}\mu = \lim_{n \to \infty} \mathrm{k}(A_n, A_n) = 0.$$

As $g \in \Delta$, this implies that $\{ g > 0 \}$ is an independent set of k and thus $\mu\,(g > 0) \leq \alpha(\mathrm{k})$. Besides, since $(\mathbb{1}_{A_n}, n \in \mathbb{N})$ converges weakly to g, we get:

$$\int_\Omega g \, \mathrm{d}\mu = \lim_{n \to \infty} \mu(A_n) = \alpha(\mathrm{k}).$$

This implies that $\mu\,(g > 0) \geq \int_\Omega g \, \mathrm{d}\mu = \alpha(\mathrm{k})$. We deduce that $\mu\,(g > 0) = \alpha(\mathrm{k})$, and since $\{ g > 0 \}$ is an independent set, it is also maximal. ☐

In the following result, we prove that maximal independent sets provide optimal Pareto strategies for the loss function R_e and the cost function C_{uni} given by (36) corresponding to the cost $c_\star = C_\star(0)$, see also Remark 6.12 for a general cost function. This property is illustrated in Figure 1 where the Pareto frontier of the non-oriented cycle graph from Example 1.2, with $N = 12$, is plotted; it is possible to prevent infections without vaccinating the whole population as $c_\star = 1/2 < 1 = c_{\max}$.

Proposition 6.11. *Let* k *be a finite double norm kernel on* Ω *such that its support,* $\{\mathrm{k} > 0\}$*, is a symmetric subset of* Ω^2 *a.s. We consider the cost* $C = C_{\mathrm{uni}}$ *given by* (36)*. For any maximal independent set* $A_\star$ *of* k*, the strategy* $\mathbb{1}_{A_\star}$ *is Pareto optimal for the loss* $R_e[\mathrm{k}]$ *and we have:*

$$c_\star = C_\star(0) = C(\mathbb{1}_{A_\star}) = 1 - \alpha(\mathrm{k}). \tag{43}$$

Remark 6.12. Definition 6.9 on maximal independent set is in fact associated to the uniform cost $C = C_{\mathrm{uni}}$. More generally, we could define the independence number $\alpha_C(\mathrm{k})$ of the kernel k with respect to a decreasing continuous cost function C (recall the convention $C(\mathbb{1}) = 0$ and $c_{\max} = C(\mathbb{0})$) as:

$$\alpha_C(\mathrm{k}) = \sup\{c_{\max} - C(\mathbb{1}_A) : \, A \text{ is an independent set of } \mathrm{k}\}.$$

The notations are consistent as $\alpha_C = \alpha$ for $C = C_{\text{uni}}$. Adapting the proof of Proposition 6.10, we get that for any kernel k on Ω, there exists an independent set A of k that is C-maximal, in the sense that $\alpha_C(\mathrm{k}) = c_{\max} - C(\mathbb{1}_A)$. Following the proof of Proposition 6.11, we then get that if the finite double norm kernel k on Ω has its support, $\{\mathrm{k} > 0\}$ which is a symmetric subset of Ω^2 a.s., then for any C-maximal independent set $A_\star$ of k, the strategy $\mathbb{1}_{A_\star}$ is Pareto optimal for the loss $R_e[\mathrm{k}]$ and the cost C. Furthermore, we have:

$$c_\star = C_\star(0) = C(\mathbb{1}_{A_\star}) = \min\{C(\mathbb{1}_A) : A \text{ is an independent set of k}\}.$$

Proof of Proposition 6.11. The existence of a maximum independent set A is given by Proposition 6.10. The effective reproduction number obviously vanishes for the strategy $\mathbb{1}_A$ with cost $1 - \alpha(\mathrm{k})$ as $(T_{\mathrm{k}\mathbb{1}_A})^2 = T_{\mathrm{k}}\, T_{\mathbb{1}_A \mathrm{k} \mathbb{1}_A} = 0$. Now, let $\eta \in \Delta$ be such that $R_e[\mathrm{k}](\eta) = 0$. To complete the proof of the proposition, it is enough to prove that $C_{\text{uni}}(\eta) \geq 1 - \alpha(\mathrm{k})$.

Since $R_e[\mathrm{k}](\eta) = 0$, the spectral radius of $T_{\mathrm{k}\eta}$ is equal to 0. Let $\varepsilon > 0$ and consider the kernel k_ε defined on Ω by:

$$\mathrm{k}_\varepsilon(x, y) = \mathbb{1}_{\{\,\mathrm{k}(x,y) > \varepsilon\,\}}.$$

Since $T_{\mathrm{k}\eta} - \varepsilon T_{\mathrm{k}_\varepsilon \eta}$ is a positive operator, we deduce from (9) that $\varepsilon \rho(T_{\mathrm{k}_\varepsilon \eta}) = \rho(\varepsilon T_{\mathrm{k}_\varepsilon \eta}) \leq \rho(T_{\mathrm{k}\eta}) = 0$ and thus $\rho(T_{\mathrm{k}_\varepsilon \eta}) = 0$. Set $\mathrm{k}' = \mathbb{1}_{\{\,\mathrm{k}>0\,\}}$. Since $\lim_{\varepsilon \to 0+} \|\mathrm{k}_\varepsilon - \mathrm{k}'\|_{p,q} = 0$, we deduce from Proposition 3.6 on the stability of R_e that $\rho(T_{\mathrm{k}'\eta}) = R_e[\mathrm{k}'](\eta) = \lim_{\varepsilon\to 0+} R_e[\mathrm{k}_\varepsilon](\eta) = \lim_{\varepsilon \to 0+} \rho(T_{\mathrm{k}_\varepsilon \eta}) = 0$. As the support of k is symmetric, we deduce that the kernel k' is symmetric. According to (12), we have:

$$\rho(T_{\mathrm{k}''}) = \rho(T_{\mathrm{k}'\eta}) = 0,$$

with $\mathrm{k}'' = \sqrt{\eta}\, \mathrm{k}' \sqrt{\eta} = \sqrt{\eta}\, \mathbb{1}_{\{\,\mathrm{k}>0\,\}} \sqrt{\eta}$. Since the kernel k'' is symmetric, non-negative and bounded by 1, this implies that $\mathrm{k}'' = 0$ $\mathrm{d}\mu^{\otimes 2}$-a.s., and thus $\{\,\eta > 0\,\}$ is an independent set for k. This gives $\mu\,(\eta > 0) \leq \alpha(\mathrm{k})$. Therefore, we have the following lower bound for the cost $C_{\text{uni}}(\eta)$:

$$C_{\text{uni}}(\eta) = 1 - \int_\Omega \eta \, \mathrm{d}\mu \geq 1 - \mu\,(\eta > 0) \geq 1 - \alpha(\mathrm{k}).$$

This ends the proof of the proposition. □

7. Pareto and anti-Pareto frontiers for reducible kernels

When the kernel k is "truly reducible" (corresponding to the set of indices I below to be such that $\sharp I \geq 2$), it is natural to ask whether the Pareto and anti-Pareto frontiers of the subsystems entirely characterize the frontiers for k, and in what sense the optimization problems can be "reduced" to the separate study of each irreducible component.

We can achieve an elementary description of the anti-Pareto frontier when the kernel is not reducible using a Frobenius decomposition, see [24, 36] and [34] or the "super diagonal" form, see [13, Part II.2]. For convenience, we follow [34], see also [5, Lemma 5.17] in the case k symmetric.

Let k be a kernel on Ω with finite double norm. Let $\mathscr{A}$ be the set of k-invariant sets, and notice that $\mathscr{A}$ is stable by countable unions and countable intersections. Let $\sigma(\mathscr{A})$ be the σ-field generated by $\mathscr{A}$, and we denote by $(\Omega_i, i \in I)$ the at most countable (but possibly empty) collection of atoms with respect to the measure μ. Notice that the atoms are define up to an a.s. equivalence and can be chosen to be pair-wise disjoint. For $i \in I$, we set:

$$\mathrm{k}_i = \mathbb{1}_{\Omega_i} \mathrm{k} \mathbb{1}_{\Omega_i}, \tag{44}$$

which is a kernel on Ω with finite double norm. Set $\Omega_0 = (\cup_{i \in I} \Omega_i)^c$ (and assume the set of indices I has been chosen so that it does not contain 0). Thanks to [34, Lemma 12] or [36, Section II], there exists a total order, say $\preccurlyeq$, on I (not unique in general) such that for all $i, j \in I$:

(i) $j \prec i$ implies $\mathrm{k}(\Omega_i, \Omega_j) = 0$. In the epidemiology setting, $j \prec i$ means that the sub-population Ω_j can not infect the sub-population Ω_i.
(ii) $\mu(\Omega_i) > 0$ and k restricted to Ω_i is irreducible and has positive spectral radius, that is k_i is quasi-irreducible, and $R_e[\mathrm{k}](\mathbb{1}_{\Omega_i}) = R_0[\mathrm{k}_i] > 0$.
(iii) k reduced to Ω_0 is quasi-nilpotent, that is $R_e[\mathrm{k}](\mathbb{1}_{\Omega_0}) = 0$.
(iv) For all $\lambda \in \mathbb{C}^*$:

$$\mathrm{m}(\lambda, \mathrm{k}) = \sum_{i \in I} \mathrm{m}(\lambda, \mathrm{k}_i). \tag{45}$$

The next remark gives some elementary results related to the Frobenius decomposition.

Remark 7.1. Recall $R_0[\mathrm{k}]$ denote the spectral radius of the integral operator with kernel k and that $\{\mathrm{k} \equiv 0\} = \{x \in \Omega : \ \mathrm{k}(x, \Omega) + \mathrm{k}(\Omega, x) = 0\}$. We have:

(i) If the spectral radius of the kernel k is positive, then I is non-empty.
(ii) If the kernel k is quasi-irreducible, then $\Omega_0 = \{\mathrm{k} \equiv 0\}$ and I is a singleton.
(iii) The kernel k is monatomic if and only if I is a singleton, say $I = \{\mathrm{a}\}$. Then the set Ω_a is the atom of k.
(iv) If A invariant implies A^c invariant, then we have $\Omega_0 = \{\mathrm{k} \equiv 0\}$ and $\mathrm{k} = \sum_{i \in I} \mathrm{k}_i$ (k reduced to Ω_0 is zero and intuitively k is block diagonal).
(v) The cardinal of the set of indices $i \in I$ such that $R_0[\mathrm{k}_i] = R_0[\mathrm{k}]$ is exactly equal to the multiplicity of $R_0[\mathrm{k}]$ for T_k, that is $\mathrm{m}(R_0[\mathrm{k}], \mathrm{k})$.
(vi) An eigenvalue λ of T_k is *distinguished* if its distinguished multiplicity $\sharp\{i \in I : \ R_0[\mathrm{k}_i] = \lambda\}$ is positive. Notice that $R_0[\mathrm{k}]$ is distinguished with its distinguished multiplicity equal to its multiplicity. Indeed if $R_0[\mathrm{k}]$ is an eigenvalue of k_i, then it is its spectral radius and thus has multiplicity one as k_i is quasi-irreducible. We also deduce that $\mathrm{m}(R_0[\mathrm{k}], \mathrm{k}_i) \in \{0, 1\}$ for all $i \in I$.

For $i \in I$ and $\eta \in \Delta$, we set $\eta_i = \eta \mathbb{1}_{\Omega_i}$ and recall that $\mathrm{k}_i = \mathbb{1}_{\Omega_i} \mathrm{k} \mathbb{1}_{\Omega_i}$. We now give the decomposition of $R_e[\mathrm{k}]$ according to the quasi-irreducible components $(\mathrm{k}_i, i \in I)$ of k.

Lemma 7.2. *Let* k *be a finite double norm kernel on* Ω *such that* $R_0 = R_0[\mathrm{k}] > 0$. *We have for* $\eta \in \Delta$*:*

$$R_e[\mathrm{k}](\eta) = \max_{i \in I} R_e[\mathrm{k}_i](\eta_i) = \max_{i \in I} R_e[\mathrm{k}](\eta \mathbb{1}_{\Omega_i}). \tag{46}$$

Proof. For $A \in \mathscr{F}$, recall $\mathrm{m}(\lambda, \mathrm{k}, A)$ denotes the multiplicity (possibly equal to 0) of the eigenvalue $\lambda \in \mathbb{C}^*$ for the integral operator $T_{\mathrm{k}\mathbb{1}_A}$ associated to the kernel $\mathrm{k}\mathbb{1}_A$. Let $A, B \in \mathscr{F}$ be such that $A \cap B = \emptyset$ a.s. and $\mathrm{k}(B, A) = 0$. Let $\eta \in \Delta$. Clearly we have $(\mathrm{k}\eta)(B, A) = 0$, and thus Lemma 6.7 gives that for all $\eta \in \Delta$:

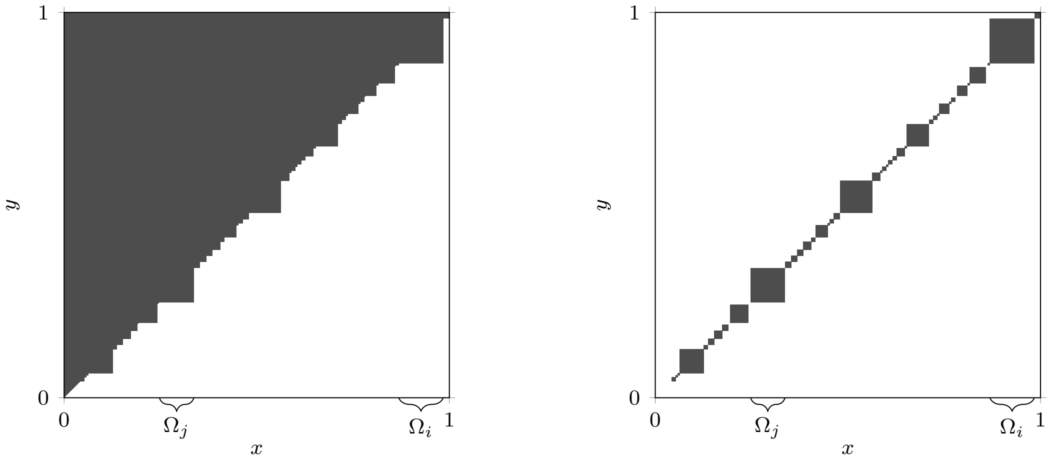
$$\mathrm{m}(\lambda, \mathrm{k}\eta, A \cup B) = \mathrm{m}(\lambda, \mathrm{k}\eta, A) + \mathrm{m}(\lambda, \mathrm{k}\eta, B).$$

Then, an immediate adaptation of the proof of [34, Theorem 7] gives that for all $\lambda \in \mathbb{C}^*$:

$$\mathrm{m}(\lambda, \mathrm{k}\eta, \Omega) = \sum_{i \in I} \mathrm{m}(\lambda, \mathrm{k}\eta, \Omega_i). \tag{47}$$

By definition of $\mathrm{m}(\lambda, \cdot, \cdot)$, we get $R_e[\mathrm{k}](\eta) = \max\{|\lambda| : \ \mathrm{m}(\lambda, \mathrm{k}\eta, \Omega) > 0\}$ and $R_e[\mathrm{k}\mathbb{1}_{\Omega_i}](\eta) = \max\{|\lambda| : \ \mathrm{m}(\lambda, \mathrm{k}\eta, \Omega_i) > 0\}$. This gives that:

$$R_e[\mathrm{k}](\eta) = \max_{i \in I} R_e[\mathrm{k}\mathbb{1}_{\Omega_i}](\eta).$$



(A) A representation of the kernel k.

(B) A representation of the kernel $\tilde{\mathrm{k}} = \sum_{i\in I} \mathrm{k}_i$. We have $\mathrm{Spec}[\mathrm{k}] = \mathrm{Spec}[\tilde{\mathrm{k}}]$ and thus $R_e[\mathrm{k}] = R_e[\tilde{\mathrm{k}}]$.

FIGURE 7. Example of a kernel k with the white zone included in $\{\mathrm{k} = 0\}$ and the kernel $\tilde{\mathrm{k}} = \sum_{i\in I} \mathrm{k}_i$, with $\mathrm{k}_i = \mathbb{1}_{\Omega_i} \mathrm{k} \mathbb{1}_{\Omega_i}$ and $\mathrm{k}(\Omega_i, \Omega_j) = 0$ for $j \prec i$.

To conclude, notice that $R_e[\mathrm{k}](\eta \mathbb{1}_{\Omega_i}) = R_e[\mathrm{k}\mathbb{1}_{\Omega_i}](\eta) = R_e[\mathbb{1}_{\Omega_i}\mathrm{k}\mathbb{1}_{\Omega_i}](\eta) = R_e[\mathrm{k}_i](\eta_i)$, where we used Lemma 4.1 (i) for the second equality. □

From Lemma 7.2, we deduce the following result.

Lemma 7.3. *Let* k *be a finite double norm kernel on* Ω *such that* $R_0 = R_0[\mathrm{k}] > 0$*. If the function* $R_e[\mathrm{k}]$ *is concave on* Δ*, then the kernel* k *is monatomic.*

Proof. Since $R_0[\mathrm{k}]$ is positive, we deduce that k is not quasi-nilpotent. Suppose that k is not monatomic. This means that the cardinal of the at most countable set I in the decomposition (46) is at least 2. So let k_1 and k_2 be two quasi-irreducible components of k, where we assume that $\{1, 2\} \subset I$. Let Ω_1 and Ω_2 denote their respective atoms. Without loss of generality, we can suppose that $R_0[\mathrm{k}_2] \geq R_0[\mathrm{k}_1] > 0$. Consider the strategies $\eta_1 = \mathbb{1}_{\Omega_1}$ and $\eta_2 = R_0[\mathrm{k}_1]\, R_0[\mathrm{k}_2]^{-1}\, \mathbb{1}_{\Omega_2}$ (which both belong to Δ). For $\theta \in [0, 1]$, we deduce from (46) and the homogeneity of the spectral radius that $R_e[\mathrm{k}](\theta\eta_1 + (1-\theta)\eta_2) = R_e[\mathrm{k}_1] \max(\theta, 1-\theta)$. Since $\theta \mapsto \max(\theta, 1-\theta)$ is not concave, we deduce that $R_e[\mathrm{k}]$ is not concave on Δ. □

Set $\tilde{\mathrm{k}} = \sum_{i\in I} \mathrm{k}_i$. As a consequence of (47), we have that:

$$\mathrm{Spec}[\mathrm{k}] = \mathrm{Spec}[\tilde{\mathrm{k}}] \quad \text{and} \quad R_e[\mathrm{k}] = R_e[\tilde{\mathrm{k}}]. \tag{48}$$

In view of Section 4, (48) gives an other transformation of the kernel k which leaves the function Spec[k] unchanged. We represent in Figure 7(A) an example of a kernel k with its atomic decomposition using $\preccurlyeq$ as a partial order on Ω and in Figure 7(B) the corresponding kernel $\tilde{\mathrm{k}}$.

We set $R_0 = R_0[\mathrm{k}]$. For $i \in I$, we consider the loss $R_e[\mathrm{k}_i]$ and the corresponding optimal loss function $R_i^\star$ defined on $[0, c_{\max}]$ and optimal cost function $C_i^\star$. For convenience the function $C_i^\star$ which is defined on $[0, R_0[\mathrm{k}_i]]$ is extended to $[0, R_0]$ by setting $C_i^\star = 0$ on $(R_0[\mathrm{k}_i], R_0]$. Notice also that $\{\mathrm{k}_i \equiv 0\} = \Omega_i^c$. Recall that $c_{\max} = C(\mathbb{1})$. We now state the main result of this section, which in particular gives a description of the anti-Pareto frontier.

Corollary 7.4. *Suppose that the cost function C is continuous decreasing with $C(\mathbb{1}) = 0$ and consider the loss function $R_e[\mathrm{k}]$, with k a finite double norm kernel on Ω such that $R_0 = R_0[\mathrm{k}] > 0$. We have:*

$$R_e^\star = \max_{i\in I} R_i^\star \quad (\textit{on } [0, c_{\max}]), \textit{ and } \quad C^\star = \max_{i\in I} C_i^\star \quad (\textit{on } [0, R_0]);$$

the maximal cost of totally inefficient strategies is given by:

$$c^\star = C^\star(R_0) = \max_{i\in I}\{C(\mathbb{1}_{\Omega_i}) : \, R_0[\mathrm{k}_i] = R_0[\mathrm{k}]\};$$

and the anti-Pareto frontier is given by:

$$\mathcal{F}^{\mathrm{Anti}} = \left\{ \left(\max_{i\in I} C_i^\star(\ell), \ell \right) \, : \, \ell \in [0, R_0] \right\}. \tag{49}$$

Furthermore, we have for $\ell \in [0, R_0]$:

$$C_\star(\ell) = C\left(\eta_\star\right) \quad \textit{with} \quad \eta_\star = \mathbb{1}_{\Omega_0} + \sum_{i\in I} \eta_{i,\star},$$

where $\eta_\star$ is Pareto optimal with $R_e[\mathrm{k}](\eta_\star) = \ell$, and, for $i \in I$, the strategy $\eta_{i,\star} = \eta_\star \mathbb{1}_{\Omega_i}$ restricted to Ω_i is Pareto optimal for the kernel k_i restricted to Ω_i, with $R_e[\mathrm{k}_i](\eta_{i,\star}) = \min(\ell, R_0[\mathrm{k}_i])$. We also have an upper bound for the minimal cost which ensures that no infection occurs at all:

$$c_\star = C_\star(0) \leq C(\mathbb{1}_{\Omega_0}).$$

Remark 7.5. We easily deduce from the previous corollary that $C_\star(0)$ is in fact equal to the cost of $\mathbb{1}_{\Omega_0 \cup A}$ where $A = \cup_{i\in I} A_i$ and, for all $i \in I$, $A_i \subset \Omega_i$ is a C-maximal independent set associated to the kernel k_i, see Remark 6.12.

Remark 7.6. If k is not monatomic, then Assumption 7 in [10] (that is any local maximum of the loss function is also a global maximum) may or may not be satisfied for the loss function $R_e = R_e[\mathrm{k}]$, see the case of the two population model in [8]. In the former case the function $C^\star$ is continuous and the anti-Pareto frontier is connected, whereas in the latter case the function $C^\star$ may have jumps and then the anti-Pareto frontier has more than one connected component.

Proof. Equation (46) and the definition of $R_e^\star$ readily implies that $R_e^\star = \max_{i\in I} R_i^\star$.

We set $R_0 = R_0[\mathrm{k}]$ and recall that $R_e[\mathrm{k}_i](\mathbb{1}) = R_0[\mathrm{k}_i]$. Let $\ell \in (0, R_0]$. Notice that (45) implies that there is a finite number of indices $i \in I$ such that $R_0[\mathrm{k}_i] \geq \ell$. This and (46) readily implies that $C^\star(\ell) = \max_{i\in I} C_i^\star(\ell)$ for $\ell > 0$. Use that $C^\star(0) = C_i^\star(0) = c_{\max}$ to deduce that the equality $C^\star = \max_{i\in I} C_i^\star$ holds on $[0, R_0]$. The formula for $c^\star = C^\star(R_0)$ is a consequence of (46), Lemma [10, Lemma 5.14] and Remark 7.1 (v). The formula (49) for $\mathcal{F}^{\mathrm{Anti}}$ is then a consequence of (41).

Eventually, if $\eta_\star$ is Pareto optimal with $R_e[\mathrm{k}](\eta_\star) = \ell$, we deduce from (46) that $R_e[\mathrm{k}](\eta_\star \mathbb{1}_{\Omega_0^c})$ is also equal to ℓ, and since C is decreasing, this implies that $\eta_\star \geq \mathbb{1}_{\Omega_0}$ and thus $\eta_\star = \mathbb{1}_{\Omega_0} + \sum_{i\in I} \eta_{i,\star}$ with $\eta_{i,\star} = \eta_\star \mathbb{1}_{\Omega_i}$. Now if $\eta_{i,\star}$ were not Pareto optimal for the kernel k_i restricted to Ω_i or if $R_e[\mathrm{k}_i](\eta_{i,\star}) < \min(\ell, R_0[\mathrm{k}_i])$, we could increase $\eta_\star$ on Ω_i without changing the value of $R_e[\mathrm{k}]$, and thus $\eta_\star$ would not be Pareto optimal. Thus, we get that $\eta_{i,\star}$ is Pareto optimal for the kernel k_i restricted to Ω_i, that is, $\eta_{i,\star} + \mathbb{1}_{\Omega_0}$ is Pareto optimal for the kernel k_i, and that $R_e[\mathrm{k}_i](\eta_{i,\star}) = \min(\ell, R_0[\mathrm{k}_i])$. From the inequality $\eta_\star \geq \mathbb{1}_{\Omega_0}$, we deduce that $c_\star = C_\star(0) \leq C(\mathbb{1}_{\Omega_0})$. $\square$

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