

CENTRAL LIMIT THEOREM FOR BIFURCATING MARKOV CHAINS UNDER POINTWISE ERGODIC CONDITIONS

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Bifurcating Markov chains (BMC) are Markov chains indexed by a full binary tree representing the evolution of a trait along a population where each individual has two children. We provide a central limit theorem for general additive functionals of BMC, and prove the existence of three regimes. This corresponds to a competition between the reproducing rate (each individual has two children) and the ergodicity rate for the evolution of the trait. This is in contrast with the work of Guyon (*Ann. Appl. Probab.* **17** (2007) 1538–1569), where the considered additive functionals are sums of martingale increments, and only one regime appears. Our result can be seen as a discrete time version, but with general trait evolution, of results in the time continuous setting of branching particle system from Adamczak and Miłoś (*Electron. J. Probab.* **20** (2015) 42), where the evolution of the trait is given by an Ornstein–Uhlenbeck process.

1. Introduction. Bifurcating Markov chains are a class of stochastic processes indexed by a regular binary tree and which satisfy the branching Markov property (see below for a precise definition). This model represents the evolution of a trait along a population where each individual has two children. To the best of our knowledge, the term bifurcating Markov chain (BMC) appears for the first time in the work of Basawa and Zhou [4]. But, it was Guyon who, in [16], highlighted and developed a theory of asymmetric bifurcating Markov chains. Since the works of Guyon, BMC theory has been enriched from a probabilistic and statistical point of view and several extensions and models using BMC have been studied; we can cite the works (see also the references therein) of Bercu, de Saporta and Gégout-Petit [5], Delmas and Marsalle [13], Bitseki, Djellout and Guillin [8], Bitseki, Hoffmann and Olivier [9], Doumic, Hoffmann, Krell and Robert [15], Bitseki and Olivier [10, 21] and Hoffmann and Marguet [18].

The recent study of BMC models was motivated by the understanding of the cell division mechanism (where the trait of an individual is given by its growth rate). The first model of BMC, named “symmetric” bifurcating auto-regressive process (BAR) was introduced by Cowan and Staudte [12] in order to analyze cell lineage data. Since the works of Cowan and Staudte, many extensions of their model were studied in Markovian and non-Markovian settings (see, e.g., [21] and references therein). In particular, in [16], Guyon has studied “asymmetric” BAR in order to prove statistical evidence of aging in *Escherichia Coli*, giving a new approach to the problem studied in [24]. Let us also note that BMC have been used recently in several statistical works to study the estimator of the cell division rate [9, 15, 18]. Moreover, other studies, such as [14], can be generalized using the BMC theory (we refer to the conclusion therein).

In this paper, our objective is to establish a central limit theorem for additive functionals of BMC. With respect to this objective, notice that asymptotic results for BMC have been

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studied in [16] (law of large numbers and central limit theorem) and in [8] (moderate deviations principle and strong law of large numbers). See [13] for the law of large numbers and central limit theorem for BMC on a Galton–Watson tree. Notice also that recently, limit theorems, in particular law of large numbers, have been studied for branching Markov process, see [19] and [11], and that large values of parameters in stable BAR process allows to exhibit two regimes, see [3]. However, the central limit theorems which appear in [5, 13, 16] are written for additive functionals using increments of martingale, which implies in particular that the functions considered depend on the traits of the mother and its two daughters. The study of the case where the functions depend only on the trait of a single individual has not yet been treated for BMC (in this case it is not useful to solve the Poisson equation and to write additive functional as sums of martingale increments as the error term on the last generation is not negligible in general). For such functions, the central limit theorems have been studied recently for branching Markov processes and for superprocesses [1, 20, 22, 23, 25]. Our results can be seen as a discrete version of those given in the previous works, but with general ergodic hypothesis on the evolution of the trait. Unlike the results given in [5, 13, 16], we observe three regimes (subcritical, critical and super-critical), which correspond to a competition between the reproducing rate (here a mother has two daughters) and the ergodicity rate for the evolution of the trait along a lineage taken uniformly at random. This phenomenon already appears in the works of Athreya [2]. For BMC models, we stress that the three regimes already appear for moderate deviations and deviation inequalities in [6–8].

We follow the approach of [13, 16] and consider ergodic theorem with respect to the pointwise convergence. However, unlike the latter papers, we provide a different normalization for the fluctuations according to the regime being critical, subcritical and super-critical, see respectively Corollaries 3.3, 3.6 and 3.13. See also Remark 3.7 on the possibility of the limit to be degenerated in the critical case and the discussion with [22] on branching Markov process in continuous time. We shall be explicit in a forthcoming paper, regarding how the present results allow to recover the one regime result from [16] for additive functionals given by a sum of martingale increments.

The paper is organized as follows. We introduce the BMC model in Section 2.1 and consider the sets of assumptions in the spirit of [16] in Section 2.2. The main results are presented in Section 3: see Section 3.1 for results in the subcritical case, with technical proofs in Section 4; see Section 3.2 for results in the critical case, with technical proofs in Section 5; and see Section 3.3 for results in the super-critical case, with technical proofs in Section 6. The proof relies essentially on explicit second moments computations and precise upper bounds of fourth moments for BMC, which are recalled in Section 7.

2. Models and assumptions.

2.1. Bifurcating Markov chain: The model. We denote by \mathbb{N} the set of nonnegative integers and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. If (E, \mathcal{E}) is a measurable space, then $\mathcal{B}(E)$ (resp. $\mathcal{B}_b(E)$, resp. $\mathcal{B}_+(E)$) denotes the set of (resp. bounded, resp. nonnegative) \mathbb{R} -valued measurable functions defined on E . For $f \in \mathcal{B}(E)$, we set $\|f\|_\infty = \sup\{|f(x)|, x \in E\}$. For a finite measure λ on (E, \mathcal{E}) and $f \in \mathcal{B}(E)$ we shall write $\langle \lambda, f \rangle$ for $\int f(x) d\lambda(x)$ whenever this integral is well defined. For $n \in \mathbb{N}^*$, the product space E^n is endowed with the product σ -field $\mathcal{E}^{\otimes n}$. If (E, d) is a metric space, then \mathcal{E} will denote its Borel σ -field and the set $\mathcal{C}_b(E)$ (resp. $\mathcal{C}_+(E)$) denotes the set of bounded (resp. nonnegative) \mathbb{R} -valued continuous functions defined on E .

Let (S, \mathcal{S}) be a measurable space. Let Q be a probability kernel on $S \times \mathcal{S}$, that is, $Q(\cdot, A)$ is measurable for all $A \in \mathcal{S}$, and $Q(x, \cdot)$ is a probability measure on (S, \mathcal{S}) for all $x \in S$. For any $f \in \mathcal{B}_b(S)$, we set, for $x \in S$,

$$(1) \quad (Qf)(x) = \int_S f(y) Q(x, dy).$$

We define (Qf) , or simply Qf , for $f \in \mathcal{B}(S)$ as soon as the integral (1) is well defined, and we have $Qf \in \mathcal{B}(S)$. For $n \in \mathbb{N}$, we denote by Q^n the n th iterate of Q defined by $Q^0 = Id$, the identity map on $\mathcal{B}(S)$, and $Q^{n+1}f = Q^n(Qf)$ for $f \in \mathcal{B}_b(S)$.

Let P be a probability kernel on $S \times \mathcal{S}^{\otimes 2}$, that is, $P(\cdot, A)$ is measurable for all $A \in \mathcal{S}^{\otimes 2}$, and $P(x, \cdot)$ is a probability measure on $(S^2, \mathcal{S}^{\otimes 2})$ for all $x \in S$. For any $g \in \mathcal{B}_b(S^3)$ and $h \in \mathcal{B}_b(S^2)$, we set for $x \in S$,

$$(2) \quad (Pg)(x) = \int_{S^2} g(x, y, z)P(x, dy, dz) \quad \text{and} \quad (Ph)(x) = \int_{S^2} h(y, z)P(x, dy, dz).$$

We define (Pg) (resp. (Ph)), or simply Pg for $g \in \mathcal{B}(S^3)$ (resp. Ph for $h \in \mathcal{B}(S^2)$), as soon as the corresponding integral (2) is well defined, and we have that Pg and Ph belong to $\mathcal{B}(S)$.

We now introduce some notation related to the regular binary tree. We set $\mathbb{T}_0 = \mathbb{G}_0 = \{\emptyset\}$, $\mathbb{G}_k = \{0, 1\}^k$ and $\mathbb{T}_k = \bigcup_{0 \leq r \leq k} \mathbb{G}_r$ for $k \in \mathbb{N}^*$, and $\mathbb{T} = \bigcup_{r \in \mathbb{N}} \mathbb{G}_r$. The set \mathbb{G}_k corresponds to the k th generation, \mathbb{T}_k to the tree up to the k th generation, and \mathbb{T} the complete binary tree. For $i \in \mathbb{T}$, we denote by $|i|$ the generation of i ($|i| = k$ if and only if $i \in \mathbb{G}_k$) and $iA = \{ij; j \in A\}$ for $A \subset \mathbb{T}$, where ij is the concatenation of the two sequences $i, j \in \mathbb{T}$, with the convention that $\emptyset i = i \emptyset = i$.

We recall the definition of a bifurcating Markov chain from [16].

DEFINITION 2.1. We say a stochastic process indexed by \mathbb{T} , $X = (X_i, i \in \mathbb{T})$, is a bifurcating Markov chain (BMC) on a measurable space (S, \mathcal{S}) with initial probability distribution ν on (S, \mathcal{S}) and probability kernel \mathcal{P} on $S \times \mathcal{S}^{\otimes 2}$ if:

- (Initial distribution.) The random variable X_\emptyset is distributed as ν .
- (Branching Markov property.) For a sequence $(g_i, i \in \mathbb{T})$ of functions belonging to $\mathcal{B}_b(S^3)$, we have for all $k \geq 0$,

$$\mathbb{E} \left[\prod_{i \in \mathbb{G}_k} g_i(X_i, X_{i0}, X_{i1}) \middle| \sigma(X_j; j \in \mathbb{T}_k) \right] = \prod_{i \in \mathbb{G}_k} \mathcal{P}g_i(X_i).$$

Let $X = (X_i, i \in \mathbb{T})$ be a BMC on a measurable space (S, \mathcal{S}) with initial probability distribution ν and probability kernel \mathcal{P} . We define three probability kernels P_0, P_1 and Q on $S \times \mathcal{S}$ by

$$P_0(x, A) = \mathcal{P}(x, A \times S), \quad P_1(x, A) = \mathcal{P}(x, S \times A) \quad \text{for } (x, A) \in S \times \mathcal{S} \quad \text{and}$$

$$Q = \frac{1}{2}(P_0 + P_1).$$

Notice that P_0 (resp. P_1) is the restriction of the first (resp. second) marginal of \mathcal{P} to S . Following [16], we introduce an auxiliary Markov chain $Y = (Y_n, n \in \mathbb{N})$ on (S, \mathcal{S}) with Y_0 distributed as X_\emptyset and transition kernel Q . The distribution of Y_n corresponds to the distribution of X_I , where I is chosen independently from X and uniformly at random in generation \mathbb{G}_n . We shall write \mathbb{E}_x when $X_\emptyset = x$ (i.e., the initial distribution ν is the Dirac mass at $x \in S$), and simply write \mathbb{E} otherwise (thus not stressing the dependence in ν in general).

We end this section with a useful notation. By convention, for $f, g \in \mathcal{B}(S)$, we define the function $f \otimes g \in \mathcal{B}(S^2)$ by $(f \otimes g)(x, y) = f(x)g(y)$ for $x, y \in S$ and introduce the notation

$$f \otimes_{\text{sym}} g = \frac{1}{2}(f \otimes g + g \otimes f) \quad \text{and} \quad f \otimes^2 = f \otimes f.$$

Notice that $\mathcal{P}(g \otimes_{\text{sym}} \mathbf{1}) = Q(g)$ for $g \in \mathcal{B}_+(S)$.

2.2. *Assumptions.* For a set $F \subset \mathcal{B}(S)$ of \mathbb{R} -valued functions, we write $F^2 = \{f^2; f \in F\}$, $F \otimes F = \{f_0 \otimes f_1; f_0, f_1 \in F\}$ and $P(F) = \{Pf; f \in F\}$ whenever a kernel P acts on F . Following [16], we state a structural assumption on the set of functions we shall consider.

ASSUMPTION 2.2. Let $F \subset \mathcal{B}(S)$ be a set of \mathbb{R} -valued functions such that:

- (i) F is a vector subspace which contains the constants;
- (ii) $F^2 \subset F$;
- (iii) $F \subset L^1(\nu)$;
- (iv) $F \otimes F \subset L^1(\mathcal{P}(x, \cdot))$ for all $x \in S$, and $\mathcal{P}(F \otimes F) \subset F$.

The condition (iv) implies that $P_0(F) \subset F$, $P_1(F) \subset F$ as well as $Q(F) \subset F$. Notice that if $f \in F$, then even if $|f|$ does not belong to F , using conditions (i) and (ii), we get, with $g = (1 + f^2)/2$, that $|f| \leq g$ and $g \in F$. Typically, when (S, d) is a metric space, the set F can be the set $\mathcal{C}_b(S)$ of bounded real-valued functions, or the set of smooth real-valued functions such that all derivatives have at most polynomials growth.

Following [16], we also consider the following ergodic properties for Q .

ASSUMPTION 2.3. There exists a probability measure μ on (S, \mathcal{S}) such that $F \subset L^1(\mu)$ and for all $f \in F$, we have the pointwise convergence $\lim_{n \rightarrow \infty} Q^n f = \langle \mu, f \rangle$ and there exists $g \in F$ with

$$(3) \quad |Q^n(f)| \leq g \quad \text{for all } n \in \mathbb{N}.$$

We consider also the following geometrical ergodicity.

ASSUMPTION 2.4. There exists a probability measure μ on (S, \mathcal{S}) such that $F \subset L^1(\mu)$, and $\alpha \in (0, 1)$ such that for all $f \in F$ there exists $g \in F$ such that

$$(4) \quad |Q^n f - \langle \mu, f \rangle| \leq \alpha^n g \quad \text{for all } n \in \mathbb{N}.$$

A sequence $\mathfrak{f} = (f_\ell, \ell \in \mathbb{N})$ of elements of F satisfies uniformly (3) and (4) if there is $g \in F$ such that

$$(5) \quad |Q^n(f_\ell)| \leq g \quad \text{and} \quad |Q^n f_\ell - \langle \mu, f_\ell \rangle| \leq \alpha^n g \quad \text{for all } n, \ell \in \mathbb{N}.$$

This implies in particular that $|f_\ell| \leq g$ and $|\langle \mu, f_\ell \rangle| \leq \langle \mu, g \rangle$. Notice that (5) trivially holds if \mathfrak{f} takes finitely distinct values (i.e., the subset $\{f_\ell; \ell \in \mathbb{N}\}$ of F is finite) each satisfying (3) and (4).

EXAMPLE 2.5. Let (S, d) be a metric space, \mathcal{S} its Borel σ -field and Y a Markov chain uniformly geometrically ergodic that is, there exists $\alpha \in (0, 1)$ and a finite constant C such that for all $x \in S$:

$$(6) \quad \|Q^n(x, \cdot) - \mu\|_{TV} \leq C\alpha^n,$$

where, for a signed finite measure π on (S, \mathcal{S}) , its total variation norm is defined by $\|\pi\|_{TV} = \sup_{f \in \mathcal{B}(S), \|f\|_\infty \leq 1} |\langle \pi, f \rangle|$. Then, taking for F the set of \mathbb{R} -valued continuous bounded function $\mathcal{C}_b(S)$, we get that properties (i)–(iii) from Assumption 2.2 and Assumption 2.4 hold. In particular, equation (6) implies that (4) holds with $g = C\|f\|_\infty$.

We consider the stronger ergodic property based on a second spectral gap.

ASSUMPTION 2.6. There exists a probability measure μ on (S, \mathcal{S}) such that $F \subset L^1(\mu)$, and $\alpha \in (0, 1)$, a finite nonempty set J of indices, distinct complex eigenvalues $\{\alpha_j, j \in J\}$ of the operator \mathcal{Q} with $|\alpha_j| = \alpha$, nonzero complex projectors $\{\mathcal{R}_j, j \in J\}$ defined on $\mathbb{C}F$, the \mathbb{C} -vector space spanned by F , such that $\mathcal{R}_j \circ \mathcal{R}_{j'} = \mathcal{R}_{j'} \circ \mathcal{R}_j = 0$ for all $j \neq j'$ (so that $\sum_{j \in J} \mathcal{R}_j$ is also a projector defined on $\mathbb{C}F$) and a positive sequence $(\beta_n, n \in \mathbb{N})$ converging to 0, such that for all $f \in F$ there exists $g \in F$ and, with $\theta_j = \alpha_j/\alpha$:

$$(7) \quad \left| \mathcal{Q}^n(f) - \langle \mu, f \rangle - \alpha^n \sum_{j \in J} \theta_j^n \mathcal{R}_j(f) \right| \leq \beta_n \alpha^n g \quad \text{for all } n \in \mathbb{N}.$$

Without loss of generality, we shall assume that the sequence $(\beta_n, n \in \mathbb{N})$ in Assumption 2.6 is nonincreasing and bounded from above by 1.

REMARK 2.7. In [16], only the structural Assumption 2.2 and the ergodic Assumption 2.3 were assumed. If F contains a set A of bounded functions which is separating (i.e., two probability measures which coincide on A are equal), then Assumptions 2.2 and 2.3 imply in particular that μ is the only invariant measure of \mathcal{Q} . Notice that the geometric ergodicity Assumption 2.4 implies Assumption 2.3, and that Assumption 2.6 implies Assumption 2.4 (with the same α but possibly different function g).

EXAMPLE 2.8. We consider the real-valued Gaussian symmetric bifurcating autoregressive process (BAR) $X = (X_u, u \in \mathbb{T})$ where, for all $u \in \mathbb{T} \setminus \{\emptyset\}$,

$$X_u = aX_v + \varepsilon_u,$$

where v is the parent of u , that is, $u = v0$ or $u = v1$, $a \in (-1, 1)$ and $(\varepsilon_v, v \in \mathbb{T})$ are independent Gaussian random variables $\mathcal{N}(0, \sigma^2)$ with $\sigma > 0$. We obtain:

$$\mathcal{P}(x, dy, dz) = \mathcal{Q}(x, dy)\mathcal{Q}(x, dz) \quad \text{with } \mathcal{Q}f(x) = \mathbb{E}[f(ax + \sigma G)],$$

where G is a standard $\mathcal{N}(0, 1)$ Gaussian random variable. More generally we have $\mathcal{Q}^n f(x) = \mathbb{E}[f(a^n x + \sqrt{1 - a^{2n}}\sigma_a G)]$, where $\sigma_a = \sigma(1 - a^2)^{-1/2}$. The kernel \mathcal{Q} admits a unique invariant probability measure μ , which is Gaussian $\mathcal{N}(0, \sigma_a^2)$. The operator \mathcal{Q} (on $L^2(\mu)$) is a symmetric integral Hilbert–Schmidt operator whose eigenvalues are given by $\sigma_p(\mathcal{Q}) = (a^n, n \in \mathbb{N})$, their algebraic multiplicity is one and the corresponding eigenfunctions $(\bar{g}_n(x), n \in \mathbb{N})$ are defined for $n \in \mathbb{N}$ by $\bar{g}_n(x) = g_n(\sigma_a^{-1}x)$, where g_n is the Hermite polynomial of degree n . In particular, we have $\bar{g}_0 = 1$ and $\bar{g}_1(x) = \sigma_a^{-1}x$. Let \mathcal{R} be the orthogonal projection on the vector space generated by \bar{g}_1 , that is, $\mathcal{R}f = \langle \mu, f\bar{g}_1 \rangle \bar{g}_1$ or equivalently, for $x \in \mathbb{R}$,

$$(8) \quad \mathcal{R}f(x) = \sigma_a^{-1}x \mathbb{E}[Gf(\sigma_a G)].$$

Consider F the set of functions $f \in \mathcal{C}^2(\mathbb{R})$ such that f, f' and f'' have at most polynomial growth. Assume that the probability distribution ν has all its moments, which is equivalent to say that $F \subset L^1(\nu)$. Then the set F satisfies Assumption 2.2. We also have that $F \subset L^1(\mu)$. Then, it is not difficult to check directly that Assumption 2.6 also holds with $J = \{j_0\}$, $\alpha_{j_0} = \alpha = |a|$, $\beta_n = |a|^n$ and $\mathcal{R}_{j_0} = \mathcal{R}$ (and also Assumptions 2.3 and 2.4 hold).

2.3. *Notation for average of different functions over different generations.* Let $X = (X_u, u \in \mathbb{T})$ be a BMC on (S, \mathcal{S}) with initial probability distribution ν , and probability kernel \mathcal{P} . Recall \mathcal{Q} is the induced Markov kernel. We assume that μ is an invariant probability measure of \mathcal{Q} .

For a finite set $A \subset \mathbb{T}$ and a function $f \in \mathcal{B}(S)$, we set

$$M_A(f) = \sum_{i \in A} f(X_i).$$

We shall be interested in the cases $A = \mathbb{G}_n$ (the n th generation) and $A = \mathbb{T}_n$ (the tree up to the n th generation). We recall from [16], Theorem 11 and Corollary 15, that under Assumptions 2.2 and 2.3 (resp. and also Assumption 2.4), we have for $f \in F$ the following convergence in $L^2(\mu)$ (resp. a.s.):

$$(9) \quad \lim_{n \rightarrow \infty} |\mathbb{G}_n|^{-1} M_{\mathbb{G}_n}(f) = \langle \mu, f \rangle \quad \text{and} \quad \lim_{n \rightarrow \infty} |\mathbb{T}_n|^{-1} M_{\mathbb{T}_n}(f) = \langle \mu, f \rangle.$$

We shall now consider the corresponding fluctuations. We will use frequently the following notation:

$$(10) \quad \boxed{\tilde{f} = f - \langle \mu, f \rangle \quad \text{for } f \in L^1(\mu).}$$

In order to study the asymptotics of $M_{\mathbb{G}_{n-\ell}}(\tilde{f})$, we shall consider the contribution of the descendants of the individual $i \in \mathbb{T}_{n-\ell}$ for $n \geq \ell \geq 0$,

$$(11) \quad N_{n,i}^\ell(f) = |\mathbb{G}_n|^{-1/2} M_{i\mathbb{G}_{n-|i|-\ell}}(\tilde{f}),$$

where $i\mathbb{G}_{n-|i|-\ell} = \{ij, j \in \mathbb{G}_{n-|i|-\ell}\} \subset \mathbb{G}_{n-\ell}$. For all $k \in \mathbb{N}$ such that $n \geq k + \ell$, we have,

$$M_{\mathbb{G}_{n-\ell}}(\tilde{f}) = |\mathbb{G}_n|^{1/2} \sum_{i \in \mathbb{G}_k} N_{n,i}^\ell(f) = |\mathbb{G}_n|^{1/2} N_{n,\emptyset}^\ell(f).$$

Let $\mathfrak{f} = (f_\ell, \ell \in \mathbb{N})$ be a sequence of elements of $L^1(\mu)$. We set for $n \in \mathbb{N}$ and $i \in \mathbb{T}_n$,

$$(12) \quad N_{n,i}(\mathfrak{f}) = \sum_{\ell=0}^{n-|i|} N_{n,i}^\ell(f_\ell) = |\mathbb{G}_n|^{-1/2} \sum_{\ell=0}^{n-|i|} M_{i\mathbb{G}_{n-|i|-\ell}}(\tilde{f}_\ell).$$

In $N_{n,i}$, we consider the contribution of the descendants of i up to generation n . We deduce that $\sum_{i \in \mathbb{G}_k} N_{n,i}(\mathfrak{f}) = |\mathbb{G}_n|^{-1/2} \sum_{\ell=0}^{n-k} M_{\mathbb{G}_{n-\ell}}(\tilde{f}_\ell)$ which gives for $k = 0$:

$$(13) \quad \boxed{N_{n,\emptyset}(\mathfrak{f}) = |\mathbb{G}_n|^{-1/2} \sum_{\ell=0}^n M_{\mathbb{G}_{n-\ell}}(\tilde{f}_\ell).}$$

In $N_{n,\emptyset}$, we consider the contribution of all the individuals from generation 0 up to generation n . We shall prove the convergence in law of $N_{n,\emptyset}(\mathfrak{f})$ in the following sections.

REMARK 2.9. We shall consider in particular the following two simple cases. Let $f \in L^1(\mu)$ and consider the sequence $\mathfrak{f} = (f_\ell, \ell \in \mathbb{N})$. If $f_0 = f$ and $f_\ell = 0$ for $\ell \in \mathbb{N}^*$, we shall write

$$(14) \quad \mathfrak{f}_0 = (f, 0, \dots),$$

and we get

$$N_{n,\emptyset}(\mathfrak{f}_0) = |\mathbb{G}_n|^{-1/2} M_{\mathbb{G}_n}(\tilde{f}).$$

If $f_\ell = f$ for $\ell \in \mathbb{N}$, then we shall write

$$(15) \quad \mathfrak{f} = (f, f, \dots),$$

and we get, as $|\mathbb{T}_n| = 2^{n+1} - 1$ and $|\mathbb{G}_n| = 2^n$:

$$N_{n,\emptyset}(\mathfrak{f}) = |\mathbb{G}_n|^{-1/2} M_{\mathbb{T}_n}(\tilde{f}) = \sqrt{2 - 2^{-n}} |\mathbb{T}_n|^{-1/2} M_{\mathbb{T}_n}(\tilde{f}).$$

Thus, we will easily deduce the fluctuations of $M_{\mathbb{T}_n}(f)$ and $M_{\mathbb{G}_n}(f)$ from the asymptotics of $N_{n,\emptyset}(\mathfrak{f})$.

To study the asymptotics of $N_{n,\emptyset}(\mathbf{f})$, it is convenient to write, for $n \geq k \geq 1$,

$$(16) \quad N_{n,\emptyset}(\mathbf{f}) = |\mathbb{G}_n|^{-1/2} \sum_{r=0}^{k-1} M_{\mathbb{G}_r}(\tilde{f}_{n-r}) + \sum_{i \in \mathbb{G}_k} N_{n,i}(\mathbf{f}).$$

For \mathbf{f} defined in (15), this becomes

$$(17) \quad N_{n,\emptyset}(\mathbf{f}) = |\mathbb{G}_n|^{-1/2} M_{\mathbb{T}_n}(\tilde{f}) = |\mathbb{G}_n|^{-1/2} M_{\mathbb{T}_{k-1}}(\tilde{f}) + \sum_{i \in \mathbb{G}_k} N_{n,i}(\mathbf{f}).$$

In the proofs, we will denote by C any unimportant finite constant which may vary from line to line (in particular C does not depend on $n \in \mathbb{N}$ nor on the considered sequence of functions $\mathbf{f} = (f_\ell, \ell \in \mathbb{N})$).

3. Main results.

3.1. *The subcritical case:* $2\alpha^2 < 1$. We shall consider, when well defined, for a sequence $\mathbf{f} = (f_\ell, \ell \in \mathbb{N})$ of measurable real-valued functions defined on S , the quantities

$$(18) \quad \Sigma^{\text{sub}}(\mathbf{f}) = \Sigma_1^{\text{sub}}(\mathbf{f}) + 2\Sigma_2^{\text{sub}}(\mathbf{f}),$$

where

$$(19) \quad \Sigma_1^{\text{sub}}(\mathbf{f}) = \sum_{\ell \geq 0} 2^{-\ell} \langle \mu, \tilde{f}_\ell^2 \rangle + \sum_{\ell \geq 0, k \geq 0} 2^{k-\ell} \langle \mu, \mathcal{P}((\mathcal{Q}^k \tilde{f}_\ell) \otimes^2) \rangle,$$

$$(20) \quad \Sigma_2^{\text{sub}}(\mathbf{f}) = \sum_{0 \leq \ell < k < \infty} 2^{-\ell} \langle \mu, \tilde{f}_k \mathcal{Q}^{k-\ell} \tilde{f}_\ell \rangle$$

$$+ \sum_{\substack{0 \leq \ell < k < \infty \\ r \geq 0}} 2^{r-\ell} \langle \mu, \mathcal{P}(\mathcal{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_\ell) \rangle.$$

We have the following result whose proof is given in Section 4.

THEOREM 3.1. *Let X be a BMC with kernel \mathcal{P} and initial distribution ν such that Assumptions 2.2 and 2.4 are in force with $\alpha \in (0, 1/\sqrt{2})$. We have the following convergence in distribution for all sequences $\mathbf{f} = (f_\ell, \ell \in \mathbb{N})$ of elements of F satisfying Assumptions 2.4 uniformly, that is, (5) for some $g \in F$:*

$$N_{n,\emptyset}(\mathbf{f}) \xrightarrow[n \rightarrow \infty]{(d)} G,$$

where G is a centered Gaussian random variable with variance $\Sigma^{\text{sub}}(\mathbf{f})$ given by (18), which is well defined and finite.

The convergence in distribution of $N_{n,\emptyset}(\mathbf{f})$ allows to recover the convergence in distribution of the average over different successive generations $|\mathbb{G}_n|^{-1/2} (M_{\mathbb{G}_n}(\tilde{f}_0), \dots, M_{\mathbb{G}_{n-k}}(\tilde{f}_k))$. Notice the limit is a Gaussian random vector (G_1, \dots, G_k) . A priori the random variables G_1, \dots, G_k are not independent because of the interaction coming from (20). In contrast, it was proved in [13], that the average over different successive generations of martingale increments converges to Gaussian independent random variables.

REMARK 3.2. For $f \in \mathcal{B}(S)$, when it is well defined, we set

$$(21) \quad \Sigma_{\mathbb{G}}^{\text{sub}}(f) = \langle \mu, \tilde{f}^2 \rangle + \sum_{k \geq 0} 2^k \langle \mu, \mathcal{P}(\mathcal{Q}^k \tilde{f} \otimes^2) \rangle \quad \text{and} \quad \Sigma_{\mathbb{T}}^{\text{sub}}(f) = \Sigma_{\mathbb{G}}^{\text{sub}}(f) + 2\Sigma_{\mathbb{T},2}^{\text{sub}}(f),$$

where

$$\Sigma_{\mathbb{T},2}^{\text{sub}}(f) = \sum_{k \geq 1} \langle \mu, \tilde{f} \mathcal{Q}^k \tilde{f} \rangle + \sum_{\substack{k \geq 1 \\ r \geq 0}} 2^r \langle \mu, \mathcal{P}(\mathcal{Q}^r \tilde{f} \otimes_{\text{sym}} \mathcal{Q}^{r+k} \tilde{f}) \rangle.$$

For f_0 defined in (14), we have $\Sigma^{\text{sub}}(f_0) = \Sigma_{\mathbb{G}}^{\text{sub}}(f)$. For \mathbf{f} defined in (15), we have $\Sigma^{\text{sub}}(\mathbf{f}) = 2\Sigma_{\mathbb{T}}^{\text{sub}}(f)$.

As a direct consequence of Remarks 3.2 and 2.9, and the more general Theorem 3.1, we get the following result.

COROLLARY 3.3. *Let X be a BMC with kernel \mathcal{P} and initial distribution ν such that Assumptions 2.2 and 2.4 are in force with $\alpha \in (0, 1/\sqrt{2})$. Let $f \in F$. Then, we have the following convergence in distribution:*

$$|\mathbb{G}_n|^{-1/2} M_{\mathbb{G}_n}(\tilde{f}) \xrightarrow[n \rightarrow \infty]{(d)} G_1 \quad \text{and} \quad |\mathbb{T}_n|^{-1/2} M_{\mathbb{T}_n}(\tilde{f}) \xrightarrow[n \rightarrow \infty]{(d)} G_2,$$

where G_1 and G_2 are centered Gaussian random variables with respective variances $\Sigma_{\mathbb{G}}^{\text{sub}}(f)$ and $\Sigma_{\mathbb{T}}^{\text{sub}}(f)$ given in (21), which are well defined and finite.

PROOF OF COROLLARY 3.3. Take f_0 , defined in (14), to deduce from Theorem 3.1 the convergence in distribution of $|\mathbb{G}_n|^{-1/2} M_{\mathbb{G}_n}(\tilde{f}) = N_{n,\emptyset}(f_0)$. Next, take \mathbf{f} , defined in (15), in Theorem 3.1 and use (17) and as well as $\lim_{n \rightarrow \infty} |\mathbb{G}_n|/|\mathbb{T}_n| = 1/2$, to get the convergence in distribution for $|\mathbb{T}_n|^{-1/2} M_{\mathbb{T}_n}(\tilde{f}) = (|\mathbb{G}_n|/|\mathbb{T}_n|)^{1/2} N_{n,\emptyset}(\mathbf{f})$. \square

3.2. The critical case $2\alpha^2 = 1$. In the critical case $\alpha = 1/\sqrt{2}$, we shall denote by \mathcal{R}_j the projector on the eigenspace associated to the eigenvalue α_j with $\alpha_j = \theta_j \alpha$, $|\theta_j| = 1$ and for j in the finite set of indices J . Since \mathcal{Q} is a real operator, we get that if α_j is a nonreal eigenvalue, so is $\bar{\alpha}_j$. We shall denote by $\bar{\mathcal{R}}_j$ the projector associated to $\bar{\alpha}_j$. Recall that the sequence $(\beta_n, n \in \mathbb{N})$ in Assumption 2.6 can (and will) be chosen nonincreasing and bounded from above by 1. For all measurable real-valued function f defined on S , we set, when this is well defined,

$$(22) \quad \boxed{\hat{f} = \tilde{f} - \sum_{j \in J} \mathcal{R}_j(f) \quad \text{with} \quad \tilde{f} = f - \langle \mu, f \rangle.}$$

We shall consider, when well defined, for a sequence $\mathbf{f} = (f_\ell, \ell \in \mathbb{N})$ of measurable real-valued functions defined on S , the quantities

$$(23) \quad \Sigma^{\text{crit}}(\mathbf{f}) = \Sigma_1^{\text{crit}}(\mathbf{f}) + 2\Sigma_2^{\text{crit}}(\mathbf{f}),$$

where

$$(24) \quad \Sigma_1^{\text{crit}}(\mathbf{f}) = \sum_{k \geq 0} 2^{-k} \langle \mu, \mathcal{P} f_{k,k}^* \rangle = \sum_{k \geq 0} 2^{-k} \sum_{j \in J} \langle \mu, \mathcal{P}(\mathcal{R}_j(f_k) \otimes_{\text{sym}} \bar{\mathcal{R}}_j(f_k)) \rangle,$$

$$(25) \quad \Sigma_2^{\text{crit}}(\mathbf{f}) = \sum_{0 \leq \ell < k} 2^{-(k+\ell)/2} \langle \mu, \mathcal{P} f_{k,\ell}^* \rangle,$$

with, for $k, \ell \in \mathbb{N}$,

$$(26) \quad f_{k,\ell}^* = \sum_{j \in J} \theta_j^{\ell-k} \mathcal{R}_j(f_k) \otimes_{\text{sym}} \bar{\mathcal{R}}_j(f_\ell).$$

Notice that $f_{k,\ell}^* = f_{\ell,k}^*$ and that $f_{k,\ell}^*$ is real-valued as $\overline{\theta_j^{\ell-k} \mathcal{R}_j(f_k) \otimes \overline{\mathcal{R}}_j(f_\ell)} = \theta_{j'}^{\ell-k} \times \mathcal{R}_{j'}(f_k) \otimes \overline{\mathcal{R}}_{j'}(f_\ell)$ for j' such that $\alpha_{j'} = \overline{\alpha}_j$ and thus $\mathcal{R}_{j'} = \overline{\mathcal{R}}_j$.

We shall consider sequences $\mathfrak{f} = (f_\ell, \ell \in \mathbb{N})$ of elements of F which satisfy Assumption 2.6 uniformly, that is, such that there exists $g \in F$ with

$$(27) \quad |\mathcal{Q}^n(f_\ell)| \leq g, \quad |\mathcal{Q}^n(\tilde{f}_\ell)| \leq \alpha^n g \quad \text{and} \quad |\mathcal{Q}^n(\hat{f}_\ell)| \leq \beta_n \alpha^n g \quad \text{for all } n, \ell \in \mathbb{N}.$$

We deduce that there exists a finite constant c_J depending only on $\{\alpha_j, j \in J\}$ such that for all $\ell \in \mathbb{N}, n \in \mathbb{N}, j_0 \in J$,

$$(28) \quad \begin{aligned} |f_\ell| \leq g, \quad |\tilde{f}_\ell| \leq g, \quad |\langle \mu, f_\ell \rangle| \leq \langle \mu, g \rangle, \\ \left| \sum_{j \in J} \theta_j^n \mathcal{R}_j(f_\ell) \right| \leq 2g \quad \text{and} \quad |\mathcal{R}_{j_0}(f_\ell)| \leq c_J g, \end{aligned}$$

where for the last inequality, we used that the Vandermonde matrix $(\theta_j^n; j \in J, n \in \{0, \dots, |J| - 1\})$ is invertible. Notice that (27) holds in particular if (7) holds for all $f \in F$ and $\mathfrak{f} = (f_n, n \in \mathbb{N})$ takes finitely distinct values in F (i.e., the set $\{f_\ell; \ell \in \mathbb{N}\} \subset F$ is finite). The proof of the following result is given in Section 5.

THEOREM 3.4. *Let X be a BMC with kernel \mathcal{P} and initial distribution ν . Assume that Assumptions 2.2 and 2.6 hold with $\alpha = 1/\sqrt{2}$. We have the following convergence in distribution for all sequences $\mathfrak{f} = (f_\ell, \ell \in \mathbb{N})$ of elements of F satisfying Assumptions 2.6 uniformly, that is, (27) for some $g \in F$:*

$$n^{-1/2} N_{n,\emptyset}(\mathfrak{f}) \xrightarrow[n \rightarrow \infty]{(d)} G,$$

where G is a Gaussian real-valued random variable with variance $\Sigma^{\text{crit}}(\mathfrak{f})$ given by (23), which is well defined and finite.

REMARK 3.5. For $f \in \mathcal{B}(S)$, when it is well defined, we set

$$(29) \quad \Sigma_{\mathbb{G}}^{\text{crit}}(f) = \sum_{j \in J} \langle \mu, \mathcal{P}(\mathcal{R}_j(f) \otimes_{\text{sym}} \overline{\mathcal{R}}_j(f)) \rangle \quad \text{and} \quad \Sigma_{\mathbb{T}}^{\text{crit}}(f) = \Sigma_{\mathbb{G}}^{\text{crit}}(f) + 2\Sigma_{\mathbb{T},2}^{\text{crit}}(f),$$

where

$$\Sigma_{\mathbb{T},2}^{\text{crit}}(f) = \sum_{j \in J} \frac{1}{\sqrt{2}\theta_j - 1} \langle \mu, \mathcal{P}(\mathcal{R}_j(f) \otimes_{\text{sym}} \overline{\mathcal{R}}_j(f)) \rangle.$$

For \mathfrak{f}_0 defined in (14), we have $\Sigma^{\text{crit}}(\mathfrak{f}_0) = \Sigma_{\mathbb{G}}^{\text{crit}}(f)$. For \mathfrak{f} defined in (15), we have $\Sigma^{\text{crit}}(\mathfrak{f}) = 2\Sigma_{\mathbb{T}}^{\text{crit}}(f)$.

As a direct consequence of Remarks 3.5 and 2.9, and the more general Theorem 3.4, we get the following result. The proof which mimics the proof of Corollary 3.3 is left to the reader.

COROLLARY 3.6. *Let X be a BMC with kernel \mathcal{P} and initial distribution ν such that Assumptions 2.2 and 2.6 are in force with $\alpha = 1/\sqrt{2}$. Let $f \in F$. Then, we have the following convergence in distribution:*

$$(n|\mathbb{G}_n|)^{-1/2} M_{\mathbb{G}_n}(\tilde{f}) \xrightarrow[n \rightarrow \infty]{(d)} G_1 \quad \text{and} \quad (n|\mathbb{T}_n|)^{-1/2} M_{\mathbb{T}_n}(\tilde{f}) \xrightarrow[n \rightarrow \infty]{(d)} G_2,$$

where G_1 and G_2 are centered Gaussian real-valued random variables with respective variance $\Sigma_{\mathbb{G}}^{\text{crit}}(f)$ and $\Sigma_{\mathbb{T}}^{\text{crit}}(f)$ given in (29), which are well defined and finite.

REMARK 3.7. We comment on the limit in Corollary 3.6 to be 0.

1. We stress that the variances $\Sigma_{\mathbb{T}}^{\text{crit}}(f)$ and $\Sigma_{\mathbb{G}}^{\text{crit}}(f)$ can take the value 0. This is the case in particular if the projection of f on the eigenspace corresponding to the eigenvalues α_j equal 0: $\mathcal{R}_j(f) = 0$ for all $j \in J$.

2. In the symmetric BAR model developed in Example 2.8 where J is reduced to a singleton and the projector is given by (8), we deduce that if $a = \alpha = 1/\sqrt{2}$ then $\Sigma_{\mathbb{G}}^{\text{crit}}(f) = \Sigma_{\mathbb{T}}^{\text{crit}}(f) = 0$ if $\mathbb{E}[Gf(\sigma_a G)] = 0$, where G is a standard $\mathcal{N}(0, 1)$ Gaussian random variable. This is in particular the case if f is even. In fact, one should consider the first integer, say n_0 , such that $\langle f, \bar{g}_n \rangle \neq 0$, where \bar{g}_n are the eigenfunctions given in Example 2.8, to identify the good normalization for the fluctuations.

3. We refer to the work of Ren, Song and Zhang [22] in the setting of a branching Markov process in continuous time for a detailed study of the good normalizations in the three regimes such that the fluctuations have a nontrivial limit. Let us stress that their results rely on a strong hypothesis on the transition kernel for the trait. In our setting this would amount to assume that the kernel \mathcal{Q} defines a compact operator on L^2 , which would give the existence of a complete spectral decomposition of the kernel \mathcal{Q} . (The Gaussian symmetric BAR model developed in Example 2.8 satisfies in particular this assumption.) This latter assumption is of course stronger than the ergodic Assumption 2.6.

3.3. *The super-critical case $2\alpha^2 > 1$.* We consider the super-critical case $\alpha \in (1/\sqrt{2}, 1)$. We shall assume that Assumption 2.6 holds. Recall (7) with the eigenvalues $\{\alpha_j = \theta_j \alpha, j \in J\}$ of \mathcal{Q} , with modulus equal to α (i.e., $|\theta_j| = 1$) and the projector \mathcal{R}_j on the eigenspace associated to eigenvalue α_j . Recall that the sequence $(\beta_n, n \in \mathbb{N})$ in Assumption 2.6 can (and will) be chosen nonincreasing and bounded from above by 1.

We shall consider the filtration $\mathcal{H} = (\mathcal{H}_n, n \in \mathbb{N})$ defined by $\mathcal{H}_n = \sigma(X_i, i \in \mathbb{T}_n)$. The next lemma, whose the proof is given in Section 6.1, exhibits martingales related to the projector \mathcal{R}_j .

LEMMA 3.8. *Let X be a BMC with kernel \mathcal{P} and initial distribution ν . Assume that Assumptions 2.2 and 2.6 hold with $\alpha \in (1/\sqrt{2}, 1)$ in (7). Then, for all $j \in J$ and $f \in F$, the sequence $M_j(f) = (M_{n,j}(f), n \in \mathbb{N})$, with*

$$M_{n,j}(f) = (2\alpha_j)^{-n} M_{\mathbb{G}_n}(\mathcal{R}_j(f)),$$

is a \mathcal{H} -martingale which converges a.s. and in L^2 to a random variable, say $M_{\infty,j}(f)$.

Now, we state the main result of this section, whose proof is given in Section 6.2. Recall that $\theta_j = \alpha_j/\alpha$ and $|\theta_j| = 1$ and $M_{\infty,j}$ is defined in Lemma 3.8.

THEOREM 3.9. *Let X be a BMC with kernel \mathcal{P} and initial distribution ν . Assume that Assumptions 2.2 and 2.6 hold with $\alpha \in (1/\sqrt{2}, 1)$ in (7). We have the following convergence in probability for all sequences $\mathfrak{f} = (f_\ell, \ell \in \mathbb{N})$ of elements of F satisfying Assumptions 2.6 uniformly, that is, (27) holds for some $g \in F$:*

$$(2\alpha^2)^{-n/2} N_{n,\emptyset}(\mathfrak{f}) - \sum_{\ell \in \mathbb{N}} (2\alpha)^{-\ell} \sum_{j \in J} \theta_j^{n-\ell} M_{\infty,j}(f_\ell) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

REMARK 3.10. We stress that if for all $\ell \in \mathbb{N}$, the orthogonal projection of f_ℓ on the eigenspaces corresponding to the eigenvalues 1 and $\alpha_j, j \in J$, equal 0, then $M_{\infty,j}(f_\ell) = 0$ for all $j \in J$ and in this case, we have

$$(2\alpha^2)^{-n/2} N_{n,\emptyset}(\mathfrak{f}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

As a direct consequence of Theorem 3.9 and Remark 2.9, we deduce the following results.

COROLLARY 3.11. *Under the assumptions of Theorem 3.9, we have for all $f \in F$,*

$$(2\alpha)^{-n} M_{\mathbb{T}_n}(\tilde{f}) - \sum_{j \in J} \theta_j^n (1 - (2\alpha\theta_j)^{-1})^{-1} M_{\infty,j}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0$$

$$(2\alpha)^{-n} M_{\mathbb{G}_n}(\tilde{f}) - \sum_{j \in J} \theta_j^n M_{\infty,j}(f) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

PROOF. We first take \mathbf{f} , defined in (15), and next \mathbf{f}_0 , defined in (15), in Theorem 3.9, and then use (13). \square

We directly deduce the following two corollaries.

COROLLARY 3.12. *Under the hypothesis of Theorem 3.9, if α is the only eigenvalue of \mathcal{Q} with modulus equal to α (and thus J is reduced to a singleton), then we have*

$$(2\alpha^2)^{-n/2} N_{n,\emptyset}(\mathbf{f}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \sum_{\ell \in \mathbb{N}} (2\alpha)^{-\ell} M_{\infty}(f_{\ell}),$$

where, for $f \in F$, $M_{\infty}(f) = \lim_{n \rightarrow \infty} (2\alpha)^{-n} M_{\mathbb{G}_n}(\mathcal{R}(f))$, and \mathcal{R} is the projection on the eigenspace associated to the eigenvalue α .

The next Corollary is a direct consequence of Corollary 3.12.

COROLLARY 3.13. *Let X be a BMC with kernel \mathcal{P} and initial distribution ν . Assume that Assumptions 2.2 and 2.6 hold with $\alpha \in (1/\sqrt{2}, 1)$ in (7). Assume α is the only eigenvalue of \mathcal{Q} with modulus equal to α (and thus J is reduced to a singleton), then we have for $f \in F$,*

$$(2\alpha)^{-n} M_{\mathbb{G}_n}(\tilde{f}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} M_{\infty}(f) \quad \text{and} \quad (2\alpha)^{-n} M_{\mathbb{T}_n}(\tilde{f}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \frac{2\alpha}{2\alpha - 1} M_{\infty}(f),$$

where $M_{\infty}(f)$ is a random variable defined in Corollary 3.12.

4. Proof of Theorem 3.1. We first give some common notation to the proof of Theorems 3.1 and 3.4. Let $i, j \in \mathbb{T}$. We write $i \preceq j$ if $j \in i\mathbb{T}$. We denote by $i \wedge j$ the most recent common ancestor of i and j , which is defined as the only $u \in \mathbb{T}$ such that if $v \in \mathbb{T}$ and $v \preceq i$, $v \preceq j$ then $v \preceq u$. We also define the lexicographic order $i \leq j$ if either $i \preceq j$ or $v0 \preceq i$ and $v1 \preceq j$ for $v = i \wedge j$. Let $X = (X_i, i \in \mathbb{T})$ be a BMC with kernel \mathcal{P} and initial measure ν . For $i \in \mathbb{T}$, we define the σ -field:

$$\mathcal{F}_i = \{X_u; u \in \mathbb{T} \text{ such that } u \leq i\}.$$

By construction, the σ -fields $(\mathcal{F}_i; i \in \mathbb{T})$ are nested as $\mathcal{F}_i \subset \mathcal{F}_j$ for $i \leq j$.

Let $(p_n, n \in \mathbb{N})$ be a nondecreasing sequence of elements of \mathbb{N}^* such that, for all $\lambda > 0$,

$$(30) \quad p_n < n, \lim_{n \rightarrow \infty} p_n/n = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} n - p_n - \lambda \log(n) = +\infty.$$

We define for $n \in \mathbb{N}$, $i \in \mathbb{G}_{n-p_n}$ and $\mathbf{f} \in F^{\mathbb{N}}$ the martingale increments

$$(31) \quad \Delta_{n,i}(\mathbf{f}) = N_{n,i}(\mathbf{f}) - \mathbb{E}[N_{n,i}(\mathbf{f})|\mathcal{F}_i] \quad \text{and} \quad \Delta_n(\mathbf{f}) = \sum_{i \in \mathbb{G}_{n-p_n}} \Delta_{n,i}(\mathbf{f}).$$

Thanks to (12), we have

$$\sum_{i \in \mathbb{G}_{n-p_n}} N_{n,i}(f) = |\mathbb{G}_n|^{-1/2} \sum_{\ell=0}^{p_n} M_{\mathbb{G}_{n-\ell}}(\tilde{f}_\ell) = |\mathbb{G}_n|^{-1/2} \sum_{k=n-p_n}^n M_{\mathbb{G}_k}(\tilde{f}_{n-k}).$$

Using the branching Markov property, and (12), we get for $i \in \mathbb{G}_{n-p_n}$,

$$\mathbb{E}[N_{n,i}(f)|\mathcal{F}_i] = \mathbb{E}[N_{n,i}(f)|X_i] = |\mathbb{G}_n|^{-1/2} \sum_{\ell=0}^{p_n} \mathbb{E}_{X_i}[M_{\mathbb{G}_{p_n-\ell}}(\tilde{f}_\ell)].$$

We deduce from (16) with $k = n - p_n$ that

$$(32) \quad N_{n,\emptyset}(f) = \Delta_n(f) + R_0(n) + R_1(n),$$

with

$$(33) \quad R_0(n) = |\mathbb{G}_n|^{-1/2} \sum_{k=0}^{n-p_n-1} M_{\mathbb{G}_k}(\tilde{f}_{n-k}) \quad \text{and} \quad R_1(n) = \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E}[N_{n,i}(f)|\mathcal{F}_i].$$

We now go on to the proof of Theorem 3.1. We have the following elementary lemma.

LEMMA 4.1. *Under the assumptions of Theorem 3.1, we have the following convergence:*

$$\lim_{n \rightarrow \infty} \mathbb{E}[R_0(n)^2] = 0.$$

PROOF. For all $k \geq 1$ and $x \in S$, we have

$$\begin{aligned} \mathbb{E}_x[M_{\mathbb{G}_k}(\tilde{f}_{n-k})^2] &\leq 2^k g_1(x) + \sum_{\ell=0}^{k-1} 2^{k+\ell} \alpha^{2\ell} Q^{k-\ell-1}(\mathcal{P}(g_2 \otimes g_2))(x) \\ &\leq 2^k g_1(x) + 2^k \sum_{\ell=0}^{k-1} (2\alpha^2)^\ell g_3(x) \\ &\leq 2^k g_4(x), \end{aligned}$$

with $g_1, g_2, g_3, g_4 \in F$ and where we used (73), (5) twice and (3) twice (with f and g replaced by $2(g^2 + \langle \mu, g \rangle^2)$ and g_1 , and with f and g replaced by g and g_2) for the first inequality, (3) (with f and g replaced by $\mathcal{P}(g_2 \otimes g_2)$ and g_3) for the second, and that $2\alpha^2 < 1$ and $g_4 = g_1 + (1 - 2\alpha^2)^{-1} g_3$ for the last. As $g_4 \in F \subset L^1(\nu)$, this implies that $\mathbb{E}[M_{\mathbb{G}_k}(\tilde{f}_{n-k})^2] \leq c^2 2^k$ for some finite constant c which does not depend on n or k . We can take c large enough, so that this upper bound holds also for $k = 0$ and all $n \in \mathbb{N}$, thanks to (5). We deduce from (33) that:

$$(34) \quad \begin{aligned} (\mathbb{E}[R_0(n)^2])^{1/2} &\leq |\mathbb{G}_n|^{-1/2} \sum_{k=0}^{n-p_n-1} (\mathbb{E}[M_{\mathbb{G}_k}(\tilde{f}_{n-k})^2])^{1/2} \\ &\leq c 2^{-n/2} \sum_{k=0}^{n-p_n-1} 2^{k/2} \leq 3c 2^{-p_n/2}. \end{aligned}$$

Use that $\lim_{n \rightarrow \infty} p_n = \infty$ to conclude. \square

We have the following lemma.

LEMMA 4.2. *Under the assumptions of Theorem 3.1, we have the following convergence:*

$$\lim_{n \rightarrow \infty} \mathbb{E}[R_1(n)^2] = 0.$$

PROOF. We set for $p_n \geq \ell \geq 0$,

$$(35) \quad R_1(\ell, n) = \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E}[N_{n,i}^\ell(f_\ell) | \mathcal{F}_i],$$

so that, thanks to (12), $R_1(n) = \sum_{\ell=0}^{p_n} R_1(\ell, n)$. We have for $i \in \mathbb{G}_{n-p_n}$,

$$\begin{aligned} |\mathbb{G}_n|^{1/2} \mathbb{E}[N_{n,i}^\ell(f_\ell) | \mathcal{F}_i] &= \mathbb{E}[M_{i\mathbb{G}_{p_n-\ell}}(\tilde{f}_\ell) | X_i] \\ &= \mathbb{E}_{X_i}[M_{\mathbb{G}_{p_n-\ell}}(\tilde{f}_\ell)] = |\mathbb{G}_{p_n-\ell}| \mathcal{Q}^{p_n-\ell} \tilde{f}_\ell(X_i), \end{aligned}$$

where we used definition (11) of $N_{n,i}^\ell$ for the first equality, the Markov property of X for the second and (72) for the third. We deduce that

$$R_1(\ell, n) = |\mathbb{G}_n|^{-1/2} |\mathbb{G}_{p_n-\ell}| M_{\mathbb{G}_{n-p_n}}(\mathcal{Q}^{p_n-\ell} \tilde{f}_\ell).$$

Using (73), we get

$$\begin{aligned} \mathbb{E}_x[R_1(\ell, n)^2] &= |\mathbb{G}_n|^{-1} |\mathbb{G}_{p_n-\ell}|^2 \mathbb{E}_x[(M_{\mathbb{G}_{n-p_n}}(\mathcal{Q}^{p_n-\ell} \tilde{f}_\ell))^2] \\ &= |\mathbb{G}_n|^{-1} |\mathbb{G}_{p_n-\ell}|^2 2^{n-p_n} \mathcal{Q}^{n-p_n} ((\mathcal{Q}^{p_n-\ell} \tilde{f}_\ell)^2)(x) \\ &\quad + |\mathbb{G}_n|^{-1} |\mathbb{G}_{p_n-\ell}|^2 \sum_{k=0}^{n-p_n-1} 2^{n-p_n+k} \mathcal{Q}^{n-p_n-k-1} (\mathcal{P}(\mathcal{Q}^{k+p_n-\ell} \tilde{f}_\ell \otimes^2))(x). \end{aligned}$$

We deduce that

$$\begin{aligned} \mathbb{E}_x[R_1(\ell, n)^2] &\leq \alpha^{2(p_n-\ell)} 2^{p_n-2\ell} \mathcal{Q}^{n-p_n}(g^2)(x) \\ &\quad + 2^{p_n-2\ell} \sum_{k=0}^{n-p_n-1} \alpha^{2(k+p_n-\ell)} 2^k \mathcal{Q}^{n-p_n-k-1} (\mathcal{P}(g \otimes g)) \\ (36) \quad &\leq \alpha^{2(p_n-\ell)} 2^{p_n-2\ell} \left(g_1(x) + \sum_{k=0}^{n-p_n-1} (2\alpha^2)^k g_2(x) \right) \\ &\leq (2\alpha^2)^{p_n} (2\alpha)^{-2\ell} g_3(x), \end{aligned}$$

with $g_1, g_2, g_3 \in F$ and where we used (5) for the first inequality, (3) twice (with f and g replaced by g^2 and g_1 and by $\mathcal{P}(g \otimes g)$ and g_2) for the second, and that $2\alpha^2 < 1$ for the last. Since $g_3 \in F \subset L^1(\nu)$, this gives that $\mathbb{E}[R_1(\ell, n)^2] \leq (2\alpha^2)^{p_n} (2\alpha)^{-2\ell} \langle \nu, g_3 \rangle$. We deduce that

$$\mathbb{E}[R_1(n)^2]^{1/2} \leq \sum_{\ell=0}^{p_n} \mathbb{E}[R_1(\ell, n)^2]^{1/2} \leq a_{1,n} \langle \nu, g_3 \rangle^{1/2},$$

with the sequence $(a_{1,n}, n \in \mathbb{N})$ defined by

$$a_{1,n} = (2\alpha^2)^{p_n/2} \sum_{\ell=0}^{p_n} (2\alpha)^{-\ell}.$$

Notice the sequence $(a_{1,n}, n \in \mathbb{N})$ converges to 0 since $\lim_{n \rightarrow \infty} p_n = \infty$, $2\alpha^2 < 1$ and

$$\sum_{\ell=0}^{p_n} (2\alpha)^{-\ell} \leq \begin{cases} 2\alpha / (2\alpha - 1) & \text{if } 2\alpha > 1, \\ p_n + 1 & \text{if } 2\alpha = 1, \\ (2\alpha)^{-p_n} / (1 - 2\alpha) & \text{if } 2\alpha < 1. \end{cases}$$

We conclude that $\lim_{n \rightarrow \infty} \mathbb{E}[R_1(n)^2] = 0$. \square

We now study the bracket of Δ_n :

$$(37) \quad V(n) = \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E}[\Delta_{n,i}(f)^2 | \mathcal{F}_i].$$

Using (12) and (31), we write

$$(38) \quad V(n) = |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E}_{X_i} \left[\left(\sum_{\ell=0}^{p_n} M_{\mathbb{G}_{p_n-\ell}}(\tilde{f}_\ell) \right)^2 \right] - R_2(n) = V_1(n) + 2V_2(n) - R_2(n),$$

with

$$\begin{aligned} V_1(n) &= |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p_n}} \sum_{\ell=0}^{p_n} \mathbb{E}_{X_i} [M_{\mathbb{G}_{p_n-\ell}}(\tilde{f}_\ell)^2], \\ V_2(n) &= |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p_n}} \sum_{0 \leq \ell < k \leq p_n} \mathbb{E}_{X_i} [M_{\mathbb{G}_{p_n-\ell}}(\tilde{f}_\ell) M_{\mathbb{G}_{p_n-k}}(\tilde{f}_k)], \\ R_2(n) &= \sum_{i \in \mathbb{G}_{n-p_n}} (\mathbb{E}[N_{n,i}(f) | X_i])^2. \end{aligned}$$

LEMMA 4.3. *Under the assumptions of Theorem 3.1, we have the following convergence:*

$$\lim_{n \rightarrow \infty} \mathbb{E}[R_2(n)] = 0.$$

PROOF. Recall p_n defined in (30). We define the sequence $(a_{2,n}, n \in \mathbb{N})$ for $n \in \mathbb{N}$ by

$$a_{2,n} = 2^{-p_n} \left(\sum_{\ell=0}^{p_n} (2\alpha)^\ell \right)^2.$$

Notice that the sequence $(a_{2,n}, n \in \mathbb{N})$ converges to 0 since $\lim_{n \rightarrow \infty} p_n = \infty, 2\alpha^2 < 1$ and

$$\sum_{\ell=0}^{p_n} (2\alpha)^\ell \leq \begin{cases} (2\alpha)^{p_n+1} / (2\alpha - 1) & \text{if } 2\alpha > 1, \\ p_n + 1 & \text{if } 2\alpha = 1, \\ 1 / (1 - 2\alpha) & \text{if } 2\alpha < 1. \end{cases}$$

We now compute $\mathbb{E}_x[R_2(n)]$:

$$\begin{aligned} \mathbb{E}_x[R_2(n)] &= |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E}_x \left[\left(\mathbb{E}_x \left[\sum_{\ell=0}^{p_n} M_{i\mathbb{G}_{p_n-\ell}}(\tilde{f}_\ell) \mid X_i \right] \right)^2 \right] \\ &= |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E}_x \left[\left(\sum_{\ell=0}^{p_n} \mathbb{E}_{X_i} [M_{\mathbb{G}_{p_n-\ell}}(\tilde{f}_\ell)] \right)^2 \right] \\ (39) \quad &= |\mathbb{G}_n|^{-1} |\mathbb{G}_{n-p_n}| \mathcal{Q}^{n-p_n} \left(\left(\sum_{\ell=0}^{p_n} |\mathbb{G}_{p_n-\ell}| \mathcal{Q}^{p_n-\ell} \tilde{f}_\ell \right)^2 \right)(x) \\ &\leq 2^{-p_n} \left(\sum_{\ell=0}^{p_n} (2\alpha)^{p_n-\ell} \right)^2 \mathcal{Q}^{n-p_n}(g^2)(x) \\ &\leq a_{2,n} g_1(x), \end{aligned}$$

with $g_1 \in F$ and where we used the definition of $N_{n,i}(f)$ for the first equality, the Markov property of X for the second, (72) for the third, (5) for the first inequality, and (3) (with f

and g replaced by g^2 and g_1) for the last. We conclude that $\lim_{n \rightarrow \infty} \mathbb{E}[R_2(n)] = 0$, using that $\langle \nu, g_1 \rangle$ is finite as $g_1 \in F \subset L^1(\nu)$. \square

We have the following technical lemma.

LEMMA 4.4. *Under the assumptions of Theorem 3.1, we have that $\Sigma_2^{\text{sub}}(f)$ defined in (20) is well defined and finite, and that a.s. $\lim_{n \rightarrow \infty} V_2(n) = \Sigma_2^{\text{sub}}(f) < +\infty$.*

PROOF. Using (74), we get

$$(40) \quad V_2(n) = V_5(n) + V_6(n),$$

with

$$V_5(n) = |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p_n}} \sum_{0 \leq \ell < k \leq p_n} 2^{p_n - \ell} Q^{p_n - k} (\tilde{f}_k Q^{k - \ell} \tilde{f}_\ell)(X_i),$$

$$V_6(n) = |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p_n}} \sum_{0 \leq \ell < k < p_n} \sum_{r=0}^{p_n - k - 1} 2^{p_n - \ell + r}$$

$$\times Q^{p_n - 1 - (r+k)} (\mathcal{P}(Q^r \tilde{f}_k \otimes_{\text{sym}} Q^{k - \ell + r} \tilde{f}_\ell))(X_i).$$

We consider the term $V_6(n)$. We have

$$(41) \quad V_6(n) = |\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}}(H_{6,n}),$$

with

$$H_{6,n} = \sum_{\substack{0 \leq \ell < k < \infty \\ r \geq 0}} h_{k,\ell,r}^{(n)} \mathbf{1}_{\{r+k < p_n\}} \quad \text{and}$$

$$h_{k,\ell,r}^{(n)} = 2^{r-\ell} Q^{p_n - 1 - (r+k)} (\mathcal{P}(Q^r \tilde{f}_k \otimes_{\text{sym}} Q^{k - \ell + r} \tilde{f}_\ell)).$$

Using (4) and since $\mathcal{P}(Q^r(F) \otimes Q^{k-\ell+r}(F)) \subset F$ and $\lim_{n \rightarrow \infty} p_n = +\infty$, we have that

$$\lim_{n \rightarrow \infty} h_{k,\ell,r}^{(n)} = h_{k,\ell,r},$$

where the constant $h_{k,\ell,r}$ is equal to $2^{r-\ell} \langle \mu, \mathcal{P}(Q^r \tilde{f}_k \otimes_{\text{sym}} Q^{k-\ell+r} \tilde{f}_\ell) \rangle$. Using (4), we also have that

$$|h_{k,\ell,r}^{(n)}| \leq 2^{r-\ell} \alpha^{k-\ell+2r} Q^{p_n - 1 - (r+k)} (\mathcal{P}(g \otimes g))$$

$$\leq 2^{r-\ell} \alpha^{k-\ell+2r} g_*,$$

with $g_* \in F$ (which does not depend on n, r, k and ℓ) and where we used (5) for the first inequality and (3) (with f and g replaced by $\mathcal{P}(g \otimes g)$ and g_*). Taking the limit, we also deduce that

$$|h_{k,\ell,r}| \leq 2^{r-\ell} \alpha^{k-\ell+2r} g_*.$$

Define the constant

$$H_6(f) = \sum_{\substack{0 \leq \ell < k < \infty \\ r \geq 0}} h_{k,\ell,r} = \sum_{\substack{0 \leq \ell < k < \infty \\ r \geq 0}} 2^{r-\ell} \langle \mu, \mathcal{P}(Q^r \tilde{f}_k \otimes_{\text{sym}} Q^{k-\ell+r} \tilde{f}_\ell) \rangle$$

which is finite as

$$(42) \quad \sum_{0 \leq \ell < k < \infty, r \geq 0} 2^{r-\ell} \alpha^{k-\ell+2r} = \frac{2\alpha}{(1-\alpha)(1-2\alpha^2)} < +\infty.$$

Using (4) (with f and g replaced by $\mathcal{P}(\mathcal{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_\ell)$ and $g_{k,\ell,r}$), we deduce that

$$|h_{k,\ell,r}^{(n)} - h_{k,\ell,r}| \leq 2^{r-\ell} \alpha^{p_n-1-(r+k)} g_{k,\ell,r}.$$

Set $r_0 \in \mathbb{N}^*$ and $g_{r_0} = \sum_{0 \leq \ell < k < \infty; r \geq 0; k \vee r \leq r_0} g_{k,\ell,r}$. Notice that g_{r_0} belongs to F and is non-negative. Furthermore, we have

$$\begin{aligned} |H_{6,n} - H_6(f)| &\leq \sum_{\substack{0 \leq \ell < k < \infty \\ r \geq 0 \\ k \vee r \leq r_0}} 2^{r-\ell} \alpha^{p_n-1-(r+k)} g_{r_0} + \sum_{\substack{0 \leq \ell < k < \infty \\ r \geq 0 \\ r \vee k > r_0}} (|h_{k,\ell,r}^{(n)}| \mathbf{1}_{\{r+k < p_n\}} + |h_{k,\ell,r}|) \\ &\leq (r_0 + 1)^2 2^{r_0+1} \alpha^{p_n-1-2r_0} g_{r_0} + \gamma_1(r_0) g_*, \end{aligned}$$

with

$$\gamma_1(r_0) = \sum_{\substack{0 \leq \ell < k < \infty \\ r \geq 0 \\ r \vee k > r_0}} 2^{r-\ell} \alpha^{k-\ell+2r}.$$

Using (9) with n replaced by $n - p_n$ and f replaced by g_* and g_{r_0} , and that $\lim_{n \rightarrow \infty} \alpha^{p_n} = 0$ as well as $\lim_{n \rightarrow \infty} n - p_n = \infty$, we deduce that

$$\limsup_{n \rightarrow \infty} |\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}}(|H_{6,n} - H_6(f)|) \leq \gamma_1(r_0) \langle \mu, g_* \rangle.$$

Thanks to (42), we get by dominated convergence that $\lim_{r_0 \rightarrow \infty} \gamma_1(r_0) = 0$. This implies that

$$\lim_{n \rightarrow \infty} |\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}}(|H_{6,n} - H_6(f)|) = 0.$$

Since $|\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}}(\cdot)$ is a probability measure, we deduce from (41) that a.s.

$$\begin{aligned} \lim_{n \rightarrow \infty} V_6(n) &= \lim_{n \rightarrow \infty} |\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}}(H_{6,n}) \\ &= H_6(f) = \sum_{\substack{0 \leq \ell < k < \infty \\ r \geq 0}} 2^{r-\ell} \langle \mu, \mathcal{P}(\mathcal{Q}^r \tilde{f}_k \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r} \tilde{f}_\ell) \rangle. \end{aligned}$$

Similarly, we get that a.s. $\lim_{n \rightarrow \infty} V_5(n) = H_5(f)$, with the finite constant $H_5(f)$ defined by

$$H_5(f) = \sum_{0 \leq \ell < k < \infty} 2^{-\ell} \langle \mu, \tilde{f}_k \mathcal{Q}^{k-\ell} \tilde{f}_\ell \rangle.$$

Notice that $\Sigma_2^{\text{sub}}(f) = H_5(f) + H_6(f)$ is finite thanks to (5) and (42). This finishes the proof. □

Using similar arguments as in the proof of Lemma 4.4, we get the following result.

LEMMA 4.5. *Under the assumptions of Theorem 3.1, we have that $\Sigma_1^{\text{sub}}(f)$ in (19) is well defined and finite, and that a.s. $\lim_{n \rightarrow \infty} V_1(n) = \Sigma_1^{\text{sub}}(f)$.*

PROOF. Using (73), we get

$$(43) \quad V_1(n) = V_3(n) + V_4(n),$$

with

$$\begin{aligned} V_3(n) &= |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p_n}} \sum_{\ell=0}^{p_n} 2^{p_n-\ell} \mathcal{Q}^{p_n-\ell}(\tilde{f}_\ell^2)(X_i), \\ V_4(n) &= |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p_n}} \sum_{\ell=0}^{p_n-1} \sum_{k=0}^{p_n-\ell-1} 2^{p_n-\ell+k} \mathcal{Q}^{p_n-1-(\ell+k)}(\mathcal{P}(\mathcal{Q}^k \tilde{f}_\ell \otimes^2))(X_i). \end{aligned}$$

We consider the term $V_4(n)$. We have

$$(44) \quad V_4(n) = |\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}}(H_{4,n}),$$

with

$$(45) \quad H_{4,n} = \sum_{\ell \geq 0, k \geq 0} h_{\ell,k}^{(n)} \mathbf{1}_{\{\ell+k < p_n\}} \quad \text{and} \quad h_{\ell,k}^{(n)} = 2^{k-\ell} \mathcal{Q}^{p_n-1-(\ell+k)} (\mathcal{P}(\mathcal{Q}^k \tilde{f}_\ell \otimes^2)).$$

Using (4), we have that

$$\lim_{n \rightarrow \infty} h_{\ell,k}^{(n)} = h_{\ell,k},$$

where the constant $h_{\ell,k}$ is equal to $2^{k-\ell} \langle \mu, \mathcal{P}(\mathcal{Q}^k \tilde{f}_\ell \otimes^2) \rangle$. We also have that

$$\begin{aligned} |h_{\ell,k}^{(n)}| &\leq 2^{k-\ell} \alpha^{2k} \mathcal{Q}^{p_n-1-(\ell+k)} (\mathcal{P}(g \otimes g)) \\ &\leq 2^{k-\ell} \alpha^{2k} g_*, \end{aligned}$$

with $g_* \in F$ (which does not depend on n, ℓ and k) and where we used (5) for the first inequality and (3) (with f and g replaced by $\mathcal{P}(g \otimes g)$ and g_*). Taking the limit, we also deduce that

$$|h_{\ell,k}| \leq 2^{k-\ell} \alpha^{2k} g_*.$$

Define the constant

$$H_4(f) = \sum_{\ell \geq 0, k \geq 0} h_{\ell,k},$$

which is finite a.s.

$$(46) \quad \sum_{\ell \geq 0, k \geq 0} 2^{k-\ell} \alpha^{2k} = 2/(1 - 2\alpha^2) < +\infty.$$

Using (4) (with f and g replaced by $\mathcal{P}(\mathcal{Q}^k \tilde{f}_\ell \otimes^2)$ and $g_{\ell,k}$), we deduce that

$$|h_{\ell,k}^{(n)} - h_{\ell,k}| \leq 2^{k-\ell} \alpha^{p_n-1-(\ell+k)} g_{\ell,k}.$$

Set $r_0 \in \mathbb{N}$ and $g_{r_0} = \sum_{\ell \vee k \leq r_0} g_{\ell,k}$. Notice that g_{r_0} belongs to F . Furthermore, we have

$$\begin{aligned} |H_{4,n} - H_4(f)| &\leq \sum_{\ell \vee k \leq r_0} 2^{k-\ell} \alpha^{p_n-1-(\ell+k)} g_{r_0} + \sum_{\ell \vee k > r_0} (|h_{\ell,k}^{(n)}| \mathbf{1}_{\{\ell+k \leq p_n-1\}} + |h_{\ell,k}|) \\ &\leq (r_0 + 1)^2 2^{r_0} \alpha^{p_n-1-2r_0} g_{r_0} + \gamma_2(r_0) g_*, \end{aligned}$$

with $\gamma_2(r_0) = 2 \sum_{\ell \vee k > r_0} 2^{k-\ell} \alpha^{2k}$. Using (9) with n replaced by $n - p_n$ and f replaced by g_* and g_{r_0} , and that $\lim_{n \rightarrow \infty} \alpha_n^p = 0$ as well as $\lim_{n \rightarrow \infty} n - p_n = \infty$, we deduce that

$$\limsup_{n \rightarrow \infty} |\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}} (|H_{4,n} - H_4(f)|) \leq \gamma_2(r_0) \langle \mu, g_* \rangle.$$

Thanks to (46), we get by dominated convergence that $\lim_{r_0 \rightarrow \infty} \gamma_2(r_0) = 0$. We deduce that

$$\lim_{n \rightarrow \infty} |\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}} (|H_{4,n} - H_4(f)|) = 0.$$

Since $|\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}}(\cdot)$ is a probability measure, we deduce from (44) that a.s.

$$\lim_{n \rightarrow \infty} V_4(n) = \lim_{n \rightarrow \infty} |\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}}(H_{4,n}) = H_4(f) = \sum_{\ell \geq 0, k \geq 0} 2^{k-\ell} \langle \mu, \mathcal{P}(\mathcal{Q}^k \tilde{f}_\ell \otimes^2) \rangle.$$

Similarly, we get that a.s. $\lim_{n \rightarrow \infty} V_3(n) = H_3(f)$ with the finite constant $H_3(f)$ defined by

$$H_3(f) = \sum_{\ell \geq 0} 2^{-\ell} \langle \mu, \tilde{f}_\ell^2 \rangle.$$

Notice that $\Sigma_1^{\text{sub}}(f) = H_3(f) + H_4(f)$ is finite thanks to (5) and (46). This finishes the proof. \square

The next lemma is a direct consequence of (38) and Lemmas 4.3, 4.4 and 4.5.

LEMMA 4.6. *Under the assumptions of Theorem 3.1, we have the following convergence in probability $\lim_{n \rightarrow \infty} V(n) = \Sigma^{\text{sub}}(f)$, where, with $\Sigma_1^{\text{sub}}(f)$ and $\Sigma_2^{\text{sub}}(f)$ defined by (19) and (20):*

$$\Sigma^{\text{sub}}(f) = \Sigma^{\text{sub}}(f) = \Sigma_1^{\text{sub}}(f) + 2\Sigma_2^{\text{sub}}(f).$$

We now check the Lindeberg condition using a fourth moment condition. We set

$$(47) \quad R_3(n) = \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E}[\Delta_{n,i}(f)^4].$$

LEMMA 4.7. *Under the assumptions of Theorem 3.1, we have that $\lim_{n \rightarrow \infty} R_3(n) = 0$.*

PROOF. We have

$$\begin{aligned} R_3(n) &\leq 16 \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E}[N_{n,i}(f)^4] \\ &\leq 16(p_n + 1)^3 \sum_{\ell=0}^{p_n} \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E}[N_{n,i}^\ell(\tilde{f}_\ell)^4], \end{aligned}$$

where we used that $(\sum_{k=0}^r a_k)^4 \leq (r + 1)^3 \sum_{k=0}^r a_k^4$ for the two inequalities (resp. with $r = 1$ and $r = p_n$) and also the Jensen inequality and (31) for the first and (12) for the last. Using (11), we get

$$\mathbb{E}[N_{n,i}^\ell(\tilde{f}_\ell)^4] = |\mathbb{G}_n|^{-2} \mathbb{E}[h_{n,\ell}(X_i)] \quad \text{with } h_{n,\ell}(x) = \mathbb{E}_x[M_{\mathbb{G}_{p_n-\ell}}(\tilde{f}_\ell)^4].$$

Thanks to the fourth moment bound given in Lemma 7.2, the uniform bounds from (5) and the structural assumption 2.2, it is easy to get there exists $g_1 \in F$ such that for all $n \geq p_n \geq \ell \geq 0$,

$$(48) \quad |h_{n,\ell}| \leq 2^{2(p_n-\ell)} g_1.$$

We deduce that

$$\begin{aligned} R_3(n) &\leq 16n^3 \sum_{\ell=0}^{p_n} \sum_{i \in \mathbb{G}_{n-p_n}} |\mathbb{G}_n|^{-2} 2^{2(p_n-\ell)} \mathbb{E}[g_1(X_i)] \\ &\leq 16n^3 2^{-2(n-p_n)} \mathbb{E}[M_{\mathbb{G}_{n-p_n}}(g_1)] \\ &\leq 16n^3 2^{-(n-p_n)} \langle \nu, \mathcal{Q}^{n-p_n} g_1 \rangle, \end{aligned}$$

where we used (72) for the third inequality. Since g_1 belongs to F , we deduce from (3) that $\mathcal{Q}^{n-p_n} g_1 \leq g_2$ for some $g_2 \in F$ and all $n \geq p_n \geq 0$. This gives that

$$R_3(n) \leq 16n^3 2^{-(n-p_n)} \langle \nu, g_2 \rangle.$$

This ends the proof as $\lim_{n \rightarrow \infty} p_n = \infty$ and $\lim_{n \rightarrow \infty} n - p_n - \lambda \log(n) = +\infty$ for all $\lambda > 0$. \square

We can now use Theorem 3.2 and Corollary 3.1, page 58, and the remark, page 59, from [17] to deduce from Lemmas 4.6 and 4.7 that $\Delta_n(f)$ converges in distribution towards a Gaussian real-valued random variable with deterministic variance $\Sigma^{\text{sub}}(f)$ given by (18). Using (32) and Lemmas 4.1 and 4.2, we then deduce Theorem 3.1.

5. Proof of Theorem 3.4. We keep notation from Section 4. Let $(p_n, n \in \mathbb{N})$ be an increasing sequence of elements of \mathbb{N} such that (30) holds. Recall the definitions of $\Delta_n(f)$ and $N_{n,\emptyset}(f)$ from (31) and (32), as well as $R_0(n)$ and $R_1(n)$ from (33). We have the following elementary lemma.

LEMMA 5.1. *Under the assumptions of Theorem 3.4, we have the following convergence:*

$$\lim_{n \rightarrow \infty} n^{-1} \mathbb{E}[R_0(n)^2] = 0.$$

PROOF. Following the proof of Lemma 4.1, and using that $2\alpha^2 = 1$ so that $\sum_{\ell=0}^{k-1} (2\alpha^2)^\ell = k$, we get there exists some finite constant c depending on f such that $\mathbb{E}[M_{\mathbb{G}_k}(\tilde{f}_{n-k})^2] \leq c^2(k+1)2^k$ for all $k \geq 0$. This implies that

$$\begin{aligned} \mathbb{E}[R_0(n)^2]^{1/2} &\leq |\mathbb{G}_n|^{-1/2} \sum_{k=0}^{n-p_n-1} \mathbb{E}[M_{\mathbb{G}_k}(\tilde{f}_{n-k})^2]^{1/2} \\ &\leq c2^{-n/2} \sum_{k=0}^{n-p_n-1} \sqrt{k+1}2^{k/2} \leq Cc\sqrt{n}2^{-p_n/2}. \end{aligned}$$

Then use that $\lim_{n \rightarrow \infty} p_n/n = 1$ to conclude. \square

We have the following lemma.

LEMMA 5.2. *Under the assumptions of Theorem 3.4, we have the following convergence:*

$$\lim_{n \rightarrow \infty} n^{-1} \mathbb{E}[R_1(n)^2] = 0.$$

PROOF. Following the proof of Lemma 4.2 with the same notation, and using that $2\alpha^2 = 1$ so that $\sum_{k=0}^{n-p_n-1} (2\alpha^2)^k = n - p_n$ in (36), we get that there exists $g_3 \in F$ such that $\mathbb{E}[R_1(\ell, n)^2] \leq (n - p_n + 1)(2\alpha)^{-2\ell} \langle v, g_3 \rangle$, where $R_1(\ell, n)$ is defined in (35). As $2\alpha = \sqrt{2}$ and $R_1(n) = \sum_{\ell=0}^{p_n} R_1(\ell, n)$, we deduce that

$$(\mathbb{E}[R_1(n)^2])^{1/2} \leq \sum_{\ell=0}^{p_n} (\mathbb{E}[R_1(\ell, n)^2])^{1/2} \leq 4\sqrt{n - p_n + 1} \langle v, g_3 \rangle^{1/2}.$$

Use that $\lim_{n \rightarrow \infty} p_n/n = 1$ to conclude. \square

Recall $\Delta_n(f)$ defined in (31), and its bracket defined by $V(n) = \sum_{i \in \mathbb{G}_{n-p_n}} \mathbb{E}[\Delta_{n,i}(f)^2 | \mathcal{F}_i]$ defined in (37). Recall, see (38), that $V(n) = V_1(n) + 2V_2(n) - R_2(n)$. We study the convergence of each term of the latter right-hand side.

LEMMA 5.3. *Under the assumptions of Theorem 3.4, we have the following convergence:*

$$\lim_{n \rightarrow \infty} n^{-1/2} \mathbb{E}[R_2(n)] = 0.$$

PROOF. Following the proof of Lemma 4.3 with the same notation and using that $2\alpha^2 = 1$ so that $\sum_{\ell=0}^{p_n} (2\alpha)^\ell \leq C2^{p_n/2}$ in (39), we get that $\mathbb{E}[R_2(n)] \leq C\langle \nu, g_1 \rangle$, with $g_1 \in F$. This gives the result. \square

Recall $f_{k,\ell}^*$ defined in (26). For $k, \ell, r \in \mathbb{N}$, we will consider the \mathbb{C} -valued functions on S^2 ,

$$(49) \quad f_{k,\ell,r} = \left(\sum_{j \in J} \theta_j^r \mathcal{R}_j(f_k) \right) \otimes_{\text{sym}} \left(\sum_{j \in J} \theta_j^{r+k-\ell} \mathcal{R}_j(f_\ell) \right) \quad \text{and} \quad f_{k,\ell,r}^\circ = f_{k,\ell,r} - f_{k,\ell}^*.$$

LEMMA 5.4. *Under the assumptions of Theorem 3.4, we have that a.s.*

$$\lim_{n \rightarrow \infty} n^{-1} V_2(n) = \Sigma_2^{\text{crit}}(f)$$

with $\Sigma_2^{\text{crit}}(f)$ defined by (25) which is well defined and finite.

PROOF. We keep the decomposition (40) of $V_2(n) = V_5(n) + V_6(n)$ given in the proof of Lemma 4.4. We first consider the term $V_6(n)$ given in (41) by

$$(50) \quad V_6(n) = |\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}}(H_{6,n}),$$

with

$$H_{6,n} = \sum_{\substack{0 \leq \ell < k \leq p_n \\ r \geq 0}} h_{k,\ell,r}^{(n)} \mathbf{1}_{\{r+k < p_n\}} \quad \text{and} \\ h_{k,\ell,r}^{(n)} = 2^{r-\ell} Q^{p_n-1-(r+k)} (\mathcal{P}(Q^r \tilde{f}_k \otimes_{\text{sym}} Q^{k-\ell+r} \tilde{f}_\ell)).$$

We set

$$\bar{H}_{6,n} = \sum_{0 \leq \ell < k \leq p_n; r \geq 0} \bar{h}_{k,\ell,r}^{(n)} \mathbf{1}_{\{r+k < p_n\}}$$

where for $0 \leq \ell < k \leq p_n$ and $0 \leq r < p_n - k$,

$$\bar{h}_{k,\ell,r}^{(n)} = 2^{r-\ell} \alpha^{k-\ell+2r} Q^{p_n-1-(r+k)} (\mathcal{P} f_{k,\ell,r}) = 2^{-(k+\ell)/2} Q^{p_n-1-(r+k)} (\mathcal{P} f_{k,\ell,r}),$$

where we used that $2\alpha^2 = 1$. We have

$$\begin{aligned} |h_{k,\ell,r}^{(n)} - \bar{h}_{k,\ell,r}^{(n)}| &\leq 2^{r-\ell} Q^{p_n-1-(r+k)} (\mathcal{P}(|Q^r \tilde{f}_k \otimes_{\text{sym}} Q^{k-\ell+r} \tilde{f}_\ell - \alpha^{k-\ell+2r} f_{k,\ell,r}|)) \\ &\leq C2^{r-\ell} \beta_r \alpha^{k-\ell+2r} Q^{p_n-1-(r+k)} (\mathcal{P}(g \otimes g)) \\ &\leq C\beta_r 2^{-(k+\ell)/2} g_1^*, \end{aligned}$$

where we wrote (with r' and f replaced by r and f_k and by $k - \ell + r$ and f_ℓ) that

$$Q^{r'} \tilde{f} = Q^{r'} \hat{f} + \alpha^{r'} \sum_{j \in J} \theta_j^{r'} \mathcal{R}_j(f)$$

and used (27), (28) and that $(\beta_n, n \in \mathbb{N})$ is nondecreasing for the second inequality and used (3) (with f and g replaced by $\mathcal{P}(g \otimes g)$ and g_1^*) for the last. We deduce that

$$|H_{6,n} - \bar{H}_{6,n}| \leq \sum_{0 \leq \ell < k \leq p_n, r \geq 0} |h_{k,\ell,r}^{(n)} - \bar{h}_{k,\ell,r}^{(n)}| \mathbf{1}_{\{r+k < p_n\}} \leq C \left(\sum_{r=0}^n \beta_r \right) g_1^*.$$

As $\lim_{n \rightarrow \infty} \beta_n = 0$, we get that $\lim_{n \rightarrow \infty} n^{-1} \sum_{r=0}^n \beta_r = 0$. We deduce from (9) that a.s.

$$(51) \quad \lim_{n \rightarrow \infty} n^{-1} |\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}}(|H_{6,n} - \bar{H}_{6,n}|) = 0.$$

We set $H_6^{[n]} = \sum_{0 \leq \ell < k \leq p_n; r \geq 0} h_{k,\ell,r} \mathbf{1}_{\{r+k < p_n\}}$ with for $0 \leq \ell < k \leq p_n$ and $0 \leq r < p_n - k$,

$$h_{k,\ell,r} = 2^{-(k+\ell)/2} \langle \mu, \mathcal{P} f_{k,\ell,r} \rangle = \langle \mu, \bar{h}_{k,\ell,r}^{(n)} \rangle.$$

Notice that

$$\begin{aligned} & |\bar{h}_{k,\ell,r}^{(n)} - h_{k,\ell,r}| \\ & \leq 2^{-(k+\ell)/2} \sum_{j,j' \in J} |\mathcal{Q}^{p_n-1-(r+k)}(\mathcal{P}(\mathcal{R}_j f_k \otimes_{\text{sym}} \mathcal{R}_{j'} f_\ell)) - \langle \mu, \mathcal{P}(\mathcal{R}_j f_k \otimes_{\text{sym}} \mathcal{R}_{j'} f_\ell) \rangle| \\ & \leq 2^{-(k+\ell)/2} \alpha^{p_n-1-(r+k)} \sum_{j,j' \in J} g_{k,\ell,j,j'} \\ & = 2^{-(k+\ell)/2} \alpha^{p_n-1-(r+k)} g_{k,\ell}, \end{aligned}$$

where we used (4) (with f and g replaced by $\mathcal{P}(\mathcal{R}_j f_k \otimes_{\text{sym}} \mathcal{R}_{j'} f_\ell)$ and $g_{k,\ell,j,j'}$) for the second inequality and $g_{k,\ell} = \sum_{j,j' \in J} g_{k,\ell,j,j'}$ for the equality. We have that $g_{k,\ell}$ belongs to F . Since $|\mathcal{P} f_{k,\ell,r}| \leq \mathcal{P}|f_{k,\ell,r}| \leq 4\mathcal{P}(g \otimes g)$, thanks to the fourth inequality in (28), we deduce from (3) (with f and g replaced by $4\mathcal{P}(g \otimes g)$ and g_2^*) that for all $0 \leq \ell < k$ and $0 \leq r < p_n - k$,

$$|\bar{h}_{k,\ell,r}^{(n)}| \leq 2^{-(k+\ell)/2} g_2^* \quad \text{and} \quad |h_{k,\ell,r}| \leq 2^{-(k+\ell)/2} \langle \mu, g_2^* \rangle.$$

Set $r_0 \in \mathbb{N}$ and $g_{r_0} = \sum_{0 \leq \ell < k \leq r_0} g_{k,\ell}$. Notice that g_{r_0} belongs to F and is nonnegative. Furthermore, we have for n large enough so that $p_n > 2r_0$,

$$\begin{aligned} |\bar{H}_{6,n} - H_6^{[n]}| & \leq \sum_{\substack{0 \leq \ell < k \leq p_n \\ r \geq 0}} |\bar{h}_{k,\ell,r}^{(n)} - h_{k,\ell,r}| \mathbf{1}_{\{r+k < p_n\}} \\ & \leq \sum_{0 \leq \ell < k \leq r_0} \sum_{r=0}^{p_n-k-1} 2^{-(k+\ell)/2} \alpha^{p_n-1-(r+k)} g_{r_0} \\ & \quad + \sum_{\substack{0 \leq \ell < k \leq p_n \\ k > r_0}} \sum_{r=0}^{p_n-k-1} (|\bar{h}_{k,\ell,r}^{(n)}| + |h_{k,\ell,r}|) \mathbf{1}_{\{r+k < p_n\}} \\ & \leq C g_{r_0} + \sum_{0 \leq \ell < k \leq p_n, k > r_0} (p_n - k) 2^{-(k+\ell)/2} (g_2^* + \langle \mu, g_2^* \rangle) \\ & \leq C g_{r_0} + C n 2^{-r_0/2} (g_2^* + \langle \mu, g_2^* \rangle). \end{aligned}$$

We deduce that

$$\limsup_{n \rightarrow \infty} n^{-1} |\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}} (|\bar{H}_{6,n} - H_6^{[n]}|) \leq C 2^{-r_0/2} \langle \mu, g_2^* \rangle.$$

Since r_0 can be arbitrary large, we get that

$$(52) \quad \lim_{n \rightarrow \infty} n^{-1} |\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}} (|\bar{H}_{6,n} - H_6^{[n]}|) = 0.$$

We set for $k, \ell \in \mathbb{N}$,

$$h_{k,\ell}^* = 2^{-(k+\ell)/2} \langle \mu, \mathcal{P}(f_{k,\ell}^*) \rangle.$$

Using the last inequality in (28) and the definition (26) of $f_{k,\ell}^*$, we deduce there exists a finite constant c independent of n such that, for all $k, \ell \in \mathbb{N}$, $|h_{k,\ell}^*| \leq c2^{-(k+\ell)/2}$. This implies that $H_0^* = \sum_{0 \leq \ell < k} (k+1)|h_{k,\ell}^*|$ is finite and (see (25)) the sum

$$H_6^*(f) = \sum_{0 \leq \ell < k < \infty} h_{k,\ell}^* = \Sigma_2^{\text{crit}}(f)$$

is well defined and finite. We write

$$h_{k,\ell,r} = h_{k,\ell}^* + h_{k,\ell,r}^\circ,$$

with

$$h_{k,\ell,r}^\circ = 2^{-(k+\ell)/2} \langle \mu, \mathcal{P} f_{k,\ell,r}^\circ \rangle,$$

where we recall that $f_{k,\ell,r}^\circ = f_{k,\ell,r} - f_{k,\ell}^*$, and

$$(53) \quad H_6^{[n]} = H_6^{[n],*} + H_6^{[n],\circ}$$

with

$$H_6^{[n],*} = \sum_{0 \leq \ell < k \leq p_n} (p_n - k) h_{k,\ell}^* \quad \text{and} \quad H_6^{[n],\circ} = \sum_{0 \leq \ell < k \leq p_n; r \geq 0} h_{k,\ell,r}^\circ \mathbf{1}_{\{r+k < p_n\}}.$$

Recall $\lim_{n \rightarrow \infty} p_n/n = 1$. We have

$$|n^{-1} H_6^{[n],*} - H_6^*(f)| \leq |n^{-1} p_n - 1| |H_6^*(f)| + n^{-1} H_0^* + \sum_{\substack{0 \leq \ell < k < \infty \\ k > p_n}} |h_{k,\ell}^*|,$$

so that $\lim_{n \rightarrow \infty} |n^{-1} H_6^{[n],*} - H_6^*(f)| = 0$ and thus

$$(54) \quad \lim_{n \rightarrow \infty} n^{-1} H_6^{[n],*} = H_6^*(f).$$

We now prove that $n^{-1} H_6^{[n],\circ}$ converges towards 0. We have

$$(55) \quad f_{k,\ell,r}^\circ = \sum_{j, j' \in J, \theta_j \theta_{j'} \neq 1} (\theta_{j'} \theta_j)^r \theta_{j'}^{k-\ell} \mathcal{R}_j f_k \otimes_{\text{sym}} \mathcal{R}_{j'} f_\ell.$$

This gives

$$\begin{aligned} |H_6^{[n],\circ}| &= \left| \sum_{0 \leq \ell < k \leq p_n, r \geq 0} 2^{-(k+\ell)/2} \langle \mu, \mathcal{P} f_{k,\ell,r}^\circ \rangle \mathbf{1}_{\{r+k < p_n\}} \right| \\ &\leq \sum_{0 \leq \ell < k \leq p_n} 2^{-(k+\ell)/2} \sum_{j, j' \in J, \theta_j \theta_{j'} \neq 1} \left| \langle \mu, \mathcal{P}(\mathcal{R}_j f_k \otimes_{\text{sym}} \mathcal{R}_{j'} f_\ell) \rangle \right| \left| \sum_{r=0}^{p_n-k-1} (\theta_{j'} \theta_j)^r \right| \\ &\leq c, \end{aligned}$$

with $c = c_J^2 \langle \mu, \mathcal{P}(g \otimes g) \rangle \sum_{0 \leq \ell < k \leq p_n} 2^{-(k+\ell)/2} \sum_{j, j' \in J, \theta_j \theta_{j'} \neq 1} |1 - \theta_j \theta_{j'}|^{-1}$, and where we used (55) for the first inequality, the last inequality of (28) for the second. Since J is finite, we deduce that c is finite. This gives that $\lim_{n \rightarrow \infty} n^{-1} H_6^{[n],\circ} = 0$. Recall that $H_6^{[n]}$ and $H_6^*(f)$ are complex numbers (i.e., constant functions). Use (53) and (54) to get that

$$\lim_{n \rightarrow \infty} n^{-1} H_6^{[n]} = H_6^*(f)$$

so that, as $|\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}}(\cdot)$ is a probability measure, a.s.

$$(56) \quad \lim_{n \rightarrow \infty} n^{-1} |\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}}(H_6^{[n]}) = H_6^*(f).$$

In conclusion, use (51), (52), (56) and the definition (50) of $V_6(n)$ to deduce that a.s.

$$\lim_{n \rightarrow \infty} n^{-1} V_6(n) = H_6^*(f) = \sum_{0 \leq \ell < k < \infty} 2^{-(k+\ell)/2} \langle \mu, \mathcal{P} f_{k,\ell}^* \rangle = \Sigma_2^{\text{crit}}(f),$$

where $f_{k,\ell}^*$ is defined in (26) and $\Sigma_2^{\text{crit}}(f)$ in (25). Recall that

$$V_5(n) = |\mathbb{G}_n|^{-1} \sum_{i \in \mathbb{G}_{n-p_n}} \sum_{0 \leq \ell < k \leq p_n} 2^{p_n-\ell} \mathcal{Q}^{p_n-k} (\tilde{f}_k \mathcal{Q}^{k-\ell} \tilde{f}_\ell)(X_i) = |\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}}(\Phi_n),$$

where

$$\Phi_n = \sum_{0 \leq \ell < k \leq p_n} 2^{-\ell} \mathcal{Q}^{p_n-k} (\tilde{f}_k \mathcal{Q}^{k-\ell} \tilde{f}_\ell).$$

We have

$$|\Phi_n| \leq \sum_{0 \leq \ell < k \leq p_n} 2^{-\ell} \alpha^{k-\ell} \mathcal{Q}^{p_n-k}(g^2) \leq \sum_{0 \leq \ell < k \leq p_n} 2^{-(k+\ell)/2} g_1 \leq C g_1,$$

where we used (4) for the first inequality and (3) (with f and g replaced by g^2 and g_1) in the second. Then, use (9) to conclude that a.s.

$$\lim_{n \rightarrow \infty} n^{-1} V_5(n) = 0.$$

This ends the proof of the lemma. \square

Using similar arguments as in the proof of Lemma 5.4, we get the following result.

LEMMA 5.5. *Under the assumptions of Theorem 3.4, we have that a.s.*

$$\lim_{n \rightarrow \infty} n^{-1} V_1(n) = \Sigma_1^{\text{crit}}(f)$$

with $\Sigma_1^{\text{crit}}(f)$ defined by (24) which is well defined and finite.

PROOF. We recall $V_1(n) = V_3(n) + V_4(n)$, see (43) and thereafter for the definition of $V_3(n)$ and $V_4(n)$. We first consider the term $V_3(n)$. Recall that $V_3(n) = |\mathbb{G}_{n-p}| M_{\mathbb{G}_{n-p_n}}(\Phi_n)$ with $\Phi_n = \sum_{\ell=0}^{p_n} 2^{-\ell} \mathcal{Q}^{p_n-\ell}(\tilde{f}_\ell^2)$. We have $\tilde{f}_\ell^2 \leq g^2$ and $\mathcal{Q}^{p_n-\ell}(g^2) \leq g_1$ for some $g_1 \in F$ and thus $|\Phi_n| \leq 2g_1$. We therefore deduce that a.s. $\lim_{n \rightarrow \infty} n^{-1} V_3(n) = 0$.

We consider the term $V_4(n) = |\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}}(H_{4,n})$ (see (44)) with $H_{4,n}$ given by (45),

$$H_{4,n} = \sum_{\ell \geq 0, k \geq 0} h_{\ell,k}^{(n)} \mathbf{1}_{\{\ell+k < p_n\}} \quad \text{and} \quad h_{\ell,k}^{(n)} = 2^{k-\ell} \mathcal{Q}^{p_n-1-(\ell+k)}(\mathcal{P}(\mathcal{Q}^k \tilde{f}_\ell \otimes^2)).$$

Recall $f_{\ell,\ell,k}$ defined in (49). We set $\bar{H}_{4,n} = \sum_{\ell \geq 0, k \geq 0} \bar{h}_{\ell,k}^{(n)} \mathbf{1}_{\{\ell+k < p_n\}}$ with

$$\bar{h}_{\ell,k}^{(n)} = 2^{k-\ell} \alpha^{2k} \mathcal{Q}^{p_n-1-(\ell+k)}(\mathcal{P} f_{\ell,\ell,k}) = 2^{-\ell} \mathcal{Q}^{p_n-1-(\ell+k)}(\mathcal{P} f_{\ell,\ell,k}),$$

where we used that $2\alpha^2 = 1$. We have

$$\begin{aligned} |h_{\ell,k}^{(n)} - \bar{h}_{\ell,k}^{(n)}| &\leq 2^{k-\ell} \mathcal{Q}^{p_n-1-(\ell+k)}(\mathcal{P}(|\mathcal{Q}^k \tilde{f}_\ell \otimes \mathcal{Q}^k \tilde{f}_\ell - \alpha^{2k} f_{\ell,\ell,k}|)) \\ &\leq C 2^{k-\ell} \beta_k \alpha^{2k} \mathcal{Q}^{p_n-1-(\ell+k)}(\mathcal{P}(g \otimes g)) \\ &\leq \beta_k 2^{-\ell} g_1^*, \end{aligned}$$

with $g_1^* \in F$, where we used (27), with the representation $\mathcal{Q}^k \tilde{f}_\ell = \mathcal{Q}^k \hat{f}_\ell + \alpha^k \sum_{j \in J} \theta_j^k \mathcal{R}_j(f_\ell)$, (28) for the second inequality and (3) for the last. We deduce that

$$|H_{4,n} - \bar{H}_{4,n}| \leq \sum_{\ell \geq 0, k \geq 0} |h_{\ell,k}^{(n)} - \bar{h}_{\ell,k}^{(n)}| \mathbf{1}_{\{\ell+k < p_n\}} \leq 2 \left(\sum_{k=0}^n \beta_k \right) g_1^*.$$

As $\lim_{n \rightarrow \infty} \beta_n = 0$, we get that $\lim_{n \rightarrow \infty} n^{-1} \sum_{k=0}^n \beta_k = 0$. We deduce from (9) that a.s.

$$(57) \quad \lim_{n \rightarrow \infty} n^{-1} |\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}} (|H_{4,n} - \bar{H}_{4,n}|) = 0.$$

We set $H_4^{[n]} = \sum_{\ell \geq 0, k \geq 0} h_{\ell,k} \mathbf{1}_{\{\ell+k < p_n\}}$ with

$$h_{\ell,k} = 2^{-\ell} \langle \mu, \mathcal{P} f_{\ell,\ell,k} \rangle.$$

Notice that

$$\begin{aligned} |\bar{h}_{\ell,k}^{(n)} - h_{\ell,k}| &\leq 2^{-\ell} \sum_{j,j' \in J} |\mathcal{Q}^{p_n-1-(\ell+k)} (\mathcal{P}(\mathcal{R}_j f_\ell \otimes_{\text{sym}} \mathcal{R}_{j'} f_\ell)) - \langle \mu, \mathcal{P}(\mathcal{R}_j f_\ell \otimes_{\text{sym}} \mathcal{R}_{j'} f_\ell) \rangle| \\ &\leq 2^{-\ell} \alpha^{p_n-1-(\ell+k)} \sum_{j,j' \in J} g_{\ell,j,j'} \\ &= 2^{-\ell} \alpha^{p_n-1-(\ell+k)} g_\ell, \end{aligned}$$

where we used (4) (with f and g replaced by $\mathcal{P}(\mathcal{R}_j f_\ell \otimes \mathcal{R}_{j'} f_\ell)$ and $g_{\ell,j,j'}$) for the second inequality and $g_\ell = \sum_{j,j' \in J} g_{\ell,j,j'}$ for the equality. We have that g_ℓ belongs to F . Since $|\mathcal{P} f_{\ell,\ell,k}| \leq \mathcal{P} |f_{\ell,\ell,k}| \leq 4\mathcal{P}(g \otimes g)$, thanks to the fourth inequality in (28), we deduce from (3) (with f and g replaced by $4\mathcal{P}(g \otimes g)$ and g_2^*) that

$$|\bar{h}_{\ell,k}^{(n)}| \leq 2^{-\ell} g_2^* \quad \text{and} \quad |h_{\ell,k}| \leq 2^{-\ell} \langle \mu, g_2^* \rangle.$$

Set $r_0 \in \mathbb{N}$ and $g_{r_0} = \sum_{0 \leq \ell \leq r_0} g_\ell$. Notice that g_{r_0} belongs to F and is nonnegative. Furthermore, we have for n large enough so that $p_n > 2r_0$,

$$\begin{aligned} |\bar{H}_{4,n} - H_4^{[n]}| &\leq \sum_{\ell \geq 0, k \geq 0} |\bar{h}_{\ell,k}^{(n)} - h_{\ell,k}| \mathbf{1}_{\{\ell+k < p_n\}} \\ &\leq \sum_{0 \leq \ell \leq r_0, k \geq 0} 2^{-\ell} \alpha^{p_n-1-(\ell+k)} g_{r_0} \mathbf{1}_{\{\ell+k < p_n\}} \\ &\quad + \sum_{\ell > r_0, k \geq 0} (|\bar{h}_{\ell,k}^{(n)}| + |h_{\ell,k}|) \mathbf{1}_{\{\ell+k < p_n\}} \\ &\leq C g_{r_0} + \sum_{\ell > r_0} (p_n - \ell) 2^{-\ell} (g_2^* + \langle \mu, g_2^* \rangle) \mathbf{1}_{\{\ell < p_n\}} \\ &\leq C g_{r_0} + n 2^{-r_0} (g_2^* + \langle \mu, g_2^* \rangle). \end{aligned}$$

We deduce that

$$\limsup_{n \rightarrow \infty} n^{-1} |\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}} (|\bar{H}_{4,n} - H_4^{[n]}|) \leq 2^{1-r_0} \langle \mu, g_2^* \rangle.$$

Since r_0 can be arbitrary large, we get that a.s.

$$(58) \quad \lim_{n \rightarrow \infty} n^{-1} |\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}} (|\bar{H}_{4,n} - H_4^{[n]}|) = 0.$$

Now we study the limit of $H_4^{[n]}$. We set for $k, \ell \in \mathbb{N}$,

$$h_\ell^* = 2^{-\ell} \langle \mu, \mathcal{P} f_{\ell,\ell}^* \rangle.$$

Using the last inequality in (28) and the definition (26) of $f_{\ell,\ell}^*$, we deduce that there exists a finite constant c independent of n (but depending on f) such that, for all $\ell \in \mathbb{N}$, $|h_\ell^*| \leq c2^{-\ell}$. This implies that $H_0^* = \sum_{\ell \geq 0} (\ell + 1)|h_\ell^*|$ is finite and the sum

$$H_4^*(f) = \sum_{\ell \geq 0} h_\ell^*$$

is well defined and finite. We write

$$h_{\ell,k} = h_\ell^* + h_{\ell,k}^\circ,$$

with $h_{\ell,k}^\circ = 2^{-\ell} \langle \mu, \mathcal{P} f_{\ell,\ell,k}^\circ \rangle$, where $f_{\ell,\ell,k}^\circ = f_{\ell,\ell,k} - f_{\ell,\ell}^*$ is defined in (49), and

$$H_4^{[n]} = H_4^{[n],*} + H_4^{[n],\circ},$$

with $H_4^{[n],*} = \sum_{\ell \geq 0} (p_n - \ell)h_\ell^*$ and $H_4^{[n],\circ} = \sum_{\ell \geq 0, k \geq 0} h_{\ell,k}^\circ \mathbf{1}_{\{\ell+k < p_n\}}$. We have

$$|n^{-1} H_4^{[n],*} - H_4^*(f)| \leq |n^{-1} p_n - 1| H_4^*(f) + n^{-1} H_0^* + \sum_{\ell > p_n} |h_\ell^*|,$$

so that $\lim_{n \rightarrow \infty} |n^{-1} H_4^{[n],*} - H_4^*(f)| = 0$ and thus

$$(59) \quad \lim_{n \rightarrow \infty} n^{-1} H_4^{[n],*} = H_4^*(f).$$

We now prove that $n^{-1} H_4^{[n],\circ}$ converges towards 0. We have

$$(60) \quad f_{\ell,\ell,k}^\circ = \sum_{j,j' \in J, \theta_j \theta_{j'} \neq 1} (\theta_{j'} \theta_j)^k \mathcal{R}_{j'} f_\ell \otimes_{\text{sym}} \mathcal{R}_j f_\ell.$$

This gives

$$\begin{aligned} |H_4^{[n],\circ}| &= \left| \sum_{\ell \geq 0, k \geq 0} 2^{-\ell} \langle \mu, \mathcal{P} f_{\ell,\ell,k}^\circ \rangle \mathbf{1}_{\{\ell+k < p_n\}} \right| \\ &\leq \sum_{\ell \geq 0} 2^{-\ell} \sum_{j,j' \in J, \theta_j \theta_{j'} \neq 1} |\langle \mu, \mathcal{P}(\mathcal{R}_{j'} f_\ell \otimes \mathcal{R}_j f_\ell) \rangle| \left| \sum_{k=0}^{p_n-\ell-1} (\theta_{j'} \theta_j)^k \right| \\ &\leq c, \end{aligned}$$

with $c = c_J^2 \langle \mu, \mathcal{P}(g \otimes g) \rangle \sum_{\ell \geq 0} 2^{-\ell} \sum_{j,j' \in J, \theta_j \theta_{j'} \neq 1} |1 - \theta_{j'} \theta_j|^{-1}$, and where we used (60) for the first inequality, the last inequality of (28) for the second. Since J is finite, we deduce that c is finite. This gives that $\lim_{n \rightarrow \infty} n^{-1} H_4^{[n],\circ} = 0$. Recall that $H_4^{[n]}$ and $H_4^*(f)$ are constant complex numbers. Use (59) to get that

$$\lim_{n \rightarrow \infty} n^{-1} H_4^{[n]} = H_4^*(f)$$

so that, as $|\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}}(\cdot)$ is a probability measure, a.s.

$$(61) \quad \lim_{n \rightarrow \infty} n^{-1} |\mathbb{G}_{n-p_n}|^{-1} M_{\mathbb{G}_{n-p_n}}(H_4^{[n]}) = H_4^*(f).$$

In conclusion, use (57), (58), (61) and the definition (44) of $V_4(n)$ to deduce that a.s.

$$\lim_{n \rightarrow \infty} n^{-1} V_4(n) = H_4^*(f) = \sum_{\ell \geq 0} 2^{-\ell} \langle \mu, \mathcal{P} f_{\ell,\ell}^* \rangle = \Sigma_1^{\text{crit}}(f),$$

where $f_{\ell,\ell}^*$ is defined in (26) and $\Sigma_1^{\text{crit}}(f)$ in (24). \square

The proof of the next lemma is a direct consequence of (38) and Lemmas 5.3, 5.5 and 5.4.

LEMMA 5.6. *Under the assumptions of Theorem 3.4, we have the following convergence in probability:*

$$\lim_{n \rightarrow \infty} n^{-1} V(n) = \Sigma_1^{\text{crit}}(f) + 2\Sigma_2^{\text{crit}}(f),$$

where $\Sigma_1^{\text{crit}}(f)$ and $\Sigma_2^{\text{crit}}(f)$, defined by (24) and (25), are well defined and finite.

We now check the Lindeberg condition. Recall $R_3(n)$ defined in (47).

LEMMA 5.7. *Under the assumptions of Theorem 3.4, we have that $\lim_{n \rightarrow \infty} n^{-2} R_3(n) = 0$.*

PROOF. Keeping the notation of Lemma 4.7, using Lemma 7.2 (with the main contribution coming from $\psi_{8,n}$ and $\psi_{9,n}$ therein), we get (compare with (48)) that for $n \geq p_n \geq \ell \geq 0$,

$$|h_{n,\ell}| \leq (p_n - \ell)^2 2^{2(p_n - \ell)} g_1,$$

with $h_{n,\ell}(x) = \mathbb{E}_x[M_{\mathbb{G}_{p_n - \ell}}(\tilde{f}_\ell)^4]$ and $g_1 \in F$. Following the proof of Lemma 4.7, we get that

$$n^{-2} R_3(n) \leq 16n^3 2^{-(n - p_n)} \langle v, g_2 \rangle.$$

This ends the proof as $\lim_{n \rightarrow \infty} p_n = \infty$ and $\lim_{n \rightarrow \infty} n - p_n - \lambda \log(p_n) = +\infty$ for all $\lambda > 0$. □

The proof of Theorem 3.4 mimics then the proof of Theorem 3.1.

6. Proof of Lemma 3.8 and of Theorem 3.9.

6.1. *Proof of Lemma 3.8.* Let $f \in F$ and $j \in J$. Use that $\mathcal{R}_j(F) \subset \mathbb{C}F$ to deduce that $\mathbb{E}[|M_{n,j}(f)|^2]$ is finite. We have, for $n \in \mathbb{N}^*$,

$$\begin{aligned} \mathbb{E}[M_{n,j}(f) | \mathcal{H}_{n-1}] &= (2\alpha_j)^{-n} \sum_{i \in \mathbb{G}_{n-1}} \mathbb{E}[\mathcal{R}_j f(X_{i_0}) + \mathcal{R}_j f(X_{i_1}) | \mathcal{H}_{n-1}] \\ &= (2\alpha_j)^{-n} \sum_{i \in \mathbb{G}_{n-1}} 2Q\mathcal{R}_j f(X_i) \\ &= (2\alpha_j)^{-(n-1)} \sum_{i \in \mathbb{G}_{n-1}} \mathcal{R}_j f(X_i) \\ &= M_{n-1,j}(f), \end{aligned}$$

where the second equality follows from branching Markov property. This gives that $M_j(f)$ is a \mathcal{H} -martingale. We also have, writing f_j for $\mathcal{R}_j(f)$,

$$\begin{aligned} \mathbb{E}[|M_{n,j}(f)|^2] &= (2\alpha)^{-2n} \mathbb{E}[M_{\mathbb{G}_n}(f_j) M_{\mathbb{G}_n}(\bar{f}_j)] \\ &= (2\alpha^2)^{-n} \langle v, Q^n(|f_j|^2) \rangle \\ &\quad + (2\alpha)^{-2n} \sum_{k=0}^{n-1} 2^{n+k} \langle v, Q^{n-k-1} \mathcal{P}(Q^k f_j \otimes_{\text{sym}} Q^k \bar{f}_j) \rangle \\ (62) \quad &= (2\alpha^2)^{-n} \langle v, Q^n(|f_j|^2) \rangle \\ &\quad + (2\alpha^2)^{-n} \sum_{k=0}^{n-1} (2\alpha^2)^k \langle v, Q^{n-k-1} \mathcal{P}(f_j \otimes_{\text{sym}} \bar{f}_j) \rangle \end{aligned}$$

$$\begin{aligned} &\leq (2\alpha^2)^{-n} \langle \nu, g_1 \rangle + (2\alpha^2)^{-n} \sum_{k=0}^{n-1} (2\alpha^2)^k \langle \nu, g_2 \rangle \\ &\leq \langle \nu, g_3 \rangle, \end{aligned}$$

where we used the definition of $M_{n,j}$ for the first equality, (74) with $m = n$ for the second equality, the fact that f_j (resp. \bar{f}_j) is an eigenfunction associated to the eigenvalue α_j (resp. $\bar{\alpha}_j$) for the third equality, (3) twice (with f and g replaced by $|f_j|^2$ and g_1 and by $\mathcal{P}(f_j \otimes_{\text{sym}} \bar{f}_j)$ and g_2) for the first inequality and $2\alpha^2 > 1$ as well as $g_3 = g_1 + g_2 / (2\alpha^2 - 1)$ for the last inequality. Since g_3 belongs to F and does not depend on n , this implies that $\sup_{n \in \mathbb{N}} \mathbb{E}[|M_{n,j}(f)|^2] < +\infty$. Thus the martingale $M_j(f)$ converges a.s. and in L^2 towards a limit.

6.2. *Proof of Theorem 3.9.* Recall the sequence $(\beta_n, n \in \mathbb{N})$ defined in Assumption 2.6 and the σ -field $\mathcal{H}_n = \sigma\{X_u, u \in \mathbb{T}_n\}$. Let $(\hat{p}_n, n \in \mathbb{N})$ be a sequence of integers such that \hat{p}_n is even and (for $n \geq 3$),

$$(63) \quad \frac{5n}{6} < \hat{p}_n < n, \quad \lim_{n \rightarrow \infty} (n - \hat{p}_n) = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \alpha^{-(n-\hat{p}_n)} \beta_{\hat{p}_n/2} = 0.$$

Notice such sequences exist. We deduce from (16) that

$$(64) \quad N_{n,\emptyset}(f) = R_0(n) + R_4(n) + T_n(f),$$

with notation from (32) and (33),

$$\begin{aligned} R_0(n) &= |\mathbb{G}_n|^{-1/2} \sum_{k=0}^{n-\hat{p}_n-1} M_{\mathbb{G}_k}(\tilde{f}_{n-k}), \\ T_n(f) = R_1(n) &= \sum_{i \in \mathbb{G}_{n-\hat{p}_n}} \mathbb{E}[N_{n,i}(f) | \mathcal{H}_{n-\hat{p}_n}], \\ R_4(n) = \Delta_n &= \sum_{i \in \mathbb{G}_{n-\hat{p}_n}} (N_{n,i}(f) - \mathbb{E}[N_{n,i}(f) | \mathcal{H}_{n-\hat{p}_n}]). \end{aligned}$$

Furthermore, using the branching Markov property, we get for all $i \in \mathbb{G}_{n-\hat{p}_n}$,

$$(65) \quad \mathbb{E}[N_{n,i}(f) | \mathcal{H}_{n-\hat{p}_n}] = \mathbb{E}[N_{n,i}(f) | X_i].$$

We have the following elementary lemma.

LEMMA 6.1. *Under the assumptions of Theorem 3.9, we have the following convergence:*

$$\lim_{n \rightarrow \infty} (2\alpha^2)^{-n} \mathbb{E}[R_0(n)^2] = 0.$$

PROOF. We follow the proof of Lemma 4.1. As $2\alpha^2 > 1$, we get that $\mathbb{E}[M_{\mathbb{G}_k}(\tilde{f}_{n-k})^2] \leq 2^k (2\alpha^2)^k \langle \nu, g \rangle$ for some $g \in F$ and all $n \geq k \geq 0$. This implies, see (34), that for some constant C which does not depend on n or \hat{p}_n ,

$$\mathbb{E}[R_0(n)^2]^{1/2} \leq C 2^{-\hat{p}_n/2} (2\alpha^2)^{(n-\hat{p}_n)/2}.$$

It follows from the previous inequality that $(2\alpha^2)^{-n} \mathbb{E}[R_0(n)^2] \leq C (2\alpha)^{-2\hat{p}_n}$. Then use $2\alpha > 1$ and $\lim_{n \rightarrow \infty} \hat{p}_n = \infty$ to conclude. \square

Next, we have the following lemma.

LEMMA 6.2. *Under the assumptions of Theorem 3.9, we have the following convergence:*

$$\lim_{n \rightarrow \infty} (2\alpha^2)^{-n} \mathbb{E}[R_4(n)^2] = 0.$$

PROOF. First we have

$$\begin{aligned} \mathbb{E}[R_4(n)^2] &= \mathbb{E}\left[\left(\sum_{i \in \mathbb{G}_{n-\hat{p}_n}} (N_{n,i}(f) - \mathbb{E}[N_{n,i}(f)|X_i])\right)^2\right] \\ (66) \quad &= \mathbb{E}\left[\sum_{i \in \mathbb{G}_{n-\hat{p}_n}} \mathbb{E}[(N_{n,i}(f) - \mathbb{E}[N_{n,i}(f)|X_i])^2 | \mathcal{H}_{n-\hat{p}_n}]\right] \\ &\leq \mathbb{E}\left[\sum_{i \in \mathbb{G}_{n-\hat{p}_n}} \mathbb{E}[N_{n,i}(f)^2 | X_i]\right], \end{aligned}$$

where we used (65) for the first equality and the branching Markov chain property for the second and the last inequality. Note that for all $i \in \mathbb{G}_{n-\hat{p}_n}$ we have

$$\begin{aligned} \mathbb{E}[\mathbb{E}[N_{n,i}(f)^2 | X_i]] &= |\mathbb{G}_n|^{-1} \mathbb{E}\left[\left(\sum_{\ell=0}^{\hat{p}_n} M_{i\mathbb{G}_{\hat{p}_n-k}}(\tilde{f}_\ell)\right)^2 | X_i\right] \\ &\leq |\mathbb{G}_n|^{-1} \left(\sum_{\ell=0}^{\hat{p}_n} \mathbb{E}_{X_i}[M_{\mathbb{G}_{\hat{p}_n-\ell}}(\tilde{f}_\ell)^2]^{1/2}\right)^2, \end{aligned}$$

where we used the definition of $N_{n,i}(f)$ for the equality and the Minkowski's inequality for the last inequality. We have

$$\begin{aligned} \mathbb{E}_{X_i}[M_{\mathbb{G}_{\hat{p}_n-\ell}}(\tilde{f}_\ell)^2] &= 2^{\hat{p}_n-\ell} Q^{\hat{p}_n-\ell}(\tilde{f}_\ell^2)(X_i) \\ &\quad + \sum_{k=0}^{\hat{p}_n-\ell-1} 2^{\hat{p}_n-\ell+k} Q^{\hat{p}_n-\ell-k-1}(\mathcal{P}(Q^k \tilde{f}_\ell \otimes Q^k \tilde{f}_\ell))(X_i) \\ &\leq 2^{\hat{p}_n-\ell} g_2(X_i) + \sum_{k=0}^{\hat{p}_n-\ell-1} 2^{\hat{p}_n-\ell+k} \alpha^{2k} Q^{\hat{p}_n-\ell-k-1}(\mathcal{P}(g_1 \otimes g_1))(X_i) \\ &\leq 2^{\hat{p}_n-\ell} g_2(X_i) + \sum_{k=0}^{\hat{p}_n-\ell-1} 2^{\hat{p}_n-\ell} (2\alpha^2)^k g_3(X_i) \\ &\leq (2\alpha)^{2(\hat{p}_n-\ell)} g_4(X_i), \end{aligned}$$

where we used (73) for the first equality, (ii) of Assumption 2.2 and (5) for the first inequality, (3) and (iv) of Assumption 2.2 for the second, and $2\alpha^2 > 1$ for the last. The latter inequality implies that, with g_5 equal to g_4 up to a finite multiplicative constant,

$$(67) \quad \mathbb{E}[N_{n,i}(f)^2 | X_i] \leq |\mathbb{G}_n|^{-1} \left(\sum_{\ell=0}^{\hat{p}_n} (2\alpha)^{(\hat{p}_n-\ell)}\right)^2 g_4(X_i) = 2^{-n} (2\alpha)^{2\hat{p}_n} g_5(X_i).$$

Using (66), (67) and (72) as well as Assumption 2.3 with $g_6 \in F$, we get

$$(2\alpha^2)^{-n} \mathbb{E}[R_4(n)^2] \leq (2\alpha^2)^{-n} 2^{n-\hat{p}_n} 2^{-n} (2\alpha)^{2\hat{p}_n} \langle v, Q^n g_5 \rangle \leq (2\alpha^2)^{-(n-\hat{p}_n)} \langle v, g_6 \rangle.$$

We then conclude using that $2\alpha^2 > 1$ and $\lim_{n \rightarrow \infty} (n - \hat{p}_n) = \infty$. \square

Now, we study the third term of the right-hand side of (64). First, note that

$$\begin{aligned} T_n(f) &= \sum_{i \in \mathbb{G}_{n-\hat{p}_n}} \mathbb{E}[N_{n,i}(f)|X_i] \\ &= \sum_{i \in \mathbb{G}_{n-\hat{p}_n}} |\mathbb{G}_n|^{-1/2} \sum_{\ell=0}^{\hat{p}_n} \mathbb{E}_{X_i}[M_{\mathbb{G}_{\hat{p}_n-\ell}}(\tilde{f}_\ell)] \\ &= |\mathbb{G}_n|^{-1/2} \sum_{i \in \mathbb{G}_{n-\hat{p}_n}} \sum_{\ell=0}^{\hat{p}_n} 2^{\hat{p}_n-\ell} Q^{\hat{p}_n-\ell}(\tilde{f}_\ell)(X_i), \end{aligned}$$

where we used (65) for the first equality, the definition (12) of $N_n(f)$ for the second equality and (72) for the last equality. Next, projecting in the eigenspace associated to the eigenvalue α_j , we get

$$T_n(f) = T_n^{(1)}(f) + T_n^{(2)}(f),$$

where, with $\hat{f} = f - \langle \mu, f \rangle - \sum_{j \in J} \mathcal{R}_j(f)$ defined in (22),

$$\begin{aligned} T_n^{(1)}(f) &= |\mathbb{G}_n|^{-1/2} \sum_{i \in \mathbb{G}_{n-\hat{p}_n}} \sum_{\ell=0}^{\hat{p}_n} 2^{\hat{p}_n-\ell} (Q^{\hat{p}_n-\ell}(\hat{f}_\ell))(X_i), \\ T_n^{(2)}(f) &= |\mathbb{G}_n|^{-1/2} \sum_{i \in \mathbb{G}_{n-\hat{p}_n}} \sum_{\ell=0}^{\hat{p}_n} 2^{\hat{p}_n-\ell} \alpha^{\hat{p}_n-\ell} \sum_{j \in J} \theta_j^{\hat{p}_n-\ell} \mathcal{R}_j(f_\ell)(X_i). \end{aligned}$$

We have the following lemma.

LEMMA 6.3. *Under the assumptions of Theorem 3.9, we have the following convergence:*

$$\lim_{n \rightarrow \infty} (2\alpha^2)^{-n/2} \mathbb{E}[|T_n^{(1)}(f)|] = 0.$$

PROOF. We have

$$\begin{aligned} (2\alpha^2)^{-n/2} \mathbb{E}[|T_n^{(1)}(f)|] &\leq (2\alpha)^{-n} \mathbb{E} \left[\sum_{i \in \mathbb{G}_{n-\hat{p}_n}} \sum_{\ell=0}^{\hat{p}_n} 2^{\hat{p}_n-\ell} |Q^{\hat{p}_n-\ell}(\hat{f}_\ell)(X_i)| \right] \\ &\leq (2\alpha)^{-n} \mathbb{E} \left[\sum_{i \in \mathbb{G}_{n-\hat{p}_n}} \sum_{\ell=0}^{\hat{p}_n} 2^{\hat{p}_n-\ell} \alpha^{\hat{p}_n-\ell} \beta_{\hat{p}_n-\ell} g(X_i) \right] \\ &= \sum_{\ell=0}^{\hat{p}_n} 2^{-\ell} \alpha^{-(n-\hat{p}_n+\ell)} \beta_{\hat{p}_n-\ell} \langle \nu, Q^{n-\hat{p}_n} g \rangle, \end{aligned}$$

where we used the definition of $T_n^{(1)}(f)$ for the first inequality, (27) for the second and (72) for the equality. Using (3) and the property (iii), we get that there is a finite positive constant C independent of n and \hat{p}_n such that $\langle \nu, Q^{n-\hat{p}_n} g \rangle < C$. We have

$$\sum_{\ell=0}^{\hat{p}_n/2} 2^{-\ell} \alpha^{-(n-\hat{p}_n+\ell)} \beta_{\hat{p}_n-\ell} \leq \alpha^{-(n-\hat{p}_n)} \beta_{\hat{p}_n/2} \sum_{\ell=0}^{\hat{p}_n/2} (2\alpha)^{-\ell}.$$

Using the third condition in (63) and that $2\alpha > 1$, we deduce the right-hand side converges to 0 as n goes to infinity. Without loss of generality, we can assume that the sequence $(\beta_n, n \in \mathbb{N}^*)$ is bounded by 1. Since $\alpha > 1/\sqrt{2}$, we also have

$$\sum_{\ell=\hat{p}_n/2}^{\hat{p}_n} 2^{-\ell} \alpha^{-(n-\hat{p}_n+\ell)} \beta_{\hat{p}_n-\ell} \leq (1-2\alpha)^{-1} 2^{-\hat{p}_n/2} \alpha^{-n+\hat{p}_n/2} \leq (1-2\alpha)^{-1} 2^{n/2-3\hat{p}_n/4}.$$

Using that $n/2 - 3\hat{p}_n/4 < -n/8$, thanks to the first condition in (63), we deduce the right-hand side converges to 0 as n goes to infinity. Thus, we get that

$$\lim_{n \rightarrow \infty} (2\alpha^2)^{-n/2} \mathbb{E}[|T_n^{(1)}(f)|] = 0. \quad \square$$

Now, we deal with the term $T_n^{(2)}(f)$ in the following result. Recall $M_{\infty,j}$ defined in Lemma 3.8.

LEMMA 6.4. *Under the assumptions of Theorem 3.9, we have the following convergence:*

$$(2\alpha^2)^{-n/2} T_n^{(2)}(f) - \sum_{\ell \in \mathbb{N}} (2\alpha)^{-\ell} \sum_{j \in J} \theta_j^{n-\ell} M_{\infty,j}(f_\ell) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

PROOF. By definition of $T_n^{(2)}(f)$, we have

$$T_n^{(2)}(f) = 2^{-n/2} \sum_{\ell=0}^{\hat{p}_n} (2\alpha)^{n-\ell} \sum_{j \in J} \theta_j^{n-\ell} M_{n,j}(f_\ell)$$

and thus

$$\begin{aligned} (2\alpha^2)^{-n/2} T_n^{(2)}(f) - \sum_{\ell \in \mathbb{N}} (2\alpha)^{-\ell} \sum_{j \in J} \theta_j^{n-\ell} M_{\infty,j}(f_\ell) &= \sum_{\ell=0}^{\hat{p}_n} (2\alpha)^{-\ell} \sum_{j \in J} \theta_j^{n-\ell} (M_{n,j}(f_\ell) - M_{\infty,j}(f_\ell)) \\ &\quad - \sum_{\ell=\hat{p}_n+1}^{\infty} (2\alpha)^{-\ell} \sum_{j \in J} \theta_j^{n-\ell} M_{\infty,j}(f_\ell). \end{aligned} \tag{68}$$

Using that $|\theta_j| = 1$, we get

$$\begin{aligned} &\mathbb{E} \left[\left| \sum_{\ell=0}^{\hat{p}_n} (2\alpha)^{-\ell} \sum_{j \in J} \theta_j^{n+\hat{p}_n-\ell} (M_{n,j}(f_\ell) - M_{\infty,j}(f_\ell)) \right|^2 \right] \\ &\leq \sum_{\ell=0}^{\hat{p}_n} (2\alpha)^{-\ell} \sum_{j \in J} \mathbb{E}[|M_{n,j}(f_\ell) - M_{\infty,j}(f_\ell)|]. \end{aligned}$$

Now, using (5), a close inspection of the proof of Lemma 3.8, see (62), reveals to us that there exists a finite constant C (depending on f) such that for all $j \in J$, we have

$$\sup_{\ell \in \mathbb{N}} \sup_{n \in \mathbb{N}} \mathbb{E}[|M_{n,j}(f_\ell)|^2] \leq C.$$

The $L^2(\nu)$ convergence in Lemma 3.8 yields that

$$(69) \quad \sup_{\ell \in \mathbb{N}} \mathbb{E}[|M_{\infty,j}(f_\ell)|^2] \leq C \quad \text{and} \quad \sup_{\ell \in \mathbb{N}} \sup_{n \in \mathbb{N}} \sum_{j \in J} \mathbb{E}[|M_{n,j}(f_\ell) - M_{\infty,j}(f_\ell)|] < 2|J|\sqrt{C}.$$

Since Lemma 3.8 implies that $\lim_{n \rightarrow \infty} \mathbb{E}[|M_{n,j}(f_\ell) - M_{\infty,j}(f_\ell)|] = 0$, we deduce, as $2\alpha > 1$ by the dominated convergence theorem that

$$(70) \quad \lim_{n \rightarrow +\infty} \mathbb{E} \left[\left| \sum_{\ell=0}^{\hat{p}_n} (2\alpha)^{-\ell} \sum_{j \in J} \theta_j^{n+\hat{p}_n-\ell} (M_{n,j}(f_\ell) - M_{\infty,j}(f_\ell)) \right| \right] = 0.$$

On the other hand, we have

$$(71) \quad \mathbb{E} \left[\left| \sum_{\ell=\hat{p}_n+1}^{\infty} (2\alpha)^{-\ell} \sum_{j \in J} \theta_j^{n-\ell} M_{\infty,j}(f_\ell) \right| \right] \leq \sum_{\ell=\hat{p}_n+1}^{\infty} (2\alpha)^{-\ell} \sum_{j \in J} \mathbb{E}[|M_{\infty,j}(f_\ell)|] \\ \leq |J| \sqrt{C} \sum_{\ell=\hat{p}_n+1}^{\infty} (2\alpha)^{-\ell},$$

where we used $|\theta_j| = 1$ for the first inequality and the Cauchy–Schwarz inequality and (69) for the second inequality. Finally, from (68), (70) and (71) (with $\lim_{n \rightarrow \infty} \sum_{\ell=\hat{p}_n+1}^{\infty} (2\alpha)^{-\ell} = 0$), we get the result of the lemma. \square

7. Moments formula for BMC. Let $X = (X_i, i \in \mathbb{T})$ be a BMC on (S, \mathcal{S}) with probability kernel \mathcal{P} . Recall that $|\mathbb{G}_n| = 2^n$ and $M_{\mathbb{G}_n}(f) = \sum_{i \in \mathbb{G}_n} f(X_i)$. We also recall that $2\mathcal{Q}(x, A) = \mathcal{P}(x, A \times S) + \mathcal{P}(x, S \times A)$ for $A \in \mathcal{S}$. We use the convention that $\sum_{\emptyset} = 0$. We recall the following well-known and easy to establish many-to-one formulas for BMC.

LEMMA 7.1. *Let $f, g \in \mathcal{B}(S)$, $x \in S$ and $n \geq m \geq 0$. Assuming that all the quantities below are well defined, we have*

$$(72) \quad \mathbb{E}_x[M_{\mathbb{G}_n}(f)] = |\mathbb{G}_n| \mathcal{Q}^n f(x) = 2^n \mathcal{Q}^n f(x),$$

$$(73) \quad \mathbb{E}_x[M_{\mathbb{G}_n}(f)^2] = 2^n \mathcal{Q}^n(f^2)(x) + \sum_{k=0}^{n-1} 2^{n+k} \mathcal{Q}^{n-k-1}(\mathcal{P}(\mathcal{Q}^k f \otimes \mathcal{Q}^k f))(x),$$

$$(74) \quad \mathbb{E}_x[M_{\mathbb{G}_n}(f)M_{\mathbb{G}_m}(g)] = 2^n \mathcal{Q}^m(g\mathcal{Q}^{n-m}f)(x) \\ + \sum_{k=0}^{m-1} 2^{n+k} \mathcal{Q}^{m-k-1}(\mathcal{P}(\mathcal{Q}^k g \otimes_{\text{sym}} \mathcal{Q}^{n-m+k}f))(x).$$

We also give an upper bound of $\mathbb{E}_x[M_{\mathbb{G}_n}(f)^4]$, which is a direct consequence of the arguments given in the proof of Theorem 2.1 in [8]. Recall that $g \otimes^2 = g \otimes g$.

LEMMA 7.2. *There exists a finite constant C such that for all $f \in \mathcal{B}(S)$, $n \in \mathbb{N}$ and ν a probability measure on S , assuming that all the quantities below are well defined, there exist functions $\psi_{j,n}$ for $1 \leq j \leq 9$ such that*

$$\mathbb{E}_\nu[M_{\mathbb{G}_n}(f)^4] = \sum_{j=1}^9 \langle \nu, \psi_{j,n} \rangle,$$

and, with $h_k = \mathcal{Q}^{k-1}(f)$ and (notice that either $|\psi_j|$ or $|\langle \nu, \psi_j \rangle|$ is bounded), writing $\nu g = \langle \nu, g \rangle$,

$$|\psi_{1,n}| \leq C 2^n \mathcal{Q}^n(f^4),$$

$$|\nu \psi_{2,n}| \leq C 2^{2n} \sum_{k=0}^{n-1} 2^{-k} |\nu \mathcal{Q}^k \mathcal{P}(\mathcal{Q}^{n-k-1}(f^3) \otimes_{\text{sym}} h_{n-k})|,$$

$$|\psi_{3,n}| \leq C2^{2n} \sum_{k=0}^{n-1} 2^{-k} \mathcal{Q}^k \mathcal{P}(\mathcal{Q}^{n-k-1}(f^2) \otimes^2),$$

$$|\psi_{4,n}| \leq C2^{4n} \mathcal{P}(|\mathcal{P}(h_{n-1} \otimes^2) \otimes^2|),$$

$$|\psi_{5,n}| \leq C2^{4n} \sum_{k=2}^{n-1} \sum_{r=0}^{k-1} 2^{-2k-r} \mathcal{Q}^r \mathcal{P}(\mathcal{Q}^{k-r-1} |\mathcal{P}(h_{n-k} \otimes^2)| \otimes^2),$$

$$|\psi_{6,n}| \leq C2^{3n} \sum_{k=1}^{n-1} \sum_{r=0}^{k-1} 2^{-k-r} \mathcal{Q}^r |\mathcal{P}(\mathcal{Q}^{k-r-1} \mathcal{P}(h_{n-k} \otimes^2) \otimes_{\text{sym}} \mathcal{Q}^{n-r-1}(f^2))|,$$

$$|\nu\psi_{7,n}| \leq C2^{3n} \sum_{k=1}^{n-1} \sum_{r=0}^{k-1} 2^{-k-r} |\nu \mathcal{Q}^r \mathcal{P}(\mathcal{Q}^{k-r-1} \mathcal{P}(h_{n-k} \otimes_{\text{sym}} \mathcal{Q}^{n-k-1}(f^2)) \otimes_{\text{sym}} h_{n-r})|,$$

$$|\psi_{8,n}| \leq C2^{4n} \sum_{k=2}^{n-1} \sum_{r=1}^{k-1} \sum_{j=0}^{r-1} 2^{-k-r-j} \\ \times \mathcal{Q}^j \mathcal{P}(|\mathcal{Q}^{r-j-1} \mathcal{P}(h_{n-r} \otimes^2)| \otimes_{\text{sym}} |\mathcal{Q}^{k-j-1} \mathcal{P}(h_{n-k} \otimes^2)|),$$

$$|\psi_{9,n}| \leq C2^{4n} \sum_{k=2}^{n-1} \sum_{r=1}^{k-1} \sum_{j=0}^{r-1} 2^{-k-r-j} \\ \times \mathcal{Q}^j |\mathcal{P}(\mathcal{Q}^{r-j-1} \mathcal{P}(h_{n-r} \otimes_{\text{sym}} \mathcal{Q}^{k-r-1} \mathcal{P}(h_{n-k} \otimes^2)) \otimes_{\text{sym}} h_{n-j})|.$$

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