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Maximum entropy copula with given diagonal section ☆

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ABSTRACT

We consider copulas with a given diagonal section and compute the explicit density of the unique optimal copula which maximizes the entropy. In this sense, this copula is the least informative among the copulas with a given diagonal section. We give an explicit criterion on the diagonal section for the existence of the optimal copula and give a closed formula for its entropy. We also provide examples for some diagonal sections of usual bivariate copulas and illustrate the differences between these copulas and the associated maximum entropy copula with the same diagonal section.

1. Introduction

Dependence of random variables can be described by copula distributions. A copula is the cumulative distribution function of a random vector $U = (U_1, \dots, U_d)$ with U_i uniformly distributed on $I = [0, 1]$. For an exhaustive overview on copulas, we refer to Nelsen [16]. The diagonal section δ of a d-dimensional copula C, defined on I as $\delta(t) = C(t, \dots, t)$ is the cumulative distribution function of $\max_{1\leq i\leq d} U_i$. The function δ is non-decreasing, d-Lipschitz, and verifies $\delta(t) \leq t$ for all $t \in I$ with $\delta(0) = 0$ and $\delta(1) = 1$. It was shown that if a function δ satisfies these properties, then there exists a copula with δ as diagonal section (see Bertino [2] or Fredricks and Nelsen [12] for $d = 2$ and Cuculescu and Theodorescu [6] for $d \geq 2$).

Copulas with a given diagonal section have been studied in different papers, as the diagonal sections are considered in various fields of application. Beyond the fact that δ is the cumulative distribution function of the maximum of the marginals, it also characterizes the tail dependence of the copula (see Joe [14, p. 33] and references in Nelsen et al. [18], Durante and Jaworski [8], Jaworski [13]) as well as the generator for Archimedean copulas (Sungur and Yang [26]). For $d = 2$, Bertino in [2] introduces the so-called Bertino copula B_δ given by $B_\delta(u, v) = u \wedge v - \min_{u\wedge v\leq t\leq u\vee v}(t - \delta(t))$ for $u, v \in I$. Fredricks and Nelsen in [12] give the example called diagonal copula defined by $K_\delta(u, v) = \min(u, v, (\delta(u) + \delta(v))/2)$ for $u, v \in I$. In Nelsen et al. [17,18] lower and upper bounds related to the pointwise partial ordering are given for copulas with a given diagonal section. They showed that if C is a symmetric copula with diagonal section δ, then for every $u, v \in I$, we have:

$$B_\delta(u, v) \leq C(u, v) \leq K_\delta(u, v).$$

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Durante et al. [10] provide another construction of copulas for a certain class of diagonal sections, called MT-copulas named after Mayor and Torrens and defined as $D_\delta(u, v) = \max(0, \delta(x \vee y) - |x - y|)$. Bivariate copulas with given sub-diagonal sections $\delta_{x_0} : [0, 1 - x_0] \to [0, 1 - x_0]$, $\delta_{x_0}(t) = C(x_0 + t, t)$ are constructed from copulas with given diagonal sections in Quesada-Molina et al. [22]. Durante et al. [9,18] introduce the technique of diagonal splicing to create new copulas with a given diagonal section based on other such copulas. According to [8] for $d = 2$ and Jaworski [13] for $d \geq 2$, there exists an absolutely continuous copula with diagonal section δ if and only if the set $\Sigma_\delta = \{t \in I; \delta(t) = t\}$ has zero Lebesgue measure. de Amo et al. [7] is an extension of [8] for given sub-diagonal sections. Further construction of possibly asymmetric absolutely continuous bidimensional copulas with a given diagonal section is provided in Erdely and González [11].

Our aim is to find the most uninformative copula with a given diagonal section δ. We choose here to maximize the relative entropy to the uniform distribution on I^d, among the copulas with given diagonal section. This is equivalent to minimizing the Kullback–Leibler divergence with respect to the independent copula. The Kullback–Leibler divergence is finite only for absolutely continuous copulas. The previously introduced bivariate copulas B_δ, K_δ and D_δ are not absolutely continuous, therefore their Kullback–Leibler divergence is infinite. Possible other entropy criteria, such as Rényi, Tsallis, etc. are considered for example in Pougaza and Mohammad-Djafari [21]. We recall that the entropy of a d-dimensional absolutely continuous random vector $X = (X_1, \ldots, X_d)$ can be decomposed as the sum of the entropy of the marginals and the entropy of the corresponding copula (see Zhao and Lin [27]):

$$H(X) = \sum_{i=1}^{d} H(X_i) + H(U),$$

where $H(Z) = -\int f_Z(z) \log f_Z(z) dz$ is the entropy of the random variable Z with density f_Z, and $U = (U_1, \ldots, U_d)$ is a random vector with U_i uniformly distributed on I, such that U has the same copula as X; namely U is distributed as $\left(F_1^{-1}(X_1), \ldots, F_d^{-1}(X_d)\right)$ with F_i the cumulative distribution function of X_i. Maximizing the entropy of X with given marginals therefore corresponds to maximizing the entropy of its copula. The maximum relative entropy approach for copulas has an extensive literature. Existence results for an optimal solution on convex closed subsets of copulas for the total variation distance can be derived from Csiszár [5]. A general discussion on abstract entropy maximization is given by Borwein et al. [3]. This theory was applied for copulas and a finite number of expectation constraints in Bedford and Wilson [1]. Some applications for various moment-based constraints include rank correlation (Meeuwissen and Bedford [15], Chu [4], Piantadosi et al. [20]) and marginal moments (Pasha and Mansoury [19]).

We shall apply the theory developed in [3] to compute the density of the maximum entropy copula with a given diagonal section. We show that there exists a copula with diagonal section δ and finite entropy if and only if δ satisfies: $\int_I |\log(t - \delta(t))| dt < +\infty$. Notice that this condition is stronger than the condition of Σ_δ having zero Lebesgue measure which is required for the existence of an absolutely continuous copula with diagonal section δ. Under this condition, and in the case of $\Sigma_\delta = \{0, 1\}$, the optimal copula's density c_δ turns out to be of the form, for $x = (x_1, \ldots, x_d) \in I^d$:

$$c_\delta(x) = b(\max(x)) \prod_{x_i \neq \max(x)} a(x_i),$$

with the notation $\max(x) = \max_{1 \leq i \leq d} x_i$, see Proposition 2.4. The optimal copula's density in the general case is given in Theorem 2.5. Notice that c_δ is symmetric: it is invariant under the permutation of the variables. This provides a new family of absolutely continuous symmetric copulas with given diagonal section enriching previous work on this subject that we discussed, see [2,8–12,18]. We also calculate the maximum entropy copula for diagonal sections that arise from well-known families of bivariate copulas.

The rest of the paper is organized as follows. Section 2 introduces the definitions and notations used later on, and gives the main theorems of the paper. In Section 3 we study the properties of the feasible solution c_δ of the problem for a special class of diagonal sections with $\Sigma_\delta = \{0, 1\}$. In Section 4, we formulate our problem as an optimization problem with linear constraints in order to apply the theory established in [3]. Then in Section 5 we give the proof for our main theorem showing that c_δ is indeed the optimal solution when $\Sigma_\delta = \{0, 1\}$. In Section 6 we extend our results for the general case when Σ_δ has zero Lebesgue measure. We give in Section 7 several examples with diagonals of popular bivariate copula families such as the Gaussian, Gumbel or Farlie–Gumbel–Morgenstern copulas among others. In the Gaussian case, we illustrate how different the Gaussian copula and the corresponding maximum entropy copula can be, by calculating conditional extreme event probabilities.

2. Main results

Let $d \geq 2$ be fixed. We recall a function C defined on I^d, with $I = [0, 1]$, is a d-dimensional copula if there exists a random vector $U = (U_1, \ldots, U_d)$ such that U_i are uniform on I and $C(u) = \mathbb{P}(U \leq u)$ for $u \in I^d$, with the convention that $x \leq y$ for $x = (x_1, \ldots .x_d)$ and $y = (y_1, \ldots, y_d)$ elements of $\mathbb{R}^d$ if and only if $x_i \leq y_i$ for all $1 \leq i \leq d$. We shall say that C is the copula of U. We refer to [16] for a monograph on copulas. The copula C is said absolutely continuous if the random variable U has a density, which we shall denote by c_C. In this case, we have for all $u \in I^d$:

$$C(u) = \int_{I^d} c_C(v) \mathbf{1}_{\{v \leq u\}} \, dv.$$

When there is no confusion, we shall write c for the density c_C associated to the copula C. We denote by $\mathcal{C}$ the set of d-dimensional copulas and by $\mathcal{C}_0$ the subset of the d-dimensional absolutely continuous copulas.

The diagonal section δ_C of a copula C is defined by: $\delta_C(t) = C(t, \dots, t)$. Let us note, for $u \in \mathbb{R}^d$, $\max(u) = \max_{1\leq i\leq d} u_i$. Notice that if C is the copula of U, then δ_C is the cumulative distribution function of $\max(U)$ as $\delta_C(t) = \mathbb{P}(\max(U) \leq t)$ for $t \in I$. We denote by $\mathcal{D} = \{\delta_C, C \in \mathcal{C}\}$ the set of diagonal sections of d-dimensional copulas and by $\mathcal{D}_0 = \{\delta_C; C \in \mathcal{C}_0\}$ the set of diagonal sections of absolutely continuous copulas. According to [12], a function δ defined on I belongs to $\mathcal{D}$ if and only if:

(i) δ is a cumulative function on $[0, 1]$: $\delta(0) = 0$, $\delta(1) = 1$ and δ is non-decreasing;
(ii) $\delta(t) \leq t$ for $t \in I$ and δ is d-Lipschitz: $|\delta(s) - \delta(t)| \leq d\,|s - t|$ for $s, t \in I$.

For $\delta \in \mathcal{D}$, we shall consider the set $\mathcal{C}^\delta = \{C \in \mathcal{C}; \delta_C = \delta\}$ of copulas with diagonal section δ, and the subset $\mathcal{C}_0^\delta = \mathcal{C}^\delta \bigcap \mathcal{C}_0$ of absolutely continuous copulas with section δ. According to [8,13], the set $\mathcal{C}_0^\delta$ is non empty if and only if the set $\Sigma_\delta = \{t \in I; \delta(t) = t\}$ has zero Lebesgue measure.

For a non-negative measurable function f defined on I^k, $k \in \mathbb{N}$, we set

$$\mathcal{I}_k(f) = \int_{I^k} f(x) \log(f(x))\, dx,$$

with the convention $0\log(0) = 0$. Since copulas are cumulative functions of probability measures, we will consider the Kullback–Leibler divergence relative to the uniform distribution as a measure of entropy, see [5]:

$$\mathcal{I}(C) = \begin{cases} \mathcal{I}_d(c) & \text{if } C \in \mathcal{C}_0, \\ +\infty & \text{if } C \notin \mathcal{C}_0, \end{cases}$$

with c the density associated to C when $C \in \mathcal{C}_0$. Notice that the Shannon-entropy introduced in [25] of the probability measure P defined on I^d with cumulative distribution function C is defined as $H(P) = -\mathcal{I}(C)$. Thus minimizing the Kullback–Leibler divergence $\mathcal{I}$ (w.r.t. the uniform distribution) is equivalent to maximizing the Shannon-entropy. It is well known that the copula Π with density $c_\Pi = 1$, which corresponds to $(U_i, 0 \leq i \leq d)$ being independent, minimizes $\mathcal{I}(C)$ over $\mathcal{C}$.

We shall minimize the Kullback–Leibler divergence $\mathcal{I}$ over the set $\mathcal{C}^\delta$ or equivalently over $\mathcal{C}_0^\delta$ of copulas with a given diagonal section $\delta \in \mathcal{D}$ (in fact for $\delta \in \mathcal{D}_0$ as otherwise $\mathcal{C}_0^\delta$ is empty). If C minimizes $\mathcal{I}$ on $\mathcal{C}^\delta$, it means that C is the least informative (or the "most random") copula with given diagonal section δ.

For $\delta \in \mathcal{D}$, let us denote:

$$\mathcal{J}(\delta) = \int_I |\log(t - \delta(t))|\, dt. \tag{1}$$

Notice that $\mathcal{J}(\delta) \in [0, +\infty]$ and it is infinite if $\delta \notin \mathcal{D}_0$. Since δ is d-Lipschitz, the derivative δ' of δ exists a.e. and since δ is non-decreasing we have a.e. $0 \leq \delta' \leq d$. This implies that $\mathcal{I}_1(\delta')$ and $\mathcal{I}_1(d - \delta')$ are well defined. Let us denote:

$$\mathcal{G}(\delta) = \mathcal{I}_1(\delta') + \mathcal{I}_1(d - \delta') - d\log(d) - (d - 1). \tag{2}$$

Since for any function f such that $0 \leq f \leq d$ we have $-1/\mathrm{e} \leq \mathcal{I}_1(f) \leq d\log(d)$, we can give a rough upper bound for $|\mathcal{G}(\delta)|$:

$$\sup_{\delta\in\mathcal{D}} |\mathcal{G}(\delta)| \leq d + d\log(d). \tag{3}$$

For $\delta \in \mathcal{D}_0$ with $\Sigma_\delta = \{0, 1\}$, we define the function c_δ as:

$$c_\delta(x) = b(\max(x)) \prod_{x_i\neq\max(x)} a(x_i) \quad \text{for a.e. } x = (x_1, \dots, x_d) \in I^d, \tag{4}$$

where the functions a and b are given by, for $r \in I$:

$$a(r) = \frac{d - \delta'(r)}{d} h(r)^{-1+1/d} \mathrm{e}^{F(r)} \quad \text{and} \quad b(r) = \frac{\delta'(r)}{d} h(r)^{-1+1/d} \mathrm{e}^{-(d-1)F(r)}, \tag{5}$$

with h and F defined as:

$$h(r) = r - \delta(r), \qquad F(r) = \frac{d-1}{d} \int_{\frac{1}{2}}^{r} \frac{1}{h(s)}\, ds. \tag{6}$$

Remark 2.1. Notice that we define F in (6) as an integral from $1/2$ to r. However, the value $1/2$ can be chosen arbitrarily on $(0, 1)$ as it will not affect the definition of the function c_δ in (4).

The following proposition shows that c_δ is an absolutely continuous copula whose diagonal section is δ. The proof of this Proposition can be found in Section 3 and Section A.1 is dedicated to the proof of (7).

Proposition 2.2. *Let $\delta \in \mathcal{D}_0$ with $\Sigma_\delta = \{0, 1\}$. The function c_δ given by (4) is the density of a symmetric copula C_δ with diagonal section δ.*

Furthermore, we have:

$$\mathcal{I}(C_\delta) = (d - 1)\mathcal{J}(\delta) + \mathcal{G}(\delta). \tag{7}$$

This and (3) readily imply the following remark.

Remark 2.3. Let $\delta \in \mathcal{D}_0$ such that $\Sigma_\delta = \{0, 1\}$. We have $\mathcal{I}(C_\delta) < +\infty$ if and only if $\mathcal{J}(\delta) < +\infty$.

We can now state our main result in the simpler case $\Sigma_\delta = \{0, 1\}$. It gives the necessary and sufficient condition for C_δ to be the unique optimal solution of the minimization problem. The proof is given in Section 5.

Proposition 2.4. *Let $\delta \in \mathcal{D}_0$ such that $\Sigma_\delta = \{0, 1\}$.*

(a) *If $\mathcal{J}(\delta) = +\infty$ then $\min_{C\in\mathcal{C}^\delta} \mathcal{I}(C) = +\infty$.*
(b) *If $\mathcal{J}(\delta) < +\infty$ then $\min_{C\in\mathcal{C}^\delta} \mathcal{I}(C) < +\infty$ and C_δ is the unique copula such that $\mathcal{I}(C_\delta) = \min_{C\in\mathcal{C}^\delta} \mathcal{I}(C)$.*

To give the answer in the general case where Σ_δ has zero Lebesgue measure, which is the necessary and sufficient condition for $\mathcal{C}_0^\delta \neq \emptyset$, we need some extra notations. Since δ is continuous, we get that $I \setminus \Sigma_\delta$ can be written as the union of non-empty open disjoint intervals $((\alpha_j, \beta_j), j \in J)$, with $\alpha_j < \beta_j$ and J at most countable. Notice that $\delta(\alpha_j) = \alpha_j$ and $\delta(\beta_j) = \beta_j$. For $J \neq \emptyset$ and $j \in J$, we set $\Delta_j = \beta_j - \alpha_j$ and for $t \in I$:

$$\delta^j(t) = \frac{\delta\left(\alpha_j + t\Delta_j\right) - \alpha_j}{\Delta_j}. \tag{8}$$

It is clear that δ^j satisfies (i) and (ii) and it belongs to $\mathcal{D}_0$ as $\Sigma_{\delta^j} = \{0, 1\}$. Let c_{δ^j} be defined by (4) with δ replaced by δ^j. For $\delta \in \mathcal{D}_0$ such that $\Sigma_\delta \neq \{0, 1\}$, we define the function c_δ by, for $u \in I^d$:

$$c_\delta(u) = \sum_{j\in J} \frac{1}{\Delta_j} c_{\delta^j}\left(\frac{u - \alpha_j\mathbf{1}}{\Delta_j}\right) \mathbf{1}_{(\alpha_j,\beta_j)^d}(u), \tag{9}$$

with $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^d$. It is easy to check that c_δ is a copula density and that is zero outside $[\alpha_j, \beta_j]^d$ for $j \in J$. We state our main result in the general case whose proof is given in Section 6.

Theorem 2.5. *Let $\delta \in \mathcal{D}$.*

(a) *If $\mathcal{J}(\delta) = +\infty$ then $\min_{C\in\mathcal{C}^\delta} \mathcal{I}(C) = +\infty$.*
(b) *If $\mathcal{J}(\delta) < +\infty$ then $\min_{C\in\mathcal{C}^\delta} \mathcal{I}(C) < +\infty$ and there exists a unique copula $C_\delta \in \mathcal{C}^\delta$ such that $\mathcal{I}(C_\delta) = \min_{C\in\mathcal{C}^\delta} \mathcal{I}(C)$. Furthermore, we have:*

$$\mathcal{I}(C_\delta) = (d-1)\mathcal{J}(\delta) + \mathcal{G}(\delta);$$

the copula C_δ is absolutely continuous, symmetric; its density c_δ is given by (4) if $\Sigma_\delta = \{0, 1\}$ or by (9) if $\Sigma_\delta \neq \{0, 1\}$.

Remark 2.6. For $\delta \in \mathcal{D}$, notice the condition $\mathcal{J}(\delta) < +\infty$ implies that Σ_δ has zero Lebesgue measure, and therefore, according to [8,13], $\delta \in \mathcal{D}_0$. And if $\delta \notin \mathcal{D}_0$, then $\mathcal{I}(C) = +\infty$ for all $C \in \mathcal{C}^\delta$. Therefore, we could replace the condition $\delta \in \mathcal{D}$ by $\delta \in \mathcal{D}_0$ in Theorem 2.5.

3. Proof of Proposition 2.2

We assume that $\delta \in \mathcal{D}_0$ and $\Sigma_\delta = \{0, 1\}$. We give the proof of Proposition 2.2, which states that C_δ, with density c_δ given by (4), is indeed a symmetric copula with diagonal section δ whose entropy is given by (7).

Recall the definition of h, F, a, b and c_δ given by (4) to (6). Notice that by construction c_δ is non-negative and well defined on I^d. In order to prove that c_δ is the density of a copula, we only have to prove that for all $1 \le i \le d$, $r \in I$:

$$\int_{I^d} c_\delta(u)\mathbf{1}_{\{u_i\le r\}}\, du = r,$$

or equivalently

$$\int_{I^d} c_\delta(u)\mathbf{1}_{\{u_i\ge r\}}\, du = 1 - r.$$

We define for $r \in I$:

$$A(r) = \int_0^r a(t)\, dt. \tag{10}$$

Elementary computations yield for $r \in (0, 1)$:

$$A(r) = h^{1/d}(r)\, \mathrm{e}^{F(r)}. \tag{11}$$

Notice that $F(0) \in [-\infty, 0]$ which implies that $A(0) = 0$. A direct integration gives:

$$d\int_I A^{d-1}(s)b(s)\mathbf{1}_{\{s\ge r\}}\, ds = 1 - \delta(r). \tag{12}$$

We also have:

$$\begin{aligned}(d-1)\int_I A^{d-2}(s)b(s)\mathbf{1}_{\{s\geq r\}}\,ds &= \frac{(d-1)}{d}\int_I \delta'(s)h^{-1/d}(s)\mathrm{e}^{-F(s)}\mathbf{1}_{\{s\geq r\}}\,ds\\ &= \left[-h^{1-1/d}(s)\mathrm{e}^{-F(s)}\right]_{s=r}^1\\ &= h^{1-1/d}(r)\mathrm{e}^{-F(r)},\end{aligned}\tag{13}$$

where we used for the last step that $h(1)=0$ and $F(1)\in[0,\infty]$. We have:

$$\begin{aligned}\int_{I^d} c_\delta(u)\mathbf{1}_{\{u_i\geq r\}}\,du &= \int_{I^d} b(\max(u))\prod_{u_j\neq\max(u)} a(u_j)\mathbf{1}_{\{u_i\geq r\}}\,du\\ &= \int_I A^{d-1}(s)b(s)\mathbf{1}_{\{s\geq r\}}\,ds + (d-1)\int_I A^{d-2}(s)b(s)(A(s)-A(r))\mathbf{1}_{\{s\geq r\}}\,ds\\ &= d\int_I A^{d-1}(s)b(s)\mathbf{1}_{\{s\geq r\}}\,ds - (d-1)A(r)\int_I A^{d-2}(s)b(s)\mathbf{1}_{\{s\geq r\}}\,ds\\ &= 1-\delta(r)-h(r)\\ &= 1-r,\end{aligned}$$

where in the second equality we separated the integral according to $\max(u)=u_i$ or not and used (10), then in the fourth equality we used (12) and (13). This implies that c_δ is indeed the density of a copula. We denote by C_δ the copula with density c_δ. We check that δ is the diagonal section of C_δ. Using (12), we get, for $r\in I$:

$$\begin{aligned}\int_{I^d} c_\delta(u)\mathbf{1}_{\{\max(u)\leq r\}}\,du &= \int_{I^d} b(\max(u))\prod_{u_i\neq\max(u)} a(u_i)\mathbf{1}_{\{\max(u)\leq r\}}\,du\\ &= d\int_I A^{d-1}(s)b(s)\mathbf{1}_{\{s\leq r\}}\,ds\\ &= \delta(r).\end{aligned}$$

The calculations which show that the entropy of C_δ is given by (7) can be found in Appendix A.1.

4. The minimization problem

Let $\delta\in\mathcal{D}_0$. As a first step we will show, using [3], that the problem of a maximum entropy copula with a given diagonal section δ has at most a unique optimal solution. To formulate this problem in the framework of [3], we introduce the continuous linear functional $\mathcal{A}=(\mathcal{A}_i, 1\leq i\leq d+1): L^1(I^d)\to L^1(I)^{d+1}$ defined by, for $1\leq i\leq d$, $f\in L^1(I^d)$ and $r\in I$,

$$\mathcal{A}_i(f)(r)=\int_{I^d} f(u)\mathbf{1}_{\{u_i\leq r\}}\,du,\quad\text{and}\quad \mathcal{A}_{d+1}(f)(r)=\int_{I^d} f(u)\mathbf{1}_{\{\max(u)\leq r\}}\,du.$$

We also define $b^\delta=(b_i, 1\leq i\leq d+1)\in L^1(I)^{d+1}$ with $b_{d+1}=\delta$ and $b_i=\mathrm{id}_I$ for $1\leq i\leq d$, with id_I the identity map on I. Notice that the conditions $\mathcal{A}_i(c)=b_i$, $1\leq i\leq d$, and $c\geq 0$ a.e. imply that c is the density of a copula $C\in\mathcal{C}_0$. If we assume further that the condition $\mathcal{A}_{d+1}(c)=b_{d+1}$ holds then the diagonal section of C is δ (thus $C\in\mathcal{C}_0^\delta$).

Since $\mathcal{I}$ is infinite outside $\mathcal{C}_0^\delta$ and the density of any copula in $\mathcal{C}_0$ belongs to $L^1(I^d)$, we get that minimizing $\mathcal{I}$ over $\mathcal{C}^\delta$ is equivalent to the optimization problem (P^δ) given by:

$$\text{minimize } \mathcal{I}_d(c) \text{ subject to } \begin{cases}\mathcal{A}(c)=b^\delta,\\ c\geq 0 \text{ a.e. and } c\in L^1(I^d).\end{cases}\tag{P^δ}$$

We say that a function f is feasible for (P^δ) if $f\in L^1(I^d)$, $f\geq 0$ a.e., $\mathcal{A}(f)=b^\delta$ and $\mathcal{I}_d(f)<+\infty$. Notice that any feasible f is the density of a copula. We say that f is an optimal solution to (P^δ) if f is feasible and $\mathcal{I}_d(f)\leq\mathcal{I}_d(g)$ for all g feasible.

Proposition 4.1. *Let $\delta\in\mathcal{D}$. If there exists a feasible c, then there exists a unique optimal solution to (P^δ) and it is symmetric.*

Proof. Since $\mathcal{A}(f)=b^\delta$ implies $\mathcal{A}_1(f)(1)=b_1(1)$ that is $\int_{I^d} f(x)\,dx=1$, we can directly apply Corollary 2.3 of [3] which states that if there exists a feasible c, then there exists a unique optimal solution to (P^δ). Since the constraints are symmetric and the functional $\mathcal{I}_d$ is also symmetric, we deduce that the unique optimal solution is also symmetric. □

The next proposition gives that the set of zeros of any non-negative solution c of $\mathcal{A}(c)=b^\delta$ contains:

$$Z_\delta=\{u\in I^d;\ \delta'(\max(u))=0 \text{ or } \exists i \text{ such that } u_i<\max(u) \text{ and } \delta'(u_i)=d\}.\tag{14}$$

Proposition 4.2. *Let $\delta\in\mathcal{D}$. If c is feasible then $c=0$ a.e. on Z_δ (that is $c\mathbf{1}_{Z_\delta}=0$ a.e.).*

Proof. Recall that $0 \leq \delta' \leq d$. Since $c \in L^1(I^d)$, the condition $\mathcal{A}_{d+1}(c) = b_{d+1}$, that is for all $r \in I$

$$\int_{I^d} c(u)\mathbf{1}_{\{\max(u)\leq r\}}\, du = \int_0^r \delta'(s)\, ds,$$

implies, by the monotone class theorem, that for all measurable subsets H of I, we have:

$$\int_{I^d} c(u)\mathbf{1}_H(\max(u))\, du = \int_H \delta'(s)\, ds.$$

Since $c \geq 0$ a.e., we deduce that a.e. $c(u)\mathbf{1}_{\{\delta'(\max(u))=0\}} = 0$.

Next, notice that for all $r \in I$:

$$\begin{aligned}\int_{I^d} c(u)\left(\sum_{i=1}^d \mathbf{1}_{\{u_i<\max(u), u_i\leq r\}}\right) du &= \sum_{i=1}^d \left(\int_{I^d} c(u)\mathbf{1}_{\{u_i\leq r\}}\, du - \int_{I^d} c(u)\mathbf{1}_{\{u_i=\max(u),\max(u)\leq r\}}\, du\right) \\ &= dr - \delta(r) \\ &= \int_0^r \left(d - \delta'(s)\right)\, ds.\end{aligned}$$

This implies that a.e. $c(u)\left(\sum_{i=1}^d \mathbf{1}_{\{u_i<\max(u),\delta'(u_i)=d\}}\right) = 0$, that is $c(u)\mathbf{1}_{\{\exists i \text{ such that } u_i<\max(u),\delta'(u_i)=d\}} = 0$. This gives the result. $\square$

We define μ to be the Lebesgue measure restricted to $Z_\delta^c = I^d \setminus Z_\delta$: $\mu(du) = \mathbf{1}_{Z_\delta^c}(u)du$. We define, for $f \in L^1(I^d, \mu)$:

$$\mathcal{I}^\mu(f) = \int_{I^d} f(u)\log(f(u))\, \mu(du).$$

From Proposition 4.2 we can deduce that if c is feasible then $\mathcal{I}^\mu(c) = \mathcal{I}_d(c)$. Let us also define, for $1 \leq i \leq d$, $r \in I$:

$$\mathcal{A}_i^\mu(c)(r) = \int_{I^d} c(u)\mathbf{1}_{\{u_i\leq r\}}\, \mu(du), \quad \text{and} \quad \mathcal{A}_{d+1}^\mu(c)(r) = \int_{I^d} c(u)\mathbf{1}_{\{\max(u)\leq r\}}\, \mu(du).$$

The corresponding optimization problem (P_μ^δ) is given by :

$$\text{minimize } \mathcal{I}^\mu(c) \text{ subject to } \begin{cases}\mathcal{A}^\mu(c) = b^\delta, \\ c \geq 0\mu\text{-a.e. and } c \in L^1(I^d, \mu),\end{cases} \tag{P_μ^δ}$$

with $\mathcal{A}^\mu = \left(\mathcal{A}_i^\mu, 1 \leq i \leq d+1\right)$. For $f \in L^1(I^d, \mu)$, we define:

$$f^\mu = \begin{cases} f & \text{on } Z_\delta^c, \\ 0 & \text{on } Z_\delta.\end{cases}$$

Using Proposition 4.2, we easily get the following corollary.

Corollary 4.3. *If c is a solution of (P_μ^δ), then c^μ is a solution of (P^δ). If c is a solution of (P^δ), then it is also a solution of (P_μ^δ).*

5. Proof of Proposition 2.4

5.1. Form of the optimal solution

Let $(\mathcal{A}^\mu)^* : L^\infty(I)^{d+1} \to L^\infty(I^d, \mu)$ be the adjoint of $\mathcal{A}^\mu$. We will use Theorem 2.9 from [3] on abstract entropy minimization, which we recall here, adapted to the context of (P_μ^δ).

Theorem 5.1 (*Borwein, Lewis and Nussbaum [3]*). *Suppose there exists $c > 0$ μ-a.e. which is feasible for (P_μ^δ). Then there exists a unique optimal solution, c^*, to (P_μ^δ). Furthermore, we have $c^* > 0$ μ-a.e. and there exists a sequence $(\lambda^n, n \in \mathbb{N})$ of elements of $L^\infty(I)^{d+1}$ such that:*

$$\int_{I^d} c^*(x)\left|(\mathcal{A}^\mu)^*(\lambda^n)(x) - \log(c^*(x))\right| \mu(dx) \xrightarrow[n\to\infty]{} 0. \tag{15}$$

We first compute $(\mathcal{A}^\mu)^*$. For $\lambda = (\lambda_i, 1 \leq i \leq d+1) \in L^\infty(I)^{d+1}$ and $f \in L^1(I^d, \mu)$, we have:

$$\begin{aligned}\langle(\mathcal{A}^\mu)^*(\lambda), f\rangle &= \langle\lambda, \mathcal{A}^\mu(f)\rangle \\ &= \sum_{i=1}^d \int_I \lambda_i(r)\int_{I^d} f(x)\mathbf{1}_{\{x_i\leq r\}}d\mu(x)dr + \int_I \lambda_{d+1}(r)\int_{I^d} f(x)\mathbf{1}_{\{\max(x)\leq r\}}d\mu(x)dr \\ &= \int_{I^d} f(x)\left(\sum_{i=1}^d \Lambda_i(x_i) + \Lambda_{d+1}(\max(x))\right) d\mu(x),\end{aligned}$$

where we used the definition of the adjoint operator for the first equality, Fubini's theorem for the second, and the following notation for the third equality:

$$\Lambda_i(x_i) = \int_I \lambda_i(r)\mathbf{1}_{\{r\geq x_i\}}\,dr, \quad \text{and} \quad \Lambda_{d+1}(t) = \int_I \lambda_{d+1}(r)\mathbf{1}_{\{r\geq t\}}\,dr.$$

Thus, we can set for $\lambda \in L^\infty(I)^{d+1}$ and $x \in I^d$:

$$(\mathcal{A}^\mu)^*(\lambda)(x) = \sum_{i=1}^d \Lambda_i(x_i) + \Lambda_{d+1}(\max(x)). \tag{16}$$

Now we are ready to prove that the optimal solution c^* of (P^δ_μ) is the product of measurable univariate functions.

Lemma 5.2. *Let $\delta \in \mathcal{D}_0$ such that $\Sigma_\delta = \{0, 1\}$. Suppose that there exists $c > 0$ μ-a.e.which is feasible for (P^δ_μ). Then there exist a^*, b^* non-negative, measurable functions defined on I such that the optimal solution c^* of (P^δ_μ) is given by, for $u = (u_1, \ldots, u_d) \in I^d$:*

$$c^*(u) = b^*(\max(u)) \prod_{u_i\neq\max(u)} a^*(u_i) \quad \mu\text{-a.e.}$$

with $a^*(s) = 0$ if $\delta'(s) = d$ and $b^*(s) = 0$ if $\delta'(s) = 0$.

Proof. According to Theorem 5.1, there exists a sequence $(\lambda^n, n \in \mathbb{N})$ of elements of $L^\infty(I)^{d+1}$ such that the optimal solution, say c^*, satisfies (15). This implies, thanks to (16), that there exist $d + 1$ sequences $(\Lambda^n_i, n \in \mathbb{N}, 1 \leq i \leq d + 1)$ of elements of $L^\infty(I)$ such that the following convergence holds in $L^1(I^d, c^*\mu)$:

$$\sum_{i=1}^d \Lambda^n_i(u_i) + \Lambda^n_{d+1}(\max(u)) \xrightarrow[n\to\infty]{} \log(c^*(u)). \tag{17}$$

Arguing as in Proposition 4.1 and since Z^c_δ, the support of μ, is symmetric, we deduce that c^* is symmetric. Therefore we shall only consider functions supported on the set $\triangle = \{u \in I^d; u_d = \max(u)\}$. The convergence (17) holds in $L^1(\triangle, c^*\mu)$. For simplicity, we introduce the functions $\Gamma^n_i \in L^\infty(I)$ defined by $\Gamma^n_i = \Lambda^n_i$ for $1 \leq i \leq d - 1$, and $\Gamma^n_d = \Lambda^n_d + \Lambda^n_{d+1}$. Then we have in $L^1(\triangle, c^*\mu)$:

$$\sum_{i=1}^d \Gamma^n_i(u_i) \xrightarrow[n\to\infty]{} \log(c^*(u)). \tag{18}$$

We first assume that there exist Γ_i, $1 \leq i \leq d$ measurable functions defined on I such that μ-a.e. on $\triangle$:

$$\sum_{i=1}^d \Gamma_i(u_i) = \log(c^*(u)). \tag{19}$$

The symmetric property of $c^*(u)$ seen in Proposition 4.1 implies we can choose $\Gamma_i = \Gamma$ for $1 \leq i \leq d - 1$ up to adding a constant to Γ_d. Set $a^* = \exp(\Gamma)$ and $b^* = \exp(\Gamma_d)$ so that μ-a.e. on $\triangle$:

$$c^*(u) = b^*(u_d) \prod_{i=1}^{d-1} a^*(u_i). \tag{20}$$

Recall $\mu(du) = \mathbf{1}_{Z^c_\delta}(u)\,du$. From the definition (14) of Z_δ, we deduce that without loss of generality, we can assume that $a^*(u_i) = 0$ if $\delta'(u_i) = d$ and $b^*(u_d) = 0$ if $\delta'(u_d) = 0$. Use the symmetry of c^* to conclude.

To complete the proof, we now show that (19) holds for Γ and Γ_d measurable functions. We introduce the notation $u_{(-i)} = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_d) \in I^{d-1}$. Let us define the probability measure $P(dx) = c^*(x)\mathbf{1}_\triangle(x)\mu(dx)/\int_\triangle c^*(y)\mu(dy)$ on I^d. We fix j, $1 \leq j \leq d - 1$. In order to apply Proposition 2 of [23], which would ensure the existence of the limiting functions Γ_i, $1 \leq i \leq d$, we first check that P is absolutely continuous with respect to $P^j_1 \otimes P^j_2$, where $P^j_1(du_{(-j)}) = \int_{u_j\in I} P(du_{(-j)}du_j)$ and $P^j_2(du_j) = \int_{u_{(-j)}\in I^{d-1}} P(du_{(-j)}du_j)$ are the marginals of P. Notice the following equivalence of measures:

$$P(du) \sim \mathbf{1}_\triangle(u) \prod_{i=1}^{d-1} \mathbf{1}_{\{\delta'(u_i)\neq d\}}\mathbf{1}_{\{\delta'(u_d)\neq 0\}}\,du. \tag{21}$$

Let $B \subset I^{d-1}$ be measurable. We have:

$$P_1(B) = 0 \iff \int_{I^d} \mathbf{1}_{\triangle}(u) \prod_{i=1}^{d-1} \mathbf{1}_{\{\delta'(u_i)\neq d\}} \mathbf{1}_{\{\delta'(u_d)\neq 0\}} \mathbf{1}_B(u_{(-j)})\, du = 0.$$

By Fubini's theorem this last equality is equivalent to:

$$\int_{I^{d-1}} \prod_{i=1, i\neq j}^{d-1} \left(\mathbf{1}_{\{\delta'(u_i)\neq d\}} \mathbf{1}_{\{u_i \leq u_d\}}\right) \mathbf{1}_{\{\delta'(u_d)\neq 0\}} \mathbf{1}_B(u_{(-j)}) \left(\int_I \mathbf{1}_{\{0\leq u_j\leq u_d\}} \mathbf{1}_{\{\delta'(u_j)\neq d\}}\, du_j\right) du_{(-j)} = 0. \tag{22}$$

Since for $\varepsilon > 0$, $\delta(\varepsilon) < \varepsilon < d\varepsilon$, we have $\int_I \mathbf{1}_{\{0\leq u_j\leq s\}} \mathbf{1}_{\{\delta'(u_j)\neq d\}}\, du_j > 0$ for all $s \in I$. Therefore (22) is equivalent to

$$\int_{I^{d-1}} \prod_{i=1, i\neq j}^{d-1} \left(\mathbf{1}_{\{\delta'(u_i)\neq d\}} \mathbf{1}_{\{u_i \leq u_d\}}\right) \mathbf{1}_{\{\delta'(u_d)\neq 0\}} \mathbf{1}_B(u_{(-j)})\, du_{(-j)} = 0.$$

This implies that there exists $h > 0$ a.e. on I^{d-1} such that

$$P_1^j(du_{(-j)}) = h(u_{(-j)}) \prod_{i=1, i\neq j}^{d-1} \left(\mathbf{1}_{\{\delta'(u_i)\neq d\}} \mathbf{1}_{\{u_i \leq u_d\}}\right) \mathbf{1}_{\{\delta'(u_d)\neq 0\}} du_{(-j)}.$$

Similarly we have for $B' \subset I$ that $P_2^j(B') = 0$ if and only if

$$\int_I \mathbf{1}_{\{\delta'(u_j)\neq d\}} \mathbf{1}_{B'}(u_j) \left(\int_{I^{d-1}} \prod_{i=1, i\neq j}^{d-1} \left(\mathbf{1}_{\{\delta'(u_i)\neq d\}} \mathbf{1}_{\{u_i \leq u_d\}}\right) \mathbf{1}_{\{\delta'(u_d)\neq 0\}}\ \mathbf{1}_{\{u_d\geq u_j\}} du_{(-j)}\right) du_j = 0. \tag{23}$$

Since, for $\varepsilon > 0$, $\delta(1) - \delta(1-\varepsilon) > 1 - (1-\varepsilon) = \varepsilon > 0$, there exists $g > 0$ a.e. on I such that $P_2^j(du_j) = g(u_j)\mathbf{1}_{\{\delta'(u_j)\neq d\}} du_j$. Therefore by (21) we deduce that P is absolutely continuous with respect to $P_1^j \otimes P_2^j$. Then according to Proposition 2 of [23], (18) implies that there exist measurable functions Φ_j and Γ_j defined respectively on I^{d-1} and I, such that $c^*\mu$-a.e. on $\triangle$:

$$\log(c^*(u)) = \Phi_j(u_{(-j)}) + \Gamma_j(u_j).$$

As μ-a.e. $c^* > 0$, this equality holds μ-a.e. on $\triangle$. Since we have such a representation for every $1 \leq j \leq d-1$, we can easily verify that there exists a measurable function Γ_d defined on I such that $\log(c^*(u)) = \sum_{i=1}^{d} \Gamma_i(u_i)$ μ-a.e. on $\triangle$. □

5.2. Calculation of the optimal solution

Now we prove that the optimal solution to (P^δ), if it exists, is indeed c_δ.

Proposition 5.3. *Let $\delta \in \mathcal{D}_0$ such that $\Sigma_\delta = \{0, 1\}$. If there exists a feasible solution c to (P^δ) such that $c > 0$ μ-a.e., then the optimal solution to (P^δ) is c_δ given by (4).*

Proof. In Lemma 5.2 we have already shown that if an optimal solution exists for (P^δ), then it is of the form $c^*(u) = b^*(\max(u)) \prod_{u_i\neq\max(u)} a^*(u_i)$. Here we will prove that the constraints of (P^δ) uniquely determine the functions a^* and b^* up to a multiplicative constant, giving $c^* = c_\delta$. We set for $r \in I$:

$$A^*(r) = \int_0^r a^*(s)\, ds$$

which take values in $[0, +\infty]$. From $\mathcal{A}_{d+1}(c^*) = b_{d+1}^\delta$, we have for $r \in I$:

$$\begin{aligned}
\delta(r) &= \int_{I^d} c^*(u)\mathbf{1}_{\{\max(u)\leq r\}}\, du \\
&= \int_{I^d} b^*(\max(u)) \prod_{u_i\neq\max(u)} a^*(u_i)\mathbf{1}_{\{\max(u)\leq r\}}\, du \\
&= d\int_I (A^*(s))^{d-1} b^*(s)\mathbf{1}_{\{s\leq r\}}\, ds.
\end{aligned} \tag{24}$$

Taking the derivative with respect to r gives a.e. on I:

$$\delta'(r) = d(A^*(r))^{d-1} b^*(r). \tag{25}$$

This implies that $A^*(r)$ is finite for all $r \in [0, 1)$ and thus $A^*(0) = 0$. Similarly, using that $\mathcal{A}_1(c^*) = b_1^\delta$, we get that for $r \in I$:

$$\begin{aligned}
1 - r &= \int_{I^d} c^*(u)\mathbf{1}_{\{u_1 \geq r\}}\, du \\
&= \int_{I^d} b^*(\max(u)) \prod_{u_i \neq \max(u)} a^*(u_i)\mathbf{1}_{\{u_1 \geq r\}}\, du \\
&= \int_{I^d} \prod_{i=2}^{d} \left(a^*(u_i)\mathbf{1}_{\{u_i \leq u_1\}}\right) b^*(u_1)\mathbf{1}_{\{u_1 \geq r\}}\, du + (d-1)\int_{I^d} a^*(u_1) \prod_{i=3}^{d} \left(a^*(u_i)\mathbf{1}_{\{u_i \leq u_2\}}\right) b^*(u_2)\mathbf{1}_{\{u_2 \geq u_1 \geq r\}}\, du \\
&= \int_I (A^*(s))^{d-1} b^*(s)\mathbf{1}_{\{s \geq r\}}\, ds + (d-1)\int_I (A^*(s))^{d-2} b^*(s)(A^*(s) - A^*(r))\mathbf{1}_{\{s \geq r\}}\, ds \\
&= d\int_I (A^*(s))^{d-1} b^*(s)\mathbf{1}_{\{s \geq r\}}\, ds - (d-1)A^*(r)\int_I (A^*(s))^{d-2} b^*(s)\mathbf{1}_{\{s \geq r\}}\, ds.
\end{aligned}$$

Using this and (24) we deduce that for $r \in I$:

$$h(r) = (d-1)A^*(r)\int_I (A^*(s))^{d-2} b^*(s)\mathbf{1}_{\{s \geq r\}}\, ds. \tag{26}$$

Since $r > \delta(r)$ on $(0, 1)$, we have that A^* and $\int_I (A^*(s))^{d-2} b^*(s)\mathbf{1}_{\{s \geq r\}}\, ds$ are positive on $(0, 1)$. Dividing (25) by (26) gives a.e. for $r \in I$:

$$\frac{d-1}{d}\frac{\delta'(r)}{h(r)} = \frac{(A^*(r))^{d-2} b^*(r)}{\int_I (A^*(r))^{d-2} b^*(s)\mathbf{1}_{\{r \leq s \leq 1\}}\, ds}.$$

We integrate both sides to get for $r \in I$:

$$\frac{d-1}{d}\left(\log\left(\frac{h(r)}{h(1/2)}\right) - \int_{1/2}^{r} \frac{1}{h(s)}\, ds\right) = \log\left(\frac{\int_I (A^*(s))^{d-2} b^*(s)\mathbf{1}_{\{r \leq s \leq 1\}}\, ds}{\int_I (A^*(s))^{d-2} b^*(s)\mathbf{1}_{\{1/2 \leq s \leq 1\}}\, ds}\right).$$

Notice that the choice for the lower bound $1/2$ of the integral was arbitrary, see Remark 2.1. Taking the exponential yields:

$$\alpha h^{(d-1)/d}(r)\mathrm{e}^{-F(r)} = \int_I (A^*(s))^{d-2} b^*(s)\mathbf{1}_{\{r \leq s \leq 1\}}\, ds, \tag{27}$$

for some positive constant α. From (26) and (27), we derive:

$$A^*(r) = \frac{1}{\alpha(d-1)} h^{1/d}(r)\mathrm{e}^{F(r)}. \tag{28}$$

This proves that the function A^* is uniquely determined up to a multiplicative constant and so is a^*. With the help of (25) and (28), we can express b^* as, for $r \in I$:

$$b^*(r) = \frac{\delta'(r)(\alpha(d-1))^{d-1}}{d}\mathrm{e}^{-(d-1)F(r)}. \tag{29}$$

The function b^* is also uniquely determined up to a multiplicative constant. Therefore (25) implies that there is a unique c^* of the form (20) which solves $\mathcal{A}(c) = b^\delta$. (Notice however that the functions a^* and b^* are defined up to a multiplicative constant.) Then according to Proposition 2.2 we get that c_δ defined by (20) with a and b defined by (5) solves $\mathcal{A}(c) = b^\delta$, implying that c^* is equal to c_δ. $\square$

5.3. Proof of Proposition 2.4

Let $\delta \in \mathcal{D}_0$ such that $\Sigma_\delta = \{0, 1\}$. By construction, we have μ-a.e. $c_\delta > 0$. According to Proposition 2.2 and Remark 2.3, if $\mathcal{J}(\delta) < +\infty$, the copula density c_δ is feasible for (P^δ). Therefore Proposition 5.3 implies that it is the optimal solution as well. When $\mathcal{J}(\delta) = +\infty$, we show that there exists no feasible solution to c_δ, see the supplementary material can be found online at http://dx.doi.org/10.1016/j.jmva.2015.01.003.

6. Proof of Theorem 2.5

We first state an elementary lemma, whose proof is left to the reader. For f a function defined on I^d and $0 \leq s < t \leq 1$, we define $f^{s,t}$ by, for $u \in I^d$:

$$f^{s,t}(u) = (t-s)f(s\mathbf{1} + u(t-s)).$$

Lemma 6.1. *If c is the density of a copula C such that $\delta_C(s) = s$ and $\delta_C(t) = t$ for some fixed $0 \le s < t \le 1$, then $c^{s,t}$ is also the density of a copula, and its diagonal section, $\delta^{s,t}$, is given by, for $r \in I$:*

$$\delta^{s,t}(r) = \frac{\delta_C(s + r(t-s)) - s}{t-s}.$$

According to Remark 2.6, it is enough to consider the case $\delta \in \mathcal{D}_0$, that is Σ_δ with zero Lebesgue measure. We shall assume that $\Sigma_\delta \neq \{0, 1\}$. Since δ is continuous, we get that $I \setminus \Sigma_\delta$ can be written as the union of non-empty open disjoint intervals $((\alpha_j, \beta_j), j \in J)$, with $\alpha_j < \beta_j$ and J non-empty and at most countable. Set $\Delta_j = \beta_j - \alpha_j$. Since Σ_δ is of zero Lebesgue measure, we have $\sum_{j\in J} \Delta_j = 1$. We define also $S = \bigcup_{j\in J}[\alpha_j, \beta_j]^d$

For $s \in \Sigma_\delta$, notice that any feasible function c of (P^δ) satisfies for all $1 \le i \le d$:

$$\int_{I^d} c(u)\mathbf{1}_{\{u_i<s\}}\mathbf{1}_{D_i^c}(u)\, du = \int_{I^d} c(u)\mathbf{1}_{\{u_i<s\}}\, du - \int_{I^d} c(u)\mathbf{1}_{\{\max(u)<s\}}\, du = s - \delta(s) = 0,$$

where $D_i = \{u \in I^d$ such that $\forall j \neq i : u_j < s\}$. This implies that $c = 0$ a.e. on $I^d \setminus S$. We set $c^j = c^{\alpha_j,\beta_j}$ for $j \in J$. We deduce that if c is feasible for (P^δ), then we have that a.e.:

$$c(u) = \sum_{j\in J} \frac{1}{\Delta_j} c^j\left(\frac{u - \alpha_j \mathbf{1}}{\Delta_j}\right) \mathbf{1}_{(\alpha_j,\beta_j)^d}(u), \tag{30}$$

and:

$$\mathcal{I}_d(c) = \sum_{j\in J} \Delta_j \left(\mathcal{I}_d(c^j) - \log(\Delta_j)\right). \tag{31}$$

Thanks to Lemma 6.1, the condition $\mathcal{A}(c) = b^\delta$ is equivalent to $\mathcal{A}(c^j) = b^{\delta^j}$ for all $j \in J$. We deduce that the optimal solution of (P^δ), if it exists, is given by (30), where the functions c^j are the optimal solutions of (P^{δ^j}) for $j \in J$. Notice that by construction $\Sigma_{\delta^j} = \{0, 1\}$. Thanks to Proposition 2.4, the optimal solution to (P^{δ^j}) exists if and only if we have $\mathcal{J}(\delta^j) < +\infty$; and if it exists it is given by c_{δ^j}. Therefore, if there exists an optimal solution to (P^δ), then it is c_δ given by (9). To conclude, we have to compute $\mathcal{I}_d(c_\delta)$. Recall that $x\log(x) \ge -1/\mathrm{e}$ for $x > 0$. We have:

$$\begin{aligned}
\mathcal{I}_d(c_\delta) &= \lim_{\varepsilon\downarrow 0} \sum_{j\in J} \Delta_j \left(\mathcal{I}_d(c^j) - \log(\Delta_j)\right) \mathbf{1}_{\{\Delta_j>\varepsilon\}} \\
&= \lim_{\varepsilon\downarrow 0} \sum_{j\in J} \Delta_j \left((d-1)\mathcal{J}(\delta^j) - \log(\Delta_j)\right) \mathbf{1}_{\{\Delta_j>\varepsilon\}} + \sum_{j\in J} \Delta_j \mathcal{G}(\delta^j) \\
&= \sum_{j\in J} \Delta_j \left((d-1)\mathcal{J}(\delta^j) - \log(\Delta_j)\right) + \sum_{j\in J} \Delta_j \mathcal{G}(\delta^j),
\end{aligned}$$

where we used the monotone convergence theorem for the first equality, (7) for the second and the fact that $\mathcal{G}(\delta)$ is uniformly bounded over $\mathcal{D}_0$ and the monotone convergence theorem for the last. Elementary computations yield:

$$(d-1)\mathcal{J}(\delta) = \sum_{j\in J} \Delta_j \left((d-1)\mathcal{J}(\delta^j) - \log(\Delta_j)\right) \quad \text{and} \quad \mathcal{G}(\delta) = \sum_{j\in J} \Delta_j \mathcal{G}(\delta^j).$$

So, we get:

$$\mathcal{I}_d(c_\delta) = (d-1)\mathcal{J}(\delta) + \mathcal{G}(\delta).$$

Since $\mathcal{G}(\delta)$ is uniformly bounded over $\mathcal{D}_0$, we get that $\mathcal{I}_d(c_\delta)$ is finite if and only if $\mathcal{J}(\delta)$ is finite. To end the proof, recall the definition of $\mathcal{I}(C_\delta)$ to conclude that $\mathcal{I}(C_\delta) = (d-1)\mathcal{J}(\delta) + \mathcal{G}(\delta)$.

7. Examples for $d = 2$

In this section we compute the density of the maximum entropy copula for various diagonal sections of popular bivariate copula families. In this Section, u and v will denote elements of I. The density for $d = 2$ is of the form $c_\delta(u, v) = a(\min(u, v))b(\max(u, v))$. For $(u, v) \in \triangle = \{(u, v) \in I^2, u \le v\}$, the formula reads:

$$c_\delta(u, v) = \frac{\delta'(u)}{2\sqrt{h(u)}} \frac{2 - \delta'(v)}{2\sqrt{h(v)}} \mathrm{e}^{-(F(v)-F(u))},$$

with h, F defined in (6). We illustrate these densities by displaying their isodensity lines or contour plots, and their diagonal cross-section φ defined as $\varphi(t) = c(t, t), t \in I$.

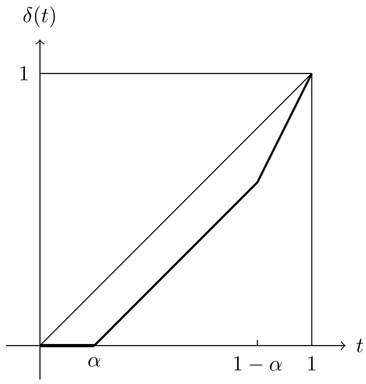
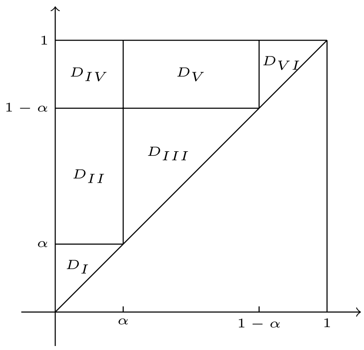
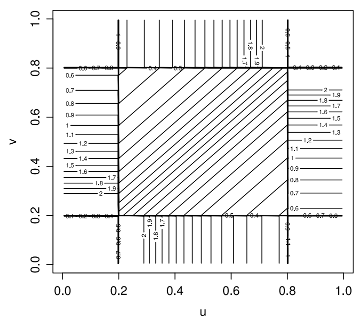


Fig. 1. Piecewise linear diagonal section (Section 7.1). Graph of δ with $\alpha = 0.2$.



(a) Partition for c_δ.



(b) Isodensity lines of c_δ.

Fig. 2. Piecewise linear diagonal section (Section 7.1). The partition and the isodensity lines of c_δ.

7.1. Maximum entropy copula for a piecewise linear diagonal section

Let $\alpha \in (0, 1/2]$. Let us calculate the density of the maximum entropy copula in the case of the following diagonal section:

$$\delta(r) = (r-\alpha)\mathbf{1}_{(\alpha,1-\alpha)}(r) + (2r-1)\mathbf{1}_{[1-\alpha,1]}(r).$$

This example was considered for example in [17]. The limiting cases $\alpha = 0$ and $\alpha = 1/2$ correspond to the Fréchet–Hoeffding upper and lower bound copulas, respectively. However for $\alpha = 0$, $\Sigma_\delta = I$, therefore every copula C with this diagonal section gives $\mathcal{I}(C) = +\infty$. (In fact the only copula that has this diagonal section is the Fréchet–Hoeffding upper bound M defined by $M(u, v) = \min(u, v)$, $u, v \in I$.) When $\alpha \in (0, 1/2]$, $\mathcal{J}(\delta) < +\infty$ is satisfied, therefore we can apply Proposition 2.4 to compute the density of the maximum entropy copula. The graph of δ can be seen in Fig. 1 for $\alpha = 0.2$. We compute the functions F, a and b:

$$F(r) = \begin{cases} \dfrac{1}{2}\log\left(\dfrac{r}{\alpha}\right) - \dfrac{1}{4\alpha} + \dfrac{1}{2} & \text{if } r \in [0, \alpha), \\ \dfrac{r}{2\alpha} - \dfrac{1}{4\alpha} & \text{if } t \in [\alpha, 1-\alpha), \\ \dfrac{1}{2}\log\left(\dfrac{\alpha}{1-r}\right) + \dfrac{1}{4\alpha} - \dfrac{1}{2} & \text{if } t \in [1-\alpha, 1], \end{cases}$$

$$a(r) = \frac{1}{\sqrt{\alpha}} e^{-\frac{1}{4\alpha}+\frac{1}{2}} \mathbf{1}_{[0,\alpha]}(r) + \frac{1}{2\sqrt{\alpha}} e^{\frac{r}{2\alpha}-\frac{1}{4\alpha}} \mathbf{1}_{(\alpha,1-\alpha)}(r),$$

and:

$$b(r) = \frac{1}{2\sqrt{\alpha}} e^{-\frac{r}{2\alpha}+\frac{1}{4\alpha}} \mathbf{1}_{(\alpha,1-\alpha)}(r) + \frac{1}{\sqrt{\alpha}} e^{-\frac{1}{4\alpha}+\frac{1}{2}} \mathbf{1}_{[1-\alpha,1]}(r).$$

The density $c_\delta(u, v)$ consists of six distinct regions on $\triangle$ as shown in Fig. 2(a) and takes the values:

$$c_\delta(u, v) = \frac{1}{2\alpha} e^{\frac{\alpha-v}{2\alpha}} \mathbf{1}_{\{(u,v)\in D_{II}\}} + \frac{1}{4\alpha} e^{\frac{u-v}{2\alpha}} \mathbf{1}_{\{(u,v)\in D_{III}\}} + \frac{1}{\alpha} e^{\frac{2\alpha-1}{2\alpha}} \mathbf{1}_{\{(u,v)\in D_{IV}\}} + \frac{1}{2\alpha} e^{\frac{u+\alpha-1}{2\alpha}} \mathbf{1}_{\{(u,v)\in D_V\}}. \tag{32}$$

Fig. 2(b) shows the isodensity lines of c_δ. In the limiting case of $\alpha = \frac{1}{2}$, the diagonal section is given by $\delta(t) = \max(0, 2t - 1)$,which is the pointwise lower bound for all elements in $\mathcal{D}$. Accordingly, it is the diagonal section of the Fréchet–Hoeffding lower bound copula W given by $W(u, v) = \max(0, u + v - 1)$ for $u, v \in I$. All copulas having this diagonal section are of the following form:

$$D_{C_1,C_2}(u, v) = \begin{cases} W(u, v) & \text{if } (u, v) \in [0, 1/2]^2 \cup [1/2, 1]^2, \\ \dfrac{1}{2} C_1(2u, 2v - 1) & \text{if } (u, v) \in [0, 1/2] \times [1/2, 1], \\ \dfrac{1}{2} C_2(2u - 1, 2v) & \text{if } (u, v) \in [1/2, 1] \times [0, 1/2], \end{cases}$$

where C_1 and C_2 are copula functions. Recall that the independent copula Π with uniform density $c_\Pi = 1$ on I^2 minimizes $\mathcal{I}(C)$ over $\mathcal{C}$. According to (32), the maximum entropy copula with diagonal section δ is $D_{\Pi,\Pi}$. This corresponds to choosing the maximum entropy copulas on $[0, 1/2] \times [1/2, 1]$ and $[1/2, 1] \times [0, 1/2]$.

7.2. Maximum entropy copula for $\delta(t) = t^\alpha$

Let $\alpha \in (1, 2]$. We consider the family of diagonal sections given by $\delta(t) = t^\alpha$. This corresponds to the Gumbel family of copulas and also to the family of Cuadras–Augé copulas. The Gumbel copula with parameter $\theta \in [1, \infty)$ is an Archimedean copula defined as, for $u, v \in I$:

$$C^G(u, v) = \varphi_\theta^{-1}(\varphi_\theta(u) + \varphi_\theta(v))$$

with generator function $\varphi_\theta(t) = (-\log(t))^\theta$. Its diagonal section is given by $\delta^G(t) = t^{2^{\frac{1}{\theta}}} = t^\alpha$ with $\alpha = 2^{\frac{1}{\theta}}$. The Cuadras–Augé copula with parameter $\gamma \in (0, 1)$ is defined as, for $u, v \in I$:

$$C^{CA}(u, v) = \min(uv^{1-\gamma}, u^{1-\gamma} v).$$

It is a subclass of the two parameter Marshall–Olkin family of copulas given by:

$$C^M(u, v) = \min(u^{1-\gamma_1} v, uv^{1-\gamma_2}).$$

The diagonal section of C^{CA} is given by $\delta(t) = t^{2-\gamma} = t^\alpha$ with $\alpha = 2-\gamma$. While the Gumbel copula is absolutely continuous, the Cuadras–Augé copula is not, although it has full support. Since $\mathcal{J}(\delta) < +\infty$, we can apply Proposition 2.4. To give the density of the maximum entropy copula, we have to calculate $F(v) - F(u)$. Elementary computations yield:

$$F(v) - F(u) = \frac{1}{2} \int_u^v \frac{ds}{s - s^\alpha} = \frac{1}{2} \log\left(\frac{v}{u}\right) - \frac{1}{2\alpha - 2} \log\left(\frac{1 - v^{\alpha-1}}{1 - u^{\alpha-1}}\right).$$

The density c_δ is therefore given by, for $(u, v) \in \triangle$:

$$c_\delta(u, v) = \frac{\alpha}{4} \frac{2 - \alpha u^{\alpha-1}}{(1 - u^{\alpha-1})^{\alpha/(2\alpha-2)}} v^{\alpha-2} (1 - v^{\alpha-1})^{(2-\alpha)/(2\alpha-2)}.$$

Fig. 3 represents the isodensity lines of the Gumbel and the maximum entropy copula c_δ with common parameter $\alpha = 2^{\frac{1}{3}}$, which corresponds to $\theta = 3$ for the Gumbel copula. We have also added a graph of the diagonal cross-section of the two densities. In the limiting case of $\alpha = 2$, the above formula gives $c_\delta(u, v) = 1$, which is the density of the independent copula Π, which also maximizes the entropy on the entire set of copulas.

7.3. Maximum entropy copula for the Farlie–Gumbel–Morgenstern diagonal section

Let $\theta \in [-1, 1]$. The Farlie–Gumbel–Morgenstern family of copulas (FGM copulas for short) is defined as:

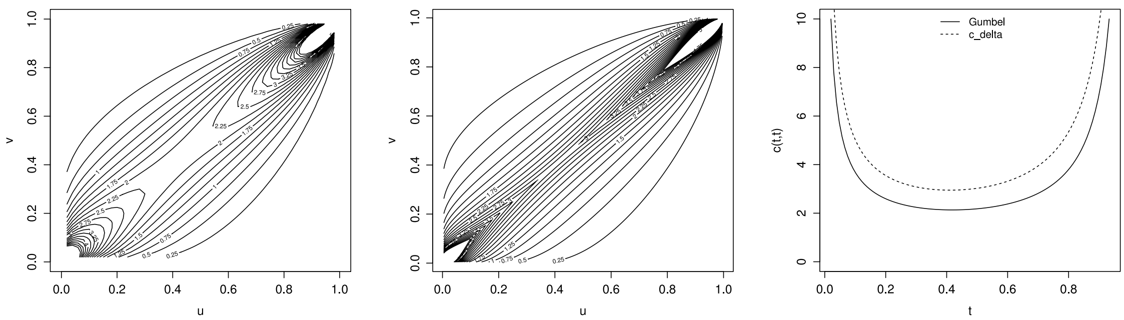
$$C(u, v) = uv + \theta uv(1 - u)(1 - v).$$

These copulas are absolutely continuous with densities $c(u, v) = 1 + \theta(1 - 2u)(1 - 2v)$. Its diagonal section δ_θ is given by:

$$\delta(t) = t^2 + \theta t^2(1 - t)^2 = \theta t^4 - 2\theta t^3 + (1 + \theta)t^2.$$

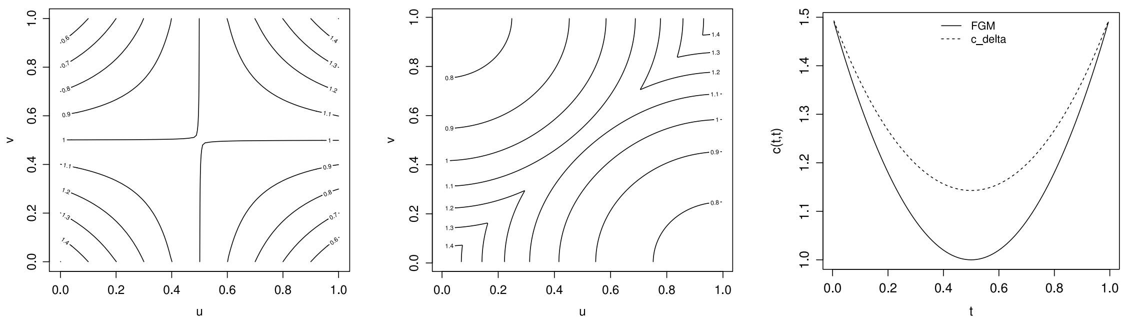
Since $\delta_\theta(t) < t$ on $(0, 1)$ and it verifies $\mathcal{J}(\delta) < +\infty$, we can apply Proposition 2.4 to calculate the density of the maximum entropy copula. For $F(r)$, we have:

$$F(r) = \begin{cases} \dfrac{1}{2} \log\left(\dfrac{r}{1 - r}\right) + \dfrac{\theta}{\sqrt{4\theta - \theta^2}} \arctan\left(\dfrac{2\theta r - \theta}{\sqrt{4\theta - \theta^2}}\right) & \text{if } \theta \in (0, 1], \\ \dfrac{1}{2} \log\left(\dfrac{r}{1 - r}\right) & \text{if } \theta = 0, \\ \dfrac{1}{2} \log\left(\dfrac{r}{1 - r}\right) - \dfrac{\theta}{\sqrt{\theta^2 - 4\theta}} \operatorname{arctanh}\left(\dfrac{2\theta r - \theta}{\sqrt{\theta^2 - 4\theta}}\right) & \text{if } \theta \in [-1, 0). \end{cases}$$



(a) Gumbel. (b) c_δ. (c) Diagonal cross-section.

Fig. 3. Power function diagonal section (Section 7.2). Isodensity lines and the diagonal cross-section of copulas with diagonal section $\delta(t) = t^\alpha$, $\alpha = 2^{\frac{1}{3}}$.



(a) FGM. (b) c_δ. (c) Diagonal cross-section.

Fig. 4. FGM diagonal section (Section 7.3). Isodensity lines and the diagonal cross-section of copulas with diagonal section $\delta(t) = \theta t^4 - 2\theta t^3 + (1+\theta)t^2$, $\theta = 0.5$.

The density c_δ is given by, for $\theta \in (0, 1]$ and $(u, v) \in \triangle$:

$$c_\delta(u, v) = \frac{\left(1 - 2\theta u^3 + 3\theta u^2 + (1+\theta)u\right)}{(1-u)\sqrt{\theta u^2 - \theta u + 1}} \frac{\left(2\theta v^2 + 3\theta v + (1+\theta)\right)}{\sqrt{\theta v^2 - \theta v + 1}} \times \exp\left(-\frac{\theta}{\sqrt{4\theta - \theta^2}}\left(\arctan\left(\frac{2\theta v - \theta}{\sqrt{4\theta - \theta^2}}\right) - \arctan\left(\frac{2\theta u - \theta}{\sqrt{4\theta - \theta^2}}\right)\right)\right).$$

Fig. 4 illustrates the isodensities of the FGM copula and the maximum entropy copula with the same diagonal section for $\theta = 0.5$ as well as the diagonal cross-section of their densities.

The case of $\theta = 0$ corresponds once again to the diagonal section $\delta(t) = t^2$, and the formula gives the density of the independent copula Π, accordingly.

7.4. Maximum entropy copula for the Ali–Mikhail–Haq diagonal section

Let $\theta \in [-1, 1]$. The Ali–Mikhail–Haq (AMH for short) family of copulas are defined as:

$$C(u, v) = \frac{uv}{1 - \theta(1-u)(1-v)}.$$

This is a family of absolutely continuous copulas whose diagonal section is given by:

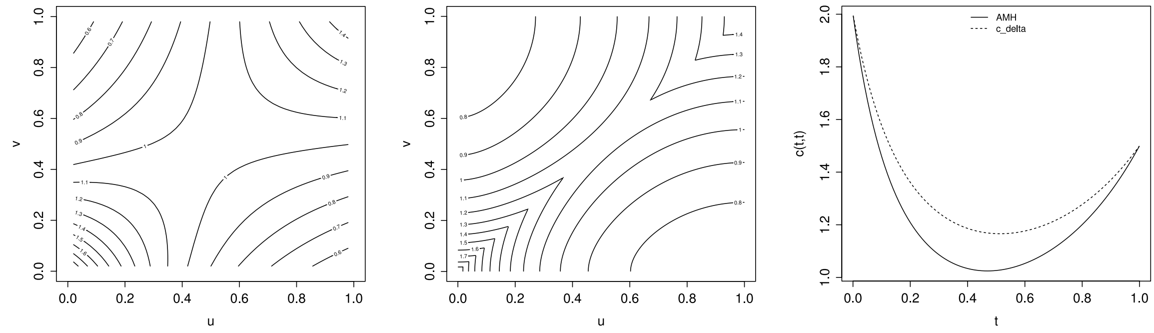
$$\delta(t) = \frac{t^2}{1 - \theta(1-t)^2}.$$

Once again, $\delta_\theta(t) < t$ on $(0, 1)$ and $\mathcal{J}(\delta) < +\infty$ is verified, so we can apply Proposition 2.4 to calculate the density of the maximum entropy copula. For $0 \le u \le v \le 1$:

$$F(v) - F(u) = \frac{1}{2}\left(\ln\left(\frac{v}{u}\right) - \ln\left(\frac{1-v}{1-u}\right) + \ln\left(\frac{\theta v + 1 - \theta}{\theta u + 1 - \theta}\right)\right).$$

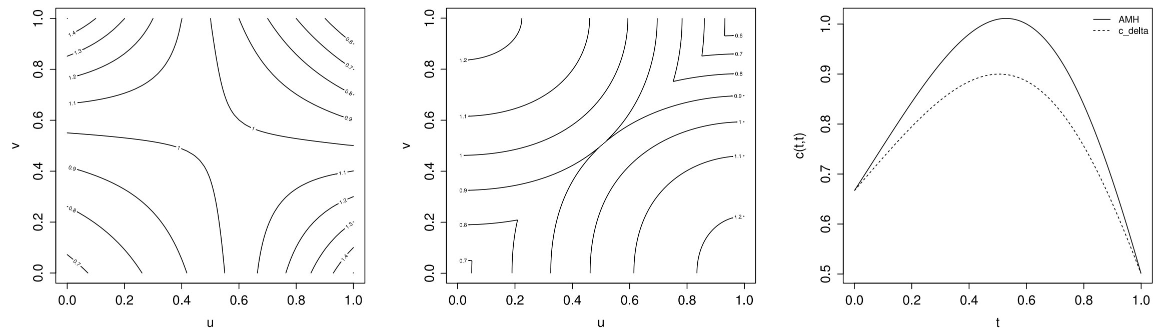
Then c_δ is given by, for $(u, v) \in \triangle$:

$$c_\delta(u, v) = \frac{1 + \theta u - 2\theta(1-u) + \theta^2(1-u)^3}{\left(1 - \theta(1-u)^2\right)^{\frac{3}{2}}} \left(1 - \theta(1-v)^2\right)^{-\frac{3}{2}}.$$



(a) AMH. (b) c_δ. (c) Diagonal cross-section.

Fig. 5. AMH diagonal section (Section 7.4). Isodensity lines and the diagonal cross-section of copulas with diagonal section $\delta(t) = \frac{t^2}{1-\theta(1-t)^2}$, $\theta = 0.5$.



(a) AMH. (b) c_δ. (c) Diagonal cross-section.

Fig. 6. AMH diagonal section (Section 7.4). Isodensity lines and the diagonal cross-section of copulas with diagonal section $\delta(t) = \frac{t^2}{1-\theta(1-t)^2}$, $\theta = -0.5$.

In the case of $\theta = 0$, the AMH copula reduces to the independent copula Π. We illustrate the density of the AMH copula and the corresponding maximum entropy copula with $\theta = 0.5$ in Fig. 5 and $\theta = -0.5$ in Fig. 6.

7.5. *Maximum entropy copula for the Gaussian diagonal section*

The Gaussian (normal) copula takes the form:

$$C_\rho(u, v) = \Phi_\rho\left(\Phi^{-1}(u), \Phi^{-1}(v)\right),$$

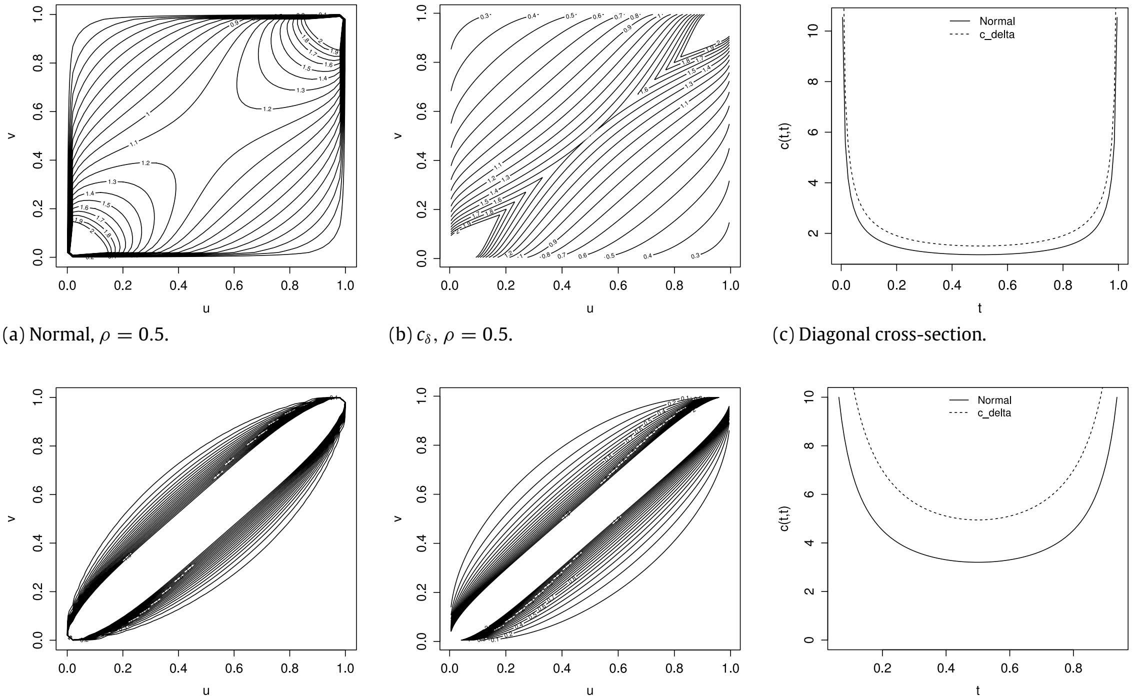
with Φ_ρ the joint cumulative distribution function of a two-dimensional normal random variable with standard normal marginals and correlation parameter $\rho \in [-1, 1]$, and Φ^{-1} the quantile function of the standard normal distribution. The density c_ρ of C_ρ can be written as:

$$c_\rho(u, v) = \frac{\varphi_\rho\left(\Phi^{-1}(u), \Phi^{-1}(v)\right)}{\varphi(\Phi^{-1}(u))\varphi(\Phi^{-1}(v))},$$

where φ and φ_ρ stand for respectively the densities of a standard normal distribution and a two-dimensional normal distribution with correlation parameter ρ, respectively. The diagonal section and its derivative are given by:

$$\delta_\rho(t) = \Phi_\rho\left(\Phi^{-1}(t), \Phi^{-1}(t)\right), \qquad \delta'_\rho(t) = 2\Phi\left(\sqrt{\frac{1-\rho}{1+\rho}}\Phi^{-1}(t)\right). \tag{33}$$

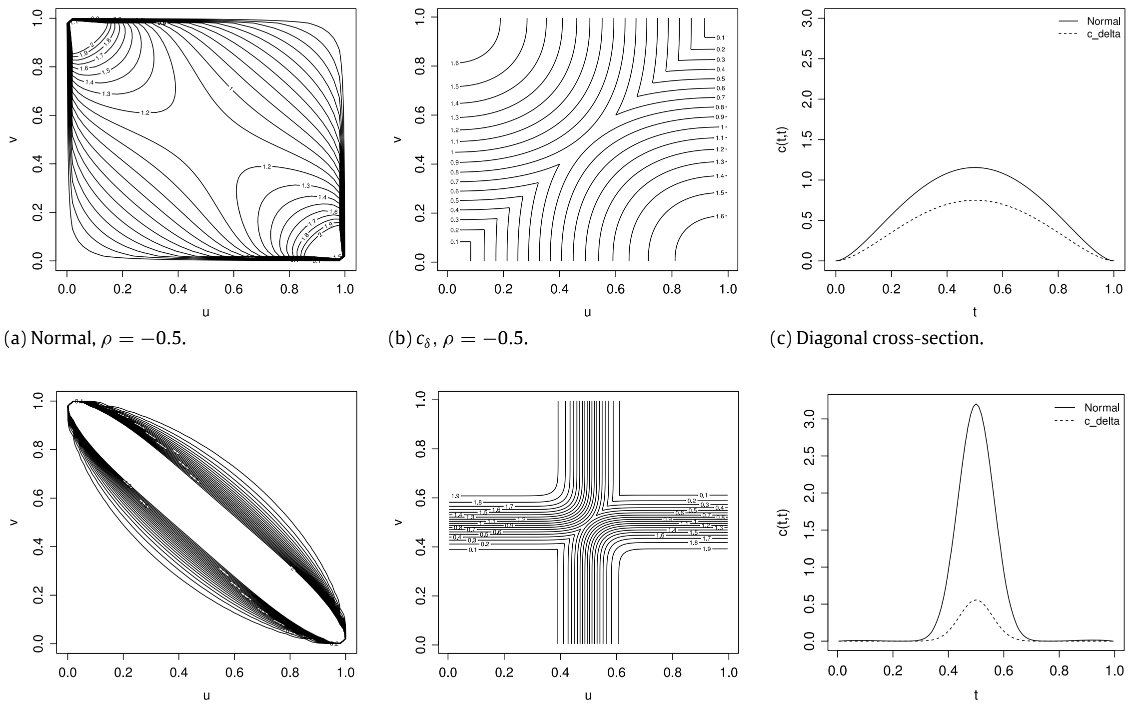
Since δ_ρ verifies $\delta_\rho(t) < t$ on $(0, 1)$ and $\mathcal{J}(\delta_\rho) < +\infty$, we can apply Proposition 2.4 to calculate the density of the maximum entropy copula. We have calculated numerically the density of the maximum entropy copula with diagonal section δ_ρ for $\rho = 0.95, 0.5, -0.5$ and -0.95. The comparison between these densities and the densities of the corresponding normal copula can be seen in Figs. 7–9. In the limiting case when ρ goes up to 1, we observe a similar behaviour of c_ρ and c_{δ_ρ}, and we get the limiting diagonal $\delta(t) = t$ of the Fréchet–Hoeffding upper bound M given by $M(u, v) = \min(u, v)$, which does not have a density. We observe a very different behaviour of c_ρ and c_{δ_ρ} in the case of $\rho < 0$. In the limiting case when ρ goes down to -1, we get the diagonal $\delta(t) = \max(0, 2t - 1)$, which we have studied earlier in Section 7.1.



(a) Normal, $\rho = 0.5$. (b) c_δ, $\rho = 0.5$. (c) Diagonal cross-section.

(d) Normal, $\rho = 0.95$. (e) c_δ, $\rho = 0.95$. (f) Diagonal cross-section.

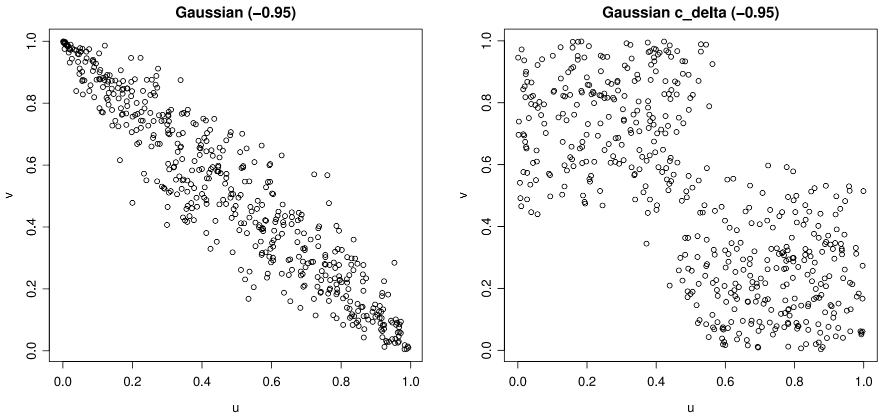
Fig. 7. Gaussian diagonal section (Section 7.5). Isodensity lines and the diagonal cross-section of copulas with diagonal section given by (33), with $\rho = 0.5$ and $\rho = 0.95$.



(a) Normal, $\rho = -0.5$. (b) c_δ, $\rho = -0.5$. (c) Diagonal cross-section.

(d) Normal, $\rho = -0.95$. (e) c_δ, $\rho = -0.95$. (f) Diagonal cross-section.

Fig. 8. Gaussian diagonal section (Section 7.5). Isodensity lines and the diagonal cross-section of copulas with diagonal section given by (33), with $\rho = -0.5$ and $\rho = -0.95$.



(a) Normal, $\rho = -0.95$.

(b) C_δ, $\rho = -0.95$.

Fig. 9. Gaussian diagonal section (Section 7.5). Sample of 500 drawn from the Gaussian copula with $\rho = -0.95$ and from the corresponding C_δ.

7.6. Comparison of conditional extreme event probabilities in the Gaussian case

We compare the conditional probabilities of extreme values of a pair of random variables (X_1, X_2) which has bivariate normal distribution with standard normal marginals and correlation coefficient ρ, with a pair of random variables (Y_1, Y_2) whose marginals are also standard normal, but has copula c_δ, where δ is the diagonal of the copula of (X_1, X_2). We compute the conditional probabilities $\mathbb{P}(X_1 \geq \alpha t | X_2 = t)$ and $\mathbb{P}(Y_1 \geq \alpha t | Y_2 = t)$ with $\alpha \geq 1$ and consider their asymptotic behaviour when t goes to infinity. This comparison is motivated by consideration of correlated defaults in mathematical finance, see Section 10.8 in [24]. (Notice however the parameters of upper tail dependence of the two copulas are the same since they have the same diagonal.)

Since by construction $\max(X_1, X_2)$ has the same distribution as $\max(Y_1, Y_2)$, and X_1, X_2, Y_1 and Y_2 have the same distribution, we deduce that $\min(X_1, X_2)$ has the same distribution as $\min(Y_1, Y_2)$. We deduce that for all $t \in \mathbb{R}$:

$$\mathbb{P}(X_1 \geq t | X_2 = t) = -\frac{\partial_t \mathbb{P}(\min(X_1, X_2) \geq t)}{\varphi(t)} = -\frac{\partial_t \mathbb{P}(\min(Y_1, Y_2) \geq t)}{\varphi(t)} = \mathbb{P}(Y_1 \geq t | Y_2 = t).$$

From now on, we shall consider $\alpha > 1$. For $k \in \mathbb{R}$, we recall the notations $h(t) = O(t^k)$ for t large which means that $\limsup_{t\to+\infty} t^{-k}|h(t)| < +\infty$, and $f(t) \ll g(t)$ for t large which means that f and g are positive for t large and $\limsup_{t\to\infty} f(t)/g(t) = 0$. The proof of the next Lemma is given in the Appendix.

Lemma 7.1. *Let $\alpha > 1$ and $\rho \in (-1, 1)$. We have for t large:*

$$\mathbb{P}(X_1 \geq \alpha t | X_2 = t) = \kappa_{\rho,\alpha} \mathbb{P}(Y_1 \geq \alpha t | Y_2 = t) \mathrm{e}^{-\Delta_{\rho,\alpha} t^2/2}(1 + O(t^{-2})), \tag{34}$$

with:

$$\kappa_{\rho,\alpha} = \frac{\alpha(1-\rho)}{(\alpha - \rho)} \quad \text{and} \quad \Delta_{\rho,\alpha} = \frac{\rho(\alpha - 1)}{1 - \rho^2}\left((\alpha + 1)\rho - 2\right).$$

We deduce that:

- for $\rho > 0$ and $\alpha > 2/\rho - 1$ or $\rho < 0$, we have $\Delta_{\rho,\alpha} > 0$ and thus $\mathbb{P}(X_1 \geq \alpha X_2 | X_2 = t) \ll \mathbb{P}(Y_1 \geq \alpha Y_2 | Y_2 = t)$ for t large,
- for $\rho > 0$ and $1 < \alpha < 2/\rho - 1$, we have $\Delta_{\rho,\alpha} < 0$ and thus $\mathbb{P}(X_1 \geq \alpha X_2 | X_2 = t) \gg \mathbb{P}(Y_1 \geq \alpha Y_2 | Y_2 = t)$ for t large.

In conclusion, in the positive correlation case, the maximum entropy copula gives more weight to the extremal conditional probabilities for large values of α.

Remark 7.2. Similar computations as in the proof of Lemma 7.1 give that for $\rho > 0$, $\rho \leq \alpha < 1$:

$$\mathbb{P}\left(\alpha t \leq X_1 \leq t | X_2 = t\right) = \bar{\Phi}\left(\frac{\alpha - \rho}{\sqrt{1-\rho^2}} t\right)\left(1 + O(t^{-2})\right),$$

$$\mathbb{P}\left(\alpha t \leq Y_1 \leq t | Y_2 = t\right) = \bar{\Phi}\left(\alpha \sqrt{\frac{1-\rho}{1+\rho}} t\right)\left(1 + O(t^{-2})\right),$$

with $\bar{\Phi} = 1 - \Phi$, the survival function of the standard Gaussian distribution. Using (42), we have $\mathbb{P}\left(\alpha t \leq X_1 \leq t | X_2 = t\right) \gg \mathbb{P}(Y_1 \geq \alpha Y_2 | Y_2 = t)$ for t large. This means that the maximum entropy copula gives less weight to the "non-worse" case, when the first variable takes also large values, but stays less than the second variable.

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Appendix

A.1. Calculation of the entropy of C_δ

In this section, we show that (7) of Proposition 2.2 holds. Let us first introduce some notations. Let $\varepsilon \in (0, 1/2)$. Since $x\log(x) \geq -1/\mathrm{e}$ for $x > 0$, we deduce by the monotone convergence theorem that:

$$\mathcal{I}(C_\delta) = \lim_{\varepsilon \downarrow 0} \mathcal{I}_\varepsilon(C_\delta), \tag{35}$$

with:

$$\mathcal{I}_\varepsilon(C_\delta) = \int_{[\varepsilon,1-\varepsilon]^d} c_\delta(x)\log(c_\delta(x))\,dx.$$

Using $\delta(t) \leq t$ and that δ is a non-decreasing, d-Lipschitz function, we get that for $t \in I$:

$$0 \leq h(t) \leq \min(t, (d-1)(1-t)) \leq (d-1)\min(t, 1-t). \tag{36}$$

We set:

$$w(t) = a(t)\mathrm{e}^{-F(t)} = \frac{d-\delta'(t)}{d} h^{-1+1/d}(t). \tag{37}$$

From the symmetric property of c_δ, we have that

$$\mathcal{I}_\varepsilon(C_\delta) = J_1(\varepsilon) + J_2(\varepsilon) - J_3(\varepsilon), \tag{38}$$

with:

$$J_1(\varepsilon) = d\int_{[\varepsilon,1-\varepsilon]^d} c_\delta(x)\mathbf{1}_{\{\max(x)=x_d\}}\left(\sum_{i=1}^{d-1}\log\left(w(x_i)\right)\right)dx,$$

$$J_2(\varepsilon) = d\int_{[\varepsilon,1-\varepsilon]^d} c_\delta(x)\mathbf{1}_{\{\max(x)=x_d\}}\log\left(\frac{\delta'(x_d)}{d}h^{-1+1/d}(x_d)\right)dx,$$

$$J_3(\varepsilon) = d\int_{[\varepsilon,1-\varepsilon]^d} c_\delta(x)\mathbf{1}_{\{\max(x)=x_d\}}\left((d-1)F(x_d) - \sum_{i=1}^{d-1}F(x_i)\right)dx.$$

We introduce $A_\varepsilon(r) = \int_\varepsilon^r a(x)\,dx$. For $J_1(\varepsilon)$, we have:

$$\begin{aligned} J_1(\varepsilon) &= d(d-1)\int_{[\varepsilon,1-\varepsilon]^d} \mathbf{1}_{\{\max(x)=x_d\}} b(x_d)\prod_{j=1}^{d-1}a(x_j)\log\left(w(x_1)\right)\,dx \\ &= d(d-1)\int_{[\varepsilon,1-\varepsilon]}\left(\int_{[t,1-\varepsilon]} A_\varepsilon^{d-2}(s)b(s)\,ds\right)a(t)\log\left(w(t)\right)\,dt. \end{aligned}$$

Notice that using (11) and (13), we have:

$$\begin{aligned} \int_{[t,1-\varepsilon]} A_\varepsilon^{d-2}(s)b(s)\,ds &= \int_{[t,1]} A^{d-2}(s)b(s)\,ds - \int_{[t,1]}\left(A^{d-2}(s) - A_\varepsilon^{d-2}(s)\right)b(s)\,ds - \int_{[1-\varepsilon,1]} A_\varepsilon^{d-2}(s)b(s)\,ds. \\ &= \frac{h(t)}{(d-1)A(t)} - \int_t^1 \left(A^{d-2}(s) - A_\varepsilon^{d-2}(s)\right)b(s)\,ds - \int_{[1-\varepsilon,1]} A_\varepsilon^{d-2}(s)b(s)\,ds. \end{aligned}$$

By Fubini's theorem, we get:

$$J_1(\varepsilon) = J_{1,1}(\varepsilon) - J_{1,2}(\varepsilon) - J_{1,3}(\varepsilon),$$

with:

$$J_{1,1}(\varepsilon) = \int_{[\varepsilon,1-\varepsilon]}(d-\delta'(t))\log\left(w(t)\right)\,dt$$

$$J_{1,2}(\varepsilon) = d(d-1)\left(\int_{[1-\varepsilon,1]} A_\varepsilon^{d-2}(s)b(s)\,ds\right)\int_{[\varepsilon,1-\varepsilon]} a(t)\log\left(w(t)\right)\,dt$$

$$J_{1,3}(\varepsilon) = d(d-1)\int_{[\varepsilon,1-\varepsilon]}\left(\int_t^1 \left(A^{d-2}(s) - A_\varepsilon^{d-2}(s)\right)b(s)\,ds\right)a(t)\log\left(w(t)\right)\,dt.$$

To study $J_{1,2}$, we first give an upper bound for the term $\int_{[1-\varepsilon,1]} A_\varepsilon^{d-2}(s)a(s)b(s)\,ds$:

$$\begin{aligned}\int_{[1-\varepsilon,1]} A_\varepsilon^{d-2}(s)b(s)\,ds &\le \int_{[1-\varepsilon,1]} A^{d-2}(s)b(s)\,ds\\ &= \frac{1}{(d-1)}h^{1-1/d}(1-\varepsilon)\mathrm{e}^{-F(1-\varepsilon)}\\ &\le (d-1)^{-1/d}\varepsilon^{1-1/d},\end{aligned} \tag{39}$$

where we used that $A_\varepsilon(s) \le A(s)$ for $s > \varepsilon$ for the first inequality, (13) for the first equality, and (36) for the last inequality. Since $t\log(t) \ge -1/\mathrm{e}$, we have, using (37):

$$\begin{aligned}J_{1,2}(\varepsilon) &\ge -\frac{d(d-1)}{\mathrm{e}}\left(\int_{[1-\varepsilon,1]} A_\varepsilon^{d-2}(s)b(s)\,ds\right)\int_{[\varepsilon,1-\varepsilon]} \mathrm{e}^{F(t)}\,dt\\ &\ge -\frac{d}{\mathrm{e}}h^{1-1/d}(1-\varepsilon)\int_{[\varepsilon,1-\varepsilon]} \mathrm{e}^{F(t)-F(1-\varepsilon)}\,dt\\ &\ge -\frac{d}{\mathrm{e}}((d-1)\varepsilon)^{1-1/d},\end{aligned}$$

where we used (13) for the second inequality, and that F is non-decreasing and (39) for the third inequality. On the other hand, we have $t\log(t) \le t^{\frac{1}{1-1/d}}$, if $t \ge 0$, which gives:

$$\begin{aligned}J_{1,2}(\varepsilon) &\le d(d-1)\left(\int_{[1-\varepsilon,1]} A_\varepsilon^{d-2}(s)b(s)\,ds\right)\int_{[\varepsilon,1-\varepsilon]} \mathrm{e}^{F(t)}\frac{\left(\frac{d-\delta'(t)}{d}\right)^{\frac{1}{1-1/d}}}{h(t)}\,dt\\ &= dh^{1-1/d}(1-\varepsilon)\int_{[\varepsilon,1-\varepsilon]} \frac{\mathrm{e}^{F(t)-F(1-\varepsilon)}}{h(t)}\,dt\\ &= dh^{1-1/d}(1-\varepsilon)\left(1-\mathrm{e}^{F(\varepsilon)-F(1-\varepsilon)}\right)\\ &\le d((d-1)\varepsilon)^{1-1/d},\end{aligned}$$

where we used (39) and $t^{\frac{1}{1-1/d}} \le 1$ for $t \in I$ for the first inequality, and that F is non-decreasing for the last. This proves that $\lim_{\varepsilon\to 0} J_{1,2}(\varepsilon) = 0$. For $J_{1,3}(\varepsilon)$, we first observe that for $s \in [\varepsilon, 1-\varepsilon]$ we have $A_\varepsilon(s) \le A(s)$ and thus:

$$\left(A^{d-2}(s) - A_\varepsilon^{d-2}(s)\right) = A(\varepsilon)\sum_{i=0}^{d-3} A^i(s)A_\varepsilon^{d-3-i}(s) \le (d-2)A(\varepsilon)A^{d-3}(s). \tag{40}$$

Using the previous inequality we obtain:

$$\begin{aligned}J_{1,3}(\varepsilon) &= d(d-1)\int_{[\varepsilon,1-\varepsilon]}\left(\int_t^1 \left(A^{d-2}(s)-A_\varepsilon^{d-2}(s)\right)b(s)\,ds\right)a(t)\log(w(t))\,dt\\ &\ge -\frac{d(d-1)}{\mathrm{e}}\int_{[\varepsilon,1-\varepsilon]}\left(\int_t^1 \left(A^{d-2}(s)-A_\varepsilon^{d-2}(s)\right)b(s)\,ds\right)\mathrm{e}^{F(t)}\,dt\\ &\ge -\frac{d(d-1)(d-2)A(\varepsilon)}{\mathrm{e}}\int_{[\varepsilon,1-\varepsilon]}\left(\int_t^1 A^{d-3}(s)b(s)\,ds\right)\mathrm{e}^{F(t)}\,dt\\ &\ge -\frac{d(d-1)(d-2)A(\varepsilon)}{\mathrm{e}}\int_{[\varepsilon,1-\varepsilon]}\frac{\left(\int_t^1 A^{d-2}(s)b(s)\,ds\right)}{A(t)}\mathrm{e}^{F(t)}\,dt\\ &= -\frac{d(d-2)A(\varepsilon)}{\mathrm{e}}\int_{[\varepsilon,1-\varepsilon]}\frac{h(t)}{A^2(t)}\mathrm{e}^{F(t)}\,dt\\ &= -\frac{d(d-2)h^{1/d}(\varepsilon)}{\mathrm{e}}\int_{[\varepsilon,1-\varepsilon]} h(t)^{1-2/d}\mathrm{e}^{F(\varepsilon)-F(t)}\,dt\\ &\ge -\frac{d(d-2)(d-1)^{1-1/d}\varepsilon^{1/d}}{\mathrm{e}},\end{aligned}$$

where we used $t\log(t) \geq -1/\mathrm{e}$ for the first inequality, (40) for the second, (11) and (13) in the following equality, and (36) to conclude. For an upper bound, we have after noticing that $t\log(t) \leq t^2$:

$$\begin{aligned}
J_{1,3}(\varepsilon) &= d(d-1)\int_{[\varepsilon,1-\varepsilon]} \left(\int_t^1 \left(A^{d-2}(s) - A_\varepsilon^{d-2}(s)\right) b(s)\, ds\right) a(t) \log\left(w(t)\right)\, dt \\
&\leq d(d-1)\int_{[\varepsilon,1-\varepsilon]} \left(\int_t^1 \left(A^{d-2}(s) - A_\varepsilon^{d-2}(s)\right) b(s)\, ds\right) \mathrm{e}^{F(t)} w^2(t)\, dt \\
&\leq d(d-1)(d-2)A(\varepsilon)\int_{[\varepsilon,1-\varepsilon]} \frac{\left(\int_t^1 A^{d-2}(s) b(s)\, ds\right)}{A(t)} \mathrm{e}^{F(t)} h^{-2+2/d}(t)\, dt \\
&= d(d-2)A(\varepsilon)\int_{[\varepsilon,1-\varepsilon]} \frac{\mathrm{e}^{-F(t)}}{h(t)}\, dt \\
&= d(d-2)h^{1/d}(\varepsilon)(1-\mathrm{e}^{F(\varepsilon)-F(1-\varepsilon)}) \\
&\leq d(d-2)(d-1)^{1/d}\varepsilon^{1/d},
\end{aligned}$$

where we used (40) and $0 \leq (d-\delta'(t))/d \leq 1$ for the second inequality; (11) and (13) in the second equality; and (36) to conclude. The results on the two bounds show that $\lim_{\varepsilon\to 0} J_{1,3}(\varepsilon) = 0$. Similarly, for $J_2(\varepsilon)$, we get:

$$\begin{aligned}
J_2(\varepsilon) &= \int_{[\varepsilon,1-\varepsilon]^d} \mathbf{1}_{\{\max(x)=x_d\}} b(x_d) \prod_{j=1}^{d-1} a(x_j) \log\left(\frac{\delta'(x_d)}{d} h^{-1+1/d}(x_d)\right) dx \\
&= d\int_{[\varepsilon,1-\varepsilon]} A_\varepsilon^{d-1}(t) b(t) \log\left(\frac{\delta'(t)}{d} h^{-1+1/d}(t)\right) dt \\
&= d\int_{[\varepsilon,1-\varepsilon]} A^{d-1}(t) b(t) \log\left(\frac{\delta'(t)}{d} h^{-1+1/d}(t)\right) dt \\
&\quad - d\int_{[\varepsilon,1-\varepsilon]} \left(A^{d-1}(t) - A_\varepsilon^{d-1}(t)\right) b(t) \log\left(\frac{\delta'(t)}{d} h^{-1+1/d}(t)\right) dt \\
&= J_{2,1}(\varepsilon) - J_{2,2}(\varepsilon)
\end{aligned}$$

with $J_{2,1}(\varepsilon)$ and $J_{2,2}(\varepsilon)$ given by, using (12):

$$\begin{aligned}
J_{2,1}(\varepsilon) &= d\int_{[\varepsilon,1-\varepsilon]} A^{d-1}(t) b(t) \log\left(\frac{\delta'(t)}{d} h^{-1+1/d}(t)\right) dt \\
J_{2,2}(\varepsilon) &= d\int_{[\varepsilon,1-\varepsilon]} \left(A^{d-1}(t) - A_\varepsilon^{d-1}(t)\right) b(t) \log\left(\frac{\delta'(t)}{d} h^{-1+1/d}(t)\right) dt.
\end{aligned}$$

By (12), we have:

$$J_{2,1}(\varepsilon) = \int_{[\varepsilon,1-\varepsilon]} \delta'(t) \log\left(\frac{\delta'(t)}{d} h^{-1+1/d}(t)\right) dt. \tag{41}$$

Similarly to $J_{1,3}(\varepsilon)$ we can show that $\lim_{\varepsilon\to 0} J_{2,2}(\varepsilon) = 0$.

Adding up $J_1(\varepsilon)$ and $J_2(\varepsilon)$ gives

$$J_1(\varepsilon) + J_2(\varepsilon) = \mathcal{J}_\varepsilon(\delta) + J_4(\varepsilon) - d\log(d)(1-2\varepsilon) - J_{1,2}(\varepsilon) - J_{1,3}(\varepsilon) - J_{2,2}(\varepsilon)$$

with

$$\begin{aligned}
\mathcal{J}_\varepsilon(\delta) &= (d-1)\int_\varepsilon^{1-\varepsilon} \left|\log\left(h(t)\right)\right|\, dt, \\
J_4(\varepsilon) &= \int_\varepsilon^{1-\varepsilon} \left(d-\delta'(t)\right) \log\left(d-\delta'(t)\right) dt + \int_\varepsilon^{1-\varepsilon} \delta'(t) \log\left(\delta'(t)\right) dt.
\end{aligned}$$

Notice that $\mathcal{J}_\varepsilon(\delta)$ is non-decreasing in $\varepsilon > 0$ and that:

$$\mathcal{J}(\delta) = \lim_{\varepsilon\to 0} \mathcal{J}_\varepsilon(\delta).$$

Since $\delta'(t) \in [0, d]$, we deduce that $(d-\delta')\log(d-\delta')$ and $\delta'\log(\delta')$ are bounded on I from above by $d\log(d)$ and from below by $-1/\mathrm{e}$ and therefore integrable on I. This implies :

$$\lim_{\varepsilon\to 0} J_4(\varepsilon) = \mathcal{I}_1(\delta') + \mathcal{I}_1(d-\delta').$$

As for $J_3(\varepsilon)$, we have by integration by parts:

$$\begin{aligned}
J_3(\varepsilon) &= d\int_{[\varepsilon,1-\varepsilon]^d} \mathbf{1}_{\{\max(x)=x_d\}} b(x_d)\prod_{i=i}^{d-1} a(x_i)\left((d-1)F(x_d)-\sum_{i=1}^{d-1}F(x_i)\right)dx\\
&= d(d-1)\int_{[\varepsilon,1-\varepsilon]} A_\varepsilon^{d-1}(t)b(t)F(t)\,dt - d(d-1)\int_{[\varepsilon,1-\varepsilon]} A_\varepsilon^{d-2}(t)b(t)\left(\int_\varepsilon^t a(s)F(s)\,ds\right)dt\\
&= d(d-1)\int_{[\varepsilon,1-\varepsilon]} A_\varepsilon^{d-1}(t)b(t)F(t)\,dt - d(d-1)\int_{[\varepsilon,1-\varepsilon]} A_\varepsilon^{d-2}(t)b(t)\left(A_\varepsilon(t)F(t)-\frac{d-1}{d}\int_\varepsilon^t \frac{A_\varepsilon(s)}{h(s)}\,ds\right)dt\\
&= (d-1)^2\int_{[\varepsilon,1-\varepsilon]}\left(\int_t^{1-\varepsilon} A_\varepsilon^{d-2}(s)b(s)\,ds\right)\frac{A_\varepsilon(t)}{h(t)}\,dt.
\end{aligned}$$

By the monotone convergence theorem, (11) and (13) we have:

$$\lim_{\varepsilon\to 0} J_3(\varepsilon) = (d-1)^2\int_I\left(\int_t^1 A^{d-2}(s)b(s)\,ds\right)\frac{A(t)}{h(t)}\,dt = d-1.$$

Summing up all the terms and taking the limit $\varepsilon = 0$ give :

$$\begin{aligned}
\mathcal{I}(C_\delta) &= (d-1)\int_I |\log(t-\delta(t))|\,dt + \mathcal{I}_1(\delta') + \mathcal{I}_1(d-\delta') - d\log(d) - (d-1)\\
&= (d-1)\mathcal{J}(\delta) + \mathcal{G}(\delta).
\end{aligned}$$

A.2. Proof of Lemma 7.1

Set $\bar\Phi(x) = 1-\Phi(x)$, the survival function of the standard Gaussian distribution. We recall the well known approximation of $\bar\Phi(t)$ for $t > 0$:

$$\bar\Phi(t) \le \frac{\varphi(t)}{t} \quad\text{and}\quad \bar\Phi(t) = \frac{\varphi(t)}{t}\left(1-\frac{1}{t^2}+g(t)\right) \quad \text{with } 0\le g(t)\le \frac{3}{t^4}. \tag{42}$$

We set $W = (X_1-\rho X_2)/\sqrt{1-\rho^2}$ so that W is standard normal and independent of X_2. We have:

$$\mathbb{P}(X_1\ge \alpha t|X_2=t) = \mathbb{P}\left(W \ge \frac{(\alpha-\rho)t}{\sqrt{1-\rho^2}}\right) = \bar\Phi\left(\frac{(\alpha-\rho)t}{\sqrt{1-\rho^2}}\right)$$

Since $\alpha\ge\rho$, this gives:

$$\mathbb{P}(X_1\ge \alpha t|X_2=t) = \frac{1}{\sqrt{2\pi}\,t}\frac{\sqrt{1-\rho^2}}{\alpha-\rho}(1+O(t^{-2}))\exp\left(-\frac{1}{2}\frac{(\alpha-\rho)^2}{(1-\rho^2)}t^2\right). \tag{43}$$

For (Y_1, Y_2), we have using notation from Section 2:

$$\mathbb{P}(Y_1\ge \alpha t|Y_2=t) = \int_{\alpha t}^{\infty} c_\delta(\Phi(x),\Phi(t))\varphi(x)\,dx = \int_{\Phi(\alpha t)}^1 b(s)a(\Phi(t))\,ds = B(\Phi(\alpha t))a(\Phi(t)),$$

with B defined for $r\in I$ as $B(r) = \int_r^1 b(s)\,ds$. Using that $B(r) = h^{1/2}(r)\mathrm{e}^{-F(r)}$ as well as the formulae (33) for δ_ρ and δ'_ρ, elementary computations give:

$$\mathbb{P}(Y_1\ge \alpha t|Y_2=t) = \bar\Phi\left(\sqrt{\frac{1-\rho}{1+\rho}}t\right)\mathrm{e}^{-\Gamma_t}, \tag{44}$$

with

$$\Gamma_t = \int_t^{\alpha t}\Phi\left(\sqrt{\frac{1-\rho}{1+\rho}}u\right)\frac{\varphi(u)}{\bar\Phi(u)-\bar\Phi_\rho(u,u)}\,du \quad\text{and}\quad \bar\Phi_\rho(u,v) = \mathbb{P}(X_1\ge u, X_2\ge v).$$

Using (42), it is easy to check that $\bar\Phi_\rho(u,u) = O(\varphi(u)u^{-5})$ for u large, and deduce that:

$$\Gamma_t = \frac{(\alpha^2-1)t^2}{2} + \log(\alpha) + O\left(t^{-2}\right).$$

Using (44) and (42), we get:

$$\mathbb{P}(Y_1\ge \alpha t|Y_2=t) = \frac{1}{\sqrt{2\pi}\,t}\frac{1}{\alpha}\sqrt{\frac{1+\rho}{1-\rho}}(1+O(t^{-2}))\exp\left(-\frac{1}{2}\left(\frac{1-\rho}{1+\rho}+\alpha^2-1\right)t^2\right).$$

Using (43), we obtain (34).

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