Pacific Journal of Mathematics

ATOMS AND ASSOCIATED SPECTRAL PROPERTIES FOR POSITIVE OPERATORS ON L^p

JEAN-FRANÇOIS DELMAS, KACEM LEFKI AND PIERRE-ANDRÉ ZITT

Volume 337 No. 1 July 2025

PACIFIC JOURNAL OF MATHEMATICS
Vol. 337, No. 1, 2025
https://doi.org/10.2140/pjm.2025.337.87

ATOMS AND ASSOCIATED SPECTRAL PROPERTIES FOR POSITIVE OPERATORS ON L^p

JEAN-FRANÇOIS DELMAS, KACEM LEFKI AND PIERRE-ANDRÉ ZITT

Inspired by Schwartz (1961) and Jang-Lewis and Victory (1993), who study generalizations of triangularizations of matrices to operators, we shall give equivalent definitions of atoms (maximal irreducible sets) for positive operators on Lebesgue spaces. We also characterize positive power compact operators having a unique nonzero atom which appear as a natural generalization of irreducible operators and are also considered in epidemiological models. Using the different characterizations of atoms, we also provide a short proof for the representation of the ascent of a positive power compact operator as the maximal length in the graph of critical atoms.

1. Introduction and main results

1.1. ***Setting and main goals.*** We consider the Lebesgue space $L^p = L^p(\Omega, \mathcal{F}, \mu)$ with $p \in (1, +\infty)$, and a state space Ω endowed with a σ-field $\mathcal{F}$ and a nonzero σ-finite measure μ. In what follows, the inclusion of measurable sets will be understood up to sets of μ-zero measure. Recall a map $f \in L^p$ is *nonnegative* if $\mu(f < 0) = 0$ and *positive* if $\mu(f \leq 0) = 0$. Let T be a positive bounded linear operator on L^p (that is, Tf is nonnegative if f is nonnegative). For $A \in \mathcal{F}$, we denote by $T(A)$ the support of $T(\mathbb{1}_A)$ which is a measurable set defined up to sets of μ-zero measure (if $\mathbb{1}_A$ does not belong to L^p, then one can replace it by $f\mathbb{1}_A$

This work is partially supported by Labex Bézout reference ANR-10-LABX-58.
MSC2020: 47A46, 47B38, 47B65.
Keywords: positive operator, power compact operator, atomic decomposition, irreducibility, distinguished eigenvalues, generalized eigenfunctions, monatomicity, ascent.

for any positive function $f \in L^p$, see the first part of Section 3 for details). Then, we say a set $A \in \mathcal{F}$ is *invariant* if $T(A) \subset A$. A set A is *coinvariant* if A^c is invariant (or equivalently if A is invariant for the dual operator $T^\star$). The collection of *admissible* sets corresponds to the σ-field $\mathcal{A} \subset \mathcal{F}$ generated by the invariant sets. We define the *atoms* as the minimal admissible sets with positive measure. An atom is *nonzero* if $\mathbb{1}_A T(\mathbb{1}_A)$ is nonzero. An atom is *critical* if the spectral radii of T and of $T_A = \mathbb{1}_A T(\mathbb{1}_A \cdot)$ are equal.

Building on works by Schwartz [27] and Jang-Lewis and Victory [19], which study in particular generalizations of triangularizations of matrices to operators, our aim in this work is threefold:

(1) Give several equivalent definitions of atoms.

(2) Describe all the nonnegative eigenfunctions of T using distinguished atoms, allowing a characterization of operators T having a unique nonzero atom.

(3) Describe all the generalized eigenfunctions of T whose eigenvalue is the spectral radius of T, and represent the ascent of T as the maximal length in the graph of critical atoms.

Except the characterization of atoms, all our results are proved under the assumption that T is power compact.

We now give details on each of these aspects, discussing the relevant literature after each statement.

1.2. ***On atoms.*** For a measurable set A, let its *future* $F(A)$ (or its *past* $P(A)$) be the smallest invariant (or coinvariant) set containing A. When T is seen as the transmission operator for an epidemic propagation, see Delmas, Dronnier and Zitt [9], the future $F(A)$ can be interpreted as the subpopulation of Ω which might be infected by an epidemic starting in A, and $P(A)$ can be interpreted as the subpopulation of Ω which may contaminate the population A. Motivated by the point of view of successive infections, we prove the following interpretation of the future in Corollary 5.7, for $A \in \mathcal{F}$:

$$\mathrm{e}^T(A) = \bigcup_{n \in \mathbb{N}} T^n(A) = F(A).$$

We say the operator T on L^p is *irreducible* if its only invariant sets are a.e. equal to $\varnothing$ or Ω; in particular $F(A) = P(A) = \Omega$ for any measurable set A with positive measure. We say that a set $A \in \mathcal{F}$ is *irreducible* if it has positive measure and the operator $T_A = \mathbb{1}_A T(\mathbb{1}_A \cdot)$ restricted to the set A is irreducible.

Motivated by the example of Volterra operator, see Example 4.4 below for details, and by an analogy with order theory, we say that an admissible set A is *convex* if $A = P(A) \cap F(A)$.

Our first result gives equivalent characterizations of atoms using convex sets and irreducible sets.

Theorem 1 (equivalent definitions of atoms). *Let T be a positive operator on L^p with $p \in (1, +\infty)$. The following properties are equivalent.*

(i) *The set A is an atom.*

(ii) *The set A is minimal among convex sets with positive measure.*

(iii) *The set A is an admissible irreducible set.*

(iv) *The set A is a maximal irreducible set.*

Notice that atoms are pairwise disjoint. Following [27], when T is power compact, we can write Ω as an at most countable partition of the atoms and a (possibly empty) set Ω_0; furthermore T is irreducible on each atom and quasinilpotent on Ω_0.

Remark 1.1 (related notions and results). Various definitions and properties of atoms already appear in the literature. Our definition of invariance and atoms are adapted from Schwartz [27], see also Victory [29; 30]. The past of a set appears in Nelson [22] (as the closure) and in Jang-Lewis and Victory [19] (as closure for bands in a Banach lattice). Irreducibility corresponds to ideal-irreducibility from Schaefer [26, Chapter 2.8]. Maximal irreducible sets appear in [22] and [29] for kernel operators (where they are called components), and Omladič and Omladič [23] for more general Banach lattices (where they are called classes). Convexity of atoms is used in the proof of [27, Lemma 12]; the irreducible bands used in the Frobenius decomposition from Jang and Victory [18] are convex irreducible sets, and the semiinvariant bands, considered by Bernik, Marcoux and Radjavi [5] are in particular convex. However, to the best of our knowledge, convexity has not been studied for its own sake in this setting, and the equivalence provided by Theorem 1 is new.

Finally, the decomposition of the space in atoms and a part where T is quasinilpotent is essentially due to Schwartz [27]. It corresponds, for nonnegative matrices, to the Frobenius normal form introduced by Victory [31], that is, a block triangularization of the matrix according to the communication classes. Notice that the triangularization of matrices has been extended to (bounded) operators in Banach spaces by Ringrose [24] using invariant spaces, see also Dowson [10, Section 2].

To conclude this section, we also stress that the atoms of T and of its power, say T^n for some fixed $n \geq 1$, might not coincide. More precisely, according to Proposition 5.9, a T^n-atom is included in a T-atom, and any T-atom is the union of d T^n-atoms where d divides n.

1.3. ***Nonnegative eigenfunctions.*** From now on we assume that the positive operator T is power compact with positive spectral radius $\rho(T) > 0$. An eigenfunction of T might be called a right eigenfunction to distinguish it from a left eigenfunction of T which corresponds to a right eigenfunction of the dual operator $T^\star$. For a (nonzero) eigenfunction v of T, let $\rho(v)$ denote the corresponding eigenvalue: $T v = \rho(v)\, v$ (and similarly for left eigenfunctions). In what follows, uniqueness of eigenfunctions is understood up to a multiplicative constant.

Let us recall briefly two key results on nonnegative eigenfunctions for positive power compact operators, see Theorem 6.2. Let $\mathrm{m}(\lambda, T)$ denote the algebraic multiplicity of $\lambda \in \mathbb{C}^*$, that is, the dimension of $\bigcup_{k\in\mathbb{N}} \mathrm{Ker}\,(T-\lambda\,\mathrm{Id})^k$. Recall that $\lambda \in \mathbb{C}^*$ is an algebraically simple eigenvalue if $\mathrm{m}(\lambda, T) = 1$. According to the Krein–Rutman theorem, $\rho(T)$ is an eigenvalue of T, and there exists corresponding nonnegative right and left eigenfunctions. Furthermore, if $\rho(T)$ is positive and if T is irreducible, the Perron–Jentzsch theorem states that the eigenvalue $\rho(T)$ is algebraically simple, and the corresponding right and left eigenfunctions are in fact positive.

Our first result of this section characterizes *monatomic* operators, that is, operators having a unique nonzero atom, in terms of nonnegative eigenfunctions. Take care that a monatomic can have many atoms, only one of them being nonzero, see Example 6.14 for a monatomic operator with two atoms.

In the next theorem, we say that there exists a unique right (or left) nonnegative eigenfunctions of T related to a nonzero eigenvalue if there exists $u \in L^p$ a right (or $u \in L^q$ a left) nonnegative eigenfunction with $\rho(u) \neq 0$ such that if u' is a right (or left) nonnegative eigenfunction with $\rho(u') \neq 0$, then $u' = c\, u$ for some $c \in \mathbb{R}$.

Theorem 2 (characterization of monatomic operators). *Let T be a positive power compact operator on L^p with $p \in (1, \infty)$. Assume that its spectral radius is positive. The following properties are equivalent.*

(i) *The operator T is a monatomic.*

(ii) *The spectral radius $\rho(T)$ is an algebraically simple eigenvalue of T and there exist a unique right and a unique left nonnegative eigenfunctions of T related to a nonzero eigenvalue.*

(iii) *There exist a unique right and a unique left nonnegative eigenfunctions of T related to a nonzero eigenvalue, say u and v, and* $\mathrm{supp}(u) \cap \mathrm{supp}(v)$ *has positive measure.*

Furthermore, when the operator T is monatomic, if u and v denote its unique right and unique left nonnegative eigenfunctions, then $\rho(u) = \rho(v) = \rho(T)$ and $\mathrm{supp}(u) \cap \mathrm{supp}(v)$ *is the nonzero atom.*

Remark 1.2 (on monatomicity). Monatomicity is a natural extension of irreducibility which generalizes the notion of quasiirreducibility defined for symmetrical operators, see Bollobás, Janson and Riordan [7, Definition 2.11]. Monatomic operators naturally appear when studying the concavity property of the function $\eta \mapsto \rho(TM_\eta)$ where η is a $[0, 1]$-valued measurable function defined on Ω and M_η

the multiplication by η operator defined on L^p, see, e.g., Delmas, Dronnier and Zitt [8, Lemma 5.10] and its discussion for additional references in epidemiology.

More generally, we may characterize nonnegative eigenfunctions in terms of the atoms appearing in their support. Let us give a few more definitions. Let $\mathfrak{A}$ denote the set of atoms (which is at most countable and might be empty); and we introduce a (partial) order $\preccurlyeq$ and the corresponding strict order $\prec$ on this set (see Proposition 4.22): for two atoms A, B, we write $B \prec A$ if $B \subset F(A) \setminus A$. A family of atoms is an *antichain* if no two atoms in the family satisfy $B \prec A$. For any atom A, let $\rho(A)$ be the spectral radius of the restriction of T_A to A. Let us say that an atom A is *distinguished* if, for any atom B, $B \prec A$ implies that $\rho(B) < \rho(A)$, and that an eigenvalue λ is *distinguished* if there exists a distinguished atom A with $\rho(A) = \lambda$. For $\lambda \in \mathbb{R}_+^*$, we consider the (finite but possible empty) set $\mathfrak{A}(\lambda)$ of atoms with spectral radius λ and the subset $\mathfrak{A}_{\text{dist}}(\lambda)$ of distinguished atoms associated to λ:

$$\mathfrak{A}(\lambda) = \{A \in \mathfrak{A} : \rho(A) = \lambda\} \quad \text{and} \quad \mathfrak{A}_{\text{dist}}(\lambda) = \{A \in \mathfrak{A}(\lambda) : A \text{ is distinguished}\}.$$

To any distinguished atom A, we may associate a unique (up to a multiplicative constant) nonnegative eigenfunction denoted w_A such that $\text{supp}(w_A) = F(A)$ and furthermore $\rho(w_A) = \rho(A)$ (see Proposition 6.11(iii)); and then the following holds.

Theorem 3 (characterization of nonnegative right eigenfunctions). *Let T be a positive power compact operator on L^p with $p \in (1, +\infty)$. Let $\lambda > 0$. We have the following properties.*

(i) *There exists a nonnegative eigenfunction of T with eigenvalue λ if and only if λ is a distinguished eigenvalue.*

(ii) *The set $\mathfrak{A}_{\text{dist}}(\lambda)$ is a (possibly empty) finite antichain of atoms, and the family $(w_A)_{A \in \mathfrak{A}_{\text{dist}}(\lambda)}$ is linearly independent.*

(iii) *The cone of nonnegative right eigenfunctions of T with eigenvalue λ is the conical hull of $\{w_A : A \in \mathfrak{A}_{\text{dist}}(\lambda)\}$, and more precisely: if v is a nonnegative eigenfunction with $\rho(v) = \lambda$, then we have*

$$v = \sum_{A \in \mathfrak{A}_{\text{dist}}(\lambda)} \mathrm{c}_A w_A,$$

where $\mathrm{c}_A \in \mathbb{R}_+$, and $\mathrm{c}_A > 0$ if and only if $A \subset \text{supp}(v)$.

Remark 1.3 (related results). The theorem is in essence a reformulation of results by Jang-Lewis and Victory. More precisely, definitions and characterization of distinguished atoms and eigenvalues appear in [17; 19; 28; 29]; statement (i) is in [19, Theorem IV.1] in the more general context of power compact operators on a Banach lattice with an order continuous norm, and (iii) appears in [29, Corollary 1] for power compact kernel operators on L^p. See also [14] which gives conditions for the kernel of $T - \rho(T)\mathrm{Id}$ to be spanned by nonnegative eigenfunctions in a more general context.

The salient point of our approach is that it leverages the decomposition of the multiplicities of the eigenvalues given in [27, Theorem 7] and our characterization of atoms from Theorem 1 to provide simpler and shorter proofs.

1.4. ***Critical atoms and generalized eigenspace.*** We now give a particular attention to the atoms associated to $\rho(T)$. We define the generalized eigenspace:

$$K(T) = \bigcup_{k\in\mathbb{N}} \mathrm{Ker}\,(T - \rho(T)\mathrm{Id})^k.$$

Following [11] and [20], we define, with the convention $\inf\varnothing = +\infty$, the *ascent* of T at its spectral radius $\rho(T)$ by

$$\alpha_T = \inf\bigl\{k \in \mathbb{N} : \mathrm{Ker}\,(T - \rho(T)\mathrm{Id})^k = \mathrm{Ker}\,(T - \rho(T)\mathrm{Id})^{k+1}\bigr\}.$$

It is well known, see [20], that when the operator T is power compact, the ascent α_T is finite.

An atom A is *critical* when we have $\rho(A) = \rho(T)$, and we denote $\mathfrak{A}_{\mathrm{crit}} = \mathfrak{A}(\rho(T))$ the set of the critical atoms. For $n \geq 1$, a *chain of length n* is a sequence $(A_0, \ldots, A_n)$ of elements of $\mathfrak{A}_{\mathrm{crit}}$ such that $A_{i+1} \prec A_i$ for all $0 \leq i < n$. The *height* $h(A)$ of a critical atom A is one plus the maximum length of a chain starting at A.

Our last result is the following.

Theorem 4 (ascent and maximal height). *Let T be a positive power compact operator on L^p with $p \in (1, +\infty)$ with positive spectral radius. Then the ascent of T at its spectral radius $\rho(T)$ is equal to the maximal height of the critical atoms*:

$$\alpha_T = \max_{A\in\mathfrak{A}_{\mathrm{crit}}} h(A).$$

This result is also stated in [18; 19] for positive power compact operators on Banach lattices with order continuous norm, see also [16] for an example where the norm is not order continuous. We provide here a shorter proof using properties of convex sets.

1.5. ***Structure of the paper.*** After recalling basic notions on Banach spaces in Section 2, we introduce the invariant/admissible/irreducible sets and the atoms associated to a positive operator T on L^p in Section 3. In Section 4, we first define convex sets, which can be characterized as the intersection of an invariant set and a coinvariant set (that is, an invariant set for the dual operator $T^\star$). We then prove our first main result, Theorem 1 (see Theorem 4.18), on equivalent definitions of atoms. We compare in Section 5 the admissible/irreducible sets and atoms of T and T^n, see Proposition 5.9.

In Section 6, considering a power compact positive operator, we characterize the cone of nonnegative eigenfunctions with the same eigenvalue, proving Theorem 3 (see Theorem 6.12), and provide the equivalent characterizations of monatomic operators from Theorem 2 (see Theorem 6.15). Section 7 is devoted to the generalized eigenfunction associated to the eigenvalue $\rho(T)$ and Theorem 4 (see Theorem 7.3 and Corollary 7.4).

2. Notation

2.1. ***Ordered set.*** In what follows, an order relation is understood to be a partial order relation. Let $(E, \leq)$ be an ordered set, also called poset. Whenever it exists, the *supremum* of $A \subset E$, denoted by $\sup(A)$, is the least upper bound of A (formally, $\sup(A) \in E$ is defined by: for all $x \in A$, $x \leq \sup(A)$ and if for some $z \in E$ one has $x \leq z$ for all $x \in A$, then $\sup(A) \leq z$). A collection $(x_i)_{i\in\mathcal{I}}$ of elements of E is an *antichain* if for all distinct $i, j \in \mathcal{I}$, the elements x_i and x_j are not comparable for the order relation. A set $D \subset E$ is a *downset* if for all $x \in D$, $y \in E$, the relation $y \leq x$ implies $y \in D$.

2.2. ***Banach space and Banach lattice.*** Let $(X, \|\cdot\|)$ be a complex Banach space not reduced to $\{0\}$. An operator T on X is a bounded linear (and thus continuous) map from X to itself. If $Y \subset X$ is a subspace of X such that $T(Y) \subset Y$, we denote by $T|Y$ the restriction of T to the subspace Y; this is an operator on the normed vector space $(Y, \|\cdot\|)$. The operator norm of T is given by

$$\|T\|_X = \sup\{\|T(x)\| : x \in X \text{ s.t. } \|x\| = 1\},$$

its spectrum by $\mathrm{Sp}(T) = \{\lambda \in \mathbb{C} : T - \lambda\,\mathrm{Id} \text{ is not bijective}\}$, where Id is the identity operator on X, and its spectral radius (see [25, Theorem 18.9]) by

$$\rho(T) = \sup\{|\lambda| : \lambda \in \mathrm{Sp}(T)\} = \lim_{n\to\infty} \|T^n\|_X^{1/n}. \tag{1}$$

By convention we set $T^0 = \mathrm{Id}$.

Let $X^\star$ denote the (topological) dual Banach space of X, that is, the set of all the bounded linear forms on X. For $x \in X$, $x^\star \in X^\star$, let $\langle x^\star, x\rangle$ denote the duality product. For an operator T, the dual operator $T^\star$ on $X^\star$ is defined by $\langle T^\star x^\star, x\rangle = \langle x^\star, Tx\rangle$ for all $x \in X$, $x^\star \in X^\star$.

If $\lambda \in \mathbb{C}$ and $v \in X\backslash\{0\}$ satisfy $T(v) = \lambda v$, we say that v is a right eigenfunction of T, λ is a right eigenvalue of T, and, in view of the forthcoming Corollary 6.9, shall write $\lambda = \rho(v)$. Any right eigenvalue (or eigenfunction) of $T^\star$ is called a left eigenvalue (or eigenfunction) of T. Unless there is an ambiguity, we shall simply write *eigenvalue* and *eigenfunction* for right eigenvalues and eigenfunctions.

An ordered real Banach space $(X, \|\cdot\|, \leq)$ is a real Banach space $(X, \|\cdot\|)$ with an order relation $\leq$ satisfying:

- For any $x, y, z \in X$, $\lambda \geq 0$ such that $x \leq y$, we have $x+z \leq y+z$ and $\lambda x \leq \lambda y$.

For any $x \in X$, we define $|x| = \sup(\{x, -x\})$ the supremum of x and $-x$ whenever it exists. Following [26, Section 2], the ordered Banach space $(X, \|\cdot\|, \leq)$ is a *Banach lattice* if:

- For any $x, y \in X$, there exists a supremum of x and y in X.
- For any $x, y \in X$ so that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$.

Let $(X, \|\cdot\|, \leq)$ be a real Banach lattice. A vector subspace Y of X is an *ideal* if

$$x \in Y,\ y \in X,\ |y| \leq |x| \quad \Longrightarrow \quad y \in Y.$$

Let T be an operator on X. A set $Z \subset X$ is T-invariant, or simply *invariant* when there is no ambiguity, if $T(Z) \subset Z$. An operator T on X is *positive* if the positive cone $X_+ = \{x \in X : x \geq 0\}$ is invariant. The operator T is *ideal-irreducible* if the only invariant closed ideals of X are $\{0\}$ and X, see [26, Definition III.8.1].

Any Banach lattice X and any operator T on X admits a natural complex extension, see [1, Section 3.2]. The spectrum of T will be identified as the spectrum of its complex extension and denoted by $\mathrm{Sp}(T)$, furthermore by [1, Lemma 6.22], the spectral radius of the complex extension is also given by $\lim_{n\to\infty} \|T^n\|_X^{1/n}$. Moreover, by [1, Corollary 3.23], if T is positive (seen as an operator on the real Banach lattice X), then T and its complex extension have the same norm. If S and T are two operators on X, we write $T \leq S$ if the operator $S - T$ is positive. If T, S and $S - T$ are positive, then we have $\rho(T) \leq \rho(S)$, see [21, Theorem 4.2].

2.3. ***Lebesgue spaces.*** Let $(\Omega, \mathcal{F}, \mu)$ be a measured space with μ a σ-finite measure such that $\mu(\Omega) > 0$. For any $\mathcal{A} \subset \mathcal{F}$, we denote by $\sigma(\mathcal{A})$ the σ-field generated by $\mathcal{A}$. If f, g are two real-valued measurable functions defined on Ω, we write $f \leq g$ a.e. (or $f = g$ a.e.) when $\mu(\{f > g\}) = 0$ (or $\mu(\{f \neq g\}) = 0$), and denote $\mathrm{supp}(f) = \{f \neq 0\}$ the support of f. We say that a real-valued measurable function f is nonnegative when $f \geq 0$ a.e., and we say that f is positive, denoted $f > 0$ a.e., when $\mu(\{f \leq 0\}) = 0$. If $A, B \subset \Omega$ are measurable sets, we write $A \subset B$ a.e. (or $A = B$ a.e.) when $\mathbb{1}_A \leq \mathbb{1}_B$ a.e. (or $\mathbb{1}_A = \mathbb{1}_B$ a.e.). For the sake of clarity, we will omit to write a.e. in the proofs. We shall consider the following definition of minimal/maximal sets.

Definition 2.1 (minimal or maximal set for a property $\mathcal{P}$). Let $\mathcal{P} \subset \mathcal{F}$ be a class of measurable sets. We say that $A \in \mathcal{F}$ is *minimal* for $\mathcal{P}$ if $A \in \mathcal{P}$ and for any $B \in \mathcal{P}$ such that $B \subset A$ a.e., we have $B = \emptyset$ a.e. or $B = A$ a.e. We say that $A \in \mathcal{F}$ is *maximal* for $\mathcal{P}$ if A^c is minimal for $\{B^c : B \in \mathcal{P}\}$.

We will usually say "minimal + property set" for a minimal (measurable) set for the corresponding property. For example, an atom of μ is any minimal measurable set with positive measure, that is, any minimal set of $\mathcal{P} = \{A \in \mathcal{F} : \mu(A) > 0\}$.

Lemma 2.2 (existence of a minimal set). *Let $\mathcal{P} \subset \mathcal{F}$ be a class of measurable sets stable by countable intersection. Then there exists a measurable set minimal for $\mathcal{P}$, and it is unique up to an a.e. equality.*

Proof. Recall from [13, Appendix A.5] (where the result is stated for μ a probability measure, but can be easily extended to a σ-finite measure) that if $\{f_i : i \in I\}$ is a (possibly uncountable) family of $[-\infty, +\infty]$-valued measurable functions defined on Ω, then there exists a $[-\infty, +\infty]$-valued measurable function f, called the essential infimum of $\{f_i : i \in I\}$ such that:

(i) For all $i \in I$, $f_i \geq f$ a.e.

(ii) If g is another $[-\infty, +\infty]$-valued measurable function satisfying (i), then a.e. $f \geq g$.

Furthermore, there exists an at most countable set $I' \subset I$ such that a.e. $f = \inf_{i \in I'} f_i$.

We consider f the essential infimum of $\{\mathbb{1}_B : B \in \mathcal{P}\}$. Thus, there exists an at most countable set $\mathcal{P}' \subset \mathcal{P}$ such that a.e. $f = \inf_{B \in \mathcal{P}'} \mathbb{1}_B$, that is, a.e. $f = \mathbb{1}_{B'}$ with $B' = \bigcap_{B \in \mathcal{P}'} B$. Since $\mathcal{P}$ is stable by countable intersection, we get that B' belongs to $\mathcal{P}$. Property (i) above on the essential infimum implies also that $B' \subset B$ a.e. for all $B \in \mathcal{P}$. Thus the set B' is minimal for $\mathcal{P}$. This provides the existence of a minimal set for $\mathcal{P}$.

Let B'' be an other minimum of $\mathcal{P}$. By property (ii), we get that $B' \subset B''$ a.e., and thus $B'' = B'$ a.e., that is, the minimum is unique up to an a.e. equality. □

For a measurable function f, we write $\mu(f) = \int f \, \mathrm{d}\mu = \int_\Omega f(x) \, \mu(\mathrm{d}x)$ the integral of f with respect to μ when it is well defined. For $p \in (1, +\infty)$, the Lebesgue space $L^p(\Omega, \mathcal{F}, \mu)$ is the set of all real-valued measurable functions f defined on Ω whose L^p-norm, $\|f\|_p = \mu(|f|^p)^{1/p}$, is finite and where functions which are a.e. equal are identified. When there is no ambiguity we shall simply write $L^p(\Omega)$ or L^p. The set $(L^p, \|\cdot\|_p)$ is a Banach space with dual $(L^q, \|\cdot\|_q)$, where $1/p + 1/q = 1$. The duality product is thus $\langle g, f\rangle = \int fg \, \mathrm{d}\mu$ for $f \in L^p$ and $g \in L^q$. The Banach space L^p endowed with the usual order $f \leq g$, that is, $\mu(\{f > g\}) = 0$, is a Banach lattice. The positive cone L^p_+ is the subset of L^p of nonnegative functions. According to [32, Section 2] and [26, Theorem II.5.14], its closed ideal are the sets

$$L^p_A = \{f \in L^p : f \mathbb{1}_{A^c} = 0\}, \tag{2}$$

where $A \subset \Omega$ is measurable. The support $\mathrm{supp}(f)$ for $f \in L^p$ is understood up to an a.e. equality, it is formally defined as a minimal set of $\mathcal{P} = \{A \in \mathcal{F} : \mathbb{1}_{A^c} f = 0\}$; and since $\mathcal{P}$ is stable by countable intersection the minimal set is unique up to an a.e. equality thanks to Lemma 2.2.

Let T be an operator on L^p. Thanks to [12, Corollary 1.3], T and its complex extension on the natural complex extension of L^p have the same L^p-norm. Let $A \subset \Omega$ be measurable. We define the projection operator of T on A, denoted T_A, by

(3) $\quad T_A = M_A T M_A, \quad$ where the operator M_A is the multiplication by $\mathbb{1}_A$,

and, if $\mu(A) > 0$, with a slight abuse of terminology, we define by $T \,|\, A$ the restriction of the operator T_A to $L^p(A)$, where the set A is endowed with the trace of $\mathcal{F}$ on A and the measure $\mu \,|\, A(\cdot) = \mu(A \cap \cdot)$. When there is no ambiguity on the operator T, we simply write $\rho(A)$ for the spectral radius of T_A (which is also the spectral radius of $T \,|\, A$). In particular, we have $\rho(\Omega) = \rho(T)$ and $\rho(A) = 0$ if $\mu(A) = 0$. If the operator T is positive, we also have that

$$A \subset B \quad \Longrightarrow \quad \rho(A) \leq \rho(B).$$

A kernel k is a measurable nonnegative function defined on $(\Omega^2, \mathcal{F}^{\otimes 2})$. We define for a real-valued measurable function f defined on Ω, such that the map $k(x, \cdot) f(\cdot)$ belongs to L^1 for a.e. $x \in \Omega$, the function $T_k(f)$ by

$$T_k(f)(x) = \int_\Omega k(x, y) f(y)\, \mu(\mathrm{d}y) \quad \text{for } x \in \Omega. \tag{4}$$

When it is well defined as an operator on L^p, we call T_k the kernel operator associated to k.

3. Invariant sets of a positive operator

We consider the Lebesgue space $L^p = L^p(\Omega, \mathcal{F}, \mu)$ with μ a nonzero σ-finite measure and $p \in (1, +\infty)$. In this preliminary section, we introduce the notion of invariant sets, atoms, future and past of sets, and link invariance to irreducibility.

3.1. ***Invariance and atoms.*** The ideal-irreducibility of a positive operator can be described in terms of sets rather than functions. We follow the presentation of Schwartz [27] (notice μ is assumed to be finite therein). Recall we say a measurable function f is positive if $\mu(f \leq 0) = 0$.

Let T be a positive operator on L^p. Let $f \in L^p$ and $g \in L^q$ be two positive functions (with $1/p + 1/q = 1$). We define the nonnegative function $k_T^{[g,f]}$ on $\mathcal{F}^2$ as, for $A, B \in \mathcal{F}$:

$$k_T^{[g,f]}(B, A) = \langle g \mathbb{1}_B, T(f \mathbb{1}_A) \rangle = \int_B g(x)\, T(f \mathbb{1}_A)(x)\, \mu(\mathrm{d}x).$$

Notice that for $A, B \in \mathcal{F}$ we have

$$k_{T^\star}^{[f,g]}(B, A) = k_T^{[g,f]}(A, B). \tag{5}$$

We shall consider the zeros of $k_T^{[g,f]}$, that is, the set

$$\mathcal{Z}_T = \{(B, A) \in \mathcal{F}^2 : k_T^{[g,f]}(B, A) = 0\}. \tag{6}$$

Let us stress that the set $\mathcal{Z}_T$ does not depend on the choice of the positive functions $f \in L^p$ and $g \in L^q$; this is indeed a direct consequence of the following equivalences:

$$k_T^{[g,f]}(B, A) = 0 \iff \mathbb{1}_B T(f\mathbb{1}_A) = 0 \text{ a.e.} \iff T^\star(g\mathbb{1}_B)\mathbb{1}_A = 0 \text{ a.e.}, \tag{7}$$

where the first equivalence is a consequence of g positive, and the second of f positive using that $T^\star$ is the dual operator and $\langle g\mathbb{1}_B, T(f\mathbb{1}_A)\rangle = \langle T^\star(g\mathbb{1}_B), f\mathbb{1}_A\rangle$. For this reason, as long as we consider the zeros of $k_T^{[g,f]}$, when there is no ambiguity, we shall simply write

$$k_T = k_T^{[g,f]}. \tag{8}$$

Notice that for any $A \in \mathcal{F}$, the maps $k_T(\cdot, A)$ and $k_T(A, \cdot)$ on $\mathcal{F}$ are σ-additive and nonnegative.

We denote by $T(A)$ the support (which is defined a.e.) of $T(f\mathbb{1}_A)$ for $f \in L^p$ a positive function, which does not depend on the choice of the positive function f. More formally, we have the following definition.

Definition 3.1 ($T(A)$). Let $A \subset \Omega$. Let T be a positive operator on L^p with $p \in (1, +\infty)$. The set $T(A)$ is the unique minimal set (up to an a.e. equality) of $\mathcal{P} = \{B \in \mathcal{F} : k_T(B^c, A) = 0\}$.

Notice the (unique) minimal set of $\mathcal{P}$ exists by Lemma 2.2, since the class $\mathcal{P}$ is stable by countable intersection. By construction, we have $T(A) = \text{supp}(T(f\mathbb{1}_A))$ a.e. for all positive $f \in L^p$.

We can now introduce the definition of invariant sets.

Definition 3.2 (invariant and coinvariant sets). Let T be a positive operator on L^p with $p \in (1, +\infty)$. A set A is T-invariant or simply *invariant* if it is measurable and $k(A^c, A) = 0$; it is T-coinvariant or simply *coinvariant* if A^c is T-invariant. We denote by $\mathcal{I}$ the class of the invariant sets.

If A is an invariant set and $B = A$ a.e., then B also is invariant. Note also that A is T-coinvariant if and only if A is $T^\star$-invariant thanks to (5), and that the following equivalences hold:

$$A \text{ is invariant} \iff \exists h \in L^p_+,\ \text{supp}(h) = A,\ \text{and}\ T(h) = 0 \text{ on } A^c. \tag{9}$$

The next lemma is a direct consequence of the σ-additivity of k_T.

Lemma 3.3 (countable union and intersection of invariant sets). *Let T be a positive operator on L^p with $p \in (1, +\infty)$. Any at most countable union or intersection of invariant* (*or coinvariant*) *sets is invariant* (*or coinvariant*).

We have the following characterization of invariance using closed ideals.

Lemma 3.4 (invariant sets and invariant closed ideals). *Let T be a positive operator on L^p with $p \in (1, +\infty)$, and A a measurable set. The set A is T-invariant if and only if the closed ideal L^p_A is T-invariant.*

Proof. We first assume that A is invariant. Let $h \in L^p_A$, that is, $h \in L^p$ and $h\mathbb{1}_{A^c} = 0$. Let $f' \in L^p$ and $g \in L^q$ be positive and set $f = f' + |h|$. Since A is invariant, we have $k_T^{[g,f]}(A^c, A) = 0$. This gives

$$0 = \langle g\mathbb{1}_{A^c}, T(f\mathbb{1}_A)\rangle \geq \langle g\mathbb{1}_{A^c}, T(|h|)\rangle \geq \langle g\mathbb{1}_{A^c}, |T(h)|\rangle \geq 0,$$

where we used the positivity of T for the inequalities. We get that $T(h)\mathbb{1}_{A^c} = 0$, that is, $T(h) \in L^p_A$. Thus the ideal L^p_A is invariant.

We now assume that the ideal L^p_A is invariant. For $f \in L^p$ and $g \in L^q$ positive, we have that $g\mathbb{1}_{A^c}T(f\mathbb{1}_A) = 0$. Therefore $k_T^{[g,f]}(A^c, A) = 0$, thus the set A is invariant. □

Example 3.5 (the Volterra operator). Consider $\big(\Omega = [0, 1], \mathcal{F} = \mathcal{B}([0, 1]), \text{Leb}\big)$, the measured space with $\mathcal{F}$ the Borel subsets of $[0, 1]$ and Leb the Lebesgue measure on $[0, 1]$, and the kernel k on $[0, 1]$ defined by

$$k(x, y) = \mathbb{1}_{\{x \geq y\}} \quad \text{for } x, y \in [0, 1].$$

The corresponding kernel operator T_k given by (4) is the so-called Volterra operator (see [4] for some spectral and compactness properties of Volterra operators). One can see that a measurable set $A \subset [0, 1]$ is T_k-invariant (or T_k-coinvariant) if and only if $A = [a, 1]$ a.e. (or $A = [0, a]$ a.e.) with $a \in [0, 1]$.

We give an immediate application of Lemma 3.4.

Lemma 3.6 (T and T^n invariant sets). *Let T be a positive operator on L^p with $p \in (1, +\infty)$ and $n \in \mathbb{N}^*$. Any T-invariant set is T^n-invariant.*

We give another example of invariant sets, which will be useful later on.

Lemma 3.7 (the support of a nonnegative eigenfunction is invariant). *Let T be a positive operator on L^p with $p \in (1, +\infty)$ and v be a nonnegative right eigenfunction of T. Then the support of v is an invariant set*: $\text{supp}(v) \in \mathcal{I}$.

Proof. Let $f \in L^p$ be positive such that $f\mathbb{1}_{\{v>0\}} = v$, and $g \in L^q$ positive. We have

$$\begin{aligned} k_T^{[g,f]}(\text{supp}(v)^c, \text{supp}(v)) &= \langle g\mathbb{1}_{\{v=0\}}, T(f\mathbb{1}_{\{v>0\}})\rangle \\ &= \langle g\mathbb{1}_{\{v=0\}}, T(v)\rangle = \rho(v)\langle g\mathbb{1}_{\{v=0\}}, v\rangle = 0, \end{aligned}$$

where we used that $f\mathbb{1}_{\{v>0\}} = v$ for the second equality and that v is an eigenfunction of T with eigenvalue $\rho(v)$ for the third one. This proves that the set $\text{supp}(v)$ is T-invariant as the zeros of the map $k_T^{[g,f]}$ does not depend on the choice of the positive functions f and g. □

In some cases, invariance is the same for an operator and its resolvent.

Lemma 3.8 (resolvent of a positive operator). *Let T be a positive operator on L^p with $p \in (1, +\infty)$. If $\lambda \in \mathbb{R}$ satisfies $\lambda > \rho(T)$, then the operator $\lambda \operatorname{Id} - T$ is invertible, and its inverse is a positive operator on L^p. Moreover, the $(\lambda \operatorname{Id} - T)^{-1}$-invariant sets are exactly the T-invariant sets.*

Proof. Since we have $\lambda > \rho(T)$, the operator $(\lambda \operatorname{Id} - T)$ is invertible, and its inverse is given by its Neumann series

$$(\lambda \operatorname{Id} - T)^{-1} = \sum_{n=0}^{+\infty} \lambda^{-n-1} T^n.$$

This proves that the operator $(\lambda \operatorname{Id} - T)^{-1}$ is positive as sum of positive operators. Thanks to Lemma 3.6, T-invariant sets are $(\lambda \operatorname{Id} - T)^{-1}$-invariant sets; and using the Neumann series, for any $(\lambda \operatorname{Id} - T)^{-1}$-invariant set A, we get

$$0 = \int \mathbb{1}_{A^c} (\lambda \operatorname{Id} - T)^{-1} (\mathbb{1}_A) \geq \lambda^{-2} \int \mathbb{1}_{A^c} T(\mathbb{1}_A) \geq 0,$$

and thus A is T-invariant. Therefore, the $(\lambda \operatorname{Id} - T)^{-1}$-invariant sets are exactly the T-invariant sets. □

Following [27], we consider the atoms associated to T.

Definition 3.9 (admissible set and atoms). Let T be a positive operator on L^p with $p \in (1, +\infty)$. A set which belongs to the σ-field $\mathcal{A} = \sigma(\mathcal{I})$ generated by the family $\mathcal{I}$ of invariant sets is called *admissible*. A minimal admissible set with positive measure is called an *atom* of the operator T or a T-atom.

Notice that a set of positive measure A is a T-atom if and only if it is an atom for the measured space $(\Omega, \mathcal{A}, \mu)$. We denote by $\mathfrak{A}$ the set of atoms:

$$\mathfrak{A} = \{A \in \mathcal{A} : A \text{ is a } T\text{-atom}\}.$$

Since atoms have positive measure and the measure μ is σ-finite, we deduce that the set $\mathfrak{A}$ is at most countable. When there is no ambiguity on the operator T, we shall simply write atom for T-atom. We present Example 3.10 below where there is no atom, and Example 3.11 where not all measurable sets are admissible.

Example 3.10 (the Volterra operator). In Example 3.5 on the Volterra operator T_k, the admissible σ-field is the Borel σ-field on $[0, 1]$: $\mathcal{A} = \mathcal{F}$. Notice that the operator T_k has no atom $\mathfrak{A} = \emptyset$.

Example 3.11 ($\mathcal{A} \neq \mathcal{F}$). Consider $\big(\Omega = [0, 1], \mathcal{F} = \mathcal{B}([0, 1]), \text{Leb}\big)$, the measured space with $\mathcal{F}$ the Borel subsets of $[0, 1]$ and Leb the Lebesgue measure on $[0, 1]$, and the kernel k on $[0, 1]$ defined by (see Figure 1, left)

$$k(x, y) = \mathbb{1}_{\{x\leq 1/2\leq y\leq x+1/2\}} + \mathbb{1}_{\{x\geq 1/2\}}\,\mathbb{1}_{\{y\leq x-1/2\}}. \tag{10}$$

Let $A \subset [0, 1]$ be a measurable set. Then A is T_k-invariant if and only if for a.e. $x \in A^c \cap [0, 1/2]$, we have $\mathrm{Leb}([1/2, x + 1/2] \cap A) = 0$ and for a.e. $x \in A^c \cap [1/2, 1]$, we have $\mathrm{Leb}([0, x - 1/2] \cap A) = 0$. Thus, A is T_k-invariant if and only if for a.e. $x \in A^c \cap [0, 1/2]$, we have $[1/2, x + 1/2] \subset A^c$ a.e. and for a.e. $x \in A^c \cap [1/2, 1]$, we have $[0, x - 1/2] \subset A^c$ a.e. Thus A is T_k-invariant if and only if $A = [a, 1/2] \cup [a + 1/2, 1]$ a.e. with $a \in [0, 1/2]$. Therefore the σ-field $\mathcal{A}$ of T_k-admissible sets consists in all the measurable sets which are a.e. equal to $A \cup (A + 1/2)$ where $A \subset [0, 1/2]$ is a Borel set. In particular, we have $\mathcal{A} \neq \mathcal{F}$. Notice the operator T_k has no atom: $\mathfrak{A} = \varnothing$.

3.2. *Future and past.*

We now consider the future and past of a set, and refer to Remark 4.1 below for an epidemiological interpretation. Recall the Definition 2.1 on minimal and maximal set.

Definition 3.12 (future and past). Let T be a positive operator on L^p with $p \in (1, +\infty)$. Let A be a measurable set. We define its *future*, $F(A)$, as the minimal invariant set containing A (that is, the minimal set of $\mathcal{P} = \{B \in \mathcal{I} : A \subset B \text{ a.e.}\}$) and its *past*, $P(A)$, as the minimal coinvariant set containing A.

We shall use later on the following notation for the future and past of a set A without A:

$$F^*(A) = F(A) \cap A^c \quad \text{and} \quad P^*(A) = P(A) \cap A^c. \tag{11}$$

The next lemma ensures the existence of the future and the past.

Lemma 3.13 (existence of future and past). *Let T be a positive operator on L^p with $p \in (1, +\infty)$. Let $A \in \mathcal{F}$, then its future and its past exist and are unique, up to an a.e. equality.*

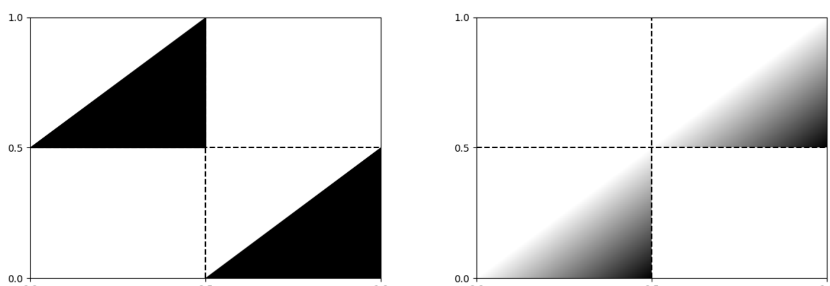

Figure 1. Example of some $[0, 1]$-valued kernels on $[0, 1]$: kernel k defined in (10) (left) and kernel $k^{\otimes 2}$ defined in (15) (right).

Proof. We only consider the future, as the proof concerning the past is similar. The set $\mathcal{P} = \{B \in \mathcal{I} : A \subset B \text{ a.e.}\}$ is stable by countable intersection thanks to Lemma 3.3. Lemma 2.2 ensures the existence of a minimal set for $\mathcal{P}$. The uniqueness is also clear. This provide the existence of the future of A. □

Let us mention that the "k-closure" of a set for a kernel operator T_k introduced by Nelson [22, p. 714] correspond to its past (with respect to the invariant sets associated to T_k). Let us gather without proof a number of elementary facts.

Lemma 3.14 (basic properties of the future of a measurable set). *Let T be a positive operator on L^p with $p \in (1, +\infty)$. For any measurable sets A and B, and for any at most countable family of measurable sets $(A_i)_{i \in I}$, the following properties hold*:

(i) *$F(\varnothing) = \varnothing$ a.e. and $F(\Omega) = \Omega$ a.e.*

(ii) *A set A is invariant if and only if $F(A) = A$ a.e.*

(iii) *If $A \subset B$ a.e., then $F(A) \subset F(B)$ a.e.*

(iv) *$F\big(\bigcup_{i \in I} A_i\big) = \bigcup_{i \in I} F(A_i)$ a.e.*

(v) *$F\big(\bigcap_{i \in I} A_i\big) \subset \bigcap_{i \in I} F(A_i)$ a.e. the reverse inclusion does not hold in general.*

(vi) *$F(F(A)) = F(A)$ a.e.*

The properties (i)–(vi) *also hold with F replaced by P.*

Futures and pasts are related by the following elementary result; by contrast, note that the inclusion $A \subset F(B)$ does not imply that $B \subset P(A)$ in general, see Example 4.2.

Lemma 3.15 (intersection of a future and a past). *Let T be a positive operator on L^p with $p \in (1, +\infty)$. Let A, B be two measurable sets. We have*

$$A \cap P(B) = \varnothing \text{ a.e.} \iff F(A) \cap P(B) = \varnothing \text{ a.e.} \iff F(A) \cap B = \varnothing \text{ a.e.}$$

Proof. If $A \cap P(B) = \varnothing$, then A is included in $P(B)^c$. Since $P(B)^c$ is invariant, we have $F(A) \subset P(B)^c$ by minimality, which means that $F(A) \cap P(B) = \varnothing$. The converse is clear since $A \subset F(A)$. The second equivalence is proved similarly. □

3.3. *Irreducibility.* Similarly to Schaefer [26, Definition III.8.1], we can define the irreducibility of an operator in terms of invariance.

Definition 3.16 (irreducible operators and invariant sets). Let T be a positive operator on L^p with $p \in (1, +\infty)$.

(i) The operator T is *irreducible* if its only invariant sets are a.e. equal to $\varnothing$ or Ω.

(ii) The measurable set A is T-irreducible or simply *irreducible* if it is measurable with positive measure and if the restricted operator $T|A$ on $L^p(A)$ is irreducible.

We stress that an irreducible positive operator on L^p is nonzero if $\dim(L^p) > 1$ (that is, if there exists $B \in \mathcal{F}$ such that $\mu(B)$ and $\mu(B^c)$ are positive).

We refer to Lemma 4.14 and Theorem 4.18 for relations between irreducible sets and atoms. See Example 5.4 for a comment on the irreducible sets of T and of T^2. We now state explicitly the relation between invariance and irreducibility from Section 2.2 and from Definitions 3.2 and 3.16. Recall the definition of the closed ideal in (2).

Lemma 3.17. *Let T be a positive operator on L^p with $p \in (1, +\infty)$. Then the operator T is irreducible if and only if it is ideal-irreducible.*

Proof. It is a direct consequence of Lemma 3.4 and the fact that the closed ideals of L^p are exactly given by L^p_A for A measurable, see [32, Section 2] and [26, Theorem II.5.14]. □

4. Convex sets and characterization of atoms

We first present a countable example with an epidemiological interpretation and also introduce the notion of convex set; then we study the properties of convex sets in a more general framework. Then provide one of the main results of this paper on equivalent characterizations of atoms of a positive bounded operator on the Lebesgue space $L^p = L^p(\Omega, \mathcal{F}, \mu)$ with μ a nonzero σ-finite measure and $p \in (1, +\infty)$. We keep notation from Section 3.

4.1. ***The countable case and an underlying preorder.*** We assume in this section only that Ω is at most countable, and without loss of generality that $\mu(\{x\}) > 0$ for all $x \in \Omega$. Let T be a positive operator on L^p. The map k_T is entirely defined by the values of $k_T(\{x\}, \{y\})$, denoted $k_T(x, y)$ for $x, y \in \Omega$. The notions of admissibility, atoms, invariance and irreducibility may in that case be completely understood by studying a particular binary relation on Ω given in terms of k_T. To see this, we write $x \preccurlyeq y$ if $x = y$ or if there exists $n \in \mathbb{N}^*$ and $(x = x_0, x_1, \dots, x_{n-1}, x_n = y) \in \Omega^{n+1}$ such that $\prod_{i=1}^n k_T(x_{i-1}, x_i) > 0$. The relation $\preccurlyeq$ defines a preorder on Ω (that is, a reflexive transitive binary relation). The relation $x \sim y$, given by $x \preccurlyeq y$ and $y \preccurlyeq x$, is then an equivalence relation. The equivalence classes of $\sim$ correspond to atoms of the operator T, and the preorder $\preccurlyeq$ naturally induces an order on them: for two atoms A, B, we have $A \preccurlyeq B$ if $x \preccurlyeq y$ for all $x \in A$ and $y \in B$. The admissible sets are the sets A that may be written as unions of atoms (the σ-field $\mathcal{A}$ is generated by the set of atoms). Furthermore, a set A is invariant if and only if the two following conditions hold:

- A is the union of atoms $\{A_i : i \in I\}$ (in particular, A is admissible).
- The family $\{A_i : i \in I\}$ is a downset for the order induced by $\preccurlyeq$ on atoms, that is, for all $i \in I$ and for all atom A, if $A \preccurlyeq A_i$ then $A \in \{A_i : i \in I\}$.

For a set A, its future corresponds to the downward closure of A, that is, the smallest downset containing A, and its future and past are given by

$$F(A) = \bigcup_{x\in A}\{y \in \Omega : y \preccurlyeq x\} \quad \text{and} \quad P(A) = \bigcup_{x\in A}\{y \in \Omega : x \preccurlyeq y\}.$$

Notice that the definition of atoms, invariant sets, future and past of a set depends only on the support $\{k_T > 0\} \subset \Omega^2$ of the kernel k_T.

Remark 4.1 (epidemiological interpretation). In the epidemiological interpretation where each element of Ω is seen as an individual or an homogeneous subpopulation and T can be assimilated to the next generation operator, we have:

- $k_T(x, y) > 0$ means that individual y can directly infect individual x.
- $x \preccurlyeq y$ when there may be a chain of infections from individual y to individual x.
- The set A is invariant if an epidemic started in A stays within A.
- The future $F(A)$ of A is the set of all individuals that might get infected by an epidemic starting at every individual of A.
- The past $P(A)$ of A is the set of all individuals that might infect an individual of A.

In Section 4.2 we consider convex sets, that is, sets A such that $A = F(A) \cap P(A)$. They have a simple representation when Ω is at most countable. Following the terminology of [6, Section I.4, p. 7] we define the *interval* $[x, y] = \{z \in \Omega : x \preccurlyeq z \preccurlyeq y\}$ for $x, y \in \Omega$, and say that a set $A \subset \Omega$ is (*order*-)*convex* if

$$x, y \in A \quad \Longrightarrow \quad [x, y] \subset A.$$

It is easily checked that an order-convex set corresponds to being the union of atoms $(A_i)_{i\in I}$ where the family $(A_i)_{i\in I}$ is order convex, that is, if A is an atom such that $A_i \preccurlyeq A \preccurlyeq A_{i'}$ for some $i, i' \in I$, then A belongs to the family $(A_i)_{i\in I}$.

Example 4.2 (a finite elementary case). We consider the finite case: $\Omega = \{1, \ldots, n\}$ with $n \in \mathbb{N}^*$, μ is the counting measure, $L^p(\Omega)$ is identified with $\mathbb{R}^n$ and operators on L^p with $n \times n$ real matrices. A matrix $M = (M_{i,j})_{1\leq i,j\leq n}$ with nonnegative entries is alternatively represented by the oriented weighted graph $G = (V, E)$ with $V = \{1, \ldots, n\}$ and with a weight $M_{i,j}$ to the edge $(j, i) \in E$.

To illustrate, consider the case $n = 6$ with the matrix given in Figure 2 (left) where the $\star$ correspond to positive terms. The corresponding communication graph (an oriented edge is represented for each positive entry of the matrix) is given in Figure 2 (right). The atoms are: $\{1, 2, 3\}$, $\{4\}$, $\{5\}$ and $\{6\}$. The invariant sets are: Ω, $\{4, 5, 6\}$, $\{4, 6\}$, $\{5, 6\}$, $\{6\}$ and $\emptyset$. For example, the sets $\{1, 2, 3\}$, $\{1, 2\}$ and $\{1\}$ are irreducible, and among those three only the first one is admissible. For example, the

$$\begin{pmatrix} 0 & \star & 0 & 0 & 0 & 0 \\ \star & 0 & \star & 0 & 0 & 0 \\ 0 & \star & 0 & 0 & 0 & 0 \\ 0 & \star & 0 & 0 & 0 & 0 \\ 0 & \star & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \star & \star & 0 \end{pmatrix}$$

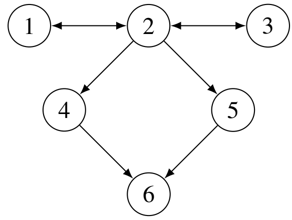

Figure 2. Matrix on $\{1, \ldots, 6\}$ with $\star > 0$ (left) and associated communication graph (right) from Example 4.2.

sets $\{1, 2, 3, 4\}$, $\{5\}$ and $\{5, 6\}$ are convex. Even though the set $\{5\}^c$ is admissible, it is not convex.

Let us notice that the inclusion in Lemma 3.14(v) is not an equality in general; indeed we have $F(\{4\} \cap \{5\}) = F(\varnothing) = \varnothing$ whereas $F(\{4\}) \cap F(\{5\}) = \{6\}$. Notice also that $\{5\}$ belongs to the future of $\{1, 2, 3, 4\}$, but the latter does not belong to the past of $\{5\}$.

The countable state space Ω and the above representation of convex sets will guide many definitions and proofs below. The general case is at the same time more technical (invariant sets are defined up to an a.e. equality), and more subtle: for example, the union of all atoms may be a strict subset of the whole space; it may even be empty, as in Example 3.10 where there exists no atom of the Volterra operator. For this reason we will work only on invariant and coinvariant sets, viewing them intuitively as down- and up-sets of an underlying order that we will not write down formally.

4.2. ***Order-convex subsets.*** By construction of the future and the past, a measurable set A is always included in $F(A) \cap P(A)$. The set A is convex when there is equality.

Definition 4.3 (order-convex subset). Let T be a positive operator on L^p with $p \in (1, +\infty)$. A set A is *order-convex* for T, or T*-convex*, if it is measurable and $A = F(A) \cap P(A)$ a.e.

When there is no ambiguity on the operator T, we shall simply write convex for T-convex.

Example 4.4 (convex sets of the Volterra operator). We continue Example 3.5 on the Volterra operator. Using the description therein of invariant and coinvariant sets, we get that a set A is convex if and only if $A = [a, b]$ a.e. with $0 \leq a \leq b \leq 1$.

Remark 4.5 (atoms, irreducibility and convexity coincide for T and $T^\star$). Notice that the admissible σ-field is the same for the operator T and its dual $T^\star$. Thanks to (5), the operator T is irreducible if and only if $T^\star$ is irreducible. Thus a set A is a T-atom (or T-irreducible, T-convex) if and only if it is a $T^\star$-atom (or $T^\star$-irreducible, $T^\star$-convex).

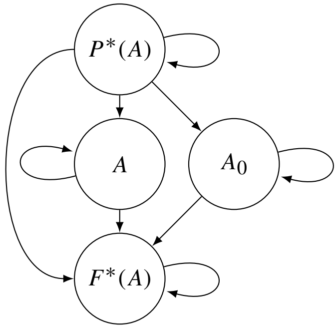

Figure 3. Past and future for a T-convex set A. The three sets A, $F^*(A)$ and $P^*(A)$ are disjoint as A is convex. Let $A_0 = (P(A) \cup F(A))^c$, so that the four sets A, $F^*(A)$, $P^*(A)$ and A_0 form a partition of Ω in admissible sets. The possible connections between the four sets are depicted in the picture: if there is no arrow from B to C then $k_T(C, B) = 0$.

Remark 4.6 (convex sets on a countable measurable set). We go back to the framework of Section 4.1, where Ω is an at most countable set. Then A is a convex set in the sense of Definition 4.3 if and only if A is order-convex in the sense of the definition of Section 4.1. Therefore the two definitions are coherent.

Recall (11), where we set $F^*(A) = F(A) \cap A^c$ and $P^*(A) = P(A) \cap A^c$.

Lemma 4.7 (characterization of convexity). *Let T be a positive operator on L^p with $p \in (1, +\infty)$. Let A be a measurable set. The following properties are equivalent*:

(i) *A is convex.*

(ii) *$F^*(A) \cap P^*(A) = \varnothing$ a.e.*

(iii) *$F^*(A)$ is invariant.*

(iv) *$P^*(A)$ is coinvariant.*

(v) *There exist an invariant set B and a coinvariant set C such that $A = B \cap C$ a.e.*

As a particular consequence of (v), if A is measurable then $F(A) \cap P(A)$ is convex. We illustrate in Figure 3 the possible connections between the sets A, $F^*(A)$, $P^*(A)$ and the complementary of their union, when A is convex.

Proof. Use the definition of convexity and that Lemma 4.7(ii) is equivalent to $P(A) \cap F(A) \cap A^c = \varnothing$ to get that (i) and (ii) are equivalent. Clearly (i) implies (v). The proofs involving (iii) are similar to the ones involving (iv), so the latter are left to the reader.

We assume (ii) and prove (iii). As $F^*(A) \cap P^*(A) = \varnothing$, the set $F^*(A)$ is a subset of $P^*(A)^c$. Therefore, the set $F^*(A) = (A \cup F^*(A)) \cap (A \cup P^*(A))^c = F(A) \cap P(A)^c$ is invariant as the intersection of two invariant sets. Thus (iii) holds.

Conversely, assuming (iii), the set $F^*(A)$ is invariant, so the set $P(A) \cap F^*(A)^c$ is a coinvariant set containing A and included in $P(A)$. By minimality of $P(A)$, this set is equal to $P(A)$, thus $P(A) \subset F^*(A)^c$. This gives (ii).

Finally let us assume (v) and prove (i). By assumption, we have $A = B \cap C$ with B invariant and C coinvariant. By minimality, we get that $F(A) \subset B$ and $P(A) \subset C$, and thus

$$A \subset F(A) \cap P(A) \subset B \cap C = A.$$

This gives that A is convex, that is, (i). □

We end this section with an auxiliary result on convexity.

Lemma 4.8 (intersection of convex and invariant sets). *Let T be a positive operator on L^p with $p \in (1, +\infty)$. Let A be a convex set and B an invariant set. Then the set $A \cap B$ is convex.*

Proof. We have $A \cap B = P(A) \cap F(A) \cap B$ by definition of convexity. So by Lemma 4.7 it is convex as the intersection of the coinvariant set $P(A)$ with the invariant set $F(A) \cap B$. □

4.3. ***Properties of the projected operators.*** Let $\Omega' \subset \Omega$ be a measurable set with positive measure. Let T be a positive operator on L^p with $p \in (1, +\infty)$. We start with a result of stability of invariant/irreducible sets and atoms by projection. Recall $T_{\Omega'}$ is the projection of T on Ω' given by (3).

Lemma 4.9 (projection and invariance/irreducibility). *Let T be a positive operator on L^p with $p \in (1, +\infty)$, $\Omega' \subset \Omega$ a measurable set with positive measure, and $T' = T_{\Omega'}$ the projection of T on Ω'. We have the following properties.*

(i) *The set Ω' is T'-invariant and T'-coinvariant.*

(ii) *Every T-invariant set is T'-invariant.*

(iii) *One can replace invariant in (ii) by coinvariant and by admissible.*

(iv) *The set $A \subset \Omega'$ is T-irreducible if and only if it is T'-irreducible.*

(v) *If Ω' is T-invariant and $A \subset \Omega'$, then A is T-invariant if and only if it is T'-invariant.*

Proof. Since $k_{T'}(\Omega'^c, \cdot\,) = k_{T'}(\,\cdot\,, \Omega'^c) = 0$, we obtain (i). Recall the definition of $\mathcal{Z}_T$, the set of zeros of k_T, given in (6). Since T is positive, we clearly have $k_T \geq k_{T'}$ and thus $\mathcal{Z}_T \subset \mathcal{Z}_{T'}$. This gives (ii) and the coinvariant case in (iii). As the invariant sets generates the σ-field of the admissible sets, we get the admissible case of (iii). Point (iv) is immediate. For $A \subset \Omega'$ we have

$$k_T(A^c, A) = k_T(A^c \cap \Omega', A) + k_T(A^c \cap \Omega'^c, A) \leq k_{T'}(A^c, A) + k_T(\Omega'^c, \Omega').$$

If Ω' is invariant, and thus $k_T(\Omega'^c, \Omega') = 0$, we deduce that if A is T'-invariant, then it is T-invariant. This and (ii) give (v). □

We now study the stability of convexity and future by projection. Let $F'(A)$ denote the future of the measurable set A for the operator $T' = T_{\Omega'}$.

Lemma 4.10 (projection and convexity/future). *Let T be a positive operator on L^p with $p \in (1, +\infty)$, $\Omega' \subset \Omega$ be a measurable set with positive measure, and $T' = T_{\Omega'}$ be the projection of T on Ω'. For any measurable set $A \subset \Omega'$, the following properties hold.*

(i) *If A is T-convex then it is T'-convex.*

(ii) *We have a.e.*

$$F(A) = F(F(A) \cap \Omega'^c) \cup F'(A). \tag{12}$$

(iii) *If Ω' is T-convex, then we have $F'(A) = F(A) \cap \Omega'$ a.e. In particular, T'-invariant subsets of Ω' are exactly the trace on Ω' of T-invariant sets.*

Proof. Let $A \subset \Omega'$ be measurable sets. As $F(A)$ is T-invariant, by Lemma 4.9(ii), we then get that the set $F(A) \cap \Omega'$ is T'-invariant, and similarly the set $P(A) \cap \Omega'$ is T'-coinvariant. Since they both contain A, we deduce by the definition of the future and past of a set, that

$$F'(A) \subset F(A) \cap \Omega' \quad \text{and} \quad P'(A) \subset P(A) \cap \Omega'. \tag{13}$$

If A is T-convex, we deduce that $A \subset P'(A) \cap F'(A) \subset P(A) \cap F(A) = A$. This implies that A is T'-convex, that is, (i).

We prove (ii). Setting $B = F(A) \cap \Omega'^c$ and $C = F(B) \cup F'(A)$, the goal is to prove that $C = F(A)$. We shall first prove that C is T-invariant. Thanks to (13), we have $F(A) \cap (\Omega' \cap F'(A)^c)^c = (F(A) \cap \Omega'^c) \cup F'(A) \subset C$, that is,

$$C^c \subset F(A)^c \cup (\Omega' \cap F'(A)^c). \tag{14}$$

We deduce that

$$\begin{aligned} k_T(C^c, C) &\leq k_T(C^c, F(B)) + k_T(F(A)^c, F'(A)) + k_T(\Omega' \cap F'(A)^c, F'(A)) \\ &\leq k_T(F(B)^c, F(B)) + k_T(F(A)^c, F(A)) + k_{T'}(F'(A)^c, F'(A)) = 0, \end{aligned}$$

where we used the additivity and monotonicity of k_T and (14) for the first inequality; the monotonicity of k_T, $F(B) \subset C$, equation (13) (twice) and the definition of T' for the second; that $F(B)$ and $F(A)$ are T-invariant, and $F'(A)$ is T'-invariant for the last equality. Thus, the set C is T-invariant. As we have $A \subset C \subset F(A)$ (use $A \subset F'(A) \subset C$ for the first inclusion, and $C \subset F(F(A)) \cup F(A) = F(A)$ for the second, see Lemma 3.14(vi) and (13)), we deduce by minimality of the future that $C = F(A)$. This gives (ii).

We now prove (iii). Since Ω' is T-convex, we have

$$F(A) \cap \Omega'^c = F(A) \cap (F(\Omega') \cap P(\Omega'))^c = F(A) \cap (F(\Omega')^c \cup P(\Omega')^c).$$

Since $F(A) \subset F(\Omega')$, we deduce that

$$F(A) \cap \Omega'^c = F(A) \cap P(\Omega')^c,$$

which is invariant as intersection of two invariant sets. Now, using (ii), we get that $F(A) = (F(A) \cap \Omega'^c) \cup F'(A)$. Taking the intersection with Ω' yields that $F(A) \cap \Omega' = F'(A)$. □

4.4. *Properties of atoms.* We first prove that atoms are convex and irreducible.

Lemma 4.11. *Let T be a positive operator on L^p with $p \in (1, +\infty)$. Then atoms are convex.*

Proof. Let A be an atom and set $B = F(A) \cap P(A)$. We consider the family of measurable sets $\mathcal{A}' = \{C \in \mathcal{F} : C \cap A = \varnothing \text{ a.e. or } B \subset C \text{ a.e.}\}$. For simplicity we do not write a.e. anymore in this proof. Let C be an invariant set. As A is a minimal admissible set, we have $C \cap A = \varnothing$ or $A \subset C$. In the latter case, by minimality of $F(A)$, as C is invariant, we deduce that $F(A) \subset C$, and thus $B \subset C$. In any case, we get that C belongs to $\mathcal{A}'$, and thus $\mathcal{A}'$ contains all the invariant sets, that is, $\mathcal{I} \subset \mathcal{A}'$. A similar argument implies that $\mathcal{A}'$ contains all the coinvariant sets, that is, the complementary of all the invariant sets.

It is clear that $\mathcal{A}'$ is stable by countable union and countable intersection. Therefore, by [2, Theorem 4.2, p. 130], $\mathcal{A}'$ contains the σ-field generated by $\mathcal{I}$, that is, $\mathcal{A} \subset \mathcal{A}'$. In particular, the set A belongs to $\mathcal{A}'$. As A is an atom it has positive measure. This gives that $B \subset A$. As $A \subset F(A) \cap P(A)$, we deduce that $B = A$, that is, the set A is convex. □

Lemma 4.12. *Let T be a positive operator on L^p with $p \in (1, +\infty)$. Then atoms are irreducible.*

Proof. Let A be an atom. It is convex according to Lemma 4.11. Set $T' = T_A$. Let $B \subset A$ be T'-invariant (and thus $T \,|\, A$-invariant), and denote its future with respect to T' by $F'(B)$. By Lemma 4.10(iii), we deduce that $B = F'(B) = F(B) \cap A$. This implies that B is T-admissible. Since A is an atom, we get that $B = A$ or $B = \varnothing$. This implies that $T \,|\, A$ on $L^p(A)$ is irreducible, that is, A is irreducible. □

We then prove that intersections of irreducible sets with admissible sets are trivial.

Lemma 4.13 (intersection of irreducible and admissible sets). *Let T be a positive operator on L^p with $p \in (1, +\infty)$. If A is admissible and B irreducible, then either $A \cap B = \varnothing$ a.e. or $B \subset A$ a.e.*

Proof. Let B be irreducible. Assume first the set A is invariant. According to Lemma 4.9(i)–(ii) with $\Omega' = B$ and Lemma 3.3, the intersection $A \cap B$ is invariant for the operator T_B, and thus also for its restricted operator $T \mid B$ on $L^p(B)$. Since B is irreducible, we deduce that $A \cap B = \varnothing$ a.e. or $A \cap B = B$ a.e. Thus the collection of sets whose intersection with B is trivial, that is,

$$\mathcal{A}' = \{C \in \mathcal{F} : C \cap B = \varnothing \text{ a.e. or } B \subset C \text{ a.e.}\},$$

contains all invariant sets.

It is clear that $\mathcal{A}'$ is stable by countable union and complement, so it contains the σ-field $\mathcal{A}$ of the admissible sets which is generated by the invariant sets, that is, $\mathcal{A} \subset \mathcal{A}'$. Thus the set A belongs to $\mathcal{A}'$ and satisfies $A \cap B = \varnothing$ or $B \subset A$. □

We directly deduce from the previous lemma the following result.

Lemma 4.14 (irreducibility and atoms, I). *Let T be a positive operator on L^p with $p \in (1, +\infty)$. Then all irreducible admissible sets are atoms.*

We then prove that any irreducible set is a subset of an atom.

Lemma 4.15 (irreducibility and atoms, II). *Let T be a positive operator on L^p with $p \in (1, +\infty)$. If A is irreducible, then $F(A) \cap P(A)$ is an atom (which contains A a.e.).*

Proof. Let A be irreducible (and thus measurable with positive measure). Set $A' = P(A) \cap F(A)$. Let $B \subset A'$ be T-invariant. Then by Lemma 4.9(i)–(ii), we obtain that $A \cap B$ is T_A-invariant, so by irreducibility of A we have either $A \subset B$ or $A \cap B = \varnothing$. If $A \subset B$, then we have $F(A) \subset F(B) = B \subset A' \subset F(A)$ as B is a T-invariant set contained in A', so we have $B = A'$. If $A \cap B = \varnothing$, then the set $P(A) \cap B^c$ is T-coinvariant and contains A, so we have $P(A) \cap B^c = P(A)$ which implies that $B = \varnothing$ as $B \subset A' \subset P(A)$ by hypothesis. This proves that A' is irreducible. Since A' is admissible, we deduce from Lemma 4.14 that A' is an atom. □

To end this section we complete the statement of Lemma 4.9 by considering atoms. Recall $T_{\Omega'}$ is the projection of T to Ω' given by (3).

Proposition 4.16 (projection and atoms). *Let T be a positive operator on L^p with $p \in (1, +\infty)$, $\Omega' \subset \Omega$ a measurable set with positive measure, and $T' = T_{\Omega'}$ = the projection of T on Ω'. Let $A \subset \Omega'$ be measurable.*

(i) *If A is a T-atom then it is a T'-atom.*

(ii) *Assume Ω' is admissible. Then A is a T'-atom if and only if it is a T-atom.*

Remark 4.17 (open question). We conjecture the following result, which would imply (ii): if Ω' is admissible, then $A \subset \Omega'$ is T'-admissible if and only if it is T-admissible.

Proof. We first prove (i). Let $A \subset \Omega'$ be a T-atom. It has a positive measure, and it is T-irreducible and T-convex by Lemmas 4.11 and 4.12. It is then T'-irreducible and T'-convex (and thus T'-admissible) by Lemmas 4.9(iv) and 4.10(i). Thus, it is a T'-atom by Lemma 4.14.

We now prove (ii). Let A be a T'-atom. It has a positive measure, and it is T'-irreducible. It is also T-irreducible by Lemma 4.9(iv). This implies that $F(A) \cap P(A)$ is a T-atom by Lemma 4.15. Since Ω' is admissible and $A \subset \Omega'$, we deduce that $F(A) \cap P(A) \subset \Omega'$. Thus $F(A) \cap P(A)$ is a T'-atom by (i). It contains A, thus it is equal to A. This proves that A is a T-atom. □

4.5. ***A characterization of atoms.*** The main goal of this subsection is to prove the following theorem, that links the definitions of atoms, convex and irreducible sets.

Theorem 4.18 (equivalent definitions of atoms). *Let T be a positive operator on L^p with $p \in (1, +\infty)$. The following properties are equivalent.*

(i) *The set A is an atom.*

(ii) *The set A is a minimal convex set with positive measure.*

(iii) *The set A is an admissible irreducible set.*

(iv) *The set A is a maximal irreducible set.*

We first give another link between convexity and irreducibility before proving the theorem.

Lemma 4.19 (convexity and irreducibility). *Let T be a positive operator on L^p with $p \in (1, +\infty)$. Then a minimal convex set with positive measure is irreducible.*

Proof. Assume that A is minimal convex. Let $B \subset A$ be a T_A-invariant set. By Lemma 4.10(iii) (with $\Omega' = A$), we have $B = F(B) \cap A$, and thus B is convex by Lemma 4.8. Therefore we have $B = A$ or $B = \varnothing$ by minimality. This proves that the set A is irreducible. □

Proof of Theorem 4.18. Assume (i), that is, the set A is an atom. By definition it has positive measure. By Lemma 4.11, it is convex. Since A is a minimal admissible set with positive measure, we get (ii).

Assume (ii), that is, the set A is minimal convex with positive measure. It is irreducible thanks to Lemma 4.19. As it is also admissible (as a convex set), we get (iii).

Notice that (iii) implies (i) by Lemma 4.14.

Assume (i) (and thus (i)–(iii) by the previous proofs). So the set A is irreducible. Let us check it is maximal irreducible. Let $A' \supset A$ be another irreducible set. As the set $F(A)$ is T-invariant, we get that $F(A) \cap A'$ is $T_{A'}$-invariant. So by irreducibility of A', we have $F(A) \cap A' = A'$ as $A \subset F(A) \cap A'$ has positive measure. We deduce that $A' \subset F(A)$, and similarly $A' \subset P(A)$. This gives $A' \subset F(A) \cap P(A) = A$ as A is convex. Therefore A is a maximal irreducible set, which proves (iv).

Assume (iv), that is, A is a maximal irreducible set. Thanks to Lemma 4.15, the set $P(A) \cap F(A)$ is an atom and thus irreducible by Lemma 4.12. By maximality of A, we have $A = P(A) \cap F(A)$, and thus A is an atom. This gives (i). □

4.6. *An intuitive order on atoms.* Nelson [22] introduced an order relation on atoms (therein called k-components, and which correspond to maximal irreducible sets, so to atoms by Theorem 4.18) using the past of measurable sets (therein k-closures). We rewrite this order relation, using futures instead of pasts for convenience.

Definition 4.20 (order relation between atoms). Let T be a positive operator on L^p with $p \in (1, +\infty)$. Let A, B be two T-atoms. We denote $A \preccurlyeq B$ if $A \subset F(B)$ a.e. (that is, if $F(A) \subset F(B)$ a.e.).

We write $A \prec B$ when $A \preccurlyeq B$ and A, B are not a.e. equal.

In the epidemiological interpretation of Remark 4.1, we have $A \preccurlyeq B$ if A may be infected by an epidemics starting on B. We first give some equivalent definitions of this relation $\preccurlyeq$. Recall $F^*(A) = F(A) \cap A^c$ and similarly for P^*.

Lemma 4.21 (equivalent definitions of $\preccurlyeq$). *Let T be a positive operator on L^p with $p \in (1, +\infty)$. Let A, B be two atoms such that A and B are not a.e. equal. The following properties are equivalent.*

(i) $A \subset F(B)$ *a.e.*

(ii) $A \subset F^*(B)$ *a.e.*

(iii) $B \subset P(A)$ *a.e.*

(iv) $B \subset P^*(A)$ *a.e.*

Proof. The equivalences between (i) and (ii) and between (iii) and (iv) are direct consequences of the fact that two atoms are always equal a.e. or disjoint a.e. We also have that $A \subset F(B)$ is equivalent to $A \cap F(B) \neq \varnothing$ as A is an atom. By Lemma 3.15, as B is also an atom, the property $A \cap F(B) \neq \varnothing$ is also equivalent to $B \subset P(A)$. □

We can now check that this indeed defines an order relation (recall that an order relation is understood to be a partial order relation.)

Proposition 4.22 ($\preccurlyeq$ is a order relation). *Let T be a positive operator on L^p with $p \in (1, +\infty)$. Then the relation $\preccurlyeq$ is a order relation on the set of atoms.*

Proof. The relation $\preccurlyeq$ is clearly reflexive and transitive by definition of $\preccurlyeq$ and by the monotony of the future, see Lemma 3.14(iii).

Let A, B be two atoms such that $A \preccurlyeq B$ and $B \preccurlyeq A$. By definition $A \subset F(B)$, which implies $F(A) \subset F(B)$. A symmetry argument yields $F(B) \subset F(A)$, so that both are equal. Similarly we have $P(A) = P(B)$. Since A and B are convex, $A = P(A) \cap F(A) = P(B) \cap F(B) = B$, so relation $\preccurlyeq$ is an order relation. □

5. Admissible/irreducible sets and atoms for T and T^n

We compare the admissible/irreducible sets and atoms of T and of T^n, with $n \geq 2$. We denote by $\mathcal{A}(S)$ the set of S-admissible sets, where S is a positive operator. We keep the notation from Sections 3 and 4.

5.1. *Elementary results on admissible sets of powers of T.* We first point out that in the next elementary lemma, one can replace T^n by e^T for example.

Lemma 5.1 (admissible sets of T^n). *Let T be a positive operator on L^p with $p \in (1, +\infty)$ and $n \in \mathbb{N}^*$.*

(i) *Any T-admissible set is T^n-admissible, that is, $\mathcal{A}(T) \subset \mathcal{A}(T^n)$.*

(ii) *Any T-convex set is T^n-convex.*

(iii) *If the operator T^n is irreducible, then T is irreducible.*

Proof. Lemma 3.6 gives (i). If a set A is T-convex, then $A = F(A) \cap P(A)$. We use Lemma 3.6 to deduce that $F(A)$ (or $P(A)$) is T^n-invariant (or T^n-coinvariant) and then Lemma 4.7(v) to get that A is thus T^n-convex. Point (iii) is immediate using Lemma 3.6. □

We illustrate in the next example that the operator T and its powers may have different atoms.

Example 5.2 (different atoms of T and T^2). We consider the finite state space $\Omega = \{1, 2\}$ endowed with the uniform probability μ, and the kernel operator T_k associated to the kernel (or matrix as the space is finite), given in Figure 4 (left). The operator T_k has only one atom $\{1, 2\}$, whereas its square T_k^2 admits two atoms $\{1\}$ and $\{2\}$. The fact that $\{1, 2\}$ may be partitioned in T^2-atoms is in fact generic, see Proposition 5.9 below.

The admissible sets of T and its power might differ even if there is no atom.

Example 5.3 (no atoms and $\mathcal{A}(T) \neq \mathcal{A}(T^2)$). We continue Example 3.11. The operator T_k^2 is a kernel operator with a kernel $k^{\otimes 2}$ on $[0, 1]$, see Figure 1 (right), defined by

$$k^{\otimes 2}(x, y) = (x - y)(\mathbb{1}_{\{y \leq x \leq 1/2\}} + \mathbb{1}_{\{1/2 \leq y \leq x\}}). \tag{15}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

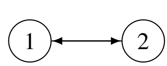

Figure 4. Example of matrix (left) and associated communication graph (right) on $\Omega = \{1, 2\}$ for which the atoms of the matrix and its square are distinct.

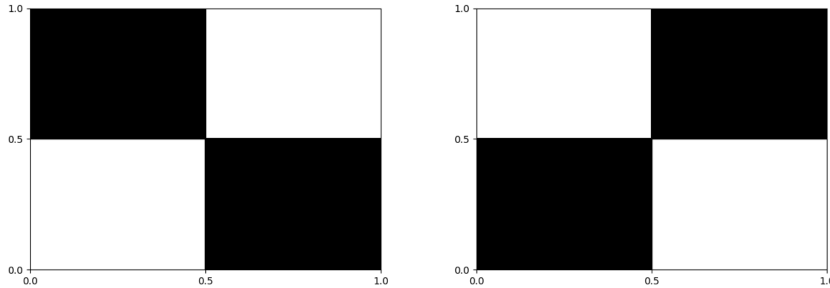

Figure 5. Support of some $\{0, 1\}$-valued kernels: kernel k defined in (16) and kernel $2k^{\otimes 2}$ (right).

The T_k^2-invariants sets are a.e. equal to $[a, 1/2] \cup [b, 1]$ with $a \in [0, 1/2]$ and $b \in [1/2, 1]$, whereas the T_k invariant sets, see Example 3.11, corresponds to those sets with $b = a + 1/2$. Therefore the σ-field of the T_k^2 admissible sets is exactly the Borel σ-field of $[0, 1]$; it does not coincide with the σ-field of the T_k admissible sets given in Example 3.11.

We now check that the irreducible sets of T and those of T^2 are not always the same.

Example 5.4 (T^2-irreducibility does not imply T-irreducibility). We consider the measured space $(\Omega = [0, 1], \mathcal{F}, \text{Leb})$, with $\mathcal{F}$ the Borel subsets of $[0, 1]$ and Leb the Lebesgue measure on $[0, 1]$, and the kernel k on $[0, 1]$ defined by (see Figure 5, left)

$$k(x, y) = \mathbb{1}_{\{x \leq 1/2 \leq y\}} + \mathbb{1}_{\{y \leq 1/2 \leq x\}}. \tag{16}$$

Then T_k^2 is a kernel operator with kernel $k^{\otimes 2}$ given by (see Figure 5, right)

$$k^{\otimes 2}(x, y) = 2^{-1}\mathbb{1}_{\{\max(x,y) \leq 1/2\}} + 2^{-1}\mathbb{1}_{\{\min(x,y) \geq 1/2\}}.$$

Then the set $[0, 1/2]$ is T_k^2-irreducible, T_k^2-admissible (and thus a T_k^2-atom), and T_k^2 invariant, but it is neither T_k-irreducible (as $T_{[0,1/2]} = 0$) nor T_k-admissible (as $[0, 1]$ is a T_k-atom).

Let S be a positive operator on L^p and A a measurable set. Recall that $S(A)$ denote the support of $S(f\mathbb{1}_A)$ for $f \in L^p$ a positive function, which does not depend on the choice of the positive function f, see Definition 3.1. We now state some corresponding preliminary properties in the next two lemmas.

Lemma 5.5 (basic properties of $T(A)$). *Let T, S be positive operators on L^p with $p \in (1, +\infty)$, and A a measurable set. We have the following properties.*

(i) $\text{supp}(T(f)) = T(\text{supp}(f))$ *a.e. for any* $f \in L^p_+$. *In particular, if* $\mathbb{1}_A$ *belongs to* L^p, *then we have* $T(A) = \text{supp}(T(\mathbb{1}_A))$ *a.e.*

(ii) $T(S(A)) = (TS)(A)$ *a.e. and* $(T + S)(A) = T(A) \cup S(A)$ *a.e.*

(iii) *If $A \subset B$ a.e., with B a measurable set, then we have $T(A) \subset T(B)$ a.e.*

(iv) *Let $(A_i)_{i\in I}$ be an at most countable family of measurable sets. We have*

$$T\Big(\bigcup_{i\in I} A_i\Big) = \bigcup_{i\in I} T(A_i) \text{ a.e.} \quad \text{and} \quad T\Big(\bigcap_{i\in I} A_i\Big) \subset \bigcap_{i\in I} T(A_i) \text{ a.e.}$$

Proof. Let $f' \in L^p$ such that $f' > 0$ and $\mathbb{1}_{\text{supp}(f)} f' = f$. Then, by (7), for any measurable set B we have $k_T(B, \text{supp}(f)) = 0$ if and only if $B \cap \text{supp}(T(\mathbb{1}_{\text{supp}(f)} f')) = \varnothing$. This gives (i). Point (ii) is a direct consequence of (i) applied to $f \mathbb{1}_A$ for any positive function $f \in L^p$. Point (iii) is a direct consequence of the positivity of T.

We now prove (iv). Let B be a measurable set. As the map $k_T(B, \cdot)$ is nondecreasing and σ-additive on $\mathcal{F}$, we have $k_T\big(B, \bigcup_{i\in I} A_i\big) = 0$ if and only if for all $i \in I$, we have $k_T(B, A_i) = 0$. Thus the maximal set B that satisfies $k_T\big(B, \bigcup_{i\in I} A_i\big) = 0$ is $\bigcap_{i\in I} T(A_i)^c$, that is, $T\big(\bigcup_{i\in I} A_i\big) = \bigcup_{i\in I} T(A_i)$. The property $T\big(\bigcap_{i\in I} A_i\big) \subset \bigcap_{i\in I} T(A_i)$ is a direct consequence of (iii). We thus have (iv). □

Lemma 5.6 ($T^k(A)$ and invariance/irreducibility). *Let T be a positive operator on L^p with $p \in (1, +\infty)$. Let A be a measurable set, and $n \in \mathbb{N}^*$. We have the following properties.*

(i) *The set A is T-invariant if and only if $T(A) \subset A$ a.e.*

(ii) *If the set A is T^n-invariant, then for all $k \in \mathbb{N}$, the set $T^k(A)$ is T^n-invariant.*

(iii) *If T is a nonzero irreducible operator and $\mu(A) > 0$, then we have $\mu(T(A)) > 0$. Moreover, we have $T(\Omega) = \Omega$ a.e.*

Proof. By definition the set A is T-invariant if and only if $A^c \cap TA = \varnothing$; this gives (i). Let the set A be T^n-invariant and $k \in \mathbb{N}$. Then by Lemma 5.5(ii), we have $T^n(T^k(A)) = T^k(T^n(A))$. Since we have $T^n(A) \subset A$, we deduce that $T^n(T^k(A)) \subset T^k(A)$. This gives (ii).

Assume that T is a nonzero irreducible operator and that $\mu(T(A)) = 0$. The latter condition implies that A is T-invariant, and by irreducibility of T, that $A = \varnothing$ or $A = \Omega$. As T is a nonzero operator, we get the latter case is impossible and thus we have $\mu(A) = 0$. As the set $T(\Omega)$ is T-invariant with positive measure, we deduce that $T(\Omega) = \Omega$ by the previous argument. This gives (iii). □

5.2. *Future sets of* e^T *and atoms of powers of* T. The following corollary provides an interesting link between the future of a set and the exponential of T.

Corollary 5.7 (future and e^T). *Let T be a positive operator on L^p with $p \in (1, +\infty)$ and A a measurable set. We have*

$$\mathrm{e}^T(A) = \bigcup_{n\in\mathbb{N}} T^n(A) = F(A) \quad \text{a.e.}$$

Proof. Using the same arguments as for Lemma 5.5(ii) the first equality is elementary. Now we prove the second equality. The set $\bigcup_{n\in\mathbb{N}} T^n(A)$ is clearly T-invariant by Lemma 5.6(i) and contains A, so $F(A) \subset \bigcup_{n\in\mathbb{N}} T^n(A)$. As $F(A)$ is a T-invariant set, it is a T^n-invariant set for any $n \in \mathbb{N}$ by Lemma 3.6. We get $T^n(F(A)) \subset F(A)$ for any $n \in \mathbb{N}$ by Lemma 5.6(i), and thus $\bigcup_{n\in\mathbb{N}} T^n(A) \subset \bigcup_{n\in\mathbb{N}} T^n(F(A)) \subset F(A)$. This gives the second equality. □

We give the following result on the projection of T^n on a convex set.

Lemma 5.8 (power of a projected operator on a convex set). *Let T be a positive operator on L^p for $p \in (1, +\infty)$ and A a convex set. Then we have $(T_A)^n = (T^n)_A$ for any $n \in \mathbb{N}^*$.*

For $n \in \mathbb{N}^*$, we use the notation T_A^n for $(T_A)^n = (T^n)_A$ when A is a convex set.

Proof. Let $n \in \mathbb{N}^*$. We have

$$(T^n)_A = M_A T^n M_A = M_A T^{n-1} M_A T M_A + M_A T^{n-1} M_{F^*(A)} T M_A = (T^{n-1})_A T_A,$$

where we used that $T(A) \subset F(A) = A \cup F^*(A)$ for the second equality, and that $F^*(A)$ is T-invariant (as A is convex, see Lemma 4.7) and thus T^{n-1}-invariant, so that $M_A T^{n-1} M_{F^*(A)} = 0$ for the last. We conclude by iteration. □

The following result on the decomposition of atoms is also related to [27, Theorem 8] which states that the eigenvalues of T (when T is compact) whose modulus are equal to the spectral radius of T are roots of unity multiplied by the spectral radius. We say that a family of measurable sets $(A_i)_{i\in I}$ forms an *a.e. partition* of a measurable set B if we have: $A_i \cap A_j = \varnothing$ a.e. for any $i \neq j$, and $B = \bigcup_{i\in I} A_i$ a.e.

Proposition 5.9 (atoms of powers of T). *Let T be a positive operator on L^p with $p \in (1, +\infty)$ and $n \in \mathbb{N}^*$. We have the following properties.*

(i) *If A is a T^n-atom, then there exists a T-atom B such that $A \subset B$.*

(ii) *Let B be a T-atom. There exists a T^n-atom $A \subset B$ and a divisor d of n such that the family $(A_k)_{0\leq k\leq \mathrm{d}-1}$, where $A_k = T^k(A) \cap B$, forms an a.e. partition of A in T^n-atoms.*

The second point is slightly more technical; its proof is given separately.

Proof. (i) Let A be a T^n atom. The family $\mathcal{P} = \{B \in \mathcal{A}(T) : A \subset B\}$ of measurable sets is clearly stable by countable intersection. Let A' denote a minimal set for $\mathcal{P}$, given by Lemma 2.2. Let $B \in \mathcal{A}(T)$ such that $B \subset A'$. As $B \in \mathcal{A}(T^n)$ by Lemma 5.1(i), we get that either $A \subset B$ or $A \cap B = \varnothing$. By the minimality of A', we deduce in the former case that $A' = B$ and in the latter case that $A' \cap B = \varnothing$, and thus $B = \varnothing$. This gives that A' is a T-atom which contains A.

(ii) Thanks to Lemma 5.8 (with A replaced by B), it is enough to consider the case where Ω is a T-atom, that is, T is irreducible. The case $T=0$ being trivial, we shall assume in this section only that T is a positive irreducible operator on L^p for $p\in(1,+\infty)$ and $T\neq 0$. In particular, we have $T(\Omega)=\Omega$ a.e. (see Lemma 5.6(iii)) and $F(A)=\Omega$ a.e. for any measurable set A with positive measure. Motivated by Corollary 5.7, we define, for any measurable set A with positive measure, the quantity

$$n_A=\inf\left\{m\in\mathbb{N}^*\cup\{\infty\}:\bigcup_{j=0}^{m-1}T^j(A)=\Omega\text{ a.e.}\right\}.$$

If A is a T^n invariant set with positive measure, the set $\bigcup_{j=0}^{n-1}T^jA$ is T-invariant and contains A; by irreducibility it must be equal to Ω, so $n_A\leq n$. It is also elementary to check that if $A\subset B$ a.e. for a measurable set B, then $n_A\geq n_B\geq 1$. □

Let $\mathcal{I}_n^*$ be the family of T^n-invariant sets with positive measure. This set is nonempty as it contains Ω, and we have $n\geq n_A\geq 1$ for all $A\in\mathcal{I}_n^*$. We have the following technical properties.

Lemma 5.10 (elementary properties). *Let $n\in\mathbb{N}^*$ and $A\in\mathcal{I}_n^*$ (i.e., a nontrivial T^n-invariant set).*

(i) *Let $\ell\in\mathbb{N}$. For $k\in\mathbb{N}^*$ we have*

$$\bigcup_{j=\ell}^{k+\ell-1}T^j(A)=\Omega\ \textit{a.e.}\quad\Longleftrightarrow\quad n_A\leq k.$$

In particular, we have $n_{T^\ell(A)}=n_A$.

(ii) *Set $B=A\bigcap\big(\bigcup_{j=1}^{n_A-1}T^j(A)\big)$ (notice the indices j are positive). We have*

$$\mu(B)>0\quad\Longrightarrow\quad n_B>n_A.$$

Proof. We prove (i). The set $B=\bigcup_{j=0}^{k-1}T^j(A)$ is T^n-invariant as union of T^n-invariant sets, see Lemma 5.6(ii), and thus $T^n(B)\subset B$. If $T^\ell(B)=\Omega$, then we get, as $(\ell+1)n-\ell\geq 0$ and $T(\Omega)=\Omega$,

$$\Omega=T^{(\ell+1)n-\ell}(\Omega)=T^{(\ell+1)n}(B)\subset B,$$

and thus $B=\Omega$ and $n_A\leq k$. On the other hand, if $n_A\leq k$, then we have $B=\Omega$ and $T^\ell(B)=\Omega$.

We prove (ii). The set $B=A\bigcap\big(\bigcup_{j=1}^{n_A-1}T^j(A)\big)$ is T^n-invariant, and thus belongs to $\mathcal{I}_n^*$ as $\mu(B)>0$. Using $B\subset A$ and thus $T^j(B)\subset T^j(A)$ for all the terms $j\geq 0$, we get

$$\bigcup_{j=0}^{n_A-1}T^j(B)\subset\bigcup_{j=1}^{n_A-1}T^j(A).$$

By (i) (with $\ell = 1$), the latter set is not a.e. equal to Ω, which in turns, using (i) again (but with $\ell = 0$), implies that $n_B > n_A$. □

Let $n \geq 2$. The supremum $n_{\max} = \sup\{n_A : A \in \mathcal{I}_n^*\}$ is less or equal than n and is thus a maximum.

We can directly deduce Proposition 5.9(ii) from the next lemma.

Lemma 5.11. *Let A be a T^n-invariant set with positive measure such that $n_A = n_{\max}$. We have, with $A_k = T^k(A)$ for $k \in \mathbb{N}$:*

(i) *n_A is a divisor of n.*

(ii) *$T^{n_A}(A_k) = A_k$ a.e. for all $k \in \mathbb{N}$.*

(iii) *$A_k \cap A_\ell = \varnothing$ a.e. for all $k \neq \ell$ in $\{0, \dots, n_A - 1\}$.*

(iv) *The sets $(A_k)_{k\in\{0,\dots n_A-1\}}$ are T^n-atoms.*

Proof. Let A be T^n-invariant such that $n_A = n_{\max}$. Set $A_k^* = \bigcup_{j\in\{0,\dots,n_A-1\}\setminus\{k\}} A_j$ for $k \in \{0, \dots, n_A - 1\}$ (so that $A_k \cup A_k^* = \Omega$ by definition of n_A) and $B = A \cap A_0^*$. The set B is invariant. We assume that $\mu(B) > 0$. Since $B \subset A$, we get $n_B \geq n_A$ and thus $n_B = n_A$ by maximality of n_A. Then, Lemma 5.10(ii) implies that $\mu(B) = 0$. By contradiction, we deduce that $\mu(B) = 0$, that is,

$$A \cap A_0^* = \varnothing.$$

Using that $T(\Omega) = \Omega$ as T is irreducible, we get

$$A \sqcup A_0^* = \Omega = T(\Omega) = T^{n_A}(A) \cup A_0^*.$$

This implies $A \subset T^{n_A}(A)$. Writing $n = kn_A + r$ with $r \in \{0, \dots, n_A - 1\}$, we get

$$T^r(A) \subset T^{r+n_A}(A) \subset T^{r+kn_A}(A) = T^n(A) \subset A.$$

If $r > 0$, this would imply that $n_A \leq r$. As $r < n_A$, we deduce that $r = 0$, that is, (i), and then that $A = T^{n_A}(A)$. This gives (ii) for $k = 0$ and thus for any k, as the T^n-invariant set A_k is also maximal in the sense that $n_{A_k} = n_A = n_{\max}$ by Lemma 5.10(i).

Using again that A_k is maximal and that $T^{n_A}(A_j) = A_j$, we can apply the previous argument to get that $A_k \cap A_k^* = \varnothing$ for all $k \in \{0, \dots, n_A - 1\}$. This readily implies that the A_k for $k \in \{0, \dots, n_A - 1\}$ are pairwise disjoint, that is, (iii).

To conclude, it is enough to check (iv) for $k = 0$. As A is T^n-invariant, to prove it is a T^n-atom, it is enough to check that if $B \subset A$ is a T^n-invariant set with positive measure, then $B = A$. Consider such a set B. Notice that n_B is finite (as $B \in \mathcal{I}_n^*$) and that $n_B \geq n_A$, that is $n_B = n_A$ by maximality of n_A. We thus have

$$A \sqcup A_0^* = \Omega = B \bigcup \left(\bigcup_{j=1}^{n_A-1} T^j(B) \right) \subset B \cup A_0^*.$$

This readily implies that $A \subset B$ and thus $B = A$. □

6. Atoms and nonnegative eigenfunctions

Until the end of this section, T is a power compact (that is, there exists $k \in \mathbb{N}^*$ such that the operator T^k is compact) positive operator on L^p, where $p \in (1, +\infty)$ and $(\Omega, \mathcal{F}, \mu)$ is a measured space with μ σ-finite and nonzero. The purpose of this section is to study the intricate links between the ordered set of atoms and spectral properties of T. Especially, we study links between atoms and nonnegative eigenfunctions of T. We also provide some criteria of monatomicity of T. The power compactness hypothesis opens access to different results, giving the existence and uniqueness under irreducibility of nonnegative eigenfunctions for a positive operator.

6.1. ***On positive power compact operators.*** Recall that $\rho(T)$ defined in (1) denote the spectral radius of the operator T. The algebraic multiplicity of $\lambda \in \mathbb{C}$ of T is defined by

$$\mathrm{m}(\lambda, T) = \dim\Big(\bigcup_{k\in\mathbb{N}^*} \mathrm{Ker}\,(T - \lambda\,\mathrm{Id})^k\Big). \tag{17}$$

The complex number $\lambda \in \mathbb{C}$ is an eigenvalue of T when $\mathrm{m}(\lambda, T) \geq 1$, it is algebraically simple when $\mathrm{m}(\lambda, T) = 1$. When T is power compact, the multiplicity $\mathrm{m}(\lambda, T)$ is finite for $\lambda \in \mathbb{C}^*$, see [20, Theorem p. 21]. For power compact operators the multiplicity of $\lambda \in \mathbb{C}^*$ is also the dimension of the range of the spectral projection (which is the definition used in [11] and [27]) thanks to [11, Theorems VII.4.5–6].

For a measurable set $A \subset \Omega$, when there is no ambiguity on the operator T, we simply write $\rho(A) = \rho(T_A)$, see Section 2.3, and $\mathrm{m}(\lambda, A) = \mathrm{m}(\lambda, T_A)$ for the spectral radius and multiplicity of λ for $T_A = M_A T M_A$, see (3), the operator T projected to A.

The following lemma proves that the projection of a power compact operator is also power compact.

Lemma 6.1 (projection of a power compact operator). *Let T be a positive power compact operator on L^p. Then there exists $k \in \mathbb{N}^*$ such that for any measurable set Ω', the operator $(T_{\Omega'})^k$ is compact.*

Proof. Let $n \in \mathbb{N}^*$ such that T^n is compact. We have $0 \leq (T_{\Omega'})^n \leq T^n$. Since T^n is compact and L^p is reflexive (and thus the norms on L^p and its dual are order continuous by [3, Exercise 4.1.19]), thanks to [3, Theorem 5.20] we get that $(T_{\Omega'})^n$ is compact. □

We say that the atom $A \subset \Omega$ is *nonzero* if $\rho(A) > 0$, and denote by $\mathfrak{A}^*$ be the (at most countable) set of nonzero atoms:

$$\mathfrak{A}^* = \{A \in \mathfrak{A} : \rho(A) > 0\}. \tag{18}$$

Notice that $\mathrm{m}(\lambda, A) = 0$ for all atoms $A \in \mathfrak{A} \setminus \mathfrak{A}^*$ and $\lambda \in \mathbb{C}^*$.

We recall in our framework the classical results related to power compact operators. In what follows, uniqueness of eigenfunctions is understood up to a multiplicative constant.

Theorem 6.2. *Let T be a positive power compact operator on L^p with $p \in (1, +\infty)$.*

(i) ***Krein–Rutman***: *If $\rho(T)$ is positive then $\rho(T)$ is an eigenvalue of T, and there exists a corresponding nonnegative right eigenfunction denoted v_T.*

(ii) ***de Pagter***: *If T is irreducible then $\rho(T)$ is positive unless $T = 0$ and $\dim(L^p) = 1$, that is, if A is measurable then either $\mu(A) = 0$ or $\mu(A^c) = 0$.*

(iii) ***Perron–Jentzsch***: *If T is irreducible with $\rho(T) > 0$, then $\rho(T)$ is algebraically simple, v_T is positive a.e., and v_T is the unique nonnegative right eigenfunction of T: if $Tv = \lambda v$ for some $\lambda \in \mathbb{C}$ and v nonnegative, then $v = c v_T$ for some $c \in \mathbb{R}_+$.*

(iv) ***Schwartz***: *For $\lambda \in \mathbb{C}^*$ we have*

$$\mathrm{m}(\lambda, T) = \sum_{A \in \mathfrak{A}^*} \mathrm{m}(\lambda, A) \quad \textit{and} \quad \rho(T) = \max_{A \in \mathfrak{A}^*} \rho(A). \tag{19}$$

Remark that, using Theorem 6.2(ii), an atom A has a zero spectral radius if and only if it satisfies $\dim(L^p_A) = 1$ and $T_A = 0$.

Proof. We first recall the vocabulary used in [15]. For any $v \in L^p$, the smallest closed ideal (therefore the smallest subspace of the form L^p_A with $A \in \mathcal{F}$) that contains v is $L^p_{\mathrm{supp}(v)}$. We say that $v \in L^p_+$ is quasiinterior if the closure of $L^p_{\mathrm{supp}(v)}$ is equal to L^p, that is, if $v > 0$ a.e.

Point (i) is given by [15, Theorem 3], and (ii) by [15, Theorem 12(1)]. To prove (iii), by [15, Theorem 12(1)], since T is irreducible, $\rho(T)$ is an algebraically simple eigenvalue and the corresponding eigenfunction is a quasiinterior point of L^p, that is, a positive eigenfunction. By [26, Theorem V.5.2(iv)] (that can be applied as T is power compact, see [26, Corollary, p. 329]), $\rho(T)$ is the only eigenvalue related to a nonnegative eigenfunction. As $\rho(T)$ is algebraically simple, v_T is the unique nonnegative eigenfunction of T.

Point (iv) is an extension of [27, Theorem 7] (stated for μ finite and T compact), and its proof is very similar. We provide a short proof for completeness. Let $h \in L^1$ with $1 \geq h > 0$ a.e.; thus the measure $h.\mu$, defined by $h.\mu(A) = \int_A h(s)\,\mu(\mathrm{d}s)$ for $A \in \mathcal{F}$, is finite. Following the proof of [27, Theorem 7], it is enough to check that Lemmas 4, 11 and 12 therein also hold by replacing μ by $h.\mu$ in their statement and when the operator T is power compact.

For Lemma 11, the proof given by [27] is also valid when the operator V given therein is power compact, as every point of $\mathrm{Sp}(V) \backslash \{0\}$ is isolated and as for any $\lambda \neq 0$, the quantity $\mathrm{m}(\lambda, V)$ is finite, see [11, Section VII.4]. For Lemma 12, the proof given by [27] holds for any positive operator, and also holds when we replace μ in the statement by the finite measure $h.\mu$.

Lemma 4 of [27] states that if μ is finite and T is a positive compact operator, then for all $\lambda > 0$ there exists $\delta > 0$ such that for all measurable set $A \in \mathcal{F}$ such that $\mu(A) < \delta$ we have $\rho(T_A) < \lambda$. An elementary adaptation of the proof of [27, Lemma 4], gives that the result also holds if μ is σ-finite provided we replace the condition $\mu(A) < \delta$ by $h.\mu(A) < \delta$. We now assume that the operator T is power compact, and let $k \in \mathbb{N}^*$ be such that the operator T^k is compact. For $\lambda > 0$, there exists $\delta > 0$ such that for all measurable set $A \in \mathcal{F}$ with $h.\mu(A) < \delta$ we have $\rho((T^k)_A) < \lambda^k$. Since $0 \leq (T_A)^k \leq (T^k)_A$, we deduce that $\rho((T_A)^k) \leq \rho((T^k)_A) < \lambda^k$, that is, $\rho(T_A) < \lambda$ thanks to [20, Theorem p. 21]. This readily gives the extension of [27, Lemma 4] to μ σ-finite and T positive power compact. This concludes the proof of (iv). □

Let us stress that Theorem 6.2 also applies to $T^\star$. Indeed, the operator T is irreducible (or positive, or power compact) if and only if the operator $T^\star$ is irreducible (or positive, or power compact). By [20, Theorem, p. 21], when T is power compact, we have $\rho(T^\star) = \rho(T)$ and $\mathrm{m}(\lambda, T^\star) = \mathrm{m}(\lambda, T)$ for all $\lambda \in \mathbb{C}^*$.

The following result is a direct consequence of Theorem 6.2, as any atom is irreducible by Theorem 4.18. The function v_A below will be called the Perron-like eigenfunction of T_A.

Corollary 6.3 (Perron-like eigenfunctions for T_A). *Let T be a positive power compact operator on L^p with $p \in (1, +\infty)$ and A a nonzero atom. Then $\rho(A)$ is an algebraically simple positive eigenvalue of T_A and there exists a unique nonnegative right eigenfunction of T_A, say v_A; furthermore its support is A, that is,* $\mathrm{supp}(v_A) = A$ *a.e., and we have $\rho(v_A) = \rho(A)$: $T_A v_A = \rho(A) v_A$.*

For $\lambda > 0$, let $\mathfrak{A}(\lambda)$ be the set of atoms with spectral radius λ:

$$\mathfrak{A}(\lambda) = \{A \in \mathfrak{A}^* : \rho(A) = \lambda\}. \tag{20}$$

We have the following elementary result, with the convention $\max \varnothing = 0$.

Lemma 6.4 (spectral radius of projected operators). *Let T be a positive power compact operator on L^p with $p \in (1, +\infty)$.*

(i) *For any $\lambda > 0$, there exists at most a finite number of atoms with spectral radius larger than λ.*

(ii) *If Ω' is admissible, then we have*

$$\rho(\Omega') = \max_{A \in \mathfrak{A}^*,\, A \subset \Omega'} \rho(A). \tag{21}$$

(iii) *If $\rho(T)$ is positive, then we have* $\mathrm{m}(\rho(T), T) = \mathrm{card}(\mathfrak{A}(\rho(T)))$.

Proof. By Corollary 6.3, any atom with a spectral radius $\rho(A) > 0$ satisfies $\mathrm{m}(\rho(A), A) = 1$. If λ is positive, then the set $\{z \in \mathbb{C}, |z| \geq \lambda, \mathrm{m}(z, T) \neq 0\}$ is finite by [11] (notice that $\mathrm{m}(z, T) \in \mathbb{N}$ by [20, Theorem, p. 21]). Therefore, by Theorem 6.2(iv), only a finite number of atoms A may satisfy $\rho(A) \geq \lambda$, that is, (i). Point (ii) then follows from (19), since the atoms of $T_{\Omega'}$ are precisely the atoms of T that are included in Ω', by Proposition 4.16(ii).

Finally, for any atom A, we have $\rho(A) \leq \rho(T)$, therefore the only atoms with $\mathrm{m}(\rho(T), A) > 0$ are exactly those with $\rho(A) = \rho(T)$. By Corollary 6.3, these atoms satisfy $\mathrm{m}(\lambda, A) = 1$, thus we deduce (iii) from (19). □

We directly deduce from (ii) the following result.

Lemma 6.5 (the operator is quasinilpotent outside the nonzero atoms). *Let T be a positive power compact operator on L^p. The projection $T_{\Omega'}$ of T to Ω', the complement set of $\bigcup_{A \in \mathfrak{A}^*} A$, is quasinilpotent, that is, $\rho(\Omega') = 0$.*

6.2. *Nonnegative eigenfunctions.* The goal of this section is to describe exactly the set of nonnegative eigenfunctions and prove Theorem 3. We start by two elementary results.

Lemma 6.6. *Let T be a positive operator on L^p. If C is convex and* $\mathrm{supp}(v) \subset F(C)$, *then we have $T_C v = \mathbb{1}_C T v$.*

Proof. Since C is convex, we have $F(C) = C \sqcup F^*(C)$ where $F^*(C)$ is invariant by Lemma 4.7. Since $\mathrm{supp}(v) \subset F(C)$, we have $v = v\mathbb{1}_C + v\mathbb{1}_{F^*(C)}$. The statement follows by checking that, by Lemma 3.4, $\mathbb{1}_C T(v\mathbb{1}_{F^*(C)}) = 0$. □

Lemma 6.7 (nonnegative eigenfunctions on an atom). *Let T be a positive operator on L^p for $p \in (1, +\infty)$ and A a nonzero atom. If v is a nonnegative right eigenfunction with $A \subset$* $\mathrm{supp}(v) \subset F(A)$, *then v coincides on A with the Perron like right eigenfunction*: $\mathbb{1}_A v = \mathrm{c} v_A$ *for some* $\mathrm{c} > 0$, *and* $\rho(v) = \rho(A)$, *that is,* $Tv = \rho(A) v$.

Proof. Let $\lambda \geq 0$ with $Tv = \lambda v$. Since $\mathrm{supp}(v) \subset F(A)$, we may apply Lemma 6.6 to the atom A (convex by Theorem 4.18), to get $T_A(\mathbb{1}_A v) = T_A v = \mathbb{1}_A T v = \lambda \mathbb{1}_A v$, that is, $\mathbb{1}_A v$ is a nonnegative eigenfunction of T_A. Since $A \subset \mathrm{supp}(v)$, we get $\mathbb{1}_A v$ is nonzero. By Corollary 6.3, we have $\lambda = \rho(A)$ and $\mathbb{1}_A v = \mathrm{c} v_A$ for some $\mathrm{c} > 0$, as claimed. □

We give a short proof of the following standard result on subsolutions to the eigenvalue equation, that is, functions f satisfying

$$Tf \leq \lambda f. \tag{22}$$

Proposition 6.8 (nonnegative subsolutions are Perron eigenfunctions). *Let T be a positive power compact irreducible operator on L^p with $p \in (1, +\infty)$. If $f \in L^p_+$ satisfies (22) for some $\lambda \in (0, \rho(T)]$, then we have $Tf = \rho(T) f$.*

Proof. Let $f \in L^p_+$ be a solution of (22). Without loss of generality we may assume $\lambda = \rho(T)$. By the Perron–Jentzsch theorem (see Theorem 6.2(iii)), there exists a nonnegative left eigenfunction $h \in L^q_+$ with left eigenvalue $\rho(T)$ such that $h > 0$ a.e. Taking the bracket of (22) with the nonnegative function h, and using the fact that it is a left eigenfunction of T, we get

$$\rho(T)\langle h, f\rangle = \langle h, Tf\rangle \leq \langle h, \rho(T)f\rangle = \rho(T)\langle h, f\rangle,$$

where the inequality holds by positivity of T and nonnegativity of f and h. Therefore we have $\langle h, Tf\rangle = \langle h, \rho(T)f\rangle$, so $\langle h, \rho(T)f - Tf\rangle = 0$. Since $\rho(T)f - Tf$ is nonnegative and $h > 0$ a.e., this implies $Tf = \rho(T)f$. □

As a first consequence, we give details on which atoms may appear in the support of a nonnegative eigenfunction. Recall that, for a nonzero atom A, the Perron-like eigenfunction v_A is the right eigenfunction of T_A given by Corollary 6.3. For $v \in L^p_+$ a nonnegative eigenfunction of T, we consider the following subset of the atoms $\mathfrak{A}(\rho(v))$:

$$\mathfrak{A}_m(v) := \{A \in \mathfrak{A} : A \subset \operatorname{supp}(v) \text{ and } \rho(v) = \rho(A)\}.$$

Corollary 6.9 (a dichotomy for atoms and nonnegative eigenfunctions). *Let T be a positive power compact operator on L^p with $p \in (1, +\infty)$. Let $v \in L^p_+$ be a nonnegative eigenfunction of T with $\lambda = \rho(v) > 0$.*

(i) *For any atom A with $A \subset \operatorname{supp}(v)$ a.e., exactly one of the following holds*:
- $\rho(A) < \lambda$.
- $\rho(A) = \lambda$, *that is*, $A \in \mathfrak{A}_m(v)$, $\mathbb{1}_A v = \mathrm{c} v_A$ *for some* $\mathrm{c} > 0$ *and* $\operatorname{supp}(v) \cap P^*(A) = \emptyset$ *a.e.*

(ii) *The set of atoms $\mathfrak{A}_m(v)$ is a nonempty finite antichain, and*

$$\rho(v) = \rho(\operatorname{supp}(v)).$$

(iii) *If $A \in \mathfrak{A}_m(v)$, $B \in \mathfrak{A}$ and $B \prec A$, then we have $\rho(B) < \rho(A)$.*

Proof. We start by proving (i). Let v, λ satisfy the hypotheses, and consider an atom A such that $A \subset \operatorname{supp}(v)$. If $\rho(A) < \lambda$ we are in the first case and there is nothing to prove. We now assume $\lambda \leq \rho(A)$. Since T is a positive operator and v is nonnegative, we have

$$T_A(v\mathbb{1}_A) = \mathbb{1}_A T(v\mathbb{1}_A) \leq \mathbb{1}_A T(v\mathbb{1}_A) + \mathbb{1}_A T(v\mathbb{1}_{A^c}) = \mathbb{1}_A Tv = \lambda \mathbb{1}_A v. \tag{23}$$

Since $\lambda \leq \rho(A)$ and A is irreducible, Proposition 6.8 applied to $T|A$ implies $T_A v_A = \rho(A) v_A$. Since we have $A \subset \operatorname{supp}(v)$, v_A is not the zero function, thus, by Corollary 6.3, we have $\lambda = \rho(A)$ and $\mathbb{1}_A v = \mathrm{c} v_A$ for some $\mathrm{c} > 0$. Going back to (23), we see that the inequality there is in fact an equality, so $\mathbb{1}_A T(v\mathbb{1}_{A^c}) = 0$.

By (7), we thus have $k_T(A, \operatorname{supp}(v) \cap A^c) = 0$. By Lemma 3.7, the set $\operatorname{supp}(v)$ is invariant, thus by additivity of the kernel we also have

$$k_T(A \cup \operatorname{supp}(v)^c, \operatorname{supp}(v) \cap A^c) = 0,$$

so that $\operatorname{supp}(v) \cap A^c$ is invariant. We then write

$$F(\operatorname{supp}(v) \cap A^c) \cap A = (\operatorname{supp}(v) \cap A^c) \cap A = \varnothing,$$

which by Lemma 3.15 implies that

$$\operatorname{supp}(v) \cap P^*(A) = \operatorname{supp}(v) \cap A^c \cap P(A) = \varnothing. \tag{24}$$

This completes the proof of (i)

We now turn to the proof of (ii). If two atoms A and B are in $\mathfrak{A}_m(v)$, equation (24) shows that B cannot be a subset of $P^*(A)$; symmetrically A cannot be included in $P^*(B)$. By the alternate formulation of $\preccurlyeq$ from Lemma 4.21, A and B are not comparable, so $\mathfrak{A}_m(v)$ is an antichain. It is finite by Lemma 6.4(i). Moreover, as $T(v) = \lambda v$, we get that $T_{\operatorname{supp}(v)}(v) = \lambda v$, and thus $\rho(\operatorname{supp}(v)) \geq \lambda$. As the set $\operatorname{supp}(v)$ is invariant by Lemma 3.7 (and thus admissible), by (21), there exists an atom $A \subset \operatorname{supp}(v)$ with $\rho(A) \geq \lambda$, and thus $\rho(A) = \lambda$ by (i). This implies that the finite antichain $\mathfrak{A}_m(v)$ is not empty.

Finally, if $A \in \mathfrak{A}_m(v)$ and $B \prec A$, then we get $B \subset F(A) \subset \operatorname{supp}(v)$ since $\operatorname{supp}(v)$ is invariant. Applying the dichotomy from (i), and noting that B cannot be in $\mathfrak{A}_m(v)$ since it is an antichain, we deduce that $\rho(B) < \rho(A) = \lambda$. □

The last statement of Corollary 6.9 motivate the following definition, we refer to Figure 6 for a pictorial representation.

Definition 6.10 (distinguished atoms and eigenvalues). Let T be a positive power compact operator on L^p with $p \in (1, +\infty)$. A nonzero atom A of T is called right *distinguished* if $\rho(B) < \rho(A)$ for any atom B such that $B \prec A$.

The set of right distinguished atoms of radius $\lambda > 0$ is denoted by $\mathfrak{A}_{\text{dist}}(\lambda)$.

An eigenvalue λ is called right distinguished if $\mathfrak{A}_{\text{dist}}(\lambda) \neq \varnothing$.

One has a similar definition for left distinguished atoms/eigenvalues. When there is no ambiguity, we shall simply write distinguished for right distinguished.

By Corollary 6.9(ii), if v is a nonnegative eigenvalue, all atoms in $\mathfrak{A}_m(v)$ are distinguished:

$$\mathfrak{A}_m(v) \subset \mathfrak{A}_{\text{dist}}. \tag{25}$$

In the other direction, we now show that for any distinguished atom, we may associate a nonnegative eigenfunction. For a nonzero atom A, recall that v_A denotes the Perron-like eigenfunction of T_A given by Corollary 6.3.

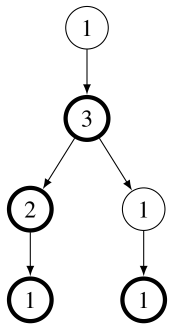

Figure 6. Diagram of the ordered set of distinguished atoms. Following the classical convention (see [6, p. 4]), each circle represents an atom A, and is labeled with its radius $\rho(A)$. An arrow from atom A to atom B signifies that $B \prec A$ and there is no atom in between. The distinguished atoms are those circled in a thick line. Note that a family of similar "finite" pictures may always be drawn in the general case, by considering only atoms with radius larger than a positive constant λ.

Proposition 6.11 (nonnegative eigenfunctions associated to distinguished atoms). *Let T be a positive power compact operator on L^p with $p \in (1, +\infty)$, and A a nonzero atom. The following statements are equivalent*:

(i) *A is a distinguished atom.*

(ii) $\rho(F^*(A)) < \rho(A)$.

(iii) *There exists a nonnegative eigenfunction $w_A \in L^p_+$ such that* $\operatorname{supp}(w_A) = F(A)$ *and* $\mathbb{1}_A w_A = v_A$.

If they hold, then we have $\rho(w_A) = \rho(A)$.

The condition $\mathbb{1}_A w_A = v_A$ in (iii) corresponds to a particular choice of normalizing constant, see Lemma 6.7.

Proof. Suppose that (iii) holds, and let w_A be a nonnegative eigenfunction with $\operatorname{supp}(w_A) = F(A)$. By Lemma 6.7, we have $\rho(w_A) = \rho(A)$, so $A \in \mathfrak{A}_m(w_A)$, and by (25), it is distinguished. Therefore (iii) implies (i).

Suppose that (i) holds. By (21), either $\rho(F^*(A)) = 0$, or there exists an atom $B \subset F^*(A)$ such that $\rho(F^*(A)) = \rho(B)$. By Lemma 4.21, this B satisfies $B \prec A$. Since A is distinguished, $\rho(B) < \rho(A)$, so (ii) holds.

We now prove that (ii) implies (iii). We set $B = F^*(A)$. By assumption, the invariant set B satisfies $\rho(B) < \rho(A)$. By Lemma 3.8, the operator $(\rho(A)\mathrm{Id} - T_B)$ is invertible and its inverse is a positive operator. Let $w_A = v_A + f_B$, where $f_B = (\rho(A)\mathrm{Id} - T_B)^{-1}(\mathbb{1}_B T v_A)$. Note that, by the expression of $(\rho(A)\mathrm{Id} - T_B)^{-1}$ as a Neumann series, we have $\operatorname{supp}(f_B) \subset B$, and thus $\mathbb{1}_A w_A = v_A$. Then we have

$$T w_A = T v_A + T f_B = \mathbb{1}_A T v_A + \mathbb{1}_{A^c} T v_A + T f_B. \tag{26}$$

As $\operatorname{supp}(f_B)$ is a subset of the invariant set B, by Lemma 3.4 we get $Tf_B = T_B f_B$. Moreover, as $\operatorname{supp}(v_A) \subset A$, we have $\mathbb{1}_A T v_A = T_A v_A = \rho(A)\, v_A$ by definition of v_A. Finally, as the set $F(A)$ is invariant and as we have $\operatorname{supp}(v_A) \subset A \subset F(A)$, we have $\mathbb{1}_{F(A)^c} T v_A = 0$, thus $\mathbb{1}_{A^c} T v_A = \mathbb{1}_B T v_A$. Plugging this in (26) yields

$$\begin{aligned} T w_A &= \rho(A)\, v_A + \mathbb{1}_B T v_A + \rho(A) f_B - \rho(A) f_B + T_B f_B \\ &= \rho(A)\, w_A + \mathbb{1}_B T v_A - (\rho(A)\mathrm{Id} - T_B)(f_B) \\ &= \rho(A)\, w_A \end{aligned}$$

by definition of f_B. So w_A is a nonnegative eigenfunction (with $\rho(w_A) = \rho(A)$). In particular, $\operatorname{supp}(w_A)$ is an invariant set that contains A, so $F(A) \subset \operatorname{supp}(w_A)$. Since $\operatorname{supp}(v_A)$ and $\operatorname{supp}(f_B) \subset B$ are both subsets of $F(A)$, we get $F(A) = \operatorname{supp}(w_A)$. This proves (iii). □

The previous result shows that, to any distinguished λ, we may associate a family $(w_A)_{A \in \mathfrak{A}_{\text{dist}}(\lambda)}$ composed of nonnegative eigenfunctions. We now completely describe the set of nonnegative eigenfunctions associated to λ, say $V_+(\lambda)$, as the conical hull of this family (that is, linear combinations with nonnegative coefficients).

Theorem 6.12 (characterization of nonnegative right eigenfunctions). *Let T be a positive power compact operator on L^p with $p \in (1, +\infty)$. Let $\lambda > 0$. We have the following properties.*

(i) *There exists a nonnegative eigenfunction of T associated to λ if and only if λ is a distinguished eigenvalue.*

(ii) *The set $\mathfrak{A}_{\text{dist}}(\lambda)$ is a (possibly empty) finite antichain of atoms, and the family $(w_A)_{A \in \mathfrak{A}_{\text{dist}}(\lambda)}$ is linearly independent.*

(iii) *If v is a nonnegative eigenfunction with $\rho(v) = \lambda$, then $\lambda = \rho(\operatorname{supp}(v))$ and*

$$v = \sum_{A \in \mathfrak{A}_m(v)} \mathrm{c}_A w_A \quad \text{with } \mathrm{c}_A > 0.$$

So the cone $V_+(\lambda)$ is the conical hull of $\{w_A : A \in \mathfrak{A}_{\text{dist}}(\lambda)\}$.

Remark 6.13. The last point implies that the cone $V_+(\lambda)$ is spanned by a finitely many vectors (because the set $\mathfrak{A}_{\text{dist}}(\lambda)$ is finite according to Lemma 6.4(i)), that is, it is polyhedral.

It also shows that if w is a nonnegative eigenfunction such that $\operatorname{supp}(w) = F(A)$, where A is a nonzero atom (see Lemmas 3.7 and 6.7), then A is distinguished, $\rho(w) = \rho(\operatorname{supp}(w)) = \rho(A)$ and $w = \mathrm{c} w_A$ with $\mathrm{c} > 0$.

The elementary adaptation of Theorem 6.12 to nonnegative left eigenfunction is left to the reader.

Proof. If λ is distinguished, then by definition there is an atom $A \in \mathfrak{A}_{\text{dist}}(\lambda)$, and w_A provides a nonnegative eigenfunction associated to λ. Conversely, if there is a nonnegative eigenfunction w associated to λ, then $\mathfrak{A}_m(w)$ is nonempty and consists of distinguished atoms by Corollary 6.9, so λ is distinguished. This proves (i).

Let us prove (ii). If A and B belongs to $\mathfrak{A}_{\text{dist}}(\lambda)$, then $\rho(A) = \rho(B)$, so they are not comparable by definition of distinguished atoms. Therefore $\mathfrak{A}_{\text{dist}}(\lambda)$ is an antichain. It is also finite by Lemma 6.4(i). To prove the linear independence property, assume that $\sum_{B \in \mathfrak{A}_{\text{dist}}(\lambda)} \mathrm{c}_B w_B = 0$. Multiplying by $\mathbb{1}_A$ for $A \in \mathfrak{A}_{\text{dist}}(\lambda)$ yields $\mathrm{c}_A v_A = 0$, since for $B \neq A$, $\operatorname{supp}(w_B) = F(B)$ is disjoint from A. Since v_A is positive, $\mathrm{c}_A = 0$. Since this is true for all A, the family $(w_A)_{A \in \mathfrak{A}_{\text{dist}}(\lambda)}$ is linearly independent.

We now prove (iii). Since the w_A are all in the cone $V_+(\lambda)$, their conical hull is included in $V_+(\lambda)$, so that we only need to prove the reverse inclusion. Let $v \in V_+(\lambda)$. By Corollary 6.9, there is an antichain $\mathfrak{A}_m(v) \subset \mathfrak{A}_{\text{dist}}(\lambda)$ of distinguished atoms of radius λ in the support of w, and all other atoms in this support satisfy $\rho(B) < \lambda$. Define

$$B = \operatorname{supp}(v) \cap \Bigg(\bigcup_{A \in \mathfrak{A}_m(v)} P(A) \Bigg)^c = \operatorname{supp}(v) \cap \Bigg(\bigcup_{A \in \mathfrak{A}_m(v)} A \Bigg)^c,$$

where the second equality follows from the fact that $\operatorname{supp}(v) \cap P^*(A) = \varnothing$ for all $A \in \mathfrak{A}_m(v)$ by Corollary 6.9. The first equality shows that B is invariant.

Still following Corollary 6.9, there exist $\mathrm{c}_A > 0$ such that $v\mathbb{1}_A = \mathrm{c}_A v_A$ for $A \in \mathfrak{A}_m(v)$. Consider the function $w = v - \sum_{A \in \mathfrak{A}_m(v)} \mathrm{c}_A w_A$. Since we have $\operatorname{supp}(w_A) = F(A) \subset \operatorname{supp}(v)$, $\operatorname{supp}(w)$ is included in $\operatorname{supp}(v)$. Since w vanishes by construction on all atoms $A \in \mathfrak{A}_m(v)$, we have in fact $\operatorname{supp}(w) \subset B$. Now, $Tw = \lambda w$ since v and the w_A are eigenfunctions. Since B is invariant and $\operatorname{supp}(w) \subset B$, we get that $T_B w = \lambda w$. However, by construction, B cannot contain atoms of radius greater than or equal to λ, so $\rho(B) < \lambda$. Therefore λ cannot be an eigenvalue of T_B, and w must be identically zero, so that $v = \sum_{A \in \mathfrak{A}_m(v)} \mathrm{c}_A w_A$. Since $\mathfrak{A}_m(v) \subset \mathfrak{A}_{\text{dist}}(\lambda)$, we get that v is in the conical hull of the $(w_A)_{A \in \mathfrak{A}_{\text{dist}}}$. This finishes the proof. □

6.3. ***Monatomic operators: definition and characterization.*** Here we consider positive power compact operators having only one nonzero atom, which are called *monatomic* operators (T is monatomic if $\operatorname{card} \mathfrak{A}^* = 1$ with $\mathfrak{A}^*$ defined in (18)).

Example 6.14 (a monatomic operator with two atoms). Consider $\mathbb{R}^2 = L^p(\{1, 2\}, \mu)$, with μ the counting measure and the operator

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

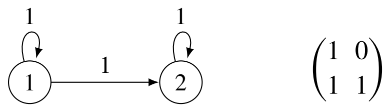

Figure 7. Example of associated graph (left) and associated matrix (right) of a kernel operator on $\Omega = \{1, 2\}$.

The set $\{1\}$ is invariant and the set $\{2\}$ is coinvariant, so both sets are atoms. The atom $\{1\}$ has spectral radius 1 and the atom $\{2\}$ has spectral radius 0, so T has only one nonzero atom and is thus monatomic.

We give in the next theorem a characterization of the monatomic positive power compact operators, see Theorem 2. Recall we say that there exists a unique right (or left) nonnegative eigenfunctions of T related to a nonzero eigenvalue if there exists $u \in L^p$ a right (or $u \in L^q$ a left) nonnegative eigenfunction with $\rho(u) \neq 0$ such that if u' is a right (or left) nonnegative eigenfunction with $\rho(u') \neq 0$, then $u' = cu$ for some $c \in \mathbb{R}$.

Theorem 6.15 (characterization of monatomic operators). *Let T be a positive power compact operator on L^p with $p \in (1, +\infty)$ such that $\rho(T) > 0$. The following properties are equivalent.*

(i) *The operator T is monatomic.*

(ii) *There exist a unique right and a unique left nonnegative eigenfunctions of T with nonzero eigenvalues, and $\rho(T)$ is an algebraically simple eigenvalue of T.*

(iii) *There exist a unique right and a unique left nonnegative eigenfunctions of T with nonzero eigenvalues, say u and v, and* $\operatorname{supp}(u) \cap \operatorname{supp}(v)$ *has positive measure.*

Furthermore, when the operator T is monatomic, we have $\rho(u) = \rho(v) = \rho(T)$ and $\operatorname{supp}(u) \cap \operatorname{supp}(v)$ *is the nonzero atom of T.*

Example 6.16 ($\rho(T)$ algebraically simple and $\operatorname{supp}(u) \cap \operatorname{supp}(v)$ with positive measure). If T has a unique right and a unique left eigenfunction, then T might not be monatomic. Indeed, consider the example given by Figure 7 with $\Omega = \{1, 2\}$ endowed with the counting measure. The positive kernel operator T associated to the matrix given in Figure 7 (right) has only one right eigenfunction $u = (0, 1)$ and one left eigenfunction $v = (1, 0)$, but it is not monatomic, as its nonzero atoms are $\{1\}$ and $\{2\}$. Here, we have $\operatorname{supp}(u) \cap \operatorname{supp}(v) = \varnothing$ and $\rho(T) = 1$ is not an algebraically simple eigenvalue.

To prove Theorem 6.15, we use the following lemma.

Lemma 6.17 (existence of minimal distinguished atoms). *Let T be a positive power compact operator on L^p. Let A be a nonzero atom. Then there exists a right* (*or left*) *distinguished atom smaller* (*or larger*) *than A for $\preccurlyeq$, say B, such that $\rho(B) \geq \rho(A)$.*

Proof. Recall that T and $T^\star$ have the same spectral radius and that they share the same atoms, so we only need to prove the lemma for right distinguished atoms for T, as it will then hold for left distinguished atoms for T as they are right distinguished atoms for $T^\star$.

Since A is a nonzero atom, $\rho(A)$ is positive. The set

$$\mathcal{A} = \{C \in \mathfrak{A}^* : \rho(C) \geq \rho(A),\ C \preccurlyeq A\}$$

is finite thanks to Lemma 6.4(i) and is nonempty as it contains A. Thus it has at least one minimal element for the order $\preccurlyeq$, say B. If an atom C satisfies $C \prec B$, then $C \preccurlyeq A$ by transitivity, but C cannot be in $\mathcal{A}$ by minimality of B, so $\rho(C) < \rho(A)$. Since $B \in \mathcal{A}$, we have $\rho(B) \geq \rho(A)$, and so $\rho(C) < \rho(B)$. Since this holds for any C such that $C \prec B$, we obtain the atom B is distinguished. □

Proof of Theorem 6.15. We assume that T is monatomic and prove (ii). Let A be the only nonzero atom. By Lemma 6.4(iii), as $\mathrm{m}(\rho(T), T) \geq 1$ and $\mathfrak{A}^*$ is reduced to $\{A\}$, we get that $\rho(T)$ is algebraically simple and $\rho(A) = \rho(T)$ by (21).

We now prove the existence and uniqueness of a nonnegative right eigenfunction. Since there is no other nonzero atom, using directly Definition 6.10 we see that A is distinguished, and is the only distinguished atom. Still by definition, $\rho(A)$ is the only distinguished eigenvalue. By Theorem 6.12, the set of nonnegative eigenfunctions is the cone $\mathbb{R}_+ w_A$, which proves uniqueness (up to a positive multiplicative constant). Applying the same proof to $T^\star$ gives (ii) and the first part of the last sentence of the theorem.

We assume (ii) and prove (iii). Since $\rho(T) > 0$ is algebraically simple, we deduce from (19) that there exists a unique atom, say A, such that $\rho(A) = \rho(T)$. In particular, all other atoms must satisfy $\rho(B) < \rho(A)$, so that A is right (and left) distinguished. Therefore, by Proposition 6.11, the unique right (or left) nonnegative eigenfunction, whose existence is given by our Assumption, is in fact w_A (or the nonnegative eigenfunction $w_A^\star$ obtained from $T^\star$). Since $\mathrm{supp}(w_A) \cap \mathrm{supp}(w_A^\star) = F(A) \cap P(A) = A$ by convexity of the atom A, we obtain (iii) and the last part of the last sentence of the theorem.

We assume (iii) and prove that the operator T is monatomic. Since $\rho(T) > 0$, there exists an atom, say A, such that $\rho(A) = \rho(T)$. Looking for a contradiction, we assume there exists an other nonzero atom B and without loss of generality that it is not smaller than B for $\preccurlyeq$ (that is, either $A \preccurlyeq B$ or A and B are not comparable), equivalently $F(A) \cap B = \varnothing$. By Lemma 3.15, this is also equivalent to $F(A) \cap P(B) = \varnothing$.

Then, using Lemma 6.17, there exists a right (or left) distinguished atom A' (or B') such that $A' \preccurlyeq A$ (or $B \preccurlyeq B'$). By Proposition 6.11, the unique nonnegative right eigenfunction v must satisfy $\mathrm{supp}(v) = F(A')$, and similarly the unique nonnegative left eigenfunction u must satisfy $\mathrm{supp}(u) = P(B')$. By construction, we have $F(A') \subset F(A)$ and $P(B') \subset P(B)$, and thus $\mathrm{supp}(v) \cap \mathrm{supp}(u) = F(A') \cap P(B') \subset F(A) \cap P(B) = \emptyset$. As this is in contradiction with the assumption of (iii), we deduce that A is the only nonzero atom, that is T is monatomic. □

7. Generalized eigenspace at the spectral radius

7.1. ***Framework and main theorem.*** The purpose of this section is to restate [19, Theorem V.1(2)] on the ascent of T in our framework of L^p-spaces, with a shorter proof based on convex sets.

Let us first recall a few classical definitions, see [11] and [20]. For T an bounded operator on a Banach space and $\lambda \in \mathbb{C}$, we call *generalized eigenspace* of T at λ, and denote by $K(\lambda, T)$, the linear subspace

$$K(\lambda, T) = \bigcup_{k\in\mathbb{N}} \mathrm{Ker}\,(T - \lambda\,\mathrm{Id})^k.$$

We now focus on the spectral radius $\lambda = \rho(T)$, and write $K(T) = K(\rho(T), T)$ the corresponding generalized eigenspace. We define the *index* of a generalized eigenvector $u \in K(T)$, as $\inf\{k \in \mathbb{N} : u \in \mathrm{Ker}\,(T - \rho(T)\mathrm{Id})^k\}$, and, with the convention $\inf \emptyset = +\infty$, the *ascent* (or Riesz index) of T at $\rho(T)$ as

$$\alpha_T = \inf\{k \in \mathbb{N} : \mathrm{Ker}\,(T - \rho(T)\mathrm{Id})^k = \mathrm{Ker}\,(T - \rho(T)\mathrm{Id})^{k+1}\}.$$

Notice that α_T is positive if $\rho(T)$ is an eigenvalue and $K(T) = \mathrm{Ker}\,(T - \rho(T)\mathrm{Id})^{\alpha_T}$ if α_T is finite. When the operator T is power compact, then the ascent α_T is finite, see [20, Lemma 1.a.2, Theorem, p. 21] (it is also equal to the descent $\delta_T = \inf\{k \in \mathbb{N} : \mathrm{Im}(T - \rho(T)\mathrm{Id})^k = \mathrm{Im}(T - \rho(T)\mathrm{Id})^{k+1}\}$).

Let T be a positive power compact operator on L^p with $p \in (1, +\infty)$, and assume $\rho(T) > 0$, and thus $\alpha_T \in \mathbb{N}^*$. By Lemma 6.4(iii), $K(T)$ is finite dimensional, and

$$\dim(K(T)) = m(\rho(T), T) = \mathrm{card}(\mathfrak{A}_{\mathrm{crit}}),$$

where $\mathfrak{A}_{\mathrm{crit}}$ is the set of *critical atoms*:

$$\mathfrak{A}_{\mathrm{crit}} = \{A \in \mathfrak{A} : \rho(A) = \rho(T)\}. \tag{27}$$

By definition of α_T, the sequence $\big(\dim(\mathrm{Ker}\,((T - \rho(T)\mathrm{Id})^k))\big)_{1\leq k\leq \alpha_T}$ is (strictly) increasing, so we have the trivial bounds

$$\dim\big(\mathrm{Ker}\,(T - \rho(T)\mathrm{Id})^k\big) \geq k \quad \text{for all } 1 \leq k \leq \alpha_T, \tag{28}$$

and in particular $\dim(K(T)) = \mathrm{card}(\mathfrak{A}_{\mathrm{crit}}) \geq \alpha_T$.

The set $\mathfrak{A}_{\text{crit}}$ may be equipped with the order $\preccurlyeq$. Recall that we write $B \prec A$ if $B \preccurlyeq A$ and $B \neq A$. We recall a few classical definitions for posets, that is, partially ordered sets (see, e.g., [6, Section I.3, p. 4]).

Definition 7.1 (covering). Let T be a positive power compact operator on L^p with $p \in (1, +\infty)$ with a spectral radius $\rho(T) > 0$. Let A and B be critical atoms. If $B \prec A$, and if there is no critical atom C such that $B \prec C \prec A$, then A is said to *cover* B.

For $n \geq 1$, a *chain of length* n is a sequence $(A_0, \dots, A_n)$ of elements of $\mathfrak{A}_{\text{crit}}$ such that $A_{i+1} \prec A_i$ for all $0 \leq i < n$. The *height* $h(A)$ of a critical atom A, is one plus the maximum length of a chain starting at A.

Remark 7.2 (terminology "off by one"). Our definition of length is consistent with the one in [6, Section I.3]. The "off by one" is due to the fact that height, in [6], is formally defined for posets with a least element. Our height coincides with Birkhoff's height on the poset $(\mathfrak{A}_{\text{crit}} \sqcup \{\mathbf{0}\}, \preccurlyeq)$ where $\mathbf{0}$ is an additional element that satisfies $\mathbf{0} \preccurlyeq A$ for all $A \in \mathfrak{A}_{\text{crit}}$.

We now restate [19, Theorem V.1(1), (2)] in our framework; its proof is given in Section 7.2. Recall v_A the Perron-like eigenfunction of T_A and the set of critical atoms $\mathfrak{A}_{\text{crit}}$ from (27). We also refer to [16, Theorems 3.2, 3.5] and the iteration procedure given in Section 3.4 therein for a similar result where the operator T is a positive quasicompact on the Banach lattice space of continuous functions vanishing at infinity. Notice that the sets $(B_i, 1 \leq i \leq N_0)$ in Theorem 3.5 therein denote the maximal critical atoms in our framework. Our approach emphasizes the graph structure of the critical atoms.

Theorem 7.3 (a basis of $K(T)$). *Let T be a positive power compact operator on L^p with $p \in (1, +\infty)$ with a spectral radius $\rho(T) > 0$. Then there exists a basis $\mathcal{W} = (w_A)_{A \in \mathfrak{A}_{\text{crit}}}$ of $K(T)$ satisfying the following properties*:

(i) *For all A, we have $A \subset \operatorname{supp}(w_A) \subset F(A)$ and $\mathbb{1}_A w_A = v_A$; moreover, if A is distinguished then w_A is the nonnegative eigenfunction introduced in Proposition 6.11.*

(ii) *If $M = (M_{A,B})$ is the matrix representing, on the basis $\mathcal{W}$, the endomorphism induced on $K(T)$ by T, then for $A, B \in \mathfrak{A}_{\text{crit}}$, we have*

$$M_{AB} = \begin{cases} 0 & \text{if } B \not\preccurlyeq A, \\ \rho(T) & \text{if } A = B, \\ > 0 & \text{if } A \text{ covers } B. \end{cases}$$

(iii) *For any $A \in \mathfrak{A}_{\text{crit}}$, the index of w_A is the height $h(A)$.*

Moreover, (ii) and (iii) hold for any basis of $K(T)$ satisfying (i).

Since the ascent is the maximum index of functions in $K(T)$, we easily get the following result.

Corollary 7.4 (ascent and maximal height). *Let T be a positive power compact operator on L^p with $p \in (1, +\infty)$ with a spectral radius $\rho(T) > 0$. The ascent of T at its spectral radius $\rho(T)$ is equal to the maximal height of the critical atoms*

$$\alpha_T = \max_{A \in \mathfrak{A}_{\mathrm{crit}}} h(A).$$

7.2. ***Existence of an adapted basis and proof of Theorem 7.3.*** We first state a key technical result.

Lemma 7.5 (generalized eigenspaces for projected operators). *Let T be a positive power compact operator on L^p with $p \in (1, +\infty)$ with a spectral radius $\rho(T) > 0$. Let A be a convex set and $\lambda \in \mathbb{C}$.*

(i) *If $v \in K(\lambda, T)$ and* $\mathrm{supp}(v) \subset F(A)$, *then we have $(\mathbb{1}_A v) \in K(\lambda, T_A)$.*

(ii) *If furthermore A is invariant, and $\lambda \neq 0$, then we have $K(\lambda, T_A) \subset K(\lambda, T)$.*

Proof. If $\mathrm{supp}(v) \subset F(A)$, then by Lemma 6.6, we have $\mathbb{1}_A T v = T_A(\mathbb{1}_A v)$. An easy induction using the identity $(T^j)_A = (T_A)^j$ from Lemma 5.8 yields that $\mathbb{1}_A T^j v = T_A^j(\mathbb{1}_A v)$ for all $j \geq 1$, and since this still holds for $j = 0$, we get

$$\mathbb{1}_A (T - \lambda\, \mathrm{Id})^j v = (T_A - \lambda\, \mathrm{Id})^j (\mathbb{1}_A v). \tag{29}$$

This proves the first item.

If $(T_A - \lambda\, \mathrm{Id})^k v = 0$, the expression $(-\lambda)^k v = -\sum_{j=1}^{k} \binom{k}{j} (-\lambda)^{k-j} T_A^j v$ shows that $\mathrm{supp}(v) \subset A$. By invariance this implies $\mathrm{supp}(T^j v) \subset A$, therefore we have $(T - \lambda\, \mathrm{Id})^k v = \mathbb{1}_A (T - \lambda\, \mathrm{Id})^k v$. We may now apply (29), as invariant sets are convex, and get $\mathbb{1}_A (T - \lambda\, \mathrm{Id})^k v = (T_A - \lambda\, \mathrm{Id})^k v = 0$, which concludes the proof. □

Corollary 7.6. *Suppose that T is a positive power compact operator on L^p with $p \in (1, +\infty)$ and a spectral radius $\rho(T) > 0$. Let $A \in \mathfrak{A}_{\mathrm{crit}}$, $B = \bigcup_{C \in \mathfrak{A}_{\mathrm{crit}}, C \prec A} C$, and $\tilde{A} = F(A) \setminus F(B)$.*

(i) *The set $\tilde{A}$ contains A, it is convex, $F^*(\tilde{A}) = F(B)$ and $F(A) = \tilde{A} \sqcup F^*(\tilde{A})$.*

(ii) *There exists a nonnegative eigenfunction $w_{\tilde{A}}$ of $T_{\tilde{A}}$ such that* $\mathrm{supp}(w_{\tilde{A}}) = \tilde{A}$, *$\mathbb{1}_A w_{\tilde{A}} = v_A$, and $\rho(w_{\tilde{A}}) = \rho(T)$.*

(iii) *If $w \in K(T)$ satisfies* $\mathrm{supp}(w) \subset F(A)$, *then there exists* $\mathrm{c} \in \mathbb{R}$ *such that* $\mathbb{1}_{\tilde{A}} w = \mathrm{c} w_{\tilde{A}}$.

Proof. The set $\tilde{A}$ is convex, since it is the intersection of the invariant set $F(A)$ with the coinvariant set $F(B)^c$. The set A cannot intersect $F(B)$, since this would imply $A \prec A$, so $\tilde{A}$ contains A. By definition of $F(B)$, $\tilde{A}$ contains no other critical atoms. Therefore A is distinguished for $T_{\tilde{A}}$, which yields the existence of $w_{\tilde{A}}$

by Proposition 6.11; moreover $K(\rho(T), T_{\tilde{A}}) = \mathrm{Vect}(w_{\tilde{A}})$ as $\rho(T)$ is algebraically simple for $T_{\tilde{A}}$. By Lemma 7.5(i), the function $\mathbb{1}_{\tilde{A}} w$ belongs to $K(\rho(T), T_{\tilde{A}})$ and is therefore proportional to $w_{\tilde{A}}$, as claimed. □

We are now in a position to prove Theorem 7.3. We proceed in several steps.

7.2.1. *Existence of a basis satisfying* (i). We prove the existence of a basis satisfying Theorem 7.3(i) by induction on the number of critical atoms of T.

If T has one critical atom A, then A is necessarily distinguished. The nonnegative eigenfunction w_A given by Proposition 6.11 is a nonzero vector in the one-dimensional vector space $K(T)$, so it is indeed a basis.

For the induction step, assume that for any positive power compact operator U on L^p with at most n critical atoms, there exists a basis of $K(U)$ satisfying (i). Let T be a positive power compact operator on L^p with $n+1$ critical atoms.

We first claim that, for each critical atom A of T, there exists $w_A \in K(T)$ such that $A \subset \mathrm{supp}(w_A) \subset F(A)$. Indeed, there are two cases. If $T_{F(A)}$ has n atoms or less, then the induction hypothesis applied to $U = T_{F(A)}$ gives the existence of $w_A \in K(U)$ such that $A \subset \mathrm{supp}(w_A) \subset F(A)$, $\mathbb{1}_A w_A = v_A$, and by Lemma 7.5(ii), w_A is in fact in $K(T)$, proving the claim in this case. If $T_{F(A)}$ has $n+1$ atoms, then all critical atoms of T are in the future of A. Notice that $\rho(T_{F(A)}) = \rho(F(A)) = \rho(T)$ and by Lemma 7.5(ii), $K(T_{F(A)}) \subset K(T)$. Furthermore, all the critical atoms of T belongs to $F(A)$ and are thus the critical atoms of $T_{F(A)}$; this implies that $\dim(K(T_{F(A)})) = \mathrm{card}(\mathfrak{A}_{\mathrm{crit}}) = \dim(K(T))$. We deduce that $K(T_{F(A)}) = K(T)$. Let $\tilde{A}$ be defined by Corollary 7.6, and let $U = T_{F^*(\tilde{A})}$. Let $w \in K(T) = K(T_{F(A)})$. We thus have $\mathrm{supp}(w) \subset F(A)$. By Corollary 7.6(ii)–(iii), if w vanishes on A, then it must be identically zero on $\tilde{A}$. Therefore we get $\mathrm{supp}(w) \subset F^*(\tilde{A})$ and $w \in K(U)$ by Lemma 7.5(i) since $F^*(\tilde{A})$ is convex. As a consequence, since by Lemma 6.4(iii), $\dim(K(T)) = n+1 > n = \dim(K(U))$, at least one element of $K(T)$ is nonzero on A. By Corollary 7.6(iii) we may assume without loss of generality that $\mathbb{1}_{\tilde{A}} w = w_{\tilde{A}}$. In particular, $\mathbb{1}_A w = v_A$, and the claim is proved.

Now, a family $\mathcal{W} = (w_A)_{A \in \mathfrak{A}_{\mathrm{crit}}}$ satisfying the claim must be linearly independent. Indeed, assume that $\sum_{A \in \mathfrak{A}_{\mathrm{crit}}} \mathrm{c}_A w_A = 0$. If the c_A do not vanish, let B be a maximal element (for $\preccurlyeq$) among the atoms for which $\mathrm{c}_B \neq 0$. For any atom $A \neq B$, either $B \not\preccurlyeq A$ and w_A is zero on B, or $B \prec A$ and $\mathrm{c}_A = 0$ by maximality of B. Therefore $0 = 0\mathbb{1}_B = \left(\sum_A \mathrm{c}_A w_A\right)\mathbb{1}_B = \mathrm{c}_B w_B \mathbb{1}_B$, so $\mathrm{c}_B = 0$, a contradiction. Therefore all c_A must vanish, and the family $\mathcal{W}$ is linearly independent.

This independence and the fact that $\mathrm{card}(\mathfrak{A}_{\mathrm{crit}}) = \dim(K(T))$ ensure that $\mathcal{W}$ is a basis: this completes the induction and proves (i).

7.2.2. *Proof of* (ii)*: the two-atoms case.* We first prove Theorem 7.3(ii) under the additional assumption that T has only two critical atoms A and B, and that $B \prec A$.

By the trivial bound (28), the ascent is either equal to 1, in which case we have that $\mathrm{Ker}\,(T-\rho(T)) = K(T)$ is two-dimensional, or equal to 2, in which case $1 = \dim(\mathrm{Ker}\,(T-\lambda\,\mathrm{Id})) < \dim(\mathrm{Ker}\,((T-\lambda\,\mathrm{Id})^2)) = \dim(K(T)) = 2$. Let (w_A, w_B) be a basis of $K(T)$ given by (i).

Note that $K(T)$ is stable by T, so there exist four coefficients such that

$$T w_A = M_{AA}\, w_A + M_{AB}\, w_B, \quad T w_B = M_{BA}\, w_A + M_{BB}\, w_B.$$

Since B is distinguished, w_B is the nonnegative eigenvector from Proposition 6.11, so $M_{BB} = \rho(T)$ and $M_{BA} = 0$.

The support of w_A is included in the future of the convex set A, so by Lemma 6.6 we get $T_A(\mathbb{1}_A w_A) = T_A(w_A) = \mathbb{1}_A T w_A = M_{AA} \mathbb{1}_A w_A$ since $w_B = 0$ on A. Since $w_A = v_A$ on A, we see that $M_{AA} = \rho(T)$. We may therefore write

$$(T - \rho(T)\mathrm{Id})\, w_A = M_{AB}\, w_B, \tag{30}$$

and establishing Theorem 7.3(ii) in this case consists in proving that M_{AB} is positive. Let $v_B^\star$ be a positive Perron eigenvector of $T_B^\star$. Since the future of B for $T^\star$ is $P(B)$, we have

$$T^\star v_B^\star = T_B^\star v_B^\star + \mathbb{1}_{P^*(B)} T^\star v_B^\star = \rho(T)\, v_B^\star + \mathbb{1}_{P^*(B)} T^\star v_B^\star.$$

Taking the scalar product with $v_B^\star$ in (30) yields

$$\begin{aligned} M_{AB}\langle v_B^\star, w_B\rangle &= \langle v_B^\star, (T-\rho(T))\, w_A\rangle \\ &= \langle T^\star v_B^\star - \rho(T)\, v_B^\star, w_A\rangle \\ &= \langle \mathbb{1}_{P^*(B)} T^\star v_B^\star, w_A\rangle \\ &= \langle v_B^\star, T(\mathbb{1}_{P^*(B)}\, w_A)\rangle. \end{aligned}$$

By Corollary 7.6, $\mathbb{1}_{P^*(B)}\, w_A$ is nonnegative, and positive on $\tilde{A} = F(A) \setminus F(B)$, so the last expression is nonnegative. Since $\langle v_B^\star, w_B\rangle$ is positive, M_{AB} is nonnegative. Assume for a moment that $M_{AB} = 0$, so that $\langle v_B^\star, T(w_A \mathbb{1}_{P^*(B)})\rangle = 0$, and by (7), $k_T(B, \tilde{A}) = 0$. Using the partition $\Omega = F(A)^c \sqcup \tilde{A} \sqcup B \sqcup F^*(B)$ and the invariance of $F(A)$, we easily check that $k_T(B \cup F(A)^c, \tilde{A} \cup F^*(B)) = 0$, so $\tilde{A} \cup F^*(B)$ is invariant. Since it contains A, it must contain $F(A)$, and therefore B, a contradiction. This shows that $M_{AB} > 0$, concluding the proof of the two-atoms case. Note that $M_{AB} \neq 0$ also shows that $w_A \notin \mathrm{Ker}\,(T - \rho(T)\mathrm{Id})$, so that the ascent is necessarily equal to two.

7.2.3. *Proof of* (ii)*: general case.* By definition, for all A, we have

$$T w_A = \sum_{B \in \mathfrak{A}_{\mathrm{crit}}} M_{AB}\, w_B = \sum_{B \in \mathfrak{A}_{\mathrm{crit}},\, B \prec A} M_{AB}\, w_B + M_{AA}\, w_A + \sum_{B \in \mathfrak{A}_{\mathrm{crit}},\, B \not\preccurlyeq A} M_{AB}\, w_B. \tag{31}$$

Since $\mathrm{supp}(w_A) \subset F(A)$, we have $w_A \in K(\rho(T), T_{F(A)})$, so (i) applied to $T_{F(A)}$ shows that $M_{AB} = 0$ if $B \not\preccurlyeq A$. Then, multiplying (31) by $\mathbb{1}_A$ and applying Corollary 7.6 yields $\rho(T) v_A = M_{AA} v_A$, so $M_{AA} = \rho(T)$.

Assume now that A covers B_0, and let C be the convex set $F(A) \cap P(B_0)$: by definition, the only critical atoms in C are A and B_0. For any other atom B, either $B \not\prec A$ and $M_{AB} = 0$, or $B \prec A$ but $B_0 \not\prec B$, so $F(B) \cap C = \varnothing$, and w_B is zero on C. Therefore, multiplying by $\mathbb{1}_C$ in (31) yields

$$\mathbb{1}_C T w_A = \rho(T) \mathbb{1}_C w_A + M_{AB_0} \mathbb{1}_C w_{B_0}.$$

Using Lemma 6.6 and the fact that $\mathbb{1}_C w_{B_0} = \mathbb{1}_{B_0} w_{B_0} = v_{B_0}$, we get the equality $T_C(\mathbb{1}_C w_A) = \rho(T)(\mathbb{1}_C w_A) + M_{AB_0} v_{B_0}$, so M_{AB_0} is a term of the matrix of T_C in the basis $(\mathbb{1}_C w_A, v_{B_0})$ of $K(T_C, \rho(T_C))$, and its positivity follows from the two-atoms case.

7.2.4. *Conclusion.* To check that (iii) of Theorem 7.3 holds, note that the matrix N of $S = T - \rho(T)\mathrm{Id}$ on the basis $\mathcal{W}$ satisfies $N_{AB} = 0$ unless $B \prec A$, and $N_{AB} > 0$ if A covers B. Thus, we get

$$(N^k)_{AB} = \sum_{A=A_0 \succ A_1 \cdots \succ A_k = B} \prod_j N_{A_j, A_{j+1}}.$$

If $k > h(A)$, there is no chain of length k starting down from A, so $N^k w_A = 0$. If $k = h(A)$, the sum is nonempty, the only chains appearing in the sum are of maximal length so A_j must cover A_{j+1}, the corresponding products are all positive, so $N^k w_A = \sum_B c_B w_B$ for some nonzero numbers c_B, and $N^k w_A \neq 0$. Therefore the index of w_A is $h(A)$.

Notice the proof of (ii) and (iii) are done under the condition that the basis only satisfies (i). This completes the proof of Theorem 7.3.

References

[1] Y. A. Abramovich and C. D. Aliprantis, *An invitation to operator theory*, Graduate Studies in Mathematics **50**, Amer. Math. Soc., Providence, RI, 2002. MR Zbl

[2] C. D. Aliprantis and K. C. Border, *Infinite dimensional analysis*: *a hitchhiker's guide*, 3rd ed., Springer, Berlin, 2006. MR Zbl

[3] C. D. Aliprantis and O. Burkinshaw, *Positive operators*, Springer, Dordrecht, 2006. MR Zbl

[4] B. A. Barnes, "Spectral properties of linear Volterra operators", *J. Operator Theory* **24**:2 (1990), 365–382. MR Zbl

[5] J. Bernik, L. W. Marcoux, and H. Radjavi, "Spectral conditions and band reducibility of operators", *J. Lond. Math. Soc.* (2) **86**:1 (2012), 214–234. MR Zbl

[6] G. Birkhoff, *Lattice theory*, 3rd ed., American Mathematical Society Colloquium Publications **25**, Amer. Math. Soc., Providence, RI, 1967. MR Zbl

[7] B. Bollobás, S. Janson, and O. Riordan, "The phase transition in inhomogeneous random graphs", *Random Structures Algorithms* **31**:1 (2007), 3–122. MR Zbl

[8] J.-F. Delmas, D. Dronnier, and P.-A. Zitt, "The effective reproduction number: convexity, concavity and invariance", preprint, 2022. To appear in *J. Eur. Math. Soc.* arXiv 2211.11862

[9] J.-F. Delmas, D. Dronnier, and P.-A. Zitt, "An infinite-dimensional metapopulation SIS model", *J. Differential Equations* **313** (2022), 1–53. MR Zbl

[10] H. R. Dowson, *Spectral theory of linear operators*, London Mathematical Society Monographs **12**, Academic Press, London, 1978. MR Zbl

[11] N. Dunford and J. T. Schwartz, *Linear operators*, *I*: *General theory*, John Wiley & Sons, New York, 1988. MR Zbl

[12] T. Figiel, T. Iwaniec, and A. Pełczyński, "Computing norms and critical exponents of some operators in L^p-spaces", *Studia Math.* **79**:3 (1984), 227–274. MR Zbl

[13] H. Föllmer and A. Schied, *Stochastic finance: an introduction in discrete time*, 4th ed., De Gruyter, Berlin, 2016. MR Zbl

[14] K.-H. Förster and B. Nagy, "On nonnegative operators and fully cyclic peripheral spectrum", *Electron. J. Linear Algebra* **3** (1998), 13–22. MR Zbl

[15] J. J. Grobler, "A note on the theorems of Jentzsch–Perron and Frobenius", *Nederl. Akad. Wetensch. Indag. Math.* **49**:4 (1987), 381–391. In Russian; translated in *Indag. Math.* **49**:4 (1987), 381–391. MR Zbl

[16] L. Herve, "Peripheral spectrum of positive quasi-compact operators on $\mathcal{C}_0(X)$", *Integral Equations Operator Theory* **32**:2 (1998), 199–215. MR

[17] R.-J. Jang, "On the peripheral spectrum of order continuous, positive operators", *Positivity* **4**:2 (2000), 119–130. MR Zbl

[18] R. Jang and H. D. Victory, Jr., "Frobenius decomposition of positive compact operators", pp. 195–224 in *Positive operators*, *Riesz spaces*, *and economics* (Pasadena, CA, 1990), edited by C. D. Aliprantis et al., Studies in Economic Theory **3**, Springer, Berlin, 1991. MR Zbl

[19] R.-J. Jang-Lewis and H. D. Victory, Jr., "On the ideal structure of positive, eventually compact linear operators on Banach lattices", *Pacific J. Math.* **157**:1 (1993), 57–85. MR Zbl

[20] H. König, *Eigenvalue distribution of compact operators*, Operator Theory: Advances and Applications **16**, Birkhäuser, Basel, 1986. MR Zbl

[21] I. Marek, "Frobenius theory of positive operators: comparison theorems and applications", *SIAM J. Appl. Math.* **19** (1970), 607–628. MR Zbl

[22] P. Nelson, Jr., "The structure of a positive linear integral operator", *J. London Math. Soc.* (2) **8** (1974), 711–718. MR Zbl

[23] M. Omladič and V. Omladič, "Positive root vectors", *Proc. Roy. Soc. Edinburgh Sect. A* **125**:4 (1995), 701–717. MR Zbl

[24] J. R. Ringrose, "Super-diagonal forms for compact linear operators", *Proc. London Math. Soc.* (3) **12** (1962), 367–384. MR Zbl

[25] W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill Book Co., New York, 1987. MR Zbl

[26] H. H. Schaefer, *Banach lattices and positive operators*, Grundl. Math. Wissen. **215**, Springer, New York, 1974. MR Zbl

[27] J. Schwartz, "Compact positive mappings in Lebesgue spaces", *Comm. Pure Appl. Math.* **14** (1961), 693–705. MR Zbl

[28] B.-S. Tam, "A cone-theoretic approach to the spectral theory of positive linear operators: the finite-dimensional case", *Taiwanese J. Math.* **5**:2 (2001), 207–277. MR Zbl

[29] H. D. Victory, Jr., "On linear integral operators with nonnegative kernels", *J. Math. Anal. Appl.* **89**:2 (1982), 420–441. MR Zbl

[30] H. D. Victory, Jr., "The structure of the algebraic eigenspaces to the spectral radius of eventually compact, nonnegative integral operators", *J. Math. Anal. Appl.* **90**:2 (1982), 484–516. MR Zbl

[31] H. D. Victory, Jr., "On nonnegative solutions of matrix equations", *SIAM J. Algebraic Discrete Methods* **6**:3 (1985), 406–412. MR Zbl

[32] H. Vogt and J. Voigt, "Bands in L_p-spaces", *Math. Nachr.* **290**:4 (2017), 632–638. MR Zbl

Received August 13, 2024. Revised March 19, 2025.

JEAN-FRANÇOIS DELMAS
CERMICS
ÉCOLE DES PONTS
MARNE-LA-VALLÉE
FRANCE
jean-francois.delmas@enpc.fr

KACEM LEFKI
UNIVERSITÉ GUSTAVE EIFFEL
UNIVERSITÉ PARIS EST CRETEIL
CNRS
MARNE-LA-VALLÉE
FRANCE
kacem.lefki@univ-eiffel.fr

PIERRE-ANDRÉ ZITT
UNIVERSITÉ GUSTAVE EIFFEL
UNIVERSITÉ PARIS EST CRETEIL
CNRS
MARNE-LA-VALLÉE
FRANCE
pierre-andre.zitt@univ-eiffel.fr