BRANCHING CAPACITY AND BROWNIAN SNAKE CAPACITY

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ABSTRACT. The branching capacity has been introduced by Zhu [25] as the limit of the hitting probability of a symmetric branching random walk in $\mathbb{Z}^d$, $d \geq 5$. Similarly, we define the Brownian snake capacity in $\mathbb{R}^d$, as the scaling limit of the hitting probability by the Brownian snake starting from afar. Then, we prove our main result on the vague convergence of the rescaled branching capacity towards this Brownian snake capacity. Our proof relies on a precise convergence rate for the approximation of the branching capacity by hitting probabilities.

1. INTRODUCTION

Let $\mathcal{T}_c$ be a critical Galton-Watson tree with offspring distribution $\mu = (\mu(i))_{i \geq 0}$ with mean 1. In other words, $\mathcal{T}_c$ is a random tree that begins with one particle, and each particle independently produces a random number of offspring according to μ. It is well-known that $\mathcal{T}_c$ is almost surely finite provided $\mu(1) < 1$.

Let θ be a probability distribution on $\mathbb{Z}^d$. For $x \in \mathbb{Z}^d$, we denote by $\mathbf{P}_x$ the law of a branching random walk (BRW) V_c in $\mathbb{Z}^d$, which is a θ-random walk indexed by the tree $\mathcal{T}_c$ constructed as follows. We pin the root of $\mathcal{T}_c$ at x, then to each edge of $\mathcal{T}_c$ we attach an independent random displacement distributed as θ. The position of a vertex of $\mathcal{T}_c$ is defined as x plus the sum of all displacements associated with the edges in the simple path from the root to that vertex. The collection of all (spatial) positions of vertices of $\mathcal{T}_c$ is called the range of V_c and is denoted by $\mathscr{R}_c$.

We consider $d \geq 5$. For $x \in \mathbb{R}^d$, let $|x|$ denote its usual Euclidean norm, and $\mathrm{B}(x, r) := \{y \in \mathbb{R}^d : |y - x| < r\}$ the open ball centered at x with radius $r > 0$. Throughout the article, we shall assume that

(1.1) μ has mean 1 and variance $\sigma^2 \in (0, \infty)$,

(1.2) θ is symmetric, irreducible with covariance matrix M_θ and

there exists a finite constant c such that for all $r > 0$: $\quad \theta\big(\mathrm{B}(0, r)^c\big) \leq c\, r^{-d}$.

The last condition in (1.2) is in particular satisfied when θ has a finite d-th moment.

To introduce the branching capacity, we will use the Green function of a random walk. Let $(S_n)_{n \geq 0}$ be a θ-random walk on $\mathbb{Z}^d$, where θ serves as the step distribution of (S_n). For $x \in \mathbb{R}^d$, we also consider its norm in relation with (S_n) given by $|x|_\theta := \sqrt{x^T M_\theta^{-1} x}$. We define the Green function of (S_n) by $g(x, y) := g(x - y)$ and for $x \in \mathbb{Z}$

(1.3) $$g(x) := \sum_{n=0}^{\infty} \mathbf{P}_0(S_n = x) = c_g\, |x|_\theta^{2-d} + O(|x|^{1-d}),$$

with

$$c_g := \frac{\Gamma(\frac{d-2}{2})}{2\pi^{d/2}\sqrt{\det M_\theta}},$$

where the asymptotic is due to Uchiyama [24, Theorem 2]. Let $K \subset \mathbb{Z}^d$ be a nonempty finite set. Following Zhu [25], we define the branching capacity of K as

$$(1.4) \qquad \mathrm{Bcap}(K) := \lim_{x\to\infty} \frac{\mathbf{P}_x(\mathscr{R}_c \cap K \neq \emptyset)}{g(x)} = \frac{1}{c_g} \lim_{x\to\infty} |x|_\theta^{d-2}\mathbf{P}_x(\mathscr{R}_c \cap K \neq \emptyset),$$

where $x \to \infty$ means $|x| \to \infty$. Moreover, $\mathrm{Bcap}(\cdot)$ viewed as a set function is non-decreasing, invariant under translations, and strictly positive when the set is not empty.

In our first result we give a rate of convergence for (1.4), which will be useful in the study of the branching capacity.

Theorem 1.1. *Let $d \geq 5$ and $\lambda > 1$. Assume (1.1) and (1.2). There exists a positive constant $C = C(d, \lambda)$ such that uniformly in $r \geq 1$, $K \subset \mathrm{B}(0, r)$ nonempty and $x \in \mathbb{Z}^d$ with $|x| \geq \lambda r$, we have, with $\alpha = (d-4)/2(d-1)$,*

$$\left| \mathrm{Bcap}(K) - \frac{\mathbf{P}_x(\mathscr{R}_c \cap K \neq \emptyset)}{g(x)} \right| \leq C \left(\frac{r}{|x|}\right)^\alpha \mathrm{Bcap}(K).$$

We do not claim that the value of α is optimal. We refer to Proposition 4.13 and Proposition 4.14 for corresponding results on the branching models V_{adj}, V_I and V_- defined in Section 4.1.

Branching capacity is a relatively new concept defined and explored in a series of works by Zhu [25, 26, 27]. In particular, Zhu [25] showed that $\mathrm{Bcap}(K)$ can also be defined through the exit probabilities from a BRW indexed by an infinite tree, see (4.3). Therefore, Le Gall and Lin [21, Proposition 3]'s ergodicity implies that $\mathrm{Bcap}(K)$ is the almost sure limit of the rescaled cardinality of the sum of K and the first n points in this BRW. Recently, Asselah, Schapira, and Sousi [1] studied the link between branching capacity and Green's function. In addition, when (S_n) is a simple random walk on $\mathbb{Z}^d$, Asselah, Okada, Schapira, and Sousi [2] demonstrated that $\mathrm{Bcap}(K)$ can be compared with the rescaled cardinality of the sum of K and the Minkowski sum of the ranges of two independent copies of (S_n). We also mention a recent study by Schapira [23] and an ongoing work [4] on the branching capacity of the range of a random walk.

It is a natural problem to investigate the scaling limit of branching capacities. Loosely speaking, the branching capacity is related to a critical BRW in the same way as the (discrete) Newtonian capacity does to a random walk. Therefore, we first define a continuous counterpart of Bcap, called Brownian snake capacity, by utilizing the Brownian snake, which is the scaling limit of BRW. The introduction of the Brownian snake capacity aims to establish a connection between the branching capacity and the hitting probabilities associated with the Brownian snake, as investigated by Le Gall [16], Dhersin and Le Gall [8], and Delmas [7].

Following Le Gall [19], let $\mathcal{N}_x(\mathrm{d}W)$ be the excursion measure of a Brownian snake $W = (W_t)_{t\geq 0}$ started from $x \in \mathbb{R}^d$, see Section 2.2 for the precise definitions. Denote by $\mathfrak{R}$ the range of the Brownian snake (see (2.6)). In the next theorem, we define $\mathrm{BScap}(A)$ the Brownian snake capacity of A for $d \geq 5$, and refer to Remark 2.6 for the case $d \leq 4$.

Theorem 1.2. *Let $d \geq 5$. For any bounded Borel set $A \subset \mathbb{R}^d$, the following limit exists and is finite:*

$$\mathrm{BScap}(A) := \lim_{x\to\infty} |x|^{d-2}\mathcal{N}_x(\mathfrak{R} \cap A \neq \emptyset).$$

In the next proposition we state some properties of the Brownian snake capacity. Choquet capacities are precisely defined in Section 2.1, see also (2.1) for the strong sub-additive

property; the Riesz capacity $\mathrm{Cap}_{d-\alpha}$ for $\alpha \in (0,d)$ is defined in Remark 2.4 below; the closure of a set $A \subset \mathbb{R}^d$ is denoted by $\overline{A}$.

Proposition 1.3. *Let $d \geq 5$.*

(1) **Choquet capacity.** *The map BScap can be extended into a strongly sub-additive Choquet capacity on $\mathcal{P}(\mathbb{R}^d)$, the power set of $\mathbb{R}^d$.*

(2) **Scaling and translation.** *For any $x \in \mathbb{R}^d$, $a \in \mathbb{R}$ and $A \subset \mathbb{R}^d$, we have*

$$\text{(1.5)} \qquad \mathrm{BScap}(x + aA) = a^{d-4}\,\mathrm{BScap}(A).$$

(3) **Comparison with Cap_{d-4}.** *There exist finite positive constants c_1 and c_2 such that for any $A \subset \mathbb{R}^d$*

$$\text{(1.6)} \qquad c_1 \mathrm{Cap}_{d-4}(A) \leq \mathrm{BScap}(A) \leq c_2 \mathrm{Cap}_{d-4}(A).$$

(4) **Regularity.** *Let $D \subset \mathbb{R}^d$ be a bounded open set. If for all $y \in \partial_{ext}D$, that is, the boundary of the unbounded connected component of D^c,*

$$\text{(1.7)} \qquad \liminf_{n \to \infty} 2^{n(d-2)}\,\mathrm{Cap}_{d-2}\left(D \cap \mathrm{B}(y, 2^{-n})\right) > 0,$$

then, we have

$$\mathrm{BScap}(D) = \mathrm{BScap}(\overline{D}).$$

(5) **Particular value.** *For $d = 6$, we have $\mathrm{BScap}(\mathrm{B}(0,1)) = 6$.*

The scaling and translation properties of the capacity BScap are consequences of its definition and the scaling properties of the Brownian snake; the comparison with the Riesz capacity Cap_{d-4} comes from estimates in [8], but it is unclear whether BScap and Cap_{d-4} are equal (up to a multiplicative constant); the regularity of BScap on open sets relies on uniqueness of solution of $\Delta u = 4u^2$ on open set with infinite boundary conditions, see [17] (this has to be compared with the Wiener test for Brownian motion given by $\sum_{n=0}^{\infty} 2^{n(d-2)}\mathrm{Cap}_{d-2}\left(D \cap \mathrm{B}(y, 2^{-n})\right) = \infty$), as mentioned in [17] the condition (1.7) is satisfied when D has a Lipschitz boundary; we are able to compute the capacity of the balls in dimension 6 by giving an explicit formula for the solution of $\Delta u = 4u^2$ outside the unit ball $\mathrm{B}(0,1)$ with infinite boundary conditions, see Remark 2.9 and the new formula (2.29) therein.

We refer to [1] for the analogue of (1.6) for branching capacity on finite subsets of $\mathbb{Z}^d$ in the case when θ has bounded jumps. To state our next result we extend the capacity Bcap on $\mathbb{Z}^d$ as a Choquet capacity on $\mathbb{R}^d$ by setting $\mathrm{Bcap}(A) = \mathrm{Bcap}(A \cap \mathbb{Z}^d)$ for all $A \subset \mathbb{R}^d$. For every set $A \subset \mathbb{R}^d$ and $\varepsilon > 0$, denote by $A^\varepsilon = \{x : d(x,A) \leq \varepsilon\}$ the closed ε-neighborhood of A, where $d(x,A) := \inf_{y \in A} |x - y|$.

On exploring the link between the discrete branching capacity and the continuous Brownian snake capacity, we have the following result. Let $\mathcal{K}$ and $\mathcal{O}$ denote the families of compact and open subsets of $\mathbb{R}^d$.

Theorem 1.4. *Let $d \geq 5$. Assume (1.1) and (1.2). We have*

$$\text{(1.8)} \qquad \limsup_{n \to \infty} \frac{\mathrm{Bcap}(nK)}{n^{d-4}} \leq c_\theta\,\mathrm{BScap}(M_\theta^{-1/2}K) \quad \text{for} \quad K \in \mathcal{K},$$

$$\text{(1.9)} \qquad \liminf_{n \to \infty} \frac{\mathrm{Bcap}(nD)}{n^{d-4}} \geq c_\theta\,\mathrm{BScap}(M_\theta^{-1/2}D) \quad \text{for} \quad D \in \mathcal{O},$$

with

$$c_\theta := \frac{2}{\sigma^2 c_g} = \frac{4\pi^{d/2}\sqrt{\det M_\theta}}{\sigma^2\,\Gamma(\frac{d-2}{2})}.$$

In the setting of Norberg [22], Equations (1.8)-(1.9) correspond to the vague convergence of the capacities $n^{-(d-4)}\mathrm{Bcap}(n \cdot)$ towards the capacity $c_\theta \mathrm{BScap}(M_\theta^{-1/2} \cdot)$.

Remark 1.5. If θ is the uniform probability distribution on the $2d$ unit vectors in $\mathbb{Z}^d$, then $M_\theta = \frac{1}{d}I$, and for any set $A \subset \mathbb{R}^d$ we have

$$c_\theta \, \mathrm{BScap}(M_\theta^{-1/2} A) = \frac{2\pi^{d/2}}{\sigma^2 \Gamma(\frac{d}{2})} \, \mathrm{BScap}(A).$$

The paper is organized as follows. In Section 2, we explore the Brownian snake, and provide proofs for Theorem 1.2 and Proposition 1.3. In Section 3, we prove Theorem 1.4 by assuming Theorem 1.1. And finally we complete the proof of Theorem 1.1 in Section 4.

For notational brevity, let $f_\delta(x), g_\delta(x)$ be two nonnegative functions depending on some parameter δ, we write $f_\delta(x) \lesssim g_\delta(x)$ when there is a constant $C > 0$, independent of x and δ, such that $f_\delta(x) \leq C g_\delta(x)$. We also denote $f_\delta(x) \asymp g_\delta(x)$ when $f_\delta(x) \lesssim g_\delta(x)$ and $g_\delta(x) \lesssim f_\delta(x)$.

We write $\mathbb{N} = \mathbb{Z} \cap [0, \infty)$ the set of non-negative integers. We denote by $\mathrm{diam}(A) := \sup_{x,y \in A} |x - y|$ the diameter of $A \subset \mathbb{R}^d$.

2. THE BRANCHING AND THE BROWNIAN SNAKE CAPACITIES

2.1. **Choquet capacity.** Let $(X, \mathcal{O})$ be an Hausdorff topological space, $\mathcal{P}(X)$ be its power set. Let $\mathcal{A} \subset \mathcal{P}(X)$ be stable by finite union and intersection. We say that a map $I : \mathcal{A} \to [0, +\infty]$ is non-decreasing if:

$$I(A) \leq I(B) \quad \text{for} \quad A, B \in \mathcal{A} \quad \text{such that} \quad A \subset B,$$

and is strongly sub-additive if

$$(2.1) \qquad I(A \cup B) + I(A \cap B) \leq I(A) + I(B) \quad \text{for} \quad A, B \in \mathcal{A}.$$

Let $\mathcal{K} \subset \mathcal{P}(X)$ denote the set of compact sets (which is stable by finite union and intersection). Following Definition 27 in [6, Section III.2], a map $I : \mathcal{P}(X) \to [0, +\infty]$ is a Choquet capacity (relative to $\mathcal{K}$) if:

(i) I is non-decreasing.
(ii) If $(A_n)_{n \in \mathbb{N}}$ is a non-decreasing sequence of subsets of X, then

$$I(\cup_n A_n) = \sup_n I(A_n).$$

(iii) If $(K_n)_{n \in \mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$, then

$$I(\cap_n K_n) = \inf_n I(K_n).$$

A set $A \subset X$ is capacitable (with respect to I) if:

$$(2.2) \qquad I(A) = \sup_{K \in \mathcal{K}, \, K \subset A} I(K).$$

Assume that $(X, \mathcal{O})$ is a second countable Hausdorff locally compact space. Then according to [6, Theorem III.13] and the comment above therein, we get that Borel sets are analytic and thus capacitable by Choquet theorem (see also Theorem 28 therein).

Given a map $J : \mathcal{K} \to [0, \infty]$ defined only on the compact sets, we can define a function $J^* : \mathcal{P}(X) \to [0, \infty]$ as follows. For every open set $D \in \mathcal{O}$, we set

$$(2.3) \qquad J^*(D) = \sup_{K \in \mathcal{K}, \, K \subset D} J(K),$$

and for any $A \subset X$,

$$(2.4) \qquad J^*(A) = \inf_{D \in \mathcal{O}, \, A \subset D} J^*(D).$$

Remark 2.1. It is not difficult, using [6, Lemme p.100], to check that if J is strongly sub-additive on $\mathcal{K}$, then J^* is also strongly sub-additive on $\mathcal{P}(X)$.

The next theorem from [6] states that J^* extends J into a Choquet capacity. We say that the map J is right continuous if for every $K \in \mathcal{K}$ and every real number $a > J(K)$, there exists an open set D such that $K \subset D$ and $J(K') < a$ for every compact set $K' \subset D$.

Theorem 2.2 ([6, Theorem III.42]). *Let $J : \mathcal{K} \to [0, \infty]$ be non-decreasing, strongly sub-additive and right continuous. Then the map J^* is a Choquet capacity (relative to $\mathcal{K}$), and it coincides with J on $\mathcal{K}$.*

Remark 2.3. On $X = \mathbb{Z}^d$ with the Euclidean distance, $\mathcal{K}$ simply consists of finite sets. Hence one can easily check that Bcap defined on $\mathcal{K}$ by (1.4) satisfies the conditions of Theorem 2.2. Hence we can extend it to a Choquet capacity, which we still denote by Bcap. Notice that Bcap is strictly positive (except for empty set) thanks to (4.4).

Remark 2.4. Let $X = \mathbb{R}^d$ and $\gamma \in (0, d)$. We consider the Riesz kernel $k_\gamma(x) = C_{d,\gamma}|x|^{-\gamma}$ with $C_{d,\gamma} = \pi^{-\gamma+d/2}\Gamma(\gamma/2)/\Gamma((d-\gamma)/2)$ (see (I.1.1.2) in [14]). For every compact set K, define

$$(2.5) \qquad \mathrm{Cap}_\gamma(K) := \left(\inf_\nu \int k_\gamma(x-y)\,\nu(\mathrm{d}x)\nu(\mathrm{d}y) \right)^{-1},$$

where the infimum is taken over all the probability measures with support in K. In particular, the case $\gamma = d - 2$ corresponds to the usual Newtonian capacity.

For $\gamma \in (0, d)$, the function Cap_γ can be extended into a Choquet capacity (relative to $\mathcal{K}$) on $\mathcal{P}(\mathbb{R}^d)$ using again (2.3) and (2.4), see [14, Section II.1-2]; we still denote this extension by Cap_γ. We recall that Cap_γ is sub-additive, see [14, Eq. (2.2.3)]:

$$\mathrm{Cap}_\gamma(A \cup B) \le \mathrm{Cap}_\gamma(A) + \mathrm{Cap}_\gamma(B), \quad A, B \subset \mathbb{R}^d.$$

Let us stress that (2.2) implies that Equation (2.5) also holds for all capacitable sets, and in particular for all Borel sets.

For $\gamma \le d - 2$, we mention that, according to [14, Section II.3], the capacity of a compact set K is equal to the capacity of its outer boundary, that is the boundary of the unbounded connected component of K^c.

For $\gamma \ge d - 2$, the function Cap_γ is strongly sub-additive by [14, Section II.1, Theorem II.2.5]; so we could also apply Theorem 2.2 to get its extension as a Choquet capacity.

2.2. **Brownian snake capacities.** Let $\mathcal{W} := \{w : [0, \zeta] \to \mathbb{R}^d \text{ is continuous}\}$ be the space of stopped continuous paths, where $\zeta = \zeta_w \ge 0$ denotes the lifetime of w. The end point, $w(\zeta)$, is denoted by $\widehat{w}$. The space $\mathcal{W}$, equipped with the metric

$$d(w, w') := \sup_{t \ge 0} |w(t \wedge \zeta_w) - w'(t \wedge \zeta_{w'})| + |\zeta_w - \zeta_{w'}|,$$

is a Polish space. Let $x \in \mathbb{R}^d$ and $\mathcal{W}_x$ be the space of stopped path starting from $w(0) = x$. The Brownian snake, denoted by $(W_t)_{t \ge 0}$, is a continuous strong Markov process taking valued in $\mathcal{W}_x$, where W_0 equals the trivial path (with $w(0) = x$ and $\zeta_w = 0$). The excursion measure outside this trivial path, $\mathcal{N}_x$, can be characterized as follows, see [19, Chapters 4 and 5]:

(i) The lifetime process $(\zeta_s)_{s\geq 0}$, under $\mathcal{N}_x$, is distributed as a positive Brownian excursion under the Itô measure. In particular, with $T_0(\zeta) := \inf\{s > 0 : \zeta_s = 0\}$ the length of the excursion,

$$\mathcal{N}_x\left(\sup_{0\leq s\leq T_0(\zeta)} \zeta_s > t\right) = \frac{1}{2t}, \qquad \mathcal{N}_x(T_0(\zeta) \in dt) = \frac{dt}{\sqrt{2\pi t^3}} \quad \text{for} \quad t > 0.$$

(ii) Conditionally on $(\zeta_s)_{s\geq 0}$, $(W_s)_{s\geq 0}$ is a time-inhomogeneous continuous Markov process such that for any $s, s' > 0$, W_s and $W_{s'}$ coincide up to $m_{s,s'} := \inf_{t\in[s,s']} \zeta_t$, and evolve as two independent Brownian motions during $[m_{s,s'}, \zeta_{W_s}]$ and $[m_{s,s'}, \zeta_{W_{s'}}]$ respectively.

Under $\mathcal{N}_x$, the range of the Brownian snake is defined by

(2.6) $$\mathfrak{R} := \{\widehat{W}_s : 0 \leq s \leq T_0(\zeta)\} = \{W_s(t) : s \in [0, T_0(\zeta)], t \in [0, \zeta_s]\}.$$

Let A be a Borel set of $\mathbb{R}^d$. We define the function u_A on A^c by:

(2.7) $$u_A(x) := \mathcal{N}_x(\mathfrak{R} \cap A \neq \emptyset), \qquad x \in A^c.$$

Using the scaling of the Brownian snake, one get the existence of a regular conditional probability distribution of the Brownian snake with respect to the excursion length:

$$\mathcal{N}_x^{(1)}(dW) = \mathcal{N}_x(dW \,|\, T_0(\zeta) = 1).$$

By the scaling property of Brownian snake, we have

$$u_A(x) = \int_0^\infty \mathcal{N}_0\Big(\mathfrak{R} \cap (A - x) \neq \emptyset \,|\, T_0(\zeta) = s\Big) \frac{ds}{\sqrt{2\pi s^3}}$$

(2.8) $$= \int_0^\infty \mathcal{N}_0^{(1)}\Big((s^{1/4}\mathfrak{R}) \cap (A - x) \neq \emptyset\Big) \frac{ds}{\sqrt{2\pi s^3}}.$$

We deduce the following scaling property

(2.9) $$u_A(x) = a^2 u_{aA}(ax) \quad \text{for} \quad a > 0.$$

Moreover, according to [8, Theorem 1], Definition (2.7) can be extended to $A \in \mathcal{P}(\mathbb{R}^d)$ and, in dimension $d \geq 5$, there exists two constants c_1 and c_2 such that, if $A \subset \mathrm{B}(0,1)$ and $|x| \geq 2$, then

(2.10) $$c_1 \,\mathrm{Cap}_{d-4}(A) \leq |x|^{d-2}\, u_A(x) \leq c_2 \,\mathrm{Cap}_{d-4}(A).$$

A set $A \subset \mathbb{R}^d$ is called $\mathfrak{R}$-polar if $\mathrm{Cap}_{d-4}(A) = 0$, or equivalently, if u_A is identically zero.

We cite the following result from [17], and give a short proof for readers' convenience.

Lemma 2.5. *Let A be a Borel set. The function u_A is a nonnegative solution of*

(2.11) $$\Delta u = 4\,u^2 \quad in \quad \overline{A}^c.$$

When $A = K$ is compact, then u_K is the maximal nonnegative solution to (2.11). Furthermore, the function u_A is strictly positive and C^∞ on $\overline{A}^c$ unless A is $\mathfrak{R}$-polar.

Proof. According to Proposition 5.3 in [17], when K is compact, u_K is the maximal nonnegative solution to (2.11).

Let A be a Borel set and D be an open ball in $\overline{A}^c$. It suffices to prove (2.11) in D. Let X^D denote the exit measure of D for the Brownian snake (see Section 2 in [18] with $\Omega = \mathbb{R}_+ \times D$). For $x \in D$, by [18, Theorem 2.4] on the spatial Markov property, we have

$$u_A(x) = \mathcal{N}_x\left(1 - e^{-\int_{\mathbb{R}^d} u_A(x)\, X^D(dx)}\right).$$

Then we deduce by [17, Corollary 4.3], that u_A is C^∞ in D and solves (2.11) in D.

Finally, we use (2.10) to conclude that the function u_A is strictly positive on $\overline{A}^c$ unless A is $\mathfrak{R}$-polar. $\qquad\square$

2.3. **Proof of Theorem 1.2.** Recall that $d \geq 5$, and $\mathrm{B}(x, r)$ denotes a ball of radius r centered at x. By the scaling property (2.9), we can assume that $A \subset \mathrm{B}(0, 1)$. Let $x \in \mathbb{R}^d$, $|x| \geq 4$. Following the ideas of [8], we consider a smooth function ψ on $\mathbb{R}^d$ such that $\psi \in [0, 1]$, $\psi = 0$ on $\mathrm{B}(0, 2)$ and $\psi = 1$ outside $\mathrm{B}(0, 3)$. Let $(\beta_t)_{t \geq 0}$ be a Brownian motion started at $\beta_0 = x$. Using Itô calculus, we get that

$$(\psi u_A)(\beta_t) = u_A(x) + \int_0^t \nabla(\psi u_A)(\beta_s) \, \mathrm{d}\beta_s + \frac{1}{2}\int_0^t \Delta(\psi u_A)(\beta_s) \, \mathrm{d}s.$$

Let $a > |x|$ and $T_a = \inf\{y \geq 0 : |\beta_t| = a\}$, which is a.s. finite. We get that for $t \geq 0$,

$$(2.12) \qquad \mathbf{E}\left[(\psi u_A)(\beta_{T_a \wedge t})\right] = u_A(x) + \frac{1}{2}\mathbf{E}\left[\int_0^{T_a \wedge t} \Delta(\psi u_A)(\beta_s) \, \mathrm{d}s\right].$$

Since T_a is finite and $\psi(\beta_{T_a}) = 1$, by (2.10), we deduce that

$$\lim_{t \to \infty} \mathbf{E}\left[(\psi u_A)(\beta_{T_a \wedge t})\right] = \mathbf{E}\left[u_A(\beta_{T_a})\right] \leq c_2 a^{2-d}\mathrm{Cap}_{d-4}(A),$$

thus

$$\lim_{a \to \infty}\lim_{t \to \infty} \mathbf{E}\left[(\psi u_A)(\beta_{T_a \wedge t})\right] = 0.$$

By Lemma 2.5, we decompose $\Delta(\psi u_A)$ into the sum of $\psi \Delta u_A = 4\psi u_A^2$ and $f := u_A \Delta \psi + 2\nabla\psi \, \nabla u_A$. By monotone convergence, we have

$$\lim_{a \to \infty}\lim_{t \to \infty} \mathbf{E}\left[\int_0^{T_a \wedge t} (\psi \Delta u_A)(\beta_s) \, \mathrm{d}s\right] = 4 \lim_{a \to \infty}\lim_{t \to \infty} \mathbf{E}\left[\int_0^{T_a \wedge t} (\psi u_A^2)(\beta_s) \, \mathrm{d}s\right]$$

$$= 4\mathbf{E}\left[\int_0^{\infty} (\psi u_A^2)(\beta_s) \, \mathrm{d}s\right]$$

$$= 4c_d \int_{\mathbb{R}^d} (\psi u_A^2)(y)|x - y|^{2-d} \, \mathrm{d}y,$$

where

$$c_d := \frac{\Gamma(\frac{d}{2} - 1)}{2\pi^{d/2}}.$$

The latter integral is finite since ψu_A^2 is bounded on $\mathbb{R}^d$, and thus on $\mathrm{B}(x, 1)$, and since

$$(2.13) \qquad (\psi u_A^2)(y) \leq c\,(1 \vee |y|)^{4-2d} \quad \text{for} \quad y \in \mathbb{R}^d$$

by (2.10), which entails that for some constant $c' > 0$,

$$(2.14) \qquad \int_{\mathrm{B}(x,1)^c} (\psi u_A^2)(y)|x - y|^{2-d} \, \mathrm{d}y \leq c' \int_0^{\infty} (1 \vee r)^{4-2d}\, r^{d-1} \, \mathrm{d}r < \infty.$$

Then, notice the function $f = u_A \Delta\psi + 2\nabla\psi \, \nabla u_A$ is continuous with support in $\overline{\mathrm{B}}(0, 3) \cap \mathrm{B}(0, 2)^c$. Since the time spent by the Brownian motion in $\mathrm{B}(0, 3)$ has a finite expectation, we deduce by dominated convergence that

$$(2.15) \quad \lim_{a \to \infty}\lim_{t \to \infty} \mathbf{E}\left[\int_0^{T_a \wedge t} f(\beta_s) \, \mathrm{d}s\right] = \mathbf{E}\left[\int_0^{\infty} f(\beta_s) \, \mathrm{d}s\right] = c_d \int_{\mathbb{R}^d} f(y)|x - y|^{2-d} \, \mathrm{d}y \in \mathbb{R}.$$

In conclusion we obtain from (2.12) that

$$(2.16) \qquad 2u_A(x) = -c_d \int_{\mathbb{R}^d} (4\psi u_A^2 + f)(y)\,|x - y|^{2-d} \, \mathrm{d}y.$$

To conclude the proof of Theorem 1.2, it is enough to prove that

$$(2.17) \qquad \lim_{x \to \infty} |x|^{d-2} u_A(x) = -\frac{c_d}{2} \int_{\mathbb{R}^d} (4\psi u_A^2 - u_A \Delta \psi)(y) \, \mathrm{d}y < \infty.$$

To begin with, since the function f is bounded with bounded support, we deduce by dominated convergence that

$$(2.18) \qquad \lim_{x \to \infty} |x|^{d-2} \int_{\mathbb{R}^d} f(y) \, |x - y|^{2-d} \, \mathrm{d}y = \int_{\mathbb{R}^d} f(y) \, \mathrm{d}y < \infty.$$

Let $\varepsilon \in (0, \frac{1}{2})$. Thanks to (2.13), we know that ψu_A^2 is integrable on $\mathbb{R}^d$. Since $y \mapsto \mathbf{1}_{\mathrm{B}(0,\varepsilon|x|)}(y) \, |x|^{d-2} |x - y|^{2-d}$ is bounded uniformly in x and converges as $x \to \infty$ to 1, we deduce by dominated convergence that

$$(2.19) \qquad \lim_{x \to \infty} |x|^{d-2} \int_{\mathrm{B}(0,\varepsilon|x|)} \psi u_A^2(y) |x - y|^{2-d} \, \mathrm{d}y = \int_{\mathbb{R}^d} \psi u_A^2(y) \, \mathrm{d}y < \infty.$$

For $y \notin \mathrm{B}(0, \varepsilon|x|)$, by (2.13), we have

$$\psi u_A^2(y) |x|^{d-2} |x - y|^{2-d} \lesssim |y|^{4-2d} |x|^{d-2} |x - y|^{2-d} =: h(x, y).$$

For $y \in \mathrm{B}(x, \varepsilon|x|)$, we have $h(x, y) \le (1 - \varepsilon)^{4-2d} |x|^{2-d} |x - y|^{2-d}$, hence

$$\int_{\mathrm{B}(x,\varepsilon|x|)} h(x, y) \, \mathrm{d}y \lesssim \varepsilon^2 |x|^{4-d}.$$

For $y \notin \mathrm{B}(x, \varepsilon|x|) \cup \mathrm{B}(0, \varepsilon|x|)$, we also get that $h(x, y) \le \varepsilon^{2-d} |y|^{4-2d}$ and thus

$$\int_{\mathrm{B}(x,\varepsilon|x|)^c \cap \mathrm{B}(0,\varepsilon|x|)^c} h(x, y) \, \mathrm{d}y \lesssim \varepsilon^{6-2d} |x|^{4-d}.$$

From the above we conclude that $\lim_{|x| \to \infty} \int_{\mathrm{B}(0,\varepsilon|x|)^c} h(x, y) \, \mathrm{d}y = 0$. This together with (2.16), (2.18) and (2.19) gives that

$$\lim_{x \to \infty} |x|^{d-2} u_A(x) = -\frac{c_d}{2} \int_{\mathbb{R}^d} (4\psi u_A^2 + f)(y) \, \mathrm{d}y < \infty.$$

Performing an integration by part and using that the derivatives of ψ are zero outside $\overline{\mathrm{B}}(0, 3)$, we deduce that $\int_{\mathbb{R}^d} (\nabla \psi \nabla u_A)(y) \, \mathrm{d}y = -\int_{\mathbb{R}^d} (u_A \Delta \psi)(y) \, \mathrm{d}y$. This gives (2.17) and concludes the proof. $\qquad \square$

Remark 2.6. We now consider low dimension case, that is, $d \in \{1, 2, 3, 4\}$. We obtain by [12, Eq. (4.7)] that for all $r > 0$

$$(2.20) \qquad \lim_{x \to \infty} \varphi_d(|x|) \, u_{\mathrm{B}(0,r)}(x) = 1,$$

where the function φ_d is defined for $t > 1$ by

$$\varphi_d(t) = \begin{cases} \frac{2}{4-d} \, t^2, & 1 \le d \le 3, \\ 2 \, t^2 \log(t), & d = 4. \end{cases}$$

Notice the function φ_d does not depend on r. Let $A \subset \mathbb{R}^d$ be a bounded Borel set with nonempty interior. There exists $x_0 \in A$ and $r_1, r_2 > 0$ such that $\mathrm{B}(x_0, r_1) \subset A \subset \mathrm{B}(x_0, r_2)$. We thus deduce from (2.20) that $\mathcal{N}_x(\mathfrak{R} \cap A \ne \emptyset) = u_A(x)$ satisfies the same asymptotic as $u_{\mathrm{B}(0,r)}(x)$:

$$\lim_{x \to \infty} \varphi_d(|x|) \, u_A(x) = 1.$$

2.4. **Proof of Proposition 1.3.** Recall $d \geq 5$ and $\mathcal{K}$ denotes the collection of compact sets in $\mathbb{R}^d$. We first prove Point 1 of Proposition 1.3 on BScap being a strongly sub-additive Choquet capacity by considering the extension J^* of $J := \mathrm{BScap}|_{\mathcal{K}}$ as in Section 2.1. Notice that for every $K \in \mathcal{K}$, $J(K)$ is finite.

Lemma 2.7. *Let $d \geq 5$. Let J^* denote the extension of J defined by using (2.3) and (2.4). Then J^* is a strongly sub-additive Choquet capacity which coincides with* BScap *on bounded Borel sets.*

Proof. For $K, K' \in \mathcal{K}$ and x large enough, we trivially deduce from (2.7) and inclusion-exclusion principle that $0 \leq u_K(x) \leq u_{K \cup K'}(x)$ and $u_{K \cup K'}(x) + u_{K \cap K'}(x) \leq u_K(x) + u_{K'}(x)$, hence J is non-decreasing and strongly sub-additive.

Set $D_n = \{x \in \mathbb{R}^d : d(x, K) < 1/n\}$ which is open and $K_n = \overline{D_n}$ which is compact. Since the map J is non-decreasing, we deduce that $J(K) \leq J(K_n)$. Moreover, $\mathfrak{R}$ is $\mathcal{N}_x$-a.e. compact, so

$$\mathbf{1}_{\{\mathfrak{R} \cap K \neq \emptyset\}} = \lim_{n \to \infty} \mathbf{1}_{\{\mathfrak{R} \cap K_n \neq \emptyset\}}, \quad \mathcal{N}_x\text{-a.e..}$$

Hence by dominated convergence in (2.7) and (2.17), we have for x large enough

$$\lim_{n \to \infty} u_{K_n}(x) = u_K(x) \quad \text{and then} \quad \lim_{n \to \infty} J(K_n) = J(K).$$

Thus for every $a > J(K)$, we can find n such that $a > J(K_n)$, and any $K' \subset D_n$ must satisfy $J(K') \leq J(K_n) < a$, showing the right continuity of J on $\mathcal{K}$. We conclude by Theorem 2.2 that J^* is a Choquet capacity which coincides with J on $\mathcal{K}$.

We also deduce from Remark 2.1 that J^* is strongly sub-additive.

Finally, we show that J^* and BScap agrees on every bounded Borel set A. Indeed, since

$$\sup_{K \in \mathcal{K}, K \subset A} \mathbf{1}_{\{\mathfrak{R} \cap K \neq \emptyset\}} \leq \mathbf{1}_{\{\mathfrak{R} \cap A \neq \emptyset\}} = \sup_{x \in A} \mathbf{1}_{\{\mathfrak{R} \cap \{x\} \neq \emptyset\}} \leq \sup_{K \in \mathcal{K}, K \subset A} \mathbf{1}_{\{\mathfrak{R} \cap K \neq \emptyset\}},$$

we deduce by dominated convergence as above that for x large enough

$$(2.21) \qquad \sup_{K \in \mathcal{K}, K \subset A} u_K(x) = u_A(x) \quad \text{and then} \quad \sup_{K \in \mathcal{K}, K \subset A} \mathrm{BScap}(K) = \mathrm{BScap}(A).$$

Since all Borel sets are capacitable (with respect to J^*), we also have

$$\sup_{K \in \mathcal{K}, K \subset A} J^*(K) = J^*(A).$$

Hence $J^*|_{\mathcal{K}} = J = \mathrm{BScap}|_{\mathcal{K}}$ implies that J^* agrees with BScap on bounded Borel sets. $\square$

We deduce Point 2 of Proposition 1.3 on the scaling and translation invariance properties for bounded Borel sets from the scaling property of the Brownian snake, see (2.9), and Point 3 on the comparison with the Riesz capacity Cap_{d-4} from (2.10), then we can extend those properties first to bounded sets using (2.4) and general sets using the monotonicity of Choquet capacities.

Point 4 corresponds to the next lemma. For $D \subset \mathbb{R}^d$ open, let $\partial_{\mathrm{ext}} D$ denote its outer boundary, that is the boundary of the unbounded open connected component of $\overline{D}^c$.

Lemma 2.8. *Let $d \geq 5$ and $D \subset \mathbb{R}^d$ be bounded open set such that, for all $y \in \partial_{\mathrm{ext}} D$, the quantity $\liminf_{n \to \infty} 2^{n(d-2)} \mathrm{Cap}_{d-2}(D \cap \mathrm{B}(y, 2^{-n}))$ is positive. Then, we have*

$$\mathrm{BScap}(D) = \mathrm{BScap}(\overline{D}).$$

Proof. In a first step, using the notion of super-regularity from [8], we shall prove that

$$(2.22) \qquad\qquad \mathcal{N}_y(\mathfrak{R} \cap D \neq \emptyset) = \infty \quad \text{for all} \quad y \in \partial_{\mathrm{ext}} D.$$

Let $y \in \partial_{\mathrm{ext}} D$ be fixed. For simplicity, write $\mathrm{B}_n = \mathrm{B}(y, 2^{-n})$ for $n \in \mathbb{N}$ and set

$$(2.23) \qquad c_0 = 4^{-1} \liminf_{n \to \infty} 2^{n(d-2)} \operatorname{Cap}_{d-2}(D \cap \mathrm{B}_n),$$

which is positive by assumption. By scaling, we have $\operatorname{Cap}_{d-2}(\mathrm{B}_n) = 2^{-n(d-2)} \operatorname{Cap}_{d-2}(\mathrm{B}_0)$. Let $k \in \mathbb{N}^*$ be such that $2^{-k(d-2)} \operatorname{Cap}_{d-2}(\mathrm{B}_0) \leq c_0$ and thus, by scaling, $\operatorname{Cap}_{d-2}(\mathrm{B}_{n+k}) \leq c_0 2^{-n(d-2)}$ for all $n \in \mathbb{N}$. We consider the spherical shell $C_{n,k} = \mathrm{B}_n \cap \mathrm{B}_{n+k}^c$. By sub-additivity and scaling of Cap_{d-2}, we have for n large enough

$$\begin{aligned}
\operatorname{Cap}_{d-2}(D \cap C_{n,k}) &\geq \operatorname{Cap}_{d-2}(D \cap \mathrm{B}_n) - \operatorname{Cap}_{d-2}(D \cap \mathrm{B}_{n+k}) \\
&\geq 2c_0\, 2^{-n(d-2)} - \operatorname{Cap}_{d-2}(\mathrm{B}_{n+k}) \\
&\geq c_0\, 2^{-n(d-2)}.
\end{aligned}$$

Since clearly $\operatorname{Cap}_{d-4}(A) \geq \operatorname{Cap}_{d-2}(A)$ for $A \subset \mathrm{B}(0,1)$, we deduce by sub-additivity that:

$$\begin{aligned}
\sum_{n=0}^{\infty} 2^{n(d-2)} \operatorname{Cap}_{d-4}(D \cap C_{n,1}) &\geq \sum_{n=0}^{\infty} 2^{n(d-2)} \operatorname{Cap}_{d-2}(D \cap C_{n,1}) \\
&\geq \sum_{\ell=0}^{\infty} 2^{\ell k(d-2)} \sum_{j=0}^{k-1} \operatorname{Cap}_{d-2}(D \cap C_{\ell k+j,1}) \\
&\geq \sum_{\ell=0}^{\infty} 2^{\ell k(d-2)} \operatorname{Cap}_{d-2}(D \cap C_{\ell k,k}) \\
&= \infty.
\end{aligned}$$

Thanks to [8, Theorem 3], we deduce that y is super-regular for D, and thanks to Proposition 2 therein with the fact that $y \notin D$, we get that (2.22) holds.

Let D' be the unbounded open connected component of $\overline{D}^c$ and $x \in D'$ be fixed. In a second step we shall prove that

$$(2.24) \qquad \mathcal{N}_x(\mathfrak{R} \cap \overline{D} \neq \emptyset) = \mathcal{N}_x(\mathfrak{R} \cap D \neq \emptyset).$$

Using for the inclusion the spatial Markov property of the Brownian snake, see [18, Theorem 2.4], we get that $\mathcal{N}_x$-a.e.

$$\{\mathfrak{R} \cap \overline{D} \neq \emptyset\} = \{\mathfrak{R} \cap D'^c \neq \emptyset\} \subset \{X^{D'} \neq 0\},$$

where $X^{D'}$ is the exit measure of D' for the Brownian snake (see Section 2 in [18] with $\Omega = \mathbb{R}_+ \times D'$). As (2.23) holds, we deduce from [19, Theorem 6.9] (by first considering $D' \cap \mathrm{B}(x,r)$ and then letting r goes to infinity and using that $\mathfrak{R}$ is compact) that

$$(2.25) \qquad \mathcal{N}_x(\mathfrak{R} \cap \overline{D} \neq \emptyset) = \mathcal{N}_x(X^{D'} \neq 0).$$

According to [10, Lemma 2.1][1], we also have that $\mathcal{N}_x$-a.e.

$$(2.26) \qquad \{X^{D'} \neq 0\} \subset \{\mathfrak{R} \cap D \neq \emptyset\}.$$

Since the support of $X^{D'}$ is a subset of $\partial D' = \partial_{\mathrm{ext}} D$, we deduce from (2.22) that $\int \mathcal{N}_y(\mathfrak{R} \cap D \neq \emptyset)\, X^{D'}(\mathrm{d}y) = \infty$ if and only if $X^{D'} \neq 0$. By the spatial Markov property, we get

---

[1]Notice that [10, Lemma 2.1] is stated under $\mathbb{P}_\mu$ with μ a finite measure on $\mathbb{R}^d$; and $\mathbb{P}_\mu$ can be seen as the distribution of the Poisson point measure $\sum_{i \in \mathrm{I}} \delta_{W^i}(\mathrm{d}W)$ of Brownian snakes with intensity $\mu(\mathrm{d}x)\mathcal{N}_x(\mathrm{d}W)$. Using again the spatial Markov property of the Brownian snake under $\mathcal{N}_x$ with the exit measure X^B of B, an open ball centered at x whose closure lies in D, and Lemma 2.1 with $\mu = X^B$, we deduce that (2.26) holds indeed under $\mathcal{N}_x$.

that

$$(2.27) \qquad \mathcal{N}_x(\mathfrak{R} \cap D \neq \emptyset) = \mathcal{N}_x\left(\int \mathcal{N}_y(\mathfrak{R} \cap D \neq \emptyset)\, X^{D'}(\mathrm{d}y) > 0\right) = \mathcal{N}_x(X^{D'} \neq 0).$$

We thus deduce (2.24) from (2.25) and (2.27). Then use the definition of BScap to conclude. $\qquad\square$

To conclude, Point 5 is a direct consequence of the next remark.

Remark 2.9. By (2.9) and symmetry, there is a function u defined on $(1,\infty)$ such that $u_{\mathrm{B}(0,r)}(x) = r^{-2}u(r^{-1}|x|)$. By Lemma 2.5, u is the maximal solution on $(1,\infty)$ of

$$(2.28) \qquad\qquad u''(t) + \frac{(d-1)}{t}\,u'(t) = 4u(t)^2.$$

As in [7, Section 5], we can expand it as $u(t) = t^{2-d}\sum_{n\geq 0} a_n t^{-n(d-4)}$, where

$$a_n = \frac{4}{n\delta(n\delta+1)}(d-2)^{-2}\sum_{k=0}^{n-1} a_k a_{n-k-1} \quad \text{for} \quad n \geq 1, \quad \text{and} \quad \delta = \frac{d-4}{d-2}.$$

Solving (2.28) then reduces to determine $a_0 > 0$, the maximum number so that the series above converges when $t > 1$. In particular, we have $a_0 = \lim_{t\to\infty} t^{d-2}u(t) = \mathrm{BScap}(\mathrm{B}(0,1))$, which is not known in general.

Let us mention that for $d = 6$,

$$(2.29) \qquad\qquad u(t) = \frac{6}{(t^2-1)^2} \quad \text{for} \quad t > 1$$

is explicit (this result seems to be new in the literature), and thus $\mathrm{BScap}(\mathrm{B}(0,1)) = 6$.

3. Convergence of the branching capacity: Proof of Theorem 1.4

To begin with, we recall a comparison result between the discrete range $\mathscr{R}_c$ of the branching random walk indexed by a critical Galton-Watson tree $\mathcal{T}_c$ and the Brownian snake range $\mathfrak{R}$ under the normalized excursion measure $\mathcal{N}^{(1)}$. We define the Hausdorff distance between two nonempty compact sets $K, K' \subset \mathbb{R}^d$ by

$$d_H(K, K') = \max\left(\sup_{x\in K} d(x, K'),\ \sup_{x\in K'} d(x, K')\right).$$

By Janson and Marckert [13, Theorem 2] (see Le Gall and Lin [20, formula (18)] for the statement in present setting), on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$ we may construct a random set $\mathfrak{R}_* \subset \mathbb{R}^d$, distributed as $\mathfrak{R}$ under $\mathcal{N}_0^{(1)}$, and a sequence $(\mathscr{R}_{(n)})_{n\geq 1}$ of random sets in $\mathbb{Z}^d$ such that for any n, $\mathscr{R}_{(n)}$ is distributed as $\mathscr{R}_c$ conditioned on $\{\#\mathcal{T}_c = n\}$ under $\mathbf{P}_0$, and almost surely,

$$(3.1) \qquad\qquad d_H\big(n^{-1/4}\mathscr{R}_{(n)}, M_*\mathfrak{R}_*\big) \to 0 \quad \text{as} \quad n \to \infty,$$

where, with σ and M_θ defined in (1.1) and (1.2),

$$M_* = \left(\frac{2}{\sigma}\right)^{1/2} M_\theta^{1/2}.$$

Now we are ready to prove Theorem 1.4 on admitting Theorem 1.1. Heuristically, we first rephrase branching capacity in terms of hitting probability. Then we show that when the (discrete) branching random walk has diameter of order n, typically there are about n^4 vertices on the tree. Finally, we compare discrete and continuous hitting events (up to the error of an ε-neighborhood) by (3.1).

We recall that $o_n(1)$ denotes a function of n which goes to 0 as n goes to infinity.

Proof of Theorem 1.4 by admitting Theorem 1.1. The proofs of (1.8) and (1.9) need some preparations. At first we may assume for simplicity that the offspring distribution μ is aperiodic, as one can easily adapt the proof line by line for the periodic case. In particular, given aperiodicity, by Dwass [9] and the local central limit theorem (see [15, Theorem 2.3.9]), we have

$$(3.2) \qquad \mathbf{P}(\#\mathcal{T}_c = n) = \left(\frac{1}{\sigma\sqrt{2\pi}} + o_n(1) \right) n^{-3/2}.$$

Secondly, let $K \subset \mathbb{R}^d$ be a given compact set. Let $x \in \mathbb{R}^d\backslash\{0\}$, with $|x|$ large enough, and $\eta \in (0, 1/2)$ be such that, with α and C from Theorem 1.1,

$$(3.3) \qquad K \subset \mathrm{B}(0, \eta'|x|) \quad \text{with} \quad \eta = C\eta'^{\alpha}.$$

By Theorem 1.1, for every $n \geq 1$ we have

$$(3.4) \qquad \frac{\mathbf{P}_{\lfloor nx \rfloor}(\mathscr{R}_c \cap nK \neq \emptyset)}{(1+\eta)g(\lfloor nx \rfloor)} \leq \mathrm{Bcap}(nK) \leq \frac{\mathbf{P}_{\lfloor nx \rfloor}(\mathscr{R}_c \cap nK \neq \emptyset)}{(1-\eta)g(\lfloor nx \rfloor)}.$$

Proof of (1.8). It suffices to show that for any K compact set and $x \neq 0$ satisfying $K \subset \mathrm{B}(0, |x|/2)$ and $\varepsilon > 0$,

$$(3.5) \qquad \limsup_{n\to\infty} n^2\, \mathbf{P}_{\lfloor nx \rfloor}(\mathscr{R}_c \cap nK \neq \emptyset) \leq \frac{1}{\sigma}\mathcal{N}_{M_*^{-1}x}\left(\mathfrak{R} \cap (M_*^{-1}K)^{\varepsilon} \neq \emptyset\right).$$

Indeed, for any $\eta \in (0, \frac{1}{2})$, let $|x|$ be large enough such that (3.3) holds. Applying the upper bound in (3.4) and using the asymptotic of g in (1.3), Equation (3.5) yields that

$$\limsup_{n\to\infty} \frac{\mathrm{Bcap}(nK)}{n^{d-4}} \leq \frac{1+\eta}{(1-\eta)\sigma c_g} \frac{|x|_\theta^{d-2}}{\left|M_*^{-1}x\right|^{d-2}} \left|M_*^{-1}x\right|^{d-2} \mathcal{N}_{M_*^{-1}x}\left(\mathfrak{R} \cap (M_*^{-1}K)^{\varepsilon} \neq \emptyset\right).$$

Recall that $M_* = \left(\frac{2}{\sigma}\right)^{1/2} M_\theta^{1/2}$. Letting $x \to \infty$ and then $\eta \to 0$, we deduce from Theorem 1.2 and the scaling properties of BScap, see (1.5), that

$$(3.6) \qquad \limsup_{n\to\infty} \frac{\mathrm{Bcap}(nK)}{n^{d-4}} \leq \frac{2}{\sigma^2 c_g}\, \mathrm{BScap}(M_\theta^{-1/2}K^{\varepsilon}).$$

Then let $\varepsilon \to 0$ and use the monotone property (iii) of the Choquet capacity to get (1.8).

It remains to show (3.5). For any $\delta \in (0, 1)$, we have

$$\mathbf{P}_{\lfloor nx \rfloor}(\mathscr{R}_c \cap nK \neq \emptyset) = \left(\sum_{j < \delta n^4} + \sum_{\delta n^4 \leq j \leq \delta^{-1}n^4} + \sum_{j > \delta^{-1}n^4} \right) \mathbf{P}_{\lfloor nx \rfloor}(\mathscr{R}_c \cap nK \neq \emptyset, \#\mathcal{T}_c = j)$$
$$(3.7) \qquad =: (3.7)_1 + (3.7)_2 + (3.7)_3.$$

We claim that when δ is small, both $(3.7)_1$ and $(3.7)_3$ are negligible in the sense that they do not contribute to the $\limsup$ term in (3.5). Indeed, by (3.2), we have

$$(3.7)_3 \leq \sum_{j > \delta^{-1}n^4} \mathbf{P}(\#\mathcal{T}_c = j) \lesssim \sum_{j > \delta^{-1}n^4} j^{-3/2} \lesssim \delta^{1/2}n^{-2},$$

yielding that

$$(3.8) \qquad \limsup_{\delta \to 0+}\limsup_{n\to\infty} \left(n^2 \times (3.7)_3\right) = 0.$$

For $(3.7)_1$, we remark that

$$\mathbf{P}_{\lfloor nx \rfloor}(\mathscr{R}_c \cap nK \neq \emptyset, \#\mathcal{T}_c = j) = \mathbf{P}_0(\mathscr{R}_c \cap (nK - \lfloor nx \rfloor) \neq \emptyset, \#\mathcal{T}_c = j).$$

Since $K \subset \mathrm{B}(0, |x|/2)$, for any $y \in nK - \lfloor nx \rfloor$, we have $|y| \geq n|x|/2 - \sqrt{d} > n|x|/3$ for all $n \geq n_0(x)$ and $n_0(x)$ large enough. Therefore, we have

$$
\begin{aligned}
(3.7)_1 \;\; &\leq \;\; \sum_{j < \delta n^4} \mathbf{P}_0\left(\max_{z \in \mathscr{R}_c} |z| \geq \frac{n|x|}{3}, \; \#\mathcal{T}_c = j \right) \\
&\lesssim \;\; \sum_{j=1}^{\delta n^4} j^{-3/2} \mathbf{P}_0\left(\max_{z \in \mathscr{R}_c} |z| \geq \frac{n|x|}{3} \;\bigg|\; \#\mathcal{T}_c = j \right),
\end{aligned}
$$

where the last line follows from (3.2). Using [3, Eq. (4.24)] and Markov's inequality, we see that for all j

$$
\mathbf{P}_0\left(\max_{z \in \mathscr{R}_c} |z| \geq \frac{n|x|}{3} \;\bigg|\; \#\mathcal{T}_c = j \right) \lesssim (n|x|)^{-5} j^{5/4}.
$$

We deduce that

$$
(3.7)_1 \lesssim \sum_{j=1}^{\delta n^4} j^{-3/2} \frac{j^{5/4}}{(n|x|)^5} \lesssim \delta^{3/4} n^{-2} |x|^{-5}.
$$

This implies that

$$
(3.9) \qquad\qquad \limsup_{\delta \to 0+} \limsup_{n \to \infty} \left(n^2 \times (3.7)_1 \right) = 0.
$$

For $(3.7)_2$, by (3.2) again, we have

$$
(3.10) \qquad (3.7)_2 = \left(\frac{1}{\sigma\sqrt{2\pi}} + o_n(1) \right) \sum_{\delta n^4 \leq j \leq \delta^{-1} n^4} j^{-3/2} \mathbf{P}\left(\mathscr{R}_{(j)} \cap (nK - \lfloor nx \rfloor) \neq \emptyset \right).
$$

By (3.1), we have

$$
\max_{\delta n^4 \leq j \leq \delta^{-1} n^4} \mathbf{P}\left(d_H(j^{-1/4} M_*^{-1} \mathscr{R}_{(j)}, \mathfrak{R}_*) > \varepsilon \delta^{1/4}/2 \right) = o_n(1),
$$

where $o_n(1)$ may depend on ε, δ but $o_n(1) \to 0$ as $n \to \infty$, and $\mathbf{P}$ is the coupling law that ensures (3.1). To shorten notations, for $K, K' \subset \mathbb{R}^d$ compact, we write

$$
d_{\min}(K, K') = \min\{|x - x'| : \; x \in K, x' \in K'\}.
$$

Notice that $d_{\min}$ is not a distance. We will use that for $K, K', K'' \in \mathcal{K}$, $x \in \mathbb{R}^d$ and $\varepsilon \geq 0$

$$
\begin{aligned}
&d_{\min}(K, K') \leq \varepsilon \iff K^\varepsilon \cap K' \neq \emptyset \iff K \cap K'^\varepsilon \neq \emptyset, \\
&d_{\min}(K, K' + x) \leq d_{\min}(K', K') + |x|, \\
&d_{\min}(K, K'') \leq d_{\min}(K', K'') + d_H(K, K').
\end{aligned}
$$

Note that for all large n and $\delta n^4 \leq j \leq \delta^{-1} n^4$,

$$
\begin{aligned}
&\mathbf{P}\left(\mathscr{R}_{(j)} \cap (nK - \lfloor nx \rfloor) \neq \emptyset \right) \\
&\quad = \mathbf{P}\left(d_{\min}(M_*^{-1} \mathscr{R}_{(j)}, M_*^{-1}(nK - \lfloor nx \rfloor)) = 0 \right) \\
&\quad \leq \mathbf{P}\left(d_{\min}(n^{-1} j^{1/4} \mathfrak{R}_*, M_*^{-1}(K - x)) \leq \varepsilon \right) + \mathbf{P}\left(d_H(M_*^{-1} \mathscr{R}_{(j)}, j^{1/4} \mathfrak{R}_*) > \varepsilon(\delta j)^{1/4}/2 \right) \\
&\quad = \mathbf{P}\left(d_{\min}(n^{-1} j^{1/4} \mathfrak{R}_*, M_*^{-1}(K - x)) \leq \varepsilon \right) + o_n(1),
\end{aligned}
$$

where in the first inequality we have used that, for $R_j := M_*^{-1} \mathscr{R}_{(j)}$, $K' := M_*^{-1}(nK - \lfloor nx \rfloor)$ and $R_* := j^{1/4} \mathfrak{R}_*$, we have

$$
d_{\min}(R_j, K') = 0 \implies d_{\min}(R_*, K') \leq d_H(R_j, R_*),
$$

and thus on $\{d_{\min}(M_*^{-1}\mathscr{R}_{(j)}, M_*^{-1}(nK - \lfloor nx \rfloor)) = 0\}$, we have for $\varepsilon n \geq 2\sqrt{d}$

$$d_H(M_*^{-1}\mathscr{R}_{(j)}, j^{1/4}\mathfrak{R}_*) \leq \varepsilon(\delta j)^{1/4}/2 \implies d_{\min}(j^{1/4}\mathfrak{R}_*, M_*^{-1}(nK - \lfloor nx \rfloor)) \leq \varepsilon(\delta j)^{1/4}/2$$
$$\implies d_{\min}(n^{-1}j^{1/4}\mathfrak{R}_*, M_*^{-1}(K - x)) \leq \varepsilon.$$

In the same way, we have

$$\mathbf{P}\big(\mathscr{R}_{(j)} \cap (nK^{2\varepsilon} - \lfloor nx \rfloor) \neq \emptyset\big) = \mathbf{P}\big(d_{\min}(M_*^{-1}\mathscr{R}_{(j)}, M_*^{-1}(nK - \lfloor nx \rfloor)) \leq 2\varepsilon\big)$$
$$\geq \mathbf{P}\big(d_{\min}(n^{-1}j^{1/4}\mathfrak{R}_*, M_*^{-1}(K - x)) \leq \varepsilon\big)$$
$$- \mathbf{P}\big(d_H(M_*^{-1}\mathscr{R}_{(j)}, j^{1/4}\mathfrak{R}_*) > \varepsilon(\delta j)^{1/4}/2\big)$$
$$(3.11) \qquad\qquad \geq \mathbf{P}\big(d_{\min}(n^{-1}j^{1/4}\mathfrak{R}_*, M_*^{-1}(K - x)) \leq \varepsilon\big) + o_n(1).$$

We will use (3.11) in the proof of (1.9). Going back to (3.10), we obtain that

$$(3.7)_2 \leq \frac{1}{\sigma\sqrt{2\pi}} \sum_{\delta n^4 \leq j \leq \delta^{-1}n^4} j^{-3/2} \mathbf{P}\big(d_{\min}(n^{-1}j^{1/4}\mathfrak{R}_*, M_*^{-1}(K - x)) \leq \varepsilon\big) + o(n^{-2})$$
$$\leq \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty t^{-3/2} \mathbf{P}\big(d_{\min}(n^{-1}t^{1/4}\mathfrak{R}_*, M_*^{-1}(K - x)) \leq \varepsilon\big) \, dt + o(n^{-2}),$$

where in the last inequality we have replaced the sum over j by the integral over t by monotonicity. Using a change of variables $t = n^4 s$ and that $\mathfrak{R}_*$ under $\mathbf{P}$ is distributed as $\mathfrak{R}$ under the normalized excursion measure $\mathcal{N}_0^{(1)}$, we get that

$$\limsup_{n\to\infty} \big(n^2 \times (3.7)_2\big) \leq \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty s^{-3/2} \mathcal{N}_0^{(1)}\big(d_{\min}(s^{1/4}\mathfrak{R}, M_*^{-1}(K - x)) \leq \varepsilon\big) ds$$
$$(3.12) \qquad\qquad = \frac{1}{\sigma} \mathcal{N}_0(d_{\min}(\mathfrak{R}, M_*^{-1}(K - x)) \leq \varepsilon)$$
$$= \frac{1}{\sigma} \mathcal{N}_{M_*^{-1}x}(\mathfrak{R} \cap (M_*^{-1}K)^\varepsilon \neq \emptyset),$$

where (3.12) is due to (2.8). This completes the proof of (3.5), and then of (1.8).

Proof of (1.9). The proof proceeds in a similar way to that of (1.8). We are going to show that for any K compact set and $x \neq 0$ satisfying $K \subset \mathrm{B}(0, |x|/2)$ and any $\varepsilon > 0$,

$$(3.13) \qquad \liminf_{n\to\infty} n^2 \mathbf{P}_{\lfloor nx \rfloor}\big(\mathscr{R}_c \cap nK^{2\varepsilon} \neq \emptyset\big) \geq \frac{1}{\sigma} \mathcal{N}_{M_*^{-1}x}\big(\mathfrak{R} \cap (M_*^{-1}K) \neq \emptyset\big).$$

Indeed, given (3.13), we let $x \to \infty$ and successively use (3.4), (1.3) and Theorem 1.2 to obtain that

$$(3.14) \qquad\qquad \liminf_{n\to\infty} \frac{\mathrm{Bcap}(nK^{2\varepsilon})}{n^{d-4}} \geq \frac{2}{\sigma^2 c_g} \mathrm{BScap}(M_\theta^{-1/2}K).$$

Let $D \subset \mathbb{R}^d$ be open. For any compact $K \subset D$, there exists $\varepsilon = \varepsilon(D, K) > 0$ such that $K^{2\varepsilon} \subset D$. Then, the lower bound (3.14) implies that

$$\liminf_{n\to\infty} \frac{\mathrm{Bcap}(nD)}{n^{d-4}} \geq \frac{2}{\sigma^2 c_g} \mathrm{BScap}(M_\theta^{-1/2}K).$$

Since BScap is a Choquet capacity and since the Borel sets are capacitable, we obtain (1.9) as a consequence of (2.2).

It remains to prove (3.13). Let $\delta \in (0, 1/2)$. By (3.2), (3.7) and (3.11), we deduce that

$$
\mathbf{P}_{\lfloor nx \rfloor}\big(\mathscr{R}_c \cap nK^{2\varepsilon} \neq \emptyset\big)
$$

$$
\geq \left(\frac{1}{\sigma\sqrt{2\pi}} + o(1)\right) \sum_{\delta n^4 \leq j \leq \delta^{-1} n^4} j^{-3/2} \mathbf{P}\big(\mathscr{R}_{(j)} \cap (nK^{2\varepsilon} - \lfloor nx \rfloor) \neq \emptyset\big)
$$

$$
\geq \frac{1}{\sigma\sqrt{2\pi}} \sum_{\delta n^4 \leq j \leq \delta^{-1} n^4} j^{-3/2} \mathbf{P}\big(d_{\min}(n^{-1} j^{1/4} \mathfrak{R}_*, M_*^{-1}(K - x)) \leq \varepsilon\big) + o(n^{-2})
$$

$$
\geq \frac{1}{\sigma\sqrt{2\pi}} \int_{\delta n^4 + 1}^{\delta^{-1} n^4} t^{-3/2} \mathbf{P}\big(d_{\min}(n^{-1} t^{1/4} \mathfrak{R}_*, M_*^{-1}(K - x)) \leq \varepsilon\big) \, \mathrm{d}t + o(n^{-2}),
$$

where for the last inequality we have used the monotonicity on t of the probability term. Using the change of variables $t = n^4 s$ and that $\mathfrak{R}_*$ under $\mathbf{P}$ is distributed as $\mathfrak{R}$ under the normalized excursion measure $\mathcal{N}_0^{(1)}$, we get that

$$
\liminf_{n \to \infty} n^2 \, \mathbf{P}_{\lfloor nx \rfloor}\big(\mathscr{R}_c \cap nK^{2\varepsilon} \neq \emptyset\big)
$$

$$
\geq \frac{1}{\sigma\sqrt{2\pi}} \int_{\delta}^{\delta^{-1}} s^{-3/2} \mathcal{N}_0^{(1)}\big(d_{\min}(s^{1/4} \mathfrak{R}, M_*^{-1}(K - x)) \leq \varepsilon\big) \, \mathrm{d}s.
$$

We let δ goes to 0 and apply the monotone convergence theorem and then (3.12). Then we get (3.13) which completes the proof of (1.9). □

4. Hitting probabilities and the branching capacity: Proof of Theorem 1.1

4.1. **Preliminaries.** The material of this subsection is derived from Zhu [25]. We introduce an adjoint Galton-Watson tree which is a random tree derived from the critical Galton-Watson tree $\mathcal{T}_c$, with the only modification being made at the root. In the adjoint tree $\mathcal{T}_{\mathrm{adj}}$ the root has offspring distribution $\widetilde{\mu} = (\widetilde{\mu}(k))_{k \geq 0}$ instead of μ, with

$$
\widetilde{\mu}(k) = \sum_{j=k+1}^{\infty} \mu(j),
$$

while all other vertices retain the original offspring distribution μ.

We then construct an infinite tree $\mathcal{T}_{\mathrm{I}}$ as follows. We begin by constructing a semi-infinite branch $\{\varnothing_0, \varnothing_1, ..., \varnothing_n, ...\}$ rooted at $\varnothing_0$, then we root a tree $\mathcal{T}_{\mathrm{adj}}^i$ on each $\varnothing_i$, $i \geq 0$, where $(\mathcal{T}_{\mathrm{adj}}^i)_{i \geq 0}$ are independent trees distributed as $\mathcal{T}_{\mathrm{adj}}$. To every finite or infinite tree $\mathcal{T}_\alpha$, with $\alpha \in \{\mathrm{adj}, \mathrm{I}\}$, we define the random walk V_α indexed by $\mathcal{T}_\alpha$ in the same way as the random walk V_c is indexed by $\mathcal{T}_c$. Under $\mathbf{P}_x$ those BRW are started from $x \in \mathbb{Z}^d$. Furthermore, we consider the sub-tree $\mathcal{T}_-$ of $\mathcal{T}_{\mathrm{I}}$ with root $\varnothing_1$ defined by $\mathcal{T}_- = \mathcal{T}_{\mathrm{I}} \backslash \mathcal{T}_{\mathrm{adj}}^0$, as shown in Fig. 1. We also consider the restriction of the BRW V_{I} to $\mathcal{T}_-$ which we denote by V_-.

We write $\mathscr{R}_\alpha$ the range of V_α for $\alpha \in \{\mathrm{adj}, \mathrm{I}, -\}$; this is consistent with the notation $\mathscr{R}_c$.

Recall $d \geq 5$. Let $K \subset \mathbb{Z}^d$ be a finite and nonempty set. We shall consider the hitting probabilities from $x \in \mathbb{Z}^d$ by

$$
(4.1) \qquad \mathbf{p}_K^{(\alpha)}(x) := \mathbf{P}_x(\mathscr{R}_\alpha \cap K \neq \emptyset) \quad \text{for} \quad \alpha \in \{c, \mathrm{I}, \mathrm{adj}, -\}.
$$

Notice that under $\mathbf{P}_x$, the BRW V_- is distributed as V_{I} started at S_1 and thus

$$
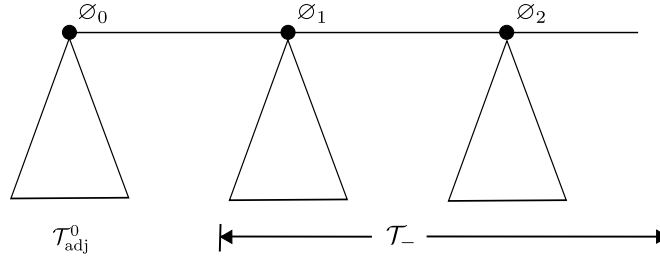(4.2) \qquad \mathbf{p}_K^{(-)}(x) = \mathbf{E}_x[\mathbf{p}_K^{(\mathrm{I})}(S_1)],
$$

FIGURE 1. An illustration of $\mathcal{T}_I = \mathcal{T}_- \cup \mathcal{T}_{adj}^0$.

where we recall that S_1 is one step of a random walk, distributed as θ. We shall consider the the escape probabilities of K by V_-

$$\mathbf{e}_K(x) := 1 - \mathbf{p}_K^{(-)}(x).$$

We deduce from Benjamini and Curien [5, Theorem 0.1] that $\mathbf{e}_K$ is not identically zero on K. It is also shown in Zhu [25, Eq. (1.1)] that

$$(4.3) \qquad \mathrm{Bcap}(K) = \sum_{a \in K} \mathbf{e}_K(a).$$

Using the monotonicity and the translation invariance of Bcap, we deduce that for any $a \in K$,

$$(4.4) \qquad \mathrm{Bcap}(K) \geq \mathrm{Bcap}(\{a\}) = \mathrm{Bcap}(\{0\}) > 0.$$

We also recall that there exists some positive constant C, only depending on d, such that for all $r \geq 1$ and $K \subset \mathrm{B}(0, r)$, we have

$$(4.5) \qquad \mathrm{Bcap}(K) \leq \mathrm{Bcap}(\mathrm{B}(0, r)) \leq C \, r^{d-4}.$$

We remark that for any $x \notin K$,

$$(4.6) \qquad \mathbf{p}_K^{(I)}(x) = \mathbf{p}_K^{(adj)}(x) + (1 - \mathbf{p}_K^{(adj)}(x))\mathbf{p}_K^{(-)}(x).$$

Moreover, these hitting probabilities are strongly related to a random walk (S^κ) with killing rate $\mathbf{p}_K^{(adj)}$. Denote by ∂ a cemetery point, and let (S^κ) be a random walk on $\mathbb{Z}^d \cup \{\partial\}$ such that for every $x \in \mathbb{Z}^d$,

$$(4.7) \qquad \mathbf{P}(S_{n+1}^\kappa = y | S_n^\kappa = x) = \begin{cases} \mathbf{p}_K^{(adj)}(x), & \text{if } y = \partial, \\ (1 - \mathbf{p}_K^{(adj)}(x))\theta(y - x), & \text{if } y \in \mathbb{Z}^d. \end{cases}$$

The Green function of S^κ is defined by

$$(4.8) \qquad G_K(x, y) := \sum_{n=0}^\infty \mathbf{P}_x(S_n^\kappa = y) = \sum_{n=0}^\infty \mathbf{E}_x\Big[\mathbf{1}_{\{S_n = y\}} \prod_{i=0}^{n-1}(1 - \mathbf{p}_K^{(adj)}(S_i))\Big].$$

By (4.6), we get that

$$(4.9) \qquad \mathbf{p}_K^{(I)}(x) = \sum_{n=0}^\infty \mathbf{E}_x\Big[\mathbf{p}_K^{(adj)}(S_n) \prod_{i=0}^{n-1}(1 - \mathbf{p}_K^{(adj)}(S_i))\Big],$$

with the convention $\prod_\emptyset := 1$. It is then immediate from (4.9) and (4.8) that

$$(4.10) \qquad \mathbf{p}_K^{(I)}(x) = \mathbf{P}_x((S^\kappa) \text{ survives}) = \sum_{y \in \mathbb{Z}^d} G_K(x, y)\mathbf{p}_K^{(adj)}(y).$$

Moreover, an important relationship between $\mathbf{p}_K^{(c)}$ and G_K has been obtained in Zhu [25, Eq. (5.4) and (5.5)]:

$$(4.11) \qquad \mathbf{p}_K^{(c)}(x) = \sum_{a \in K} G_K(x, a) = \sum_{n=0}^{\infty} \mathbf{P}_x(S_n^\kappa \in K) \quad \text{for} \quad x \in \mathbb{Z}^d.$$

We summarize some comparisons between the probabilities $\mathbf{p}_K^{(\alpha)}(x)$ for $\alpha \in \{c, I, \text{adj}, -\}$. For $x \in K$, we note that $\mathbf{p}_K^{(\alpha)}(x) = 1$ for $\alpha \in \{c, I, \text{adj}\}$, whereas $\mathbf{p}_K^{(-)}(x) \leq 1$.

Lemma 4.1. *Let $d \geq 5$ and $K \subset \mathbb{Z}^d$ be a finite nonempty set. Under (1.1) and (1.2), for any $x \in \mathbb{Z}^d \backslash K$, we have*

$$(4.12) \qquad \frac{2(1 - \mu(0))}{\sigma^2} \mathbf{p}_K^{(\text{adj})}(x) \leq \mathbf{p}_K^{(c)}(x) \leq \frac{1}{\mu(0)} \mathbf{p}_K^{(\text{adj})}(x),$$

$$(4.13) \qquad \mathbf{p}_K^{(c)}(x) \leq \mathbf{p}_K^{(I)}(x) \quad and \quad \mathbf{p}_K^{(\text{adj})}(x) \leq \frac{\sigma^2}{2} \mathbf{p}_K^{(-)}(x),$$

$$(4.14) \qquad \mathbf{p}_K^{(-)}(x) \leq \mathbf{p}_K^{(I)}(x) \leq \left(\frac{\sigma^2}{2} + 1 \right) \mathbf{p}_K^{(-)}(x).$$

Proof. The claim (4.12) follows from the construction of V_{adj}, see Zhu [25, Eq. (8.6)]. The first inequality of (4.13) is a consequence of (4.10) and (4.11). We deduce then from (4.2) that $\mathbf{E}_x[\mathbf{p}_K^{(c)}(S_1)] \leq \mathbf{E}_x[\mathbf{p}_K^{(I)}(S_1)] = \mathbf{p}_K^{(-)}(x)$. To obtain the second inequality of (4.13), use [25, Eq. (8.4)] to get for $x \notin K$ that, with $k_\varnothing$ the number of children for the root of $\mathcal{T}_{\text{adj}}$ and, for $i \in \{1, \ldots, k_\varnothing\}$, $\mathscr{R}^i$ the range of the BRW V_{adj} restricted to the descendants of the i-th child of the root,

$$\mathbf{p}_K^{(\text{adj})}(x) = \mathbf{P}_x(\mathscr{R}^i \cap K \neq \emptyset \quad \text{for } i \leq k_\varnothing) \leq \sum_{i \geq 0} i \widetilde{\mu}(i) \, \mathbf{E}_x[\mathbf{p}_K^{(c)}(S_1)] \leq \frac{\sigma^2}{2} \mathbf{p}_K^{(-)}(x).$$

Lastly, the inequalities (4.14) are a direct consequence of (4.6) and (4.13). □

We shall consider the first exit times from $B \subset \mathbb{Z}^d$ by the θ-random walk (S_n) and (S_n^κ)

$$(4.15) \qquad \tau_S(B) := \inf\{n \geq 1 : S_n \notin B\} \quad \text{and} \quad \tau_{S^\kappa}(B) := \inf\{n \geq 1 : S_n^\kappa \notin B\}.$$

Then, we define the harmonic measure of (S^κ) with respect to every nonempty set $B \subset \mathbb{Z}^d$ by, for $x, y \in \mathbb{Z}^d$

$$(4.16) \qquad H_K^B(x, y) := \sum_{n=0}^{\infty} \mathbf{P}_x\left(S_n^\kappa = y, \tau_{S^\kappa}(B) \geq n\right)$$

$$(4.17) \qquad = \sum_{n=0}^{\infty} \mathbf{E}_x\left[\prod_{i=0}^{n-1} (1 - \mathbf{p}_K^{(\text{adj})}(S_i)), S_n = y, \tau_S(B) \geq n \right].$$

We note that neither x nor y needs to be in B. We also note that for $x \in B$ and $y \notin B$

$$(4.18) \qquad H_K^B(x, y) = \mathbf{P}_x(S_{\tau_{S^\kappa}(B)}^\kappa = y) = \mathbf{E}_x\left[\prod_{i=0}^{\tau_S(B)-1} (1 - \mathbf{p}_K^{(\text{adj})}(S_i)), S_{\tau_S(B)} = y \right].$$

By the Markov property of S^κ, we easily obtain the following first entrance and the last exit decomposition (see [25, Lemma 2.1]): for any $x \in B, y \notin B$,

$$(4.19) \qquad G_K(x, y) = \sum_{z \notin B} H_K^B(x, z) G_K(z, y) = \sum_{z \in B} G_K(x, z) H_K^{B^c}(z, y),$$

$$(4.20) \qquad G_K(y, x) = \sum_{z \in B} H_K^{B^c}(y, z) G_K(z, x) = \sum_{z \notin B} G_K(y, z) H_K^B(z, x).$$

4.2. **Hitting probabilities and branching capacity.** We provide a comparison result between the hitting probabilities of a finite set K by the branching random walks and the branching capacity.

The following result is given in [25, Theorem 1.3 and Eq. (8.6)], establishing

Proposition 4.2. *Let* $d \geq 5$, $\lambda > 1$. *Under* (1.1) *and* (1.2), *uniformly in* $r \geq 1$, $K \subset \mathbb{Z}^d \cap \mathrm{B}(0, r)$ *nonempty* , $x \in \mathbb{Z}^d$ *with* $|x| \geq \lambda r$, *we have*

$$
\mathbf{p}_K^{(c)}(x) \asymp \mathbf{p}_K^{(\mathrm{adj})}(x) \asymp |x|^{2-d} \mathrm{Bcap}(K), \tag{4.21}
$$

$$
\mathbf{p}_K^{(-)}(x) \asymp \mathbf{p}_K^{(I)}(x) \asymp |x|^{4-d} \mathrm{Bcap}(K). \tag{4.22}
$$

Equation (4.21) is given in Zhu [25, Theorem 1.3 and Eq. (8.6)] and Equation (4.22) is also given in Zhu [26] when the displacements given by θ are bounded. Notice that the first part of (4.21) and (4.22) are given by (4.12) and (4.14) for $x \notin K$.

So we are left to the proof of the second part of (4.22). Under the general case (1.2), one direction of the proof of (4.22) is easy.

Lemma 4.3. *Let* $d \geq 5$. *Under* (1.1) *and* (1.2), *uniformly in* $K \subset \mathbb{Z}^d$ *finite and nonempty, and* $x \in \mathbb{Z}^d$, *we have*

$$
\mathbf{p}_K^{(I)}(x) \asymp \sum_{y \in \mathbb{Z}^d} G_K(x, y) \mathbf{p}_K^{(c)}(y). \tag{4.23}
$$

Moreover, for any $\lambda > 1$, *uniformly in* $r \geq 1$, $K \subset \mathrm{B}(0, r)$ *nonempty and* $|x| \geq \lambda r$, *we have*

$$
\mathbf{p}_K^{(I)}(x) \gtrsim |x|^{4-d} \mathrm{Bcap}(K). \tag{4.24}
$$

Proof. First, (4.23) follows immediately from (4.10) and (4.12). By [25, Lemma 12.3], uniformly in $K \subset \mathrm{B}(0, r)$ and $|x|, |y| \geq \lambda r$, we get, as $|x|_\theta \asymp |x|$, that

$$
G_K(x, y) \asymp |x - y|^{2-d}. \tag{4.25}
$$

Then by (4.10) and (4.21), we obtain that when $|x| \geq \lambda r$,

$$
\mathbf{p}_K^{(I)}(x) \geq \sum_{2|x| \leq |y| \leq 3|x|} G_K(x, y) \mathbf{p}_K^{(\mathrm{adj})}(y)
$$

$$
\asymp \sum_{2|x| \leq |y| \leq 3|x|} |x - y|^{2-d} |y|^{2-d} \mathrm{Bcap}(K)
$$

$$
\asymp |x|^{4-d} \mathrm{Bcap}(K), \tag{4.26}
$$

proving (4.24). □

The other direction of the proof of (4.22) will be presented at the end of this section after some preliminary results. We first cite a result from [26], and note that the proof provided there, under the condition that the displacements given by θ are bounded, can be adapted to our case word by word.

Lemma 4.4. [26, Eq. (4.1)] *Let* $d \geq 5$. *Under* (1.1) *and* (1.2), *uniformly in* $K \subset \mathbb{Z}^d$ *finite and nonempty,* $x \in \mathbb{Z}^d$ *and* $B \subset \mathbb{Z}^d$,

$$
\sum_{y \in B} G_K(x, y) \mathbf{p}_K^{(I)}(y) \lesssim (\mathrm{diam}(B) + 1)^2 \, \mathbf{p}_K^{(I)}(x).
$$

We set

$$
L_n^\kappa(B) := \sum_{i=0}^n \mathbf{1}_{\{S_i^\kappa \in B\}}.
$$

By the Markov property, we get that for any $B \subset \mathbb{Z}^d$,

$$\sum_{y \in B} G_K(x, y) \mathbf{p}_K^{(c)}(y) = \sum_{y \in B} \sum_{i=0}^{\infty} \mathbf{P}_x(S_i^\kappa = y) \sum_{j=0}^{\infty} \mathbf{P}_y(S_j^\kappa \in K)$$

$$(4.27) \qquad\qquad\qquad = \sum_{n=0}^{\infty} \mathbf{E}_x \left[L_n^\kappa(B) \, \mathbf{1}_{\{S_n^\kappa \in K\}} \right].$$

In particular, for $B = \mathbb{Z}^d$ in (4.27), we obtain

$$(4.28) \qquad \sum_{y \in \mathbb{Z}^d} G_K(x, y) \mathbf{p}_K^{(c)}(y) = \sum_{n=0}^{\infty} (n + 1) \mathbf{P}_x \left(S_n^\kappa \in K \right).$$

Lemma 4.5. *Let $d \geq 5$. Under (1.1) and (1.2), uniformly in $K \subset \mathbb{Z}^d$ finite and nonempty, $x \in \mathbb{Z}^d$ and $B \subset \mathbb{Z}^d$,*

$$(4.29) \qquad \sum_{y \in B} G_K(x, y) \mathbf{p}_K^{(I)}(y) \gtrsim \sum_{n=0}^{\infty} \mathbf{E}_x \left[L_n^\kappa(B)^2 \mathbf{1}_{\{S_n^\kappa \in K\}} \right].$$

Proof. Using successively (4.23), (4.11) and the Markov property, we get that

$$\sum_{y \in B} G_K(x, y) \mathbf{p}_K^{(I)}(y) \asymp \sum_{y \in B} G_K(x, y) \sum_{z \in \mathbb{Z}^d} G_K(y, z) \mathbf{p}_K^{(c)}(z)$$

$$= \sum_{y \in B} G_K(x, y) \sum_{z \in \mathbb{Z}^d} G_K(y, z) \sum_{\ell=0}^{\infty} \mathbf{P}_z(S_\ell^\kappa \in K)$$

$$= \sum_{y \in B} \sum_{i,j,\ell=0}^{\infty} \mathbf{P}_x \left(S_i^\kappa = y, S_{i+j+\ell}^\kappa \in K \right)$$

$$(4.30) \qquad\qquad = \sum_{n=0}^{\infty} \sum_{i=0}^{n} (n - i + 1) \mathbf{P}_x \left(S_i^\kappa \in B, S_n^\kappa \in K \right).$$

Note that

$$\sum_{i=0}^{n} (n - i + 1) \mathbf{1}_{\{S_i^\kappa \in B\}} = \sum_{j=0}^{n} \sum_{i=0}^{j} \mathbf{1}_{\{S_i^\kappa \in B\}} \geq \frac{1}{2} \sum_{j=0}^{n} \sum_{i=0}^{n} \mathbf{1}_{\{S_j^\kappa \in B\}} \mathbf{1}_{\{S_i^\kappa \in B\}} = \frac{1}{2} L_n^\kappa(B)^2.$$

Then (4.29) follows from (4.30). $\qquad \square$

Corollary 4.6. *Let $d \geq 5$. Under (1.1) and (1.2), uniformly in $K \subset \mathbb{Z}^d$ finite and nonempty, $x \in \mathbb{Z}^d$ and $B \subset \mathbb{Z}^d$,*

$$\sum_{y \in B} G_K(x, y) \mathbf{p}_K^{(c)}(y) \lesssim (1 + \operatorname{diam}(B)) \sqrt{\mathbf{p}_K^{(I)}(x) \, \mathbf{p}_K^{(c)}(x)}.$$

Proof. By (4.27) and the Cauchy-Schwarz inequality,

$$\sum_{y \in B} G_K(x, y) \mathbf{p}_K^{(c)}(y) = \sum_{n=0}^{\infty} \mathbf{E}_x \left[L_n^\kappa(B) \, \mathbf{1}_{\{S_n^\kappa \in K\}} \right]$$

$$\leq \sqrt{\sum_{n=0}^{\infty} \mathbf{E}_x \left[L_n^\kappa(B)^2 \, \mathbf{1}_{\{S_n^\kappa \in K\}} \right]} \sqrt{\sum_{n=0}^{\infty} \mathbf{P}_x(S_n^\kappa \in K)}.$$

We conclude by (4.11) and Lemmas 4.4 and 4.5. $\qquad \square$

Lemma 4.7. *Let $d \geq 5$, $\lambda > 1$. Under (1.1) and (1.2), uniformly in $r \geq 1$, $K \subset \mathrm{B}(0, r)$ and $|x| \geq \lambda r$, we have*

$$(4.31) \qquad \mathbf{p}_K^{(I)}(x) \lesssim |x|^{4-d}\mathrm{Bcap}(K).$$

Proof. Let $B := \mathrm{B}(0, \lambda|x|)$. By (4.21) and the fact that $G_K(x, y) \leq g(x, y)$, we get

$$\sum_{y \notin B} G_K(x, y)\mathbf{p}_K^{(c)}(y) \quad \lesssim \quad \sum_{y \notin B} |y|^{2-d}|y|^{2-d}\mathrm{Bcap}(K)$$

$$(4.32) \qquad\qquad\qquad\qquad \asymp \quad |x|^{4-d}\mathrm{Bcap}(K).$$

Then by (4.23) and Corollay 4.6, we get that

$$\mathbf{p}_K^{(I)}(x) \quad \lesssim \quad \sum_{y \in B} G_K(x, y)\mathbf{p}_K^{(c)}(y) + |x|^{4-d}\mathrm{Bcap}(K)$$

$$\lesssim \quad (1 + \mathrm{diam}(B))\sqrt{\mathbf{p}_K^{(I)}(x)\,\mathbf{p}_K^{(c)}(x)} + \sum_{y \notin B} G_K(x, y)\mathbf{p}_K^{(c)}(y)$$

$$\lesssim \quad |x|\sqrt{\mathbf{p}_K^{(I)}(x)\,\mathbf{p}_K^{(c)}(x)} + |x|^{4-d}\mathrm{Bcap}(K).$$

By (4.21), $\mathbf{p}_K^{(c)}(x) \lesssim |x|^{2-d}\mathrm{Bcap}(K)$. The conclusion follows easily. $\qquad \square$

Proof of Proposition 4.2. Recall we only need to prove the second part of (4.22). Now, this is a direct consequence of (4.24) and Lemma 4.7. $\qquad \square$

4.3. **Proof of Theorem 1.1.** In this section, we prove Theorem 1.1 following the the same ideas as [26, Proposition 2.4, Lemma 2.5] and [25, Lemma 6.1, Lemma 7.1], while keeping track of the error terms at each step to obtain the exact asymptotic.

We give below an expression of the branching capacity $\mathrm{Bcap}(K)$ using $\mathbf{p}_K^{(-)}$ and the harmonic measure H_K^B. In the proof, we shall us that the first exit time $\tau_S(B)$, of B for (S_n), see by (4.15), is a.s. finite.

Lemma 4.8. *Let $d \geq 5$. Let $K \subset B \subset \mathbb{Z}^d$ be nonempty finite sets. We have*

$$\mathrm{Bcap}(K) = \sum_{a \in K}\sum_{b \notin B} H_K^B(b, a)\,\mathbf{e}_K(b).$$

Proof. Let $a \in K$. By construction, we have

$$\mathbf{e}_K(a) \quad = \quad \mathbf{E}_a\Big[\prod_{i=1}^{\infty}(1 - \mathbf{p}_K^{(\mathrm{adj})}(S_i))\Big]$$

$$(4.33) \qquad\qquad = \quad \sum_{b \notin B}\mathbf{E}_a\Big[\prod_{i=1}^{\tau_S(B)}(1 - \mathbf{p}_K^{(\mathrm{adj})}(S_i)), S_{\tau_S(B)} = b\Big]\,\mathbf{e}_K(b),$$

where we used the strong Markov property of S for the last equality. For $b \notin B$, we have

$$(4.34) \quad \mathbf{E}_a \Big[\prod_{i=1}^{\tau_S(B)} (1 - \mathbf{p}_K^{(\mathrm{adj})}(S_i)), S_{\tau_S(B)} = b \Big]$$

$$= \sum_{n=1}^{\infty} \mathbf{E}_a \Big[\prod_{i=1}^{n} (1 - \mathbf{p}_K^{(\mathrm{adj})}(S_i)), \tau_S(B) = n, S_n = b \Big]$$

$$= \sum_{n=1}^{\infty} \mathbf{E}_b \Big[\prod_{i=0}^{n-1} (1 - \mathbf{p}_K^{(\mathrm{adj})}(S_i)), \tau_S(B) \geq n, S_n = a \Big]$$

$$= H_K^B(b, a),$$

where we used the symmetry of θ for the second equality and (4.17) for the last. Then use (4.3) and (4.33) to conclude. $\qquad\square$

Lemma 4.9. *Let $d \geq 5$. Let $K \subset B$ be two finite nonempty subsets of $\mathbb{Z}^d$ and $x \in B$. We have*
$$\sum_{a \notin B} H_K^B(x, a) \geq (1 - \mathbf{p}_K^{(\mathrm{adj})}(x)) \mathbf{e}_K(x).$$

Proof. Recall the construction of $\mathcal{T}_I$ using the trees $(\mathcal{T}_{\mathrm{adj}}^i)_{i \geq 0}$. We denote by $\mathscr{R}_{\mathrm{adj}}^i$ the range of the restriction of the BRW V_I to $\mathcal{T}_{\mathrm{adj}}^i$. Thanks to (4.6), we have for $x \in B$

$$(1 - \mathbf{p}_K^{(\mathrm{adj})}(x)) \, \mathbf{e}_K(x) = 1 - \mathbf{p}_K^{(I)}(x)$$

$$\leq \mathbf{P}_x((\cup_{0 \leq i < \tau_S(B)} \mathscr{R}_{\mathrm{adj}}^i) \cap K = \emptyset)$$

$$= \sum_{a \notin B} \mathbf{P}_x((\cup_{0 \leq i < \tau_S(B)} \mathscr{R}_{\mathrm{adj}}^i) \cap K = \emptyset, S_{\tau_S(B)} = a)$$

$$= \sum_{a \notin B} \mathbf{E}_x \Big[\prod_{i=0}^{\tau_S(B)-1} (1 - \mathbf{p}_K^{(\mathrm{adj})}(S_i)), S_{\tau_S(B)} = a \Big]$$

$$= \sum_{a \notin B} H_K^B(x, a),$$

where the last equality follows from (4.18) and the fact that $x \in B$. This ends the proof. $\qquad\square$

The main ingredient in the proof of Theorem 1.1 is the following estimate on the Green function:

Lemma 4.10. *Let $d \geq 5$. Under (1.1) and (1.2), there exists some constants $\alpha = \alpha(d), c = c(d) > 0$ such that uniformly in $r \geq 1$, $K \subset B(0, r) \cap \mathbb{Z}^d$ nonempty, and $x, y \in \mathbb{Z}^d \backslash K$, we have*

$$(4.35) \qquad 0 \leq 1 - \frac{G_K(x, y)}{g(x, y)} \leq c\, s^{-\alpha} \quad \text{with} \quad s = \frac{1}{r} \min\{|x|, |y|\}.$$

Proof. By definition $G_K(x, y) \leq g(x, y)$, see (4.8), so it is enough to show that the upper bound (4.35) holds for all $s \geq s_0$ where $s_0 = s_0(d) > 1$ denotes a large constant whose value only depends on d.

By using (4.21) with $\lambda = s_0$ there and the fact $\mathrm{Bcap}(K) \lesssim r^{d-4}$ (see (4.5)), we get that $\mathbf{p}_K^{(\mathrm{adj})}(x) \lesssim s^{2-d} r^{-2} \leq s^{2-d}$; the same holds for $\mathbf{p}_K^{(\mathrm{adj})}(y)$. Since θ is symmetric, we have

$$\frac{G_K(x, y)}{1 - \mathbf{p}_K^{(\mathrm{adj})}(x)} = \frac{G_K(y, x)}{1 - \mathbf{p}_K^{(\mathrm{adj})}(y)},$$
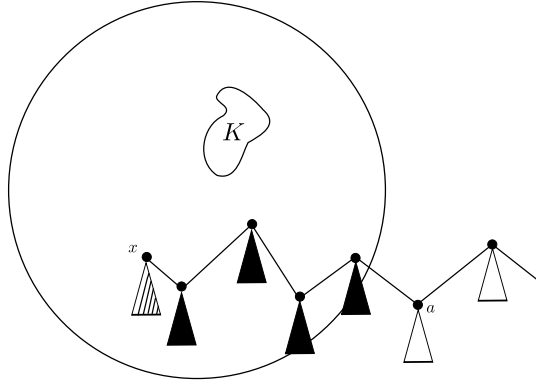
FIGURE 2. In this picture, we have a random walk starting at x and exiting B at the point a. On each point we attach an adjoint branching random walk, illustrated with the small triangles. The quantity $\sum_{a \notin B} H_K^B(x,a)$ is the probability that the shaded and solid triangles do not hit K, $\mathbf{e}_K(x) = 1 - \mathbf{p}_K^{(-)}(x) = \mathbf{P}_x(\mathscr{R}_- \cap K = \emptyset)$ is the probability that the solid and hollow triangles do not hit K, $1 - \mathbf{p}_K^{(\mathrm{adj})}(x) = \mathbf{P}_x(\mathscr{R}_{\mathrm{adj}} \cap K = \emptyset)$ is the probability that the shaded triangle does not hit K, and $1 - \mathbf{p}_K^{(I)}(x) = \mathbf{P}_x(\mathscr{R}_I \cap K = \emptyset)$ is the probability that no triangle hits K.

so that uniformly in r, K, x, y such that $s \geq s_0$, we have $G_K(y,x) = G_K(x,y)(1+O(s^{2-d}))$. Therefore, provided $\alpha \leq d - 2$, it is enough to prove the lemma for $|x| \leq |y|$.

So we assume that $|x| \leq |y|$ and thus $rs = |x|$. Moreover, let $\upsilon = \upsilon(d) \in (0, \frac{d-4}{d})$ be a small constant whose value will be fixed later, see Remark 4.11. We distinguish two cases depending on $|x|$ and $|y|$:

$$(4.36) \qquad\qquad |x| \leq |y| \leq |x| s^\upsilon,$$

and

$$(4.37) \qquad\qquad |y| > |x| s^\upsilon.$$

Case (4.36): $|x| \leq |y| \leq |x| s^\upsilon$. See the picture on the left in Fig. 3. We first modify the Green function $g(x,y)$ of the random walk S (without killing). For simplicity we write

$$B := \mathrm{B}(0, |x| s^{-\upsilon}) \quad \text{and} \quad T := \tau_S(B^c) = \inf\{n \geq 1 : S_n \in B\}$$

the first exit time of B^c, see (4.15). Let

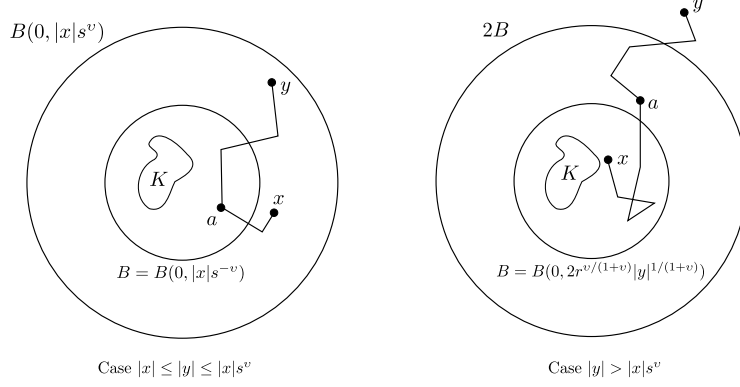$$(4.38) \qquad\qquad \widetilde{g}(x,y) := \sum_{n=0}^{s^\upsilon(1+|x-y|)^2} \mathbf{P}_x(S_n = y, T > n).$$

We claim that

$$(4.39) \qquad\qquad 0 \leq 1 - \frac{\widetilde{g}(x,y)}{g(x,y)} = O(s^{-\upsilon(d-2)/2}).$$

Indeed, we have

$$g(x,y) - \widetilde{g}(x,y) \;\leq\; \sum_{n \geq s^\upsilon(1+|x-y|)^2} \mathbf{P}_x(S_n = y) + \sum_{n=0}^{\infty} \mathbf{P}_x(S_n = y, T \leq n)$$

$$(4.40) \qquad\qquad =: \quad (4.40)_1 + (4.40)_2.$$

By the local limit theorem for the random walk S (see [15, Proposition 2.4.4]), we have $\mathbf{P}_x(S_n = y) \lesssim n^{-d/2}$ uniformly in $x, y \in \mathbb{Z}^d$. This gives that

$$(4.41) \qquad (4.40)_1 \lesssim s^{-\upsilon(d-2)/2}(1 + |x-y|)^{2-d} \lesssim s^{-\upsilon(d-2)/2} g(x,y).$$

FIGURE 3. Relative positions of a, x, y and B in the proof of Lemma 4.10.

For $(4.40)_2$, we use the Markov property at T to get that

$$(4.42) \qquad (4.40)_2 = \sum_{a \in B} \sum_{j=0}^{\infty} \mathbf{P}_x(T = j, S_j = a)g(a, y) \le \max_{a \in B} g(a, y) \, \mathbf{P}_x(T < \infty).$$

Since g is the Green function, $(g(S_n))_{n \in \mathbb{N}}$ is a supermartingale. Thus as $x \ne 0$, we get using (1.3) that

$$g(x) \ge \mathbf{E}_x[\mathbf{1}_{\{T < \infty\}} g(S_T)] \ge \mathbf{P}_x(T < \infty) \min_{z \in B} g(z) \gtrsim \mathbf{P}_x(T < \infty)|x|^{2-d} s^{-v(2-d)},$$

hence

$$(4.43) \qquad \mathbf{P}_x(T < \infty) \lesssim s^{-v(d-2)}.$$

Note that $\min_{a \in B} |a - y| \ge |y| - |x|s^{-v} \ge |y|(1 - s^{-v})$, hence $\min_{a \in B} \frac{|a-y|}{1+|x-y|} \ge \frac{1}{3}$, for all $s \ge s_0$ and s_0 large enough (depending only on v). This implies that $\max_{a \in B} g(a, y) \lesssim g(x, y)$, and therefore

$$(4.44) \qquad (4.40)_2 \lesssim s^{-v(d-2)} g(x, y).$$

This and (4.41) imply the claim in (4.39).

Now we go back to the proof of the upper bound (4.35). By (4.8), we have

$$G_K(x, y) \ge \sum_{n=0}^{s^v(1+|x-y|)^2} \mathbf{E}_x \Big[\prod_{i=0}^{n-1} (1 - \mathbf{p}_K^{(\text{adj})}(S_i)), S_n = y, T > n \Big].$$

By (4.5) and (4.21), we deduce that $\mathbf{p}_K^{(\text{adj})}(S_i) \lesssim r^{d-4}|S_i|^{2-d} \le r^{d-4}|x|^{2-d}s^{v(d-2)}$ for any $i < T$. Since $|x - y| \le 2|y| \le 2|x|s^v$, there are some positive constants C, C' (depending only on d) such that for all $n \le s^v(1 + |x - y|)^2$ and $s \ge s_0$, on $\{T > n\}$,

$$\prod_{i=0}^{n-1} (1 - \mathbf{p}_K^{(\text{adj})}(S_i)) \ge (1 - Cr^{d-4}|x|^{2-d}s^{v(d-2)})^{s^v(1+|x-y|)^2} \ge 1 - C's^{-(d-4)+v(d+1)}.$$

Using (4.39), we see that

$$(4.45) \qquad \begin{aligned} G_K(x, y) &\ge (1 - C's^{-(d-4)+v(d+1)}) \, \widetilde{g}(x, y) \\ &\ge (1 - C''s^{-(d-4)+v(d+1)} - C''s^{-v(d-2)/2})g(x, y). \end{aligned}$$

This proves (4.35) in the case (4.36) with α at most equal to

$$\alpha_1 := \min(d - 4 - v(d + 1), v(d - 2)/2).$$

Case (4.37): $|y| > |x|s^v$. See the picture on the right in Fig. 3. We choose $R := 2r^{v/(1+v)}|y|^{1/(1+v)}$ and let $B := B(0, R)$, $2B = B(0, 2R)$. Note that $R \geq 2rs = 2|x|$. By taking $s \geq s_0$ large enough (depending only on v), we have for any $a \in (2B) \backslash B$ that

$$|a| \leq |y| \leq |a| \left(\frac{|a|}{r} \right)^v.$$

This means that we may apply the already proved case (4.36) to a and y, and get from (4.45) that

$$G_K(a, y) \geq (1 - c's^{-\alpha_1})g(a, y) \geq (1 - c''s^{-\alpha_1 \wedge v})g(x, y), \qquad \forall a \in (2B) \backslash B,$$

where the last inequality follows from (1.3) and the fact that $\frac{|y|}{|x|} \geq 2\frac{|y|}{|a|} \geq \frac{|y|}{R} \geq \frac{1}{2}s^v$ so that $|x - y|_\theta = |a - y|_\theta(1 + O(s^{-v}))$ and $|x - y|^{-1} + |a - y|^{-1} \lesssim s^{-(v+1)}$. Thus using (4.19), we get

$$G_K(x, y) = \sum_{a \notin B} H_K^B(x, a)G_K(a, y)$$

(4.46)
$$\geq \sum_{a \in (2B) \backslash B} H_K^B(x, a)G_K(a, y)$$

$$\geq (1 - c''s^{-\alpha_1 \wedge v})g(x, y) \sum_{a \in (2B) \backslash B} H_K^B(x, a).$$

The proof of (4.40) will be complete once we show the existence of constants $\alpha_2 > 0$ and $c > 0$ such that

(4.47)
$$\sum_{a \in (2B) \backslash B} H_K^B(x, a) \geq 1 - cs^{-\alpha_2}.$$

To get (4.47), we first deduce from Lemma 4.9 that for all large $s \geq s_0$,

$$\sum_{a \notin B} H_K^B(x, a) \geq (1 - \mathbf{p}_K^{(\text{adj})}(x))(1 - \mathbf{p}_K^{(-)}(x))$$

$$\geq \left(1 - C\frac{r^{d-4}}{(rs)^{d-2}} \right)\left(1 - C\frac{r^{d-4}}{(rs)^{d-4}} \right)$$

(4.48)
$$\geq 1 - C's^{-(d-4)},$$

where the second inequality follows from (4.21), (4.22) and (4.4). Finally, by definition of H_K^B in (4.16) and (1.2), we get

$$\sum_{a \notin (2B)} H_K^B(x, a) \leq \sum_{a \notin (2B)} \sum_{z \in B} \sum_{n=1}^{\infty} \mathbf{P}_x\left(S_n^\kappa = a, S_{n-1}^\kappa = z \right)$$

$$\leq \sum_{z \in B} \sum_{a \notin (2B)} g(x, z)\theta(a - z)$$

$$\lesssim R^{-(d-2)}$$

$$\leq s^{-(d-2)}.$$

Together with (4.48), this imply (4.47) with $\alpha_2 := d - 4$ there. This completes the proof of Lemma 4.10. $\square$

Remark 4.11. In the proof of Lemma 4.10, by (4.45), (4.46) and (4.47), we need

$$\alpha \leq \min(\alpha_1, v, \alpha_2) = \min(d - 4 - v(d + 1), v(d - 2)/2, v, d - 4).$$

Hence the optimal choice for α in Lemma 4.10 is

$$\alpha = \upsilon = \frac{d-4}{d+2}.$$

Now we are entitled to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $s := \frac{|x|}{r} \geq \lambda$ and fix a constant $\upsilon \in (0, \frac{d-4}{d})$. By (4.21), we may assume without loss of generality that $s \geq s_0$ with a large constant s_0 which only depends on λ and d. Let $R := |x|s^{-\upsilon} = rs^{1-\upsilon}$ and $B := \mathrm{B}(0, R)$. Notice that $R < |x|$ and $K \subset B$. By (4.11) and (4.20), we get

$$\mathbf{P}_x(\mathscr{R}_c \cap K \neq \emptyset) = \mathbf{p}_K^{(c)}(x) = \sum_{a \in K} G_K(x, a)$$

$$= \sum_{a \in K} \sum_{b \notin B} G_K(x, b) H_K^B(b, a)$$

(4.49)
$$= (1 + O(s^{-(1-\upsilon)(d-4)/(d+2)})) \sum_{a \in K} \sum_{b \notin B} g(x, b) H_K^B(b, a),$$

using Lemma 4.10 (with s replaced by $s^{1-\upsilon}$) and Remark 4.11 for the last equality.

By (4.22), uniformly in $b \notin B$,

$$\mathbf{p}_K^{(-)}(b) \lesssim \frac{\mathrm{Bcap}(K)}{R^{d-4}} \lesssim \left(\frac{r}{R}\right)^{d-4} = s^{-(1-\upsilon)(d-4)},$$

which in view of Lemma 4.8 yields that

(4.50)
$$1 + O\left(s^{-(1-\upsilon)(d-4)}\right) = \frac{\mathrm{Bcap}(K)}{\sum_{a \in K} \sum_{b \notin B} H_K^B(b, a)} \leq 1.$$

Comparing (4.50) with (4.49), the proof of Theorem 1.1 reduces to show that

(4.51)
$$\sum_{a \in K} \sum_{b \notin B} g(x, b) H_K^B(b, a) = (1 + O(s^{-\alpha_3})) g(x) \sum_{a \in K} \sum_{b \notin B} H_K^B(b, a),$$

for some constant $\alpha_3 > 0$.

To prove (4.51), we decompose both sides of it into the sums over $b \in (2B) \setminus B$ and $b \notin (2B)$, with $2B = \mathrm{B}(0, 2R)$, and show that those over $b \notin (2B)$ are negligible. Indeed, by (1.3), uniformly in x, r and $b \in (2B) \setminus B$, $g(x, b) = g(x)(1 + O(s^{-\upsilon}))$ as $\upsilon \leq 1$. Then, we have

(4.52)
$$\sum_{a \in K} \sum_{b \in (2B) \setminus B} g(x, b) H_K^B(b, a) = g(x)(1 + O(s^{-\upsilon})) \sum_{a \in K} \sum_{b \in (2B) \setminus B} H_K^B(b, a).$$

Recall that for any $a \in K$, $\mathrm{Bcap}(K) \geq \mathrm{Bcap}(\{a\}) = \mathrm{Bcap}(\{0\}) > 0$. By (4.50) and (4.52), (4.51) follows once we establish the following two claims:

(4.53)
$$\sum_{a \in K} \sum_{b \notin (2B)} H_K^B(b, a) = O(s^{-\alpha_3}),$$

(4.54)
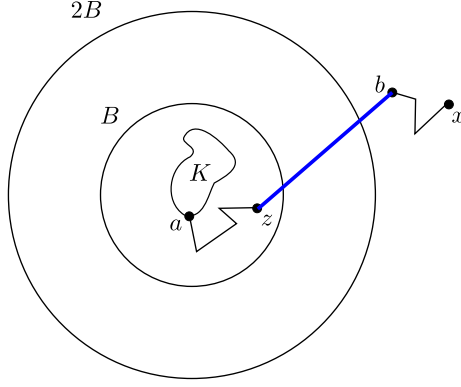$$\sum_{a \in K} \sum_{b \notin (2B)} g(x, b) H_K^B(b, a) = O(s^{-\alpha_3}) g(x).$$

FIGURE 4. An illustration for the case $b \notin (2B)$. Note that by definition of $H_K^B(b,a)$ in (4.16), S_1^κ makes a large jump (in blue) from $b \notin (2B)$ to some point $z \in B$, so that $H_K^B(b,a) \leq \sum_{z \in B} \theta(z-b) G_K(z,a)$.

Let $b \notin (2B)$. To begin with, as illustrated in Fig. 4, we deduce from the definition of $H_K^B(b,a)$ in (4.16) and (4.11) that

$$\sum_{a \in K} H_K^B(b,a) \leq \sum_{a \in K} \sum_{n=1}^{\infty} \mathbf{P}_b \Big(S_n^\kappa = a, S_1^\kappa \in B \Big)$$

$$= \sum_{a \in K} \sum_{z \in B} \sum_{n=1}^{\infty} \mathbf{P}_b(S_1^\kappa = z) \, \mathbf{P}_z(S_{n-1}^\kappa = a)$$

$$(4.55) \qquad\qquad \leq \sum_{z \in B} \theta(z-b) \, \mathbf{p}_K^{(c)}(z).$$

This gives

$$(4.56) \qquad\qquad \sum_{a \in K} \sum_{b \notin (2B)} H_K^B(b,a) \leq \sum_{z \in B} \sum_{b \notin (2B)} \theta(z-b) \mathbf{p}_K^{(c)}(z).$$

We are going to show that

$$(4.57) \qquad\qquad \sum_{z \in B} \sum_{b \notin (2B)} \theta(z-b) \mathbf{p}_K^{(c)}(z) \leq O(s^{-\alpha_4}),$$

for some $\alpha_4 \leq \alpha_3$, which in view of (4.56) yields (4.53).

To show (4.57), we use (1.2) to see that uniformly in $z \in B$,

$$(4.58) \qquad\qquad \sum_{b \notin (2B)} \theta(z-b) \lesssim R^{-d}.$$

It remains to show that

$$(4.59) \qquad\qquad R^{-d} \sum_{z \in B} \mathbf{p}_K^{(c)}(z) = O(s^{-\alpha_4}).$$

To this end, we decompose $\sum_{z \in B}$ in (4.59) into $\sum_{z \in B(0, rs^{(1-v)/2})}$ and $\sum_{z \in B \setminus B(0, rs^{(1-v)/2})}$. For $|z| \geq rs^{(1-v)/2}$, we use Proposition 4.2 to see that $\mathbf{p}_K^{(c)}(z) \lesssim |z|^{2-d} \mathrm{Bcap}(K)$. Since $K \subset B(0,r)$ and $\mathrm{Bcap}(B(0,r)) \lesssim r^{d-4}$, we get that

$$\sum_{z \in B \setminus B(0, rs^{(1-v)/2})} \mathbf{p}_K^{(c)}(z) \lesssim r^{d-4} \sum_{z \in B \setminus B(0, rs^{(1-v)/2})} |z|^{2-d} \lesssim r^{d-4} R^2 = R^{d-2} s^{-(d-4)(1-v)}.$$

For the sum $\sum_{z \in B(0, rs^{(1-\upsilon)/2})}$, we bound $\mathbf{p}_K^{(c)}(z)$ by 1 and get that

$$\sum_{z \in B(0, rs^{(1-\upsilon)/2})} \mathbf{p}_K^{(c)}(z) \lesssim r^d s^{d(1-\upsilon)/2} = R^d s^{-d(1-\upsilon)/2},$$

proving (4.59), hence (4.57). The proof of (4.53) is complete with $\alpha_3 \leq \alpha_4 := \min(d/2, (d-4)(1-\upsilon))$.

We now prove (4.54) in a similar way. By (4.55), we have

$$\sum_{a \in K} \sum_{b \notin (2B)} g(x, b) H_K^B(b, a) \leq \sum_{z \in B} \sum_{b \notin (2B)} g(x, b) \theta(z - b) \mathbf{p}_K^{(c)}(z).$$

Let $\eta > 0$ be small (depending only on d). We decompose the above sum $\sum_{b \notin (2B)}$ into $\sum_{b \notin (2B), |b-x| \leq |x|s^{-\eta}}$ and $\sum_{b \notin (2B), |b-x| > |x|s^{-\eta}}$. For $|b - x| > |x|s^{-\eta}$, we have $g(x, b) \lesssim |x|^{2-d} s^{\eta(d-2)} \lesssim g(x) s^{\eta(d-2)}$. This implies that

$$\sum_{\substack{z \in B \\ |b-x| > |x|s^{-\eta}}} \sum_{b \notin (2B)} g(x, b) \theta(z - b) \mathbf{p}_K^{(c)}(z) \lesssim g(x) s^{\eta(d-2)} \sum_{z \in B} \sum_{b \notin (2B)} \theta(z - b) \mathbf{p}_K^{(c)}(z)$$

$$(4.60) \qquad\qquad\qquad\qquad\qquad\qquad \lesssim g(x) s^{\eta(d-2)-\alpha_4},$$

where the last inequality is due to (4.57).

For the sum $\sum_{b \notin (2B), |b-x| \leq |x|s^{-\eta}}$, we bound $\mathbf{p}_K^{(c)}(z)$ by 1 and use $g(x, b) \lesssim (1 + |x - b|)^{2-d}$. Then we have

$$\sum_{\substack{z \in B \\ |b-x| \leq |x|s^{-\eta}}} \sum_{b \notin (2B)} g(x, b) \theta(z - b) \mathbf{p}_K^{(c)}(z) \lesssim \sum_{z \in B} \sum_{\substack{b \notin (2B) \\ |b-x| \leq |x|s^{-\eta}}} (1 + |x - b|)^{2-d} \theta(z - b).$$

Note that for any $|b - x| \leq |x|s^{-\eta}$ and $z \in B$, we have $|z - b| \geq |x|/2$ (assuming $s_0^{-\upsilon} + s_0^{-\eta} \leq 1/2$), hence $\sum_{z \in B} \theta(z - b) \leq \mathbf{P}_0(|S_1| \geq |x|/2) \lesssim |x|^{-d}$ by (1.2). It follows that

$$\sum_{\substack{z \in B \\ |b-x| \leq |x|s^{-\eta}}} \sum_{b \notin (2B)} g(x, b) \theta(z - b) \mathbf{p}_K^{(c)}(z) \lesssim |x|^{-d} \sum_{|b-x| \leq |x|s^{-\eta}} (1 + |x - b|)^{2-d}$$

$$\lesssim |x|^{-d} \left(|x|s^{-\eta} \right)^2$$

$$\lesssim g(x) s^{-2\eta}.$$

Let $\eta := \min(d/2, (d-4))(1-\upsilon)/d$. This combined with (4.60) yields (4.54) with

$$(4.61) \qquad\qquad \alpha_3 := \min(\alpha_4 - \eta(d-2), 2\eta).$$

This completes the proof of Theorem 1.1. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

Remark 4.12. By (4.49), (4.52), (4.57) and (4.61), we have

$$\alpha \leq \min \left(\frac{(d-4)(1-\upsilon)}{d+2}, (d-4)(1-\upsilon), \upsilon, (1-\upsilon), \frac{2(d-4)(1-\upsilon)}{d} \right)$$

for some $0 < \upsilon < \frac{d-4}{d}$ (independent from the parameter υ in Remark 4.11). The optimal choice is

$$\alpha = \upsilon = \frac{d-4}{2(d-1)}.$$

4.4. **Consequences of Theorem 1.1.** As a by-product, we can approximate $\mathrm{Bcap}(K)$ in terms of $\mathbf{p}_K^{(\alpha)}(x)$ for $\alpha \in \{\mathrm{I}, \mathrm{adj}, -\}$ with the same precision as in Theorem 1.1. These approximations hold under an additional third-moment assumption on the offspring distribution μ.

Proposition 4.13. *Let* $d \geq 5$ *and* $\lambda > 1$. *Assume* (1.1), (1.2) *and that* μ *has a finite third moment. There exists a positive constant* $C = C(d, \lambda)$ *such that, uniformly in* $r \geq 1$, $K \subset \mathrm{B}(0, r) \cap \mathbb{Z}^d$ *nonempty,* $x \in \mathbb{Z}^d$ *such that* $|x| \geq \lambda r$, *we have, with* α *as in Theorem 1.1,*

$$\left| \frac{\mathbf{p}_K^{(\mathrm{adj})}(x)}{g(x)} - \frac{\sigma^2}{2} \mathrm{Bcap}(K) \right| \leq C \left(\frac{r}{|x|} \right)^\alpha \mathrm{Bcap}(K).$$

Proof. Let $s := \frac{|x|}{r} \geq \lambda$. By (4.21), we may assume without loss of generality that $s \geq s_0$ with some large constant s_0 which only depends on d and λ. Let $\widetilde{X}$ be distributed according to $\widetilde{\mu}$, that is, the number of children of the root in $\mathcal{T}_{\mathrm{adj}}$. Since μ has finite third moment we deduce that $\mathbf{E}[\widetilde{X}^2] \leq \sum_{k \in \mathbb{N}} k^3 \mu(k)$ is finite. By definition of V_{adj}, we have

$$\mathbf{p}_K^{(\mathrm{adj})}(x) = 1 - \mathbf{E}\left[(1 - \mathbf{E}_x[\mathbf{p}_K^{(c)}(S_1)])^{\widetilde{X}} \right].$$

Since $1 - \widetilde{X}t \leq (1 - t)^{\widetilde{X}} \leq 1 - \widetilde{X}t + \widetilde{X}(\widetilde{X} - 1)t^2$ for $t \in [0, 1]$, we deduce that

$$(4.62) \qquad 0 \leq \mathbf{E}[\widetilde{X}] - \frac{\mathbf{p}_K^{(\mathrm{adj})}(x)}{\mathbf{E}_x[\mathbf{p}_K^{(c)}(S_1)]} \leq \mathbf{E}[\widetilde{X}^2]\, \mathbf{E}_x[\mathbf{p}_K^{(c)}(S_1)], \qquad \forall x \in \mathbb{Z}^d.$$

Since $\mathbf{E}[\widetilde{X}] = \sigma^2/2$ and $g(x) = O(s^{2-d})$, the proof is complete once we prove that for $|x| = rs$ and some $\alpha' < d - 2$

$$(4.63) \qquad \mathbf{E}_x[\mathbf{p}_K^{(c)}(S_1)] = g(x)\mathrm{Bcap}(K)\left(1 + O(s^{-\alpha'}) \right).$$

We decompose the left hand side according to $|S_1 - x|$ less or larger than $s^{-\upsilon}|x|$, with $\upsilon \in (0, 1)$. By (1.2), we get

$$(4.64) \qquad \mathbf{E}_x[\mathbf{p}_K^{(c)}(S_1)\mathbf{1}_{\{|S_1-x|>s^{-\upsilon}|x|\}}] \leq \mathbf{P}_x(|S_1 - x| > s^{-\upsilon}|x|) \lesssim s^{\upsilon d}|x|^{-d} \lesssim s^{\upsilon d - 2} g(x).$$

Under $\mathbf{P}_x$, on $\{|S_1 - x| \leq s^{-\upsilon}|x|\}$, we have $|S_1| \geq (1 - s^{-\upsilon})|x| \geq \lambda' r$ for some $\lambda' > 1$. Using Theorem 1.1 and $\mathbf{E}_x[g(S_1)] = g(x)$ for $x \neq 0$, we get

$$\left| \frac{\mathbf{E}_x[\mathbf{p}_K^{(c)}(S_1)\mathbf{1}_{\{|S_1-x|\leq s^{-\upsilon}|x|\}}]}{\mathrm{Bcap}(K)} - g(x) \right|$$

$$= \left| \frac{\mathbf{E}_x[\mathbf{p}_K^{(c)}(S_1)\mathbf{1}_{\{|S_1-x|\leq s^{-\upsilon}|x|\}}]}{\mathrm{Bcap}(K)} - \mathbf{E}_x[g(S_1)] \right|$$

$$\lesssim \mathbf{E}_x[g(S_1)\mathbf{1}_{\{|S_1-x|>s^{-\upsilon}|x|\}}] + \mathbf{E}_x\left[g(S_1)\left(\frac{r}{|S_1|} \right)^\alpha \mathbf{1}_{\{|S_1-x|\leq s^{-\upsilon}|x|\}} \right]$$

$$\lesssim \mathbf{P}_x(|S_1 - x| > s^{-\upsilon}|x|) + s^{-\alpha} g(x)$$

$$\lesssim s^{-\alpha'} g(x),$$

where we used (4.64) and $\alpha' \leq \min(2 - \upsilon d, \alpha)$ for the last equality. Taking, υ such that $2 - \upsilon d = \alpha$, we deduce that (4.63) holds with $\alpha' = \alpha$. $\qquad \square$

Proposition 4.14. *Let* $d \geq 5$ *and* $\lambda > 1$. *Assume* (1.1), (1.2) *and that* μ *has a finite third moment. There exists a positive constant* $C = C(d, \lambda)$ *such that, uniformly in* $r \geq 1$, $K \subset \mathrm{B}(0, r) \cap \mathbb{Z}^d$ *nonempty,* $x \in \mathbb{Z}^d$ *such that* $|x| \geq \lambda r$, *we have, with* α *as in Theorem 1.1,*

$$(4.65) \quad \left| \frac{\mathbf{p}_K^{(I)}(x)}{G(x)\mathrm{Bcap}(K)} - \frac{\sigma^2}{2} \right| \leq C \left(\frac{r}{|x|} \right)^\alpha \quad and \quad \left| \frac{\mathbf{p}_K^{(-)}(x)}{G(x)\mathrm{Bcap}(K)} - \frac{\sigma^2}{2} \right| \leq C \left(\frac{r}{|x|} \right)^\alpha,$$

where

$$G(x) = \sum_{y \in \mathbb{Z}^d} g(x,y)g(y) \asymp (1+|x|)^{4-d}.$$

We mention Schapira [23, Lemma 2.5] for an estimate on $\mathbf{p}_K^{(I)}(x)$ similar to (4.65), in the case when θ is uniformly distributed among the $2d$ unit vectors in $\mathbb{Z}^d$.

Proof. Let $s := \frac{|x|}{r} \geq \lambda$. By (4.22), we may assume without loss of generality that $s \geq s_0$ with some large constant s_0 which only depends on d and λ. Let $\lambda' := (1+\lambda)/2 > 1$. By (4.10), we have

$$\mathbf{p}_K^{(I)}(x) = \sum_{y \in \mathbb{Z}^d} G_K(x,y)\mathbf{p}_K^{(\mathrm{adj})}(y)$$

$$= \sum_{y \in \mathrm{B}(0,\lambda'r)} G_K(x,y)\mathbf{p}_K^{(\mathrm{adj})}(y) + \sum_{y \notin \mathrm{B}(0,\lambda'r)} G_K(x,y)\mathbf{p}_K^{(\mathrm{adj})}(y).$$

Using successivley (4.12), Corollary 4.6, (4.21) and (4.31), we have

$$\sum_{y \in \mathrm{B}(0,\lambda'r)} G_K(x,y)\mathbf{p}_K^{(\mathrm{adj})}(y) \lesssim \sum_{y \in \mathrm{B}(0,\lambda'r)} G_K(x,y)\mathbf{p}_K^{(c)}(y)$$

$$\lesssim r\sqrt{\mathbf{p}_K^{(c)}(x)\mathbf{p}_K^{(I)}(x)}$$

$$\lesssim r|x|^{3-d}\mathrm{Bcap}(K) = s^{-1}|x|^{4-d}\mathrm{Bcap}(K).$$

For the sum $\sum_{y \notin \mathrm{B}(0,\lambda'r)}$, we deduce from Lemma 4.10 and Proposition 4.13 that

$$\sum_{y \notin \mathrm{B}(0,\lambda'r)} G_K(x,y)\mathbf{p}_K^{(\mathrm{adj})}(y) = (1 + O(s^{-\alpha}))\frac{\sigma^2}{2}\mathrm{Bcap}(K) \sum_{y \notin \mathrm{B}(0,r)} g(x,y)g(y)$$

$$= (1 + O(s^{-\alpha} + s^{-2}))\frac{\sigma^2}{2}\mathrm{Bcap}(K)G(x),$$

where we used that

$$G(x) - \sum_{y \notin \mathrm{B}(0,r)} g(x,y)g(y) = \sum_{y \in \mathrm{B}(0,r)} g(x,y)g(y) \lesssim |x|^{2-d} \sum_{y \in \mathrm{B}(0,r)} g(y) \lesssim |x|^{4-d}s^{-2}.$$

This proves the first part of (4.65).

Finally, by (4.6) and the fact that $\frac{\mathbf{p}_K^{(\mathrm{adj})}(x)}{\mathbf{p}_K^{(I)}(x)} \lesssim |x|^{-2}$, $\mathbf{p}_K^{(\mathrm{adj})}(x) \lesssim g(x)\mathrm{Bcap}(K) \lesssim |x|^{2-d}r^{d-4}$, we immediately conclude the second part of (4.65) from its first part. $\square$

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