OPTIMAL VACCINATIONS: CORDONS SANITAIRES, REDUCIBLE POPULATION AND OPTIMAL RAYS

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Abstract. We consider the bi-objective problem of allocating doses of a (perfect) vaccine to an infinite-dimensional metapopulation in order to minimize simultaneously the vaccination cost and the effective reproduction number R_e, which is defined as the spectral radius of the effective next-generation operator.

In this general framework, we prove that a *cordon sanitaire*, that is, a strategy that effectively disconnects the non-vaccinated population, might not be optimal, but it is still better than the "worst" vaccination strategies. Inspired by graph theory, we also compute the minimal cost which ensures that no infection occurs using independent sets. Using Frobenius decomposition of the whole population into irreducible sub-populations, we give some explicit formulae for optimal ("best" and "worst") vaccinations strategies. Eventually, we provide some sufficient conditions for a scaling of an optimal strategy to still be optimal.

1. Introduction

1.1. **Vaccination in metapopulation models.** In metapopulation epidemiological models, the population is composed of N sub-populations labelled $1, \ldots, N$, of respective sizes $\mu_1, \ldots, \mu_N$. Following [12], much of the behaviour of the epidemic may be derived from the so called next-generation matrix $K = (K_{ij})_{1 \leq i,j \leq N}$, where K_{ij} corresponds to an expected number of secondary infections for people in subgroup i resulting from a single randomly selected non-vaccinated infectious person in subgroup j.

A vaccination strategy is represented by a vector $\eta \in \Delta = [0, 1]^N$, where η_i is the **fraction of non-vaccinated** individuals in the i^{th} sub-population. In particular, η_i is equal to 0 when the i^{th} sub-population is fully vaccinated, and 1 when it is not vaccinated at all. The strategy $\mathbb{1} \in \Delta$, with all its entries equal to 1, therefore corresponds to an entirely non-vaccinated population. The spectral radius (*i.e.*, the largest modulus of the eigenvalues) of $K \cdot \mathrm{Diag}(\eta)$, denoted $R_e(\eta)$, is referred to as the *effective reproduction number*, and may then be interpreted as the expected number of cases directly generated by one typical case where all non-vaccinated indivuduals are susceptible to the infection. In particular, we denote by $R_0 = R_e(\mathbb{1})$ the so-called *basic reproduction number* associated to the metapopulation epidemiological model. We refer to Section 2 for the the computation of the reproduction number for a wide-class of compartmental metapopulation models appearing in the litterature.

With this interpretation of the reproduction number in mind, it is then natural to minimize it on the space Δ under a constraint on the cost C. A natural choice for the cost function is given by the uniform cost $C_{\text{uni}}(\eta) = 1 - \sum_i \eta_i \mu_i$, which corresponds to the fraction of vaccinated individuals in the population. This constrained optimization problem appears in most of the literature for designing efficient vaccination strategies for multiple epidemic

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situation (SIR/SEIR); see [2, 8, 9, 12, 15, 16, 21]. Note that in some of these references, the effective reproduction number is defined as the spectral radius of the matrix $\mathrm{Diag}(\eta) \cdot K$. Since the eigenvalues of $\mathrm{Diag}(\eta) \cdot K$ are exactly the eigenvalues of the matrix $K \cdot \mathrm{Diag}(\eta)$, this actually defines the same function R_e.

The goal of this paper is to prove a number of properties of the optimal vaccination strategies associated to a bi-objective optimization problem with cost function C and loss function R_e, that shed a light on how to vaccinate in the best possible way. In previous works [4, 7], we introduced a general kernel framework in which the matrix formulation appears as a special finite-dimensional case. We state our results in this general framework, but for ease of the presentation, we shall stick to the matrix formulation in this introduction. We also refer the interested reader to [5] for a detailed study of R_e and its convexity/concavity property, and to [6] for various examples of kernels and optimal vaccinations.

In our previous work [7], we assumed only minimal hypothesis on the so-called loss function whose aims to measure the vulnerability of the population. Here, we choose to take the effective reproduction number as the loss. We also consider strictly decreasing cost functions (because vaccinating more costs more; see Section 3.4). These more restrictive assumptions allow us to simplify some of the statements made in [7] and to give additional specific results.

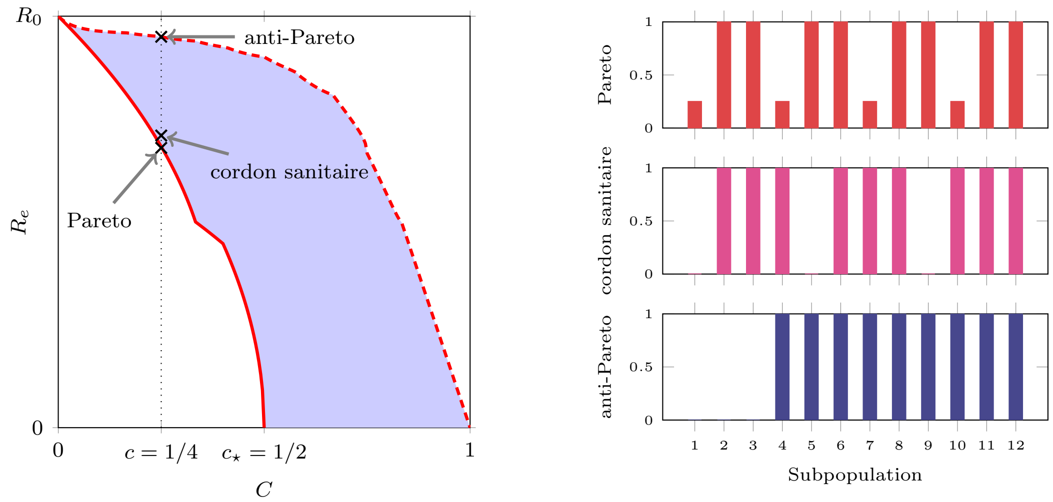
In bi-objective optimization, one can identify Pareto optimal (resp. anti-Pareto) optimal vaccinations strategies, informally “best” (resp. “worst”) vaccination strategies, in the sense that every strategy that does strictly better for one objective must do strictly worse for the other (resp. every strategy that does strictly worse for one objective must do strictly better for the other). We refer to [7, Section 5] for details. We also consider the Pareto frontier $\mathcal{F}$ (resp. anti-Pareto frontier $\mathcal{F}^{\mathrm{Anti}}$) as the outcomes $(C(\eta), R_e(\eta))$ of the Pareto (resp. anti-Pareto) optimal strategies η; see Section 3.4. In Figure 1(a), we have plotted in red the Pareto frontier and in a dashed red line the anti-Pareto frontier when the next-generation matrix is the adjacency matrix of the non-oriented cycle graph with $N = 12$ nodes from Figure 2(a) and Example 1.1; see also Example 2.1.

1.2. A *cordon sanitaire* is not the worst vaccination strategy.

Recall that a matrix K is reducible if there exists a permutation σ such that $(K_{\sigma(i)\sigma(j)})_{i,j}$ is block upper triangular, and irreducible otherwise. A *cordon sanitaire* is a vaccination strategy η such that the effective next-generation matrix $K \cdot \mathrm{Diag}(\eta)$ is reducible. Informally, such a strategy splits the effective population in at least two groups, one of which does not infect the other.

Disconnecting the population by creating a cordon sanitaire is not always the “best” choice, that is, it may not be Pareto optimal. However, we prove in Proposition 5.3 that a cordon sanitaire can never be anti-Pareto optimal; this result still holds in the general kernel framework, provided that the definition of a cordon sanitaire is generalized in an appropriate way.

Example 1.1 (Non-oriented cycle graph). Suppose that the matrix K is given by the adjacency matrix (see Figure 2(b) for a grayplot representation) of the non-oriented cycle graph with $N = 12$ nodes and μ is the uniform probability measure; see Figure 2(a). For a cost $C_{\mathrm{uni}} = 1/4$, there is a *cordon sanitaire* η that consists in vaccinating one sub-population in four; see Figure 2(c) (and Figure 2(d) for a grayplot representation of the corresponding adjacency matrix). The associated effective reproduction number is equal to $\sqrt{2}$. This strategies performs better than the anti-Pareto optimal strategy but it is not Pareto optimal as we can see in Figure 1. This example is discussed in detail in [6, Section 2.4].



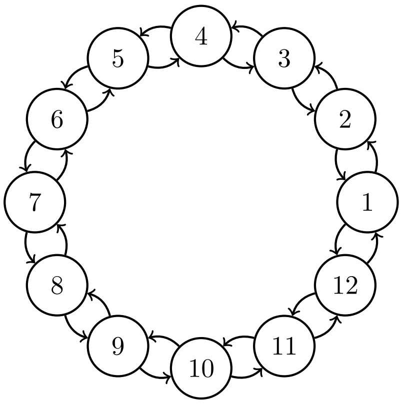
(A) Thick red line: Pareto frontier; Dashed line: anti-Pareto frontier; x marker: outcomes of various strategies; light blue: all possible outcomes $(C(\eta), R_e(\eta))$ for $\eta \in \Delta$.

(B) Profile of various strategies.

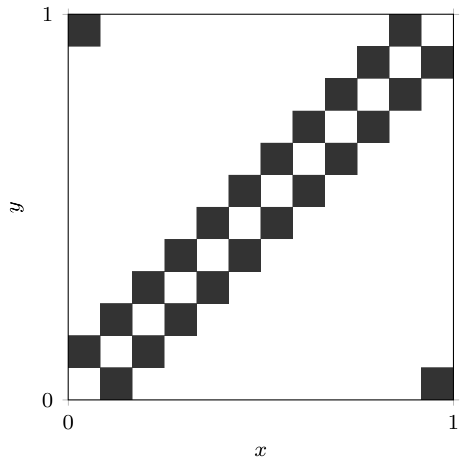
FIGURE 1. Performance of the disconnecting vaccination strategy "one in 4" for the non-oriented cycle graph with 12 nodes and uniform cost 1/4.

1.3. **Minimal cost required to completely stop the transmission of the disease.** A vaccination strategy η such that $R_e(\eta) = 0$ completely stops the transmission of the infection. The minimal cost of a vaccination that achieves this goal, denoted by $c_\star$ in the following, is introduced and discussed in [7] under general assumptions on the loss function. Section 4.2 is devoted to the characterization of this minimal cost. As our loss function is taken to be the effective reproduction number, we are able to give in Proposition 4.4 an explicit expression of this quantity in the kernel model. When the kernel is the graphon associated to a graph of size N (and thus the kernel is symmetric), μ is a probability measure and the cost is uniform, this expression corresponds to the size of maximal independent sets in this graph divided by N. We can observe this property in Figure 1(A) as the size of the maximal independent set of the non-oriented cycle graph of size N from Example 1.1 is equal to $\lfloor N/2 \rfloor$.

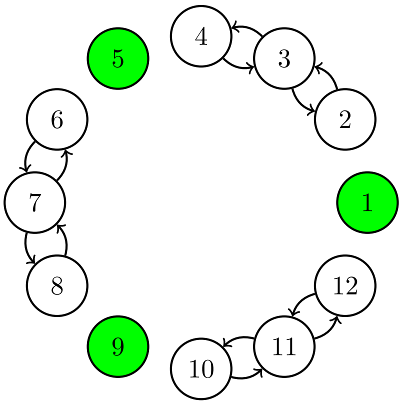
1.4. **Reducible case.** When the matrix K happens to be reducible, up to a relabeling, we may assume that it is block upper triangular. Denoting by m the number of blocks and $I_1, \ldots, I_m$ the sets of indices describing the blocks, this means that for all $\ell > k$ and $(i,j) \in I_\ell \times I_k$, we have $K_{ij} = 0$. In the epidemiological interpretation, this means that the populations with indices in I_k never infect the ones with indices in I_ℓ. One may then hope that the study of R_e can be effectively reduced to the study of the effective radius of the square sub-matrices $(K_{ij})_{i,j \in I_k}$ describing the infections within block I_k. This is indeed the case, and we give in Section 5.4 a complete picture of the Pareto and anti-Pareto frontiers of R_e, in terms of the effective reproduction numbers restricted to each irreducible component of the infection kernel or matrix. In particular, this allows a better understanding of why the anti-Pareto frontier may be discontinuous, while the Pareto frontier is always continuous. For the reduction to each irreducible component to be effective for the Pareto frontier, one has to assume that the cost function is extensive: the cost of vaccinating disjoint subsets of the population is additive. Once



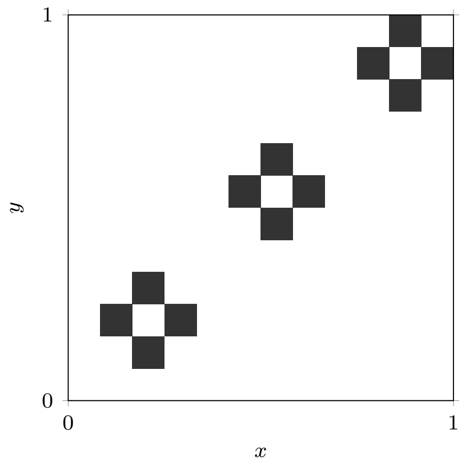
(A) The non-oriented cycle graph.



(B) Grayplot of the corresponding kernel.



(C) Cordon sanitaire corresponding to the "one in 4" vaccination strategy (in green the vaccinated groups).



(D) Grayplot of the corresponding kernel.

FIGURE 2. Example of disconnecting vaccination strategy on the non-oriented cycle graph with $N = 12$ nodes.

more, special care has to be taken with the definitions when handling the infinite dimensional kernel case.

1.5. **Optimal ray.** It is observed by Poghotanyan, Feng, Glasser and Hill in [16, Theorem 4.3], that in the finite dimensional case, under an assumption that ensures the convexity of the function R_e, and for a uniform cost, if there exists a Pareto optimal strategy η with all its entries strictly less than 1, then all the strategies $\lambda\eta$, with $\lambda \geq 0$ such that $\lambda\eta \in \Delta$, are Pareto optimal. We give a short proof on the existence of such optimal rays in Section 4.1 in a general kernel framework, when the cost function C is affine and R_e is convex on Δ.

1.6. **Organization of the paper.** We present in Section 2 different models for which the effective reproduction number associated to an epidemic model with vaccination can be seen as the spectral radius of a compact operator. In Section 3, we present the mathematical framework for the study of the effective reproduction function and the associated bi-objective problems with a general cost function as well as the Pareto and anti-Pareto frontiers. Section 4 is devoted to the description of optimal vaccination strategies which eradicate the epidemic, and the possible existence of optimal rays in the Pareto frontier. Using a Frobenius decomposition of the next generation kernel in Section 5.1, we first complete the description of the anti-Pareto frontier in the irreducible and monatomic cases in Section 5.2. We study in Section 5.3 the optimality of *cordons sanitaires* vaccination strategies and show in Section 5.4 how the optimization problem may be effectively reduced to the study on subpopulations when the next generation kernel is reducible.

2. Generality of the effective next-generation operator

In [4, 7], we developed a framework that we call the kernel model where the population is represented as an abstract measure space $(\Omega, \mathscr{F}, \mu)$, with μ non-zero σ-finite measure. Individuals are characterized by a trait $x \in \Omega$. The size of the sub-population with trait x is given by $\mu(\mathrm{d}x)$. The underlying structure described by this trait can be very diverse. Typical examples include spatial position, social contacts, susceptibility, infectiousness, characteristics of the immunological response, etc. The analogue of the next-generation matrix K is the kernel operator defined formally by:

$$T_{\mathrm{k}}(g)(x) = \int_\Omega \mathrm{k}(x,y)\, g(y)\, \mathrm{d}\mu(y);$$

where the non-negative kernel k is defined on $\Omega \times \Omega$ and $\mathrm{k}(x, y)$ still represents a strength of infection from y to x. Vaccination strategies $\eta : \Omega \to [0, 1]$ encode the **density of non-vaccinated individuals** with respect to the measure μ. So, the strategy $\eta = \mathbb{1}$, the constant function equal to 1, corresponds to no vaccination in the population, whereas the strategy $\eta = \mathbb{0}$, the constant function equal to 0, corresponds to all the population being vaccinated. The measure $\eta(y)\, \mu(\mathrm{d}y)$ may then be understood as an effective population, giving rise to an effective next-generation operator:

$$T_{\mathrm{k}\eta}(g)(x) = \int_\Omega \mathrm{k}(x,y)\, g(y)\, \eta(y)\, \mu(\mathrm{d}y).$$

The effective reproduction number is then defined by $R_e(\eta) = \rho(T_{\mathrm{k}\eta})$, where ρ stands for the spectral radius of the operator and $\mathrm{k}\eta$ for the kernel $(\mathrm{k}\eta)(x, y) = \mathrm{k}(x, y)\eta(y)$.

The results mentioned in the introduction will be given in this general framework, which is flexible enough to describe a wide range of epidemic models from the literature including the metapopulation models. We give in the following a few examples to support this claim: in each of them, the spectral radius of a given, explicit kernel operator appears as a threshold parameter, and the epidemic either "invades/survives" or "dies out" depending on the value of this parameter. Classical notations are used: S denotes the proportion of susceptible individuals, E the proportion of those who have been exposed to the disease, I the proportion of infected individuals, R the proportion of removed individuals in the population. Thus $I(t, x)$ denotes the proportion of the population with trait $x \in \Omega$ which is infected at time $t \geq 0$. In the following examples, the measure μ is assumed to be a probability measure.

Example 2.1 (Meta-population models). Recall that in metapopulation models, the population is divided into $N \geq 2$ different sub-populations of respective proportional size $\mu_1, \ldots, \mu_N$, and the reproduction number is given by $R_e(\eta) = \rho(K \cdot \mathrm{Diag}(\eta))$, where K is the next generation matrix and η belongs to $[0, 1]^N$ and gives the proportion of non-vaccinated individuals in each sub-population. To express the function R_e as the effective reproduction number of a kernel model, consider the discrete state space $\Omega_\mathrm{d} = \{1, \dots, N\}$ equipped with the probability measure μ_d defined by $\mu_\mathrm{d}(\{i\}) = \mu_i$, and let $\mathrm{k_d}$ denote the discrete kernel on Ω_d defined by:

$$\mathrm{k_d}(i,j) = K_{ij}/\mu_j. \tag{1}$$

For all $\eta \in \Delta = [0,1]^N$, the matrix $K\cdot\mathrm{Diag}(\eta)$ is the matrix representation of the endomorphism $T_{\mathrm{k_d}\eta}$ in the canonical basis of $\mathbb{R}^N$. In particular, we have: $R_e(\eta) = \rho(T_{\mathrm{k}\eta}) = \rho(K \cdot \mathrm{Diag}(\eta))$.

In Figure 2(B), we have plotted a kernel on $[0,1]$ endowed with the usual Borel σ-algebra and the Lebesgue measure. This kernel is equivalent to $\mathrm{k_d}$ when K is the adjacency matrix of the non-oriented cycle graph and all sub-populations have the same size.

Example 2.2 (An SIR model with nonlinear incidence rate and vital dynamics). In [19], Thieme proposed an SIR model in an infinite-dimensional population structure with a nonlinear incidence rate. The structure space is given by Ω a compact subset of $\mathbb{R}^N$ equipped with the normalized Lebesgue measure denoted by μ. We restrict slightly his assumptions so that the incidence rate is a linear function of the number of susceptible. Besides, we write explicitly the equation giving the evolution of the recovered compartment. It does not play a role in the long-time behavior analysis of the equations made by Thieme but it helps to understand the model when taking into account the vaccination. The dynamic of the epidemic then writes:

$$\text{For } t \geq 0, x \in \Omega, \quad \begin{cases} \partial_t S(t,x) = \Lambda(x) - \nu_S(x)S(t,x) - S(t,x) \int_\Omega f(I(t,y),x,y)\,\mu(\mathrm{d}y), \\[1em] \partial_t I(t,x) = S(t,x) \int_\Omega f(I(t,y),x,y)\,\mu(\mathrm{d}y) - (\gamma(x) + \nu_I(x))I(t,x), \\[1em] \partial_t R(t,x) = \gamma(x)I(t,x) - \nu_R(x)R(t,x), \end{cases} \tag{2}$$

where, at location $x \in \Omega$:

- $\Lambda(x)$ is the rate at which fresh susceptible individuals are recruited,
- $\nu_S(x)$, $\nu_I(x)$, $\nu_R(x)$ are the *per capita* death rate of the susceptible, infected and recovered individuals respectively,
- $\gamma(x)$ is the *per capita* recovery rate of infected individuals,
- the integral term describes the incidence at time t, *i.e.*, the rate of new infections.

The threshold parameter identified in [19], that plays the role of the reproduction number, is given by the spectral radius of the operator T_k with the kernel k given by:

$$\mathrm{k}(x,y) = \frac{\Lambda(x)}{\nu_S(x)(\gamma(x) + \nu_I(x))}\partial_I f(0,x,y), \quad x, y \in \Omega,$$

where $\partial_I f(0,x,y)$, the derivative of f with respect to its first variable I, is supposed to be non-negative.

Suppose that individuals at location x are vaccinated with probability $1 - \eta(x)$ at birth. In the corresponding model, the rate at which susceptible individuals with trait x are recruited becomes equal to $\eta(x)\Lambda(x)$ while recovered/immunized individuals are recruited at rate $(1 - \eta(x))\Lambda(x)$ at location x so that the dynamic of the recovered compartment is given by:

$$\partial_t R(t,x) = (1 - \eta(x))\Lambda(x) + \gamma(x)I(t,x) - \nu_R(x)R(t,x), \qquad x \in \Omega,\ t \geq 0.$$

The threshold parameter $R_e(\eta)$ is then given by the spectral radius of the integral operator $T_{\eta\mathrm{k}}$ with kernel $\eta\mathrm{k}$ given by $(\eta\mathrm{k})(x,y) = \eta(x)\mathrm{k}(x,y)$. According to Equation (7), we have $\rho(T_{\eta\mathrm{k}}) = \rho(T_{\mathrm{k}\eta})$, and our framework can be used for this model.

Under regularity assumptions on the parameters of the model, Thieme proved that if $R_e(\eta)$ is greater than 1, then there exists an endemic equilibrium that attracts all the solutions while if $R_e(\eta)$ is smaller than 1, then $I(t,x)$ converges to 0 for all $x \in \Omega$ as t goes to infinity.

Example 2.3 (An SEIR model without vital dynamics). In [1], Almeida, Bliman, Nadin and Perthame studied an heterogeneous SEIR model where the population is again structured with a bounded subset $\Omega \subset \mathbb{R}^N$ equipped with the normalized Lebesgue measure denoted by μ. This time however there is no birth nor death of the individuals. The dynamic of the susceptible, exposed, infected and recovered individuals writes:

$$\text{(3)} \qquad \text{For } t \geq 0,\, x \in \Omega, \qquad \begin{cases} \partial_t S(t,x) = -S(t,x) \int_\Omega k(x,y) I(t,y)\, \mu(\mathrm{d}y), \\ \partial_t E(t,x) = S(t,x) \int_\Omega k(x,y) I(t,y)\, \mu(\mathrm{d}y) - \alpha(x) E(t,x), \\ \partial_t I(t,x) = \alpha(x) E(t,x) - \gamma(x) I(t,x), \\ \partial_t R(t,x) = \gamma(x) I(t,x). \end{cases}$$

Here, the average incubation rate is denoted by $\alpha(x)$ and the average recovery rate by $\gamma(x)$; both quantities may depend upon the trait x. The function k is the transmission kernel of the disease. In this model, the basic reproduction number is given by the spectral radius of the integral operator T_{k} with kernel $\mathrm{k} = k/\gamma$ given by:

$$\text{(4)} \qquad \mathrm{k}(x,y) = k(x,y)/\gamma(y).$$

Note that the basic reproduction number does not depend on the average incubation rate α as in the one-dimensional SEIR model with constant population size; see [20, Section 2.2] with death rate $d = 0$.

Suppose that, prior to the beginning of the epidemic, the decision maker immunizes a density $1 - \eta$ of individuals. According to [1, Section 3.2], the effective reproduction number is given by $\rho(T_{\eta\mathrm{k}})$ which is also equal to $\rho(T_{\mathrm{k}\eta})$. Hence, our model is indeed suitable for designing optimal vaccination strategies in this context.

Example 2.4 (An SIS model without vital dynamic). In [4], generalizing the discrete model of Lajmanovich and Yorke [14], we introduced the following heterogeneous SIS model where the population is structured with an abstract probability space $(\Omega, \mathscr{F}, \mu)$:

$$\text{(5)} \qquad \text{For } t \geq 0,\, x \in \Omega, \qquad \begin{cases} \partial_t S(t,x) = -S(t,x) \int_\Omega k(x,y) I(t,y)\, \mu(\mathrm{d}y) + \gamma(x) I(t,x), \\ \partial_t I(t,x) = S(t,x) \int_\Omega k(x,y) I(t,y)\, \mu(\mathrm{d}y) - \gamma(x) I(t,x). \end{cases}$$

The function γ is the *per-capita* recovery rate and k is the transmission kernel. For this model, $R_e(\eta) = \rho(T_{\mathrm{k}\eta})$ where $\mathrm{k} = k/\gamma$ is defined by $\mathrm{k}(x,y) = k(x,y)/\gamma(y)$.

Suppose that, prior to the beginning of the epidemic, a density $1 - \eta$ of individuals is vaccinated with a perfect vaccine. In the same way as for the SEIR model, we proved, as t goes to infinity, that if $R_e(\eta)$ is smaller than or equal to 1, then $I(t, \cdot)$ converges to 0, and, under a connectivity assumption on the kernel k, that if $R_e(\eta)$ is greater than 1, then $I(t, \cdot)$ converges to the (unique) positive endemic equilibrium. This highlights the importance of R_e in the design of vaccination strategies.

3. Setting, notations and previous results

3.1. **Spaces, operators, spectra.** All metric spaces (S, d) are endowed with their Borel σ-field denoted by $\mathscr{B}(S)$. Let $(\Omega, \mathscr{F}, \mu)$ be a measured space, with μ a σ-finite positive and non-zero measure. For f and g real-valued functions defined on Ω, we write $\langle f, g \rangle$ or $\int_\Omega fg\, \mathrm{d}\mu$ for $\int_\Omega f(x) g(x)\, \mu(\mathrm{d}x)$ whenever the latter is meaningful. For $p \in [1, +\infty]$, we denote by $L^p = L^p(\mu) = L^p(\Omega, \mu)$ the space of real-valued measurable functions g defined on Ω such that

$\| g \|_p = \left(\int |g|^p \, \mathrm{d}\mu\right)^{1/p}$ (with the convention that $\| g \|_\infty$ is the μ-essential supremum of $|g|$) is finite, where functions which agree μ-a.e. are identified. We denote by L^p_+ the subset of L^p of non-negative functions. We define Δ as the subset of L^∞ of $[0,1]$-valued measurable functions defined on Ω. We denote by $\mathbb{1}$ (resp. $\mathbb{0}$) the constant function on Ω equal to 1 (resp. 0); both functions belong to Δ.

Let $(E, \|\cdot\|)$ be a complex Banach space. We denote by $\|\cdot\|_E$ the operator norm on $\mathcal{L}(E)$ the Banach algebra of linear bounded operators. The spectrum $\mathrm{Spec}(T)$ of $T \in \mathcal{L}(E)$ is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda\mathrm{Id}$ does not have a bounded inverse, where Id is the identity operator on E. Recall that $\mathrm{Spec}(T)$ is a compact subset of $\mathbb{C}$, and that the spectral radius of T is given by:

$$\rho(T) = \max\{|\lambda| : \lambda \in \mathrm{Spec}(T)\} = \lim_{n\to\infty} \| T^n \|_E^{1/n} . \tag{6}$$

The element $\lambda \in \mathrm{Spec}(T)$ is an eigenvalue if there exists $x \in E$ such that $Tx = \lambda x$ and $x \neq 0$.

Recall that the spectrum of a compact operator is finite or countable and has at most one accumulation point, which is 0. Furthermore, 0 belongs to the spectrum of compact operators in infinite dimension. If $A \in \mathcal{L}(E)$ is compact and $B \in \mathcal{L}(E)$, then both AB and BA are compact and:

$$\rho(AB) = \rho(BA). \tag{7}$$

We refer to [17] for an introduction to Banach lattices and positive operators; we shall only consider the real Banach lattices $L^p = L^p(\Omega, \mu)$ for $p \in [1, +\infty]$ on a measured space $(\Omega, \mathscr{F}, \mu)$ with a σ-finite non-zero measure, as well as their complex extension. (Recall that the norm of an operator on L^p or its natural complex extension is the same, see [10, Corollary 1.3]). A bounded operator A is positive if $A(L^p_+) \subset L^p_+$. If $A, B \in \mathcal{L}(L^p)$ and $A - B$ are positive operators, then:

$$\rho(A) \geq \rho(B). \tag{8}$$

If E is also a real or complex function space, for $g \in E$, we denote by M_g the multiplication operator (possibly unbounded) defined by $M_g(h) = gh$ for all $h \in E$; if furthermore g is the indicator function of a set A, we simply write M_A for $M_{\mathbb{1}_A}$.

3.2. **Kernel operators.** We define a *kernel* (resp. *signed kernel*) on Ω as a $\mathbb{R}_+$-valued (resp. $\mathbb{R}$-valued) measurable function defined on $(\Omega^2, \mathscr{F}^{\otimes 2})$. For f, g two non-negative measurable functions defined on Ω and k a kernel on Ω, we denote by $f\mathrm{k}g$ the kernel defined by:

$$f\mathrm{k}g : (x, y) \mapsto f(x)\, \mathrm{k}(x, y) g(y). \tag{9}$$

For $p \in (1, +\infty)$, we define the double norm of a signed kernel k on L^p by:

$$\| \mathrm{k} \|_{p,q} = \left(\int_\Omega \left(\int_\Omega | \mathrm{k}(x, y) |^q \; \mu(\mathrm{d}y) \right)^{p/q} \mu(\mathrm{d}x) \right)^{1/p} \quad \text{with } q \text{ given by} \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{10}$$

We say that k has a finite double norm, if there exists $p \in (1, +\infty)$ such that $\| \mathrm{k} \|_{p,q} < +\infty$. To such a kernel k, we then associate the positive integral operator T_k on L^p defined by:

$$T_\mathrm{k}(g)(x) = \int_\Omega \mathrm{k}(x, y)\, g(y)\, \mu(\mathrm{d}y) \quad \text{for } g \in L^p \text{ and } x \in \Omega. \tag{11}$$

According to [11, p. 293], the operator T_k is compact. It is well known and easy to check that:

$$\| T_\mathrm{k} \|_{L^p} \leq \| \mathrm{k} \|_{p,q} . \tag{12}$$

We define the *reproduction number* associated to the operator T_{k} as:

$$R_0[\mathrm{k}] = \rho(T_{\mathrm{k}}). \tag{13}$$

3.3. **The effective reproduction number R_e.** A *vaccination strategy* η of a vaccine with perfect efficiency is an element of Δ, where $\eta(x)$ represents the proportion of ***non-vaccinated*** individuals with feature x, so that the constant functions $\eta = \mathbb{1}$ and $\eta = \mathbb{0}$ correspond respectively to no vaccination and complete vaccination. Notice that $\eta \, \mathrm{d}\mu$ corresponds in a sense to the effective population. Let k be a kernel on Ω with finite double norm on L^p. For $\eta \in \Delta$, the operator M_η is bounded on L^p, whence the operator $T_{\mathrm{k}\eta} = T_{\mathrm{k}} M_\eta$ is compact. We define the *effective reproduction number* function $R_e[\mathrm{k}]$ from Δ to $\mathbb{R}_+$ by:

$$R_e[\mathrm{k}](\eta) = \rho(T_{\mathrm{k}\eta}), \tag{14}$$

and the corresponding reproduction number is then given by $R_0[\mathrm{k}] = R_e[\mathrm{k}](\mathbb{1})$. When there is no risk of confusion on the kernel k, we simply write R_e and R_0 for the function $R_e[\mathrm{k}]$ and the number $R_0[\mathrm{k}]$.

We can see Δ as a subset of L^∞, and consider the corresponding *weak-* topology*: a sequence $(g_n, n \in \mathbb{N})$ of elements of Δ converges weakly-* to g if for all $h \in L^1$ we have:

$$\lim_{n\to\infty} \int_\Omega h g_n \, \mathrm{d}\mu = \int_\Omega h g \, \mathrm{d}\mu. \tag{15}$$

The set Δ endowed with the weak-* topology is compact and sequentially compact; see [7, Lemma 3.1]. We also recall the properties of the effective reproduction number given in [7, Proposition 4.1 and Theorem 4.2].

Proposition 3.1. *Let* k *be a finite double norm kernel on a measured space $(\Omega, \mathscr{F}, \mu)$ where μ is a σ-finite non-zero measure Then, the function $R_e = R_e[\mathrm{k}]$ is a continuous function from Δ (endowed with the weak-* topology) to $\mathbb{R}_+$. Furthermore, the function $R_e = R_e[\mathrm{k}]$ satisfies the following properties:*

(i) $R_e(\eta_1) = R_e(\eta_2)$ *if* $\eta_1 = \eta_2$, μ *a.s., and* $\eta_1, \eta_2 \in \Delta$,
(ii) $R_e(\mathbb{0}) = 0$ *and* $R_e(\mathbb{1}) = R_0$,
(iii) $R_e(\eta_1) \le R_e(\eta_2)$ *for all* $\eta_1, \eta_2 \in \Delta$ *such that* $\eta_1 \le \eta_2$,
(iv) $R_e(\lambda\eta) = \lambda R_e(\eta)$, *for all* $\eta \in \Delta$ *and* $\lambda \in [0, 1]$.

3.4. **Pareto and anti-Pareto frontiers.** Let k be a kernel on Ω with a finite double norm. We consider the effective reproduction function $R_e = R_e[\mathrm{k}]$ defined on Δ as a loss function. We quantify the cost of the vaccination strategy $\eta \in \Delta$ by a function $C : \Delta \to \mathbb{R}^+$, and we assume that $C(\mathbb{1}) = 0$ (doing nothing costs nothing), C is continuous for the weak-* topology on Δ defined in Section 3.3 and *decreasing* (doing more costs strictly more), that is, for any $\eta_1, \eta_2 \in \Delta$:

$$\eta_1 \le \eta_2 \quad \text{and} \quad \mu(\eta_1 < \eta_2) > 0 \implies C(\eta_1) > C(\eta_2).$$

For example, when the measure μ is finite, the uniform cost function:

$$C_{\mathrm{uni}}(\eta) = \int_\Omega (1 - \eta) \, \mathrm{d}\mu. \tag{16}$$

is continuous and decreasing on Δ (recall that $1 - \eta$ represents the proportion of the population which has been vaccinated when using the strategy η.)

In [7], we formalized and study the problem of optimal allocation strategies for a perfect vaccine. This question may be viewed as a bi-objective minimization problem, where one tries

	“Best” vaccinations	“ Worst” vaccinations
Optimization problem	Pb (17): $\min_\Delta(C, R_e)$	Pb (19): $\max_\Delta(C, R_e)$
Opt. cost for a given loss defined on $[0, R_0]$, with $R_0 := \max_\Delta R_e = R_e(\mathbb{1})$.	$C_\star(\ell) := \min_{R_e \leq \ell} C$. $C_\star$ is continuous. $C_\star$ is decreasing. $C_\star(R_0) = 0$ and $c_\star := C_\star(0)$.	$C^\star(\ell) := \max_{R_e \geq \ell} C$. ✗ $C^\star$ is decreasing. $C^\star(0) = c_{\max}$ and $c^\star := C^\star(R_0)$.
Opt. loss for a given cost defined on $[0, c_{\max}]$, with $c_{\max} := \max_\Delta C = C(\mathbb{0})$.	$R_{e\star}(c) := \min_{C \leq c} R_e$. $R_{e\star}$ is continuous. $R_{e\star}$ is decreasing on $[0, c_\star]$. $R_{e\star} = 0$ on $[c_\star, c_{\max}]$. $R_{e\star}(0) = R_0$.	$R_e^\star(c) := \max_{C \geq c} R_e$. $R_e^\star$ is continuous. ✗ $R_e^\star = R_0$ on $[0, c^\star]$. $R_e^\star(c_{\max}) = 0$.
Inverse formula	$R_{e\star} \circ C_\star = \mathrm{Id}$ on $[0, R_0]$. $C_\star \circ R_{e\star} = \mathrm{Id}$ on $[0, c_\star]$.	$R_e^\star \circ C^\star = \mathrm{Id}$ on $[0, R_0]$. ✗
Optimal strategies	$\mathcal{P} := \{ C = C_\star \circ R_e \} \cap \{ R_e = R_{e\star} \circ C \}$ $= \{ C = C_\star \circ R_e \}$ $= \{ R_e = R_{e\star} \circ C,\ C \leq c_\star \}$. $\mathcal{P}$ is compact.	$\mathcal{P}^{\mathrm{Anti}} := \{ C = C^\star \circ R_e \} \cap \{ R_e = R_e^\star \circ C \}$ $= \{ C = C^\star \circ R_e \}$ $= \{ R_e = R_e^\star \circ C,\ C \geq c^\star \}$. ✗
Range of cost/loss	$\mathcal{A} := [0, c_{\max}] \times [0, R_0]$	
Possible outcomes	$\mathbf{F} := (C, R_e)(\Delta)$ $= \{ (c, \ell) \in \mathcal{A} :\ R_{e\star}(c) \leq \ell \leq R_e^\star(c) \}$ $= \{ (c, \ell) \in \mathcal{A} :\ C_\star(\ell) \leq c \leq C^\star(\ell) \}$.	
Optimal frontier	$\mathcal{F} := (C, R_e)(\mathcal{P})$ $= (C_\star, \mathrm{Id})([0, R_0])$ $= (\mathrm{Id}, R_{e\star})([0, c_\star])$. $\mathcal{F}$ is connected and compact.	$\mathcal{F}^{\mathrm{Anti}} := (C, R_e)(\mathcal{P}^{\mathrm{Anti}})$ $= (C^\star, \mathrm{Id})([0, R_0])$. ✗ ✗

The missing results, indicated by ✗, will be further completed under some additional conditions on the kernel k (see Proposition 5.1 for k positive and Corollary 5.2 for k monatomic).

TABLE 1. Summary of notation and results for the bi-objective problems.

to minimize simultaneously the cost of the vaccination and its loss given by the corresponding effective reproduction number:

$$\min_\Delta(C, R_e). \tag{17}$$

Let us now briefly summarize the results from [7]. For the reader's convenience we also collect the main points in Table 1, and provide plots of typical Pareto and anti-Pareto frontiers in Figure 5.

Note that Assumptions 4 and 5 of [7] hold, thanks to [7, Lemma 5.13].

By definition, we have $R_0 = \max_\Delta R_e$ and we set $c_{\max} = \max_\Delta C$ which is positive as C is decreasing (and μ non-zero) and finite as C is continuous and Δ compact. Related to the minimization problem (17), we shall consider $R_{e\star}$ the *optimal loss* function and $C_\star$ the *optimal cost* function defined by:

$$\begin{aligned} R_{e\star}(c) &= \min \{ R_e(\eta) :\ \eta \in \Delta,\ C(\eta) \leq c \} \quad \text{for } c \in [0, c_{\max}], \\ C_\star(\ell) &= \min \{ C(\eta) :\ \eta \in \Delta,\ R_e(\eta) \leq \ell \} \quad \text{for } \ell \in [0, R_0]. \end{aligned}$$

We have $C_\star(R_0) = 0$ and $R_{e\star}(0) = R_0$ since C is decreasing. For convenience, we write $c_\star$ for the minimal cost required to completely stop the transmission of the disease:

$$c_\star = C_\star(0) = \inf\{c \in [0, c_{\max}] : \, R_{e\star}(c) = 0\}. \tag{18}$$

The function $R_{e\star}$ is continuous, decreasing on $[0, c_\star]$ and zero on $[c_\star, 1]$; the function $C_\star$ is continuous and decreasing on $[0, R_0]$; and the functions $R_{e\star}$ and $C_\star$ are the inverse of each other, that is, $R_{e\star} \circ C_\star(\ell) = \ell$ for $\ell \in [0, R_0]$ and $C_\star \circ R_{e\star}(c) = c$ for $c \in [0, c_\star]$.

We define the Pareto optimal strategies $\mathcal{P}$ as the "best" solutions of the minimization problem (17) (we refer to [7] for a precise justification of this terminology):

$$\mathcal{P} = \{\eta \in \Delta : \, C(\eta) = C_\star(R_e(\eta)) \quad \text{and} \quad R_e(\eta) = R_{e\star}(C(\eta))\}\,.$$

We have in fact the following representation of the Pareto optimal strategies:

$$\begin{aligned}\mathcal{P} &= \{\eta \in \Delta : \, C(\eta) = C_\star(R_e(\eta))\} \\ &= \{\eta \in \Delta : \, R_e(\eta) = R_{e\star}(C(\eta)) \quad \text{and} \quad C(\eta) \leq c_\star\}\,.\end{aligned}$$

The Pareto frontier is defined as the outcomes of the Pareto optimal strategies:

$$\mathcal{F} = \{(C(\eta), R_e(\eta)) : \, \eta \in \mathcal{P}\}\,.$$

The set $\mathcal{P}$ is a non empty compact (for the weak topology) in Δ and furthermore the Pareto frontier can be easily represented using the graph of the optimal loss function or cost function:

$$\mathcal{F} = \{(C_\star(\ell), \ell) : \, \ell \in [0, R_0]\} = \{(c, R_{e\star}(c)) : \, c \in [0, c_\star]\}.$$

It is also of interest to consider the "worst" strategies which can be viewed as solutions to the bi-objective maximization problem:

$$\max_{\Delta}(C, R_e). \tag{19}$$

The next results can be found in [7, Propositions 5.8 and 5.9] (notice therein that Assumption 6 holds in general but that Assumption 7 holds under the stronger condition that the kernel k is monatomic, see Section 5.4.2). Related to the maximization problem (19), we shall consider $R_e^\star$ the *optimal loss* function and $C^\star$ the *optimal cost* function defined by:

$$\begin{aligned}R_e^\star(c) &= \max\,\{\, R_e(\eta) : \, \eta \in \Delta, \, C(\eta) \geq c \,\} \quad \text{for } c \in [0, c_{\max}], \\ C^\star(\ell) &= \max\,\{\, C(\eta) : \, \eta \in \Delta, \, R_e(\eta) \geq \ell \,\} \quad \text{for } \ell \in [0, R_0].\end{aligned}$$

We have $C^\star(0) = c_{\max}$ and $R_e^\star(c_{\max}) = 0$ since C is decreasing and $C(\mathbb{0}) = c_{\max}$. Since, for $\varepsilon \in (0, 1)$ we have $C(\varepsilon\mathbb{1}) < c_{\max}$ as C is decreasing and $R_e(\varepsilon\mathbb{1}) = \varepsilon R_0 > 0$, we deduce that $C^\star(0+) = c_{\max}$. For convenience, we write $c^\star$ for the maximal cost of totally inefficient strategies:

$$c^\star = C^\star(R_0) = \max\{c \in [0, c_{\max}] : \, R_e^\star(c) = R_0\}. \tag{20}$$

The function $C^\star$ is decreasing on $[0, R_0]$; the function $R_e^\star$ is constant equal to R_0 on $[0, c^\star]$; we have $R_e^\star \circ C^\star(\ell) = \ell$ for $\ell \in [0, R_0]$. This latter property implies that the function $R_e^\star$ is continuous.

We define the anti-Pareto optimal strategies $\mathcal{P}^{\mathrm{Anti}}$ as the "worst" strategies, that is solutions of the maximization problem (19):

$$\mathcal{P}^{\mathrm{Anti}} = \{\eta \in \Delta : \, C(\eta) = C^\star(R_e(\eta)) \quad \text{and} \quad R_e(\eta) = R_e^\star(C(\eta))\}\,.$$

We have in fact the following representation of the anti-Pareto optimal strategies:

$$\begin{aligned}\mathcal{P}^{\mathrm{Anti}} &= \{\eta \in \Delta : \, C(\eta) = C^\star(R_e(\eta))\} \\ &= \{\eta \in \Delta : \, R_e(\eta) = R_e^\star(C(\eta)) \quad \text{and} \quad C(\eta) \geq c^\star\}\,.\end{aligned}$$

The anti-Pareto frontier is defined as the outcomes of the anti-Pareto optimal strategies:

$$\mathcal{F}^{\mathrm{Anti}} = \left\{ (C(\eta), R_e(\eta)) :\ \eta \in \mathcal{P}^{\mathrm{Anti}} \right\}.$$

The set $\mathcal{P}^{\mathrm{Anti}}$ is non empty and furthermore the Pareto frontier can be easily represented using the graph of the optimal cost function:

$$\mathcal{F}^{\mathrm{Anti}} = \{ (C^\star(\ell), \ell) :\ \ell \in [0, R_0] \}. \tag{21}$$

We also have that the feasible region or set of possible outcomes for (C, R_e):

$$\mathbf{F} = \{ (C(\eta), R_e(\eta)) :\ \eta \in \Delta \}$$

is compact, path connected, and its complement is connected in $\mathbb{R}^2$. It is the whole region between the graphs of the one-dimensional value functions:

$$\begin{aligned} \mathbf{F} &= \{ (c, \ell) \in [0, c_{\max}] \times [0, R_0] :\ R_{e\star}(c) \leq \ell \leq R_e^\star(c) \} \\ &= \{ (c, \ell) \in [0, c_{\max}] \times [0, R_0] :\ C_\star(\ell) \leq c \leq C^\star(\ell) \}. \end{aligned}$$

We plotted in Figure 5 the typical Pareto and anti-Pareto frontiers for a general kernel (notice the anti-Pareto frontier is not connected *a priori*). In Section 5, we check that reducibility conditions on the kernel k provide further properties on the frontiers.

4. Optimal ray and optimal strategies which eradicate the epidemic

We introduced in Section 3.4 the bi-objective minimization/maximization problems, where one tries to minimize/maximize simultaneously the cost of the vaccination and the effective reproduction number. In Section 4.1, we derive the existence of Pareto optimal rays as soon as there exists a Pareto optimal strategy uniformly strictly bounded from above by 1; and in Section 4.2 we give a characterization of $c_\star = C_\star(0)$ using the notion of independent set from graph theory.

4.1. **Optimal ray.** If the loss function R_e is convex and if the cost function is affine, then the set $\mathcal{P}$ of Pareto optimal strategies may contain a non-trivial optimal ray $\{\lambda\eta :\ \lambda \in [0, 1]\}$. This optimal ray has already been observed in finite dimension, see [16]. We also refer to [5] for sufficient condition on the kernel k for the function $R_e[\mathrm{k}]$ to be convex or concave.

Proposition 4.1 (Optimal ray)**.** *Suppose that the cost function C takes the form:*

$$C(\eta) = c_{\max} - \int_\Omega \eta c \, \mathrm{d}\mu \quad \textit{with} \quad c_{\max} = \int_\Omega c \, \mathrm{d}\mu,$$

for a positive function $c \in L^1$, and that the loss function $R_e[\mathrm{k}]$, with k *a finite double norm kernel, is convex. If $\eta_\star \in \mathcal{P}$ is a Pareto optimal strategy that satisfies $\eta_\star < 1$, μ-a.e., then, for all $\lambda \geq 0$, the strategy $\lambda\eta_\star$ is Pareto optimal as soon as $\lambda\eta_\star \in \Delta$.*

In particular, the Pareto frontier contains the segment joining the points of coordinates $(c_{\max}, 0)$ and $(C(\eta_\star/\sup\eta_\star), R_e(\eta_\star/\sup\eta_\star))$. We also have $c_\star = c_{\max}$.

Remark 4.2. Suppose that C takes the form given in the Proposition and that $R_e[\mathrm{k}]$, with k a finite double norm kernel, is concave. With a similar proof (but for the last part which has to be replaced by the fact that $C^\star(0+) = c_{\max}$ as the set of anti-Pareto optimal strategies might not be closed), it is easy to get that if $\eta^\star$ is anti-Pareto optimal such that $\eta^\star < 1$ μ-a.e., then, for all $\lambda \geq 0$, the strategy $\lambda\eta^\star$ is anti-Pareto optimal as soon as $\lambda\eta^\star \in \Delta$.

Proof of Proposition 4.1. Assume that $\eta_\star \in \mathcal{P}$ satisfies $\eta_\star < 1$ μ-a.e., and $\xi_\star \in \Delta$ is a multiple of $\eta_\star$, say $\xi_\star = \lambda\eta_\star$. Assume for now that $\lambda > 0$. Our goal is to prove that $\xi_\star$ is Pareto optimal. Let $\xi \in \Delta$ be such that $R_e(\xi) \leq R_e(\xi_\star)$: by [7, Proposition 5.5 (ii)], it is enough to show that necessarily, $C(\xi) \geq C(\xi_\star)$, or equivalently that $\int_\Omega \xi c \, \mathrm{d}\mu \leq \int_\Omega \xi_\star c \, \mathrm{d}\mu$.

To use the optimality of $\eta_\star$, we construct an auxiliary strategy:

$$\eta_n = \min\big((1 - n^{-1})\eta_\star + n^{-1}\eta; 1\big),$$

where $n \in \mathbb{N}^*$ and $\eta = \xi/\lambda$ (note that $\eta \notin \Delta$ in general). By monotony, convexity and homogeneity of R_e, and the fact that $R_e(\xi) \leq R_e(\xi_\star)$ by hypothesis, we get:

$$\begin{aligned} R_e(\eta_n) &\leq (1 - n^{-1})R_e(\eta_\star) + n^{-1}R_e(\eta) \\ &\leq (1 - n^{-1})R_e(\eta_\star) + \frac{1}{n\lambda}R_e(\xi_\star) \\ &= R_e(\eta_\star). \end{aligned}$$

Since $\eta_\star$ is optimal, this implies $C(\eta_n) \geq C(\eta_\star)$, so $\int_\Omega \eta_\star c \, \mathrm{d}\mu \geq \int_\Omega \eta_n c \, \mathrm{d}\mu$. We now compute the right hand side, defining $u_n = (1 - n^{-1})\eta_\star + n^{-1}\eta$, we get:

$$\begin{aligned} \int_\Omega \eta_\star c \, \mathrm{d}\mu \geq \int_\Omega \eta_n c \, \mathrm{d}\mu &= \int_\Omega u_n c \, \mathrm{d}\mu - \int_\Omega (u_n - 1)\mathbb{1}_{\{u_n > 1\}} c \, \mathrm{d}\mu \\ &= (1 - n^{-1}) \int_\Omega \eta_\star c \, \mathrm{d}\mu + n^{-1} \int_\Omega \eta c \, \mathrm{d}\mu - \int_\Omega (u_n - 1)\mathbb{1}_{\{u_n > 1\}} c \, \mathrm{d}\mu. \end{aligned}$$

Rearranging the terms, we arrive at:

$$\int_\Omega \eta c \, \mathrm{d}\mu \leq \int_\Omega \eta_\star c \, \mathrm{d}\mu + n \int_\Omega (u_n - 1)\mathbb{1}_{\{u_n > 1\}} c \, \mathrm{d}\mu.$$

Elementary computations give that:

$$0 \leq n(u_n - 1)\mathbb{1}_{\{u_n > 1\}} \leq \eta \mathbb{1}_{\{n < (\eta - \eta^\star)/(1 - \eta^\star)\}}.$$

Since μ-a.e. $\eta^\star < 1$, this implies that μ-a.e. $\lim_{n\to\infty} n(u_n - 1)\mathbb{1}_{\{u_n>1\}} = 0$. By dominated convergence, we obtain $\lim_{n\to\infty} n \int_\Omega (u_n - 1)\mathbb{1}_{\{u_n>1\}} c \, \mathrm{d}\mu = 0$ and thus:

$$\int_\Omega \eta c \, \mathrm{d}\mu \leq \int_\Omega \eta_\star c \, \mathrm{d}\mu,$$

and, multiplying by λ, we get $\int_\Omega \xi c \, \mathrm{d}\mu \leq \int_\Omega \xi_\star c \, \mathrm{d}\mu$, as claimed. Finally, the statement still holds for $\xi_\star = 0$ by letting λ go down to zero and using the fact that the Pareto optimal set is closed, see [7, Corollary 5.7]. □

4.2. **A characterization of $c_\star = C_\star(0)$ when the support of k is symmetric.** We characterize the Pareto optimal strategies which minimize R_e when the kernel k has a symmetric support, and get a very simple representation of $C_\star(0)$ when μ is finite and the cost is uniform.

Let us first recall a notion from graph theory. If $G = (V, E)$ is an non-oriented graph with vertices set V and edge set E, an *independent* set of G is a subset $A \subset V$ of vertices which are pairwise not adjacent, that is, $i, j \in A$ implies $ij \notin E$.

Following [13], we generalize this definition to kernels.

Definition 4.3 (Independent sets for kernels). *Let* k *be a kernel on* Ω. *A measurable set* $A \in \mathscr{F}$ *is an independent set of* k *if* $\mathrm{k} = 0$ $\mu^{\otimes 2}$*-a.e. on* $A \times A$.

In the following result, we prove that "maximal" independent sets provide optimal Pareto strategies for the loss function R_e and the cost function C. This property is illustrated in Figure 1 with the uniform cost $C = C_{\text{uni}}$ given by (16), where the Pareto frontier of the non-oriented cycle graph from Example 1.1, with $N = 12$, is plotted; it is possible to prevent infections without vaccinating the whole population as $c_\star = 1/2 < 1 = c_{\max}$.

Proposition 4.4. *Let* k *be a finite double norm kernel on* Ω *such that its support,* $\{\mathrm{k} > 0\}$*, is a symmetric subset of* Ω^2 *a.e. We have:*

$$c_\star = C_\star(0) = \min\{C(\mathbb{1}_A) : \, A \textit{ is an independent set of } \mathrm{k}\}. \tag{22}$$

Furthermore if $\eta_\star$ *is Pareto optimal such that* $R_e[\mathrm{k}](\eta_\star) = 0$*, then* $\{\eta_\star > 0\}$ *is an independent set,* $\eta_\star = \mathbb{1}_{\{\eta_\star>0\}}$ *a.e. and* $c_\star = C(\mathbb{1}_{\{\eta_\star>0\}})$.

Proof. Let A be an independent set. The effective reproduction number obviously vanishes for the strategy $\mathbb{1}_A$ as $(T_{\mathrm{k}\mathbb{1}_A})^2 = T_\mathrm{k}\, T_{\mathbb{1}_A \mathrm{k} \mathbb{1}_A} = 0$. This gives:

$$c_\star \leq \inf\{C(\mathbb{1}_A) : \, A \text{ is an independent set of } \mathrm{k}\}. \tag{23}$$

Now, let $\eta \in \Delta$ be such that $R_e[\mathrm{k}](\eta) = 0$. We shall prove that $\{\eta > 0\}$ is an independent set. Let $f \in L^1 \cap L^\infty$ such that $0 < f \leq 1$. Notice that $f \in L^r$ for all $r \in [1, +\infty]$. Let $\varepsilon > 0$. Since $\mathrm{k}\eta \geq \varepsilon \mathrm{k}_\varepsilon$, with $\mathrm{k}_\varepsilon = (\eta f)\, \mathbb{1}_{\{\mathrm{k} \geq \varepsilon\}}\, (\eta f)$, that is:

$$\mathrm{k}_\varepsilon(x, y) = (\eta f)(x)\, \mathbb{1}_{\{\mathrm{k}(x,y) \geq \varepsilon\}}\, (\eta f)(y),$$

we get that $T_{\mathrm{k}\eta} - \varepsilon T_{\mathrm{k}_\varepsilon}$ is a positive operator, and deduce from (8) that $\varepsilon \rho(T_{\mathrm{k}_\varepsilon}) = \rho(\varepsilon T_{\mathrm{k}_\varepsilon}) \leq \rho(T_{\mathrm{k}\eta}) = 0$ and thus $R_0[\mathrm{k}_\varepsilon] = 0$. Set $\mathrm{k}' = (\eta f)\, \mathbb{1}_{\{\,\mathrm{k}>0\,\}}\, (\eta f)$, which has finite double norm in L^p. Since $\lim_{\varepsilon \to 0+} \| \mathrm{k}_\varepsilon - \mathrm{k}' \|_{p,q} = 0$, we deduce from [7, Proposition 4.3] on the stability of R_e that $R_0[\mathrm{k}'] = \lim_{\varepsilon \to 0+} R_0[\mathrm{k}_\varepsilon] = 0$. As the support of k is symmetric, we deduce that the non-negative kernel k' is symmetric. Since $f \in L^2$, we deduce that k' has finite double norm on L^2. According to Theorem 4.2.15 and Problem 2.2.9 p. 49 in [3], we get that the integral operator $T_{\mathrm{k}'}$ on L^p and the integral operator T on L^2 with (the same) kernel k' have the same spectrum, and thus their spectral radius is zero. Since T is self-adjoint with zero spectral radius, we deduce that $T = 0$ and thus a.e. $\mathrm{k}' = 0$. Since f is positive, we deduce that $\mathrm{k} = 0$ a.e. on $\{\eta > 0\} \times \{\eta > 0\}$, and thus $\{\eta > 0\}$ is an independent set.

We now prove that the inequality in (23) is an equality and that the infimum is reached. Let $\eta_\star$ be a Pareto optimal strategy such that $R_e[\mathrm{k}](\eta_\star) = 0$ and thus $c_\star = C(\eta_\star)$. We deduce from the previous argument that $\{\eta_\star > 0\}$ is an independent set; and thus $R_e[\mathrm{k}](\mathbb{1}_{\{\eta_\star>0\}}) = 0$. Using the monotonicity and continuity of the cost function, we get that $C(\eta_\star) \geq C(\mathbb{1}_{\{\eta_\star>0\}})$ since $\eta_\star \leq \mathbb{1}_{\{\eta_\star>0\}}$. This implies that $\mathbb{1}_{\{\eta_\star>0\}}$ is Pareto optimal as well as $C(\eta_\star) = C(\mathbb{1}_{\{\eta_\star>0\}})$. This gives the claim.

Using the monotonicity of C, we also deduce from the equality $C(\eta_\star) = C(\mathbb{1}_{\{\eta_\star>0\}})$ that a.e. $\eta_\star = \mathbb{1}_{\{\eta_\star>0\}}$. This ends the proof. □

Remark 4.5 (On the independence number). The *independence number* of a graph G, denoted by $\alpha(G)$, is the maximum of $\sharp A/\sharp G$, over all the independent sets A of G. Similarly, for μ a probability measure, we can define the independence number $\alpha(\mathrm{k})$ of the kernel k by:

$$\alpha(\mathrm{k}) = \sup\{\mu(A) : \, A \text{ is an independent set of } \mathrm{k}\},$$

and we say that A is a maximal independent set for k if $\mu(A) = \alpha(\mathrm{k})$. Consider the uniform cost $C = C_{\text{uni}}$ given by (16) and a finite double norm kernel k on Ω such that its support, $\{\mathrm{k} > 0\}$, is a symmetric subset of Ω^2 a.s. Then, we deduce from Proposition 4.4, that any

Pareto optimal strategy $\mathbb{1}_{A_\star}$ for the loss $R_e[\mathrm{k}]$ corresponds to a maximal independent set $A_\star$ of k and *vice versa*, and we have:

$$c_\star = C_\star(0) = C(\mathbb{1}_{A_\star}) = 1 - \alpha(\mathrm{k}).$$

5. Atomic decomposition and cordons sanitaires

Following [18] and the presentation given in [5], we recall the decomposition of the kernel into its irreducible components in Section 5.1. Then, we complete the properties related to the anti-Pareto frontier (see Table 1) in Section 5.2 for kernels having only one irreducible component; and we prove in Section 5.3 that creating a *cordon sanitaire* is not anti-Pareto optimal. Finally, considering reducible kernels in Section 5.4, we provide a decomposition of the optimal cost and loss functions (related to the anti-Pareto and Pareto frontiers) by considering the corresponding optimization problems on the irreducible components.

5.1. **Atomic decomposition.** We follow the presentation in [5, Section 5] on the atomic decomposition of positive compact operator and Remark 5.2 therein for the particular case of integral operators, see also the references therein for further results. Let k be a kernel on Ω with a finite double norm. For $A, B \in \mathscr{F}$, we write $A \subset B$ a.e. if $\mu(B^c \cap A) = 0$ and $A = B$ a.e. if $A \subset B$ a.e. and $B \subset A$ a.e. For $A, B \in \mathscr{F}$, $x \in \Omega$, we simply write $\mathrm{k}(x, A) = \int_A \mathrm{k}(x, y)\, \mu(\mathrm{d}y)$, $\mathrm{k}(B, x) = \int_B \mathrm{k}(z, x)\, \mu(\mathrm{d}z)$ and:

$$\mathrm{k}(B, A) = \int_{B \times A} \mathrm{k}(z, y)\, \mu(\mathrm{d}z)\mu(\mathrm{d}y).$$

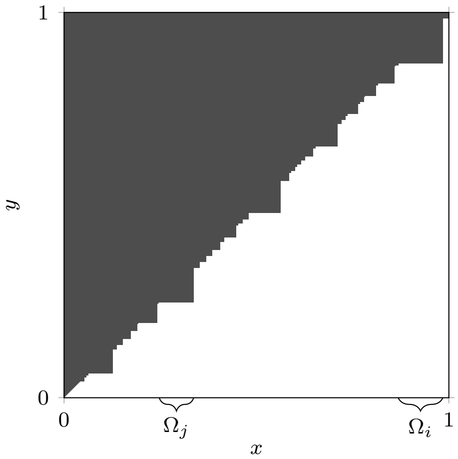
A set $A \in \mathscr{F}$ is called k-*invariant*, or simply *invariant* when there is no ambiguity on the kernel k, if $\mathrm{k}(A^c, A) = 0$. In the epidemiological setting, the set A is invariant if the sub-population A does not infect the sub-population A^c. The kernel k is *irreducible* (or *connected*) if any invariant set A is such that $\mu(A) = 0$ or $\mu(A^c) = 0$. If k is irreducible, then either $R_0[\mathrm{k}] > 0$ or $k \equiv 0$ and Ω is an atom of μ in $\mathscr{F}$ (degenerate case). A simple sufficient condition for irreducibility is for the kernel to be positive a.e.

Let $\mathscr{A}$ be the set of k-invariant sets, and notice that $\mathscr{A}$ is stable by countable unions and countable intersections. Let $\mathscr{F}_{\mathrm{inv}} = \sigma(\mathscr{A})$ be the σ-field generated by $\mathscr{A}$. Then, the operator k restricted to an atom of μ in $\mathscr{F}_{\mathrm{inv}}$ is irreducible. We shall only consider non degenerate atoms, and say the atom (of μ in $\mathscr{F}_{\mathrm{inv}}$) is non-zero if the restriction of the kernel k to this atom is non-zero (and thus the spectral radius of the corresponding integral operator is positive). We denote by $(\Omega_i, i \in I)$ the at most countable (but possibly empty) collection of non-zero atoms of μ in $\mathscr{F}_{\mathrm{inv}}$. Notice that the atoms are defined up to an a.e. equivalence and can be chosen to be pair-wise disjoint. According to [5, Lemma 5.3], we have the decomposition:

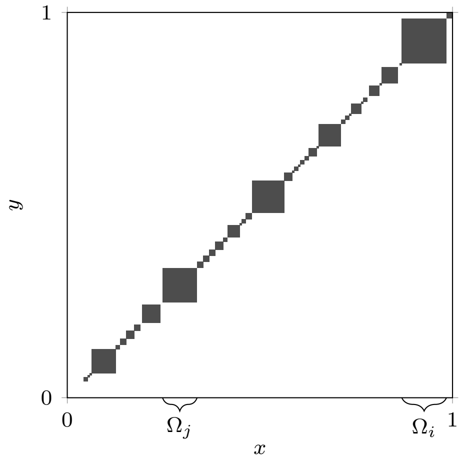
$$R_e[\mathrm{k}] = \max_{i \in I} R_e[\mathrm{k}_i] \quad \text{where} \quad \mathrm{k}_i = \mathbb{1}_{\Omega_i} \mathrm{k} \mathbb{1}_{\Omega_i}. \tag{24}$$

We represent in Figure 3(a) an example of a kernel k with its atomic decomposition using a "nice" order on Ω (so the kernel is upper block triangular: the population on the left of an atom does not infect the population on the right of an atom) in Figure 3(b) the corresponding kernel $\mathrm{k}' = \sum_{i \in I} \mathrm{k}_i$; thanks to (24), the kernels k and k' have the same effective reproduction function: $R_e[\mathrm{k}] = R_e[\mathrm{k}'] = \max_{i \in I} R_e[\mathrm{k}_i]$.

We say the kernel k is *monatomic* if there exists a unique non-zero atom ($\sharp I = 1$), and the kernel is *quasi-irreducible* if it is monatomic, with non-zero atom say Ω_{a}, and $\mathrm{k} \equiv 0$ outside $\Omega_{\mathrm{a}} \times \Omega_{\mathrm{a}}$. The quasi-irreducible property is the usual extension of the irreducible property in the setting of symmetric kernels; and the monatomic property is the natural generalization to non-symmetric kernels. We represented in Figure 4(a) a monatomic kernel k with non-zero



(A) A representation of the kernel k with the white zone included in $\{k = 0\}$.



(B) A representation of the kernel $k' = \sum_{i \in I} k_i$ with the white zone included in $\{k' = 0\}$.

FIGURE 3. Example of a kernel k on $\Omega = [0, 1]$ and the kernel $k' = \sum_{i \in I} k_i$, with $k_i(x, y) = \mathbb{1}_{\Omega_i}(x)\, k(x, y)\, \mathbb{1}_{\Omega_i}(y)$ and $(\Omega_i, i \in I)$ the non-zero atoms. We have $\mathrm{Spec}(T_k) = \mathrm{Spec}(T_{k'})$ as well as $R_e[T_k] = R_e[T_{k'}]$.

atom say Ω_a and in Figure 4(B) the quasi-irreducible kernel $k_a = \mathbb{1}_{\Omega_a} k \mathbb{1}_{\Omega_a}$ with the same atom; the set Ω being "nicely ordered" so that the representation of the kernels are upper triangular and the set Ω_i in Figure 4(A) corresponds to the sub-population infected by the atom Ω_a.

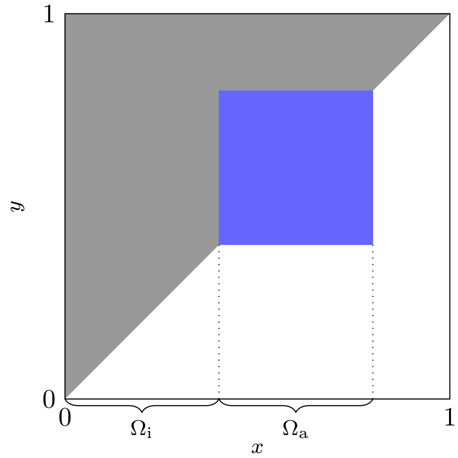
5.2. **The anti-Pareto frontier for irreducible and monatomic kernels.** We prove in the next result that for positive and/or irreducible kernels, the gaps in Table 1 may essentially be filled. We illustrate these properties in Figure 5 by plotting the typical Pareto and anti-Pareto frontiers for irreducible kernels and positive kernels. In order to avoid the degenerate irreducible kernel, we shall consider a non-zero kernel k, that is a kernel such that $k(\Omega, \Omega)$ is positive.

Proposition 5.1 (Consequences of irreducibility)**.** *Suppose that the cost function C is continuous decreasing with $C(\mathbb{1}) = 0$ and consider the loss function $R_e = R_e[k]$, with k a finite double norm irreducible non-zero kernel. Then, we have the following properties:*

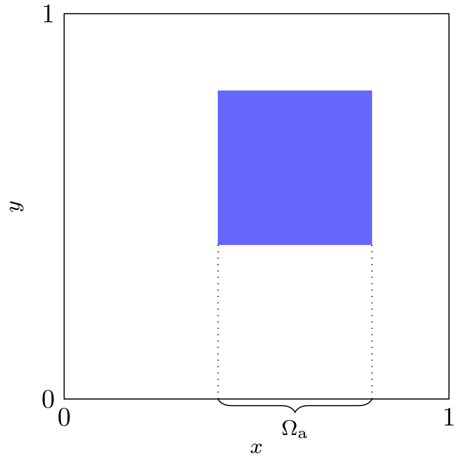
(i) *a)* $R_0 > 0$.
b) *The function $R_e^\star$ is continuous, decreasing on $[c^\star, c_{\max}]$.*
c) *The function $C^\star$ is continuous and decreasing on $[0, R_0]$.*
d) *We have $C^\star \circ R_e^\star(c) = c$ for $c \in [c^\star, c_{\max}]$.*
e) *The set $\mathcal{P}^{\mathrm{Anti}}$ is compact (for the weak-* topology), $\mathcal{F}^{\mathrm{Anti}}$ is connected and compact, and:*
$$\mathcal{F}^{\mathrm{Anti}} = \{(c, R_e^\star(c)) : \ c \in [c^\star, c_{\max}]\}.$$
f) $c^\star = 0$.

(ii) *If furthermore $k > 0$ a.e., then we also have:*
a) $c_\star = c_{\max}$.
b) *The strategy $\mathbb{1}$ (resp. $\mathbb{0}$) is the only Pareto optimal as well as the only anti-Pareto optimal strategy with cost $c = 0$ (resp. $c = 1$).*

Proof. According to [17, Theorem V.6.6], if k is an irreducible kernel with finite double norm, then, as k is non-zero, we have $R_0 = R_0[k] > 0$. This gives (i) a).



(A) A representation of a monatomic kernel.



(B) A representation of a quasi-irreducible kernel.

FIGURE 4. Example of kernels k and $\mathrm{k_a}$ of a monatomic integral operator T_{k} and the quasi-irreducible integral operator $T_{\mathrm{a}} = T_{\mathrm{k_a}}$ on $\Omega = [0, 1]$, with non-zero atom Ω_{a}. The kernels are zero on the white zone and are irreducible when restricted to the blue zone.

The other items follow from various results from [7]: Assumptions 3 and 6 from that paper hold, as well as Assumption 7, thanks to [7, Lemma 5.14]. In the notation of [7], as $\Omega_{\mathrm{a}} = \Omega$, we get $c^\star = C(\mathbb{1}) = 0$. We conclude using [7, Proposition 5.9] that items (i) b)- e) hold.

We now assume that $\mathrm{k} > 0$ a.e. As $c^\star = 0$, we deduce that the strategy $\mathbb{1}$ is anti-Pareto optimal. As C is decreasing, we also get that the strategy $\mathbb{1}$ is Pareto optimal.

Let $\eta \in \Delta$ be different from $\mathbb{0}$. The kernel $\mathrm{k}\eta$ restricted to the set of positive μ-measure $\{\eta > 0\}$ is positive, thus the kernel $\mathrm{k}\eta$ restricted to $\{\eta > 0\}$ is positive. It is therefore irreducible and its spectral radius is positive, so $R_e(\eta) > 0$. This also readily implies that $c_\star = c_{\max}$ and that the strategy $\mathbb{0}$ is Pareto optimal. As C is decreasing, we also get that the strategy $\mathbb{0}$ is anti-Pareto optimal. $\square$

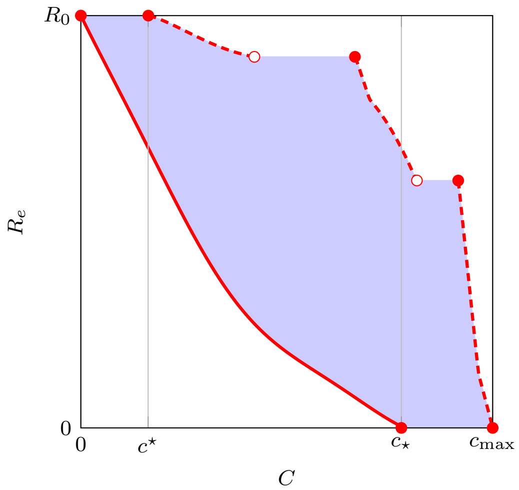
We now state the properties of the anti-Pareto frontiers for monatomic kernel.

Corollary 5.2 (Consequences of monatomicity)**.** *Suppose that the cost function C is continuous decreasing with $C(\mathbb{1}) = 0$ and consider the loss function $R_e = R_e[\mathrm{k}]$, with k a finite double norm monatomic kernel with non-zero atom Ω_{a}. Then, Properties (i) a)-e) of Proposition 5.1 hold. The strategy $\mathbb{1}_{\Omega_{\mathrm{a}}}$ is anti-Pareto optimal with cost $c^\star = C(\mathbb{1}_{\Omega_{\mathrm{a}}})$.*

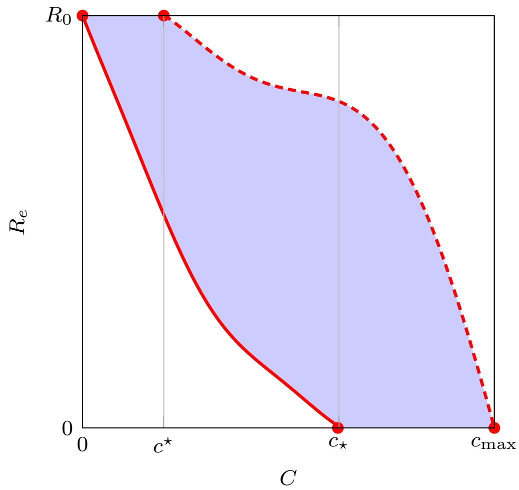
Proof. According to [7, Lemma 5.14], we get that R_0 is positive and $c^\star = C(\mathbb{1}_{\Omega_{\mathrm{a}}})$. The other results are proved as in Proposition 5.1. $\square$

Using the properties of the anti-Pareto frontiers stated in Proposition 5.1 for positive kernels and in Corollary 5.2 for monatomic kernel, we plotted in Figure 5 the typical Pareto and anti-Pareto frontiers for a general kernel (notice the anti-Pareto frontier is not connected *a priori*), a monatomic kernel (notice the anti-Pareto frontier is connected), and a positive kernel.

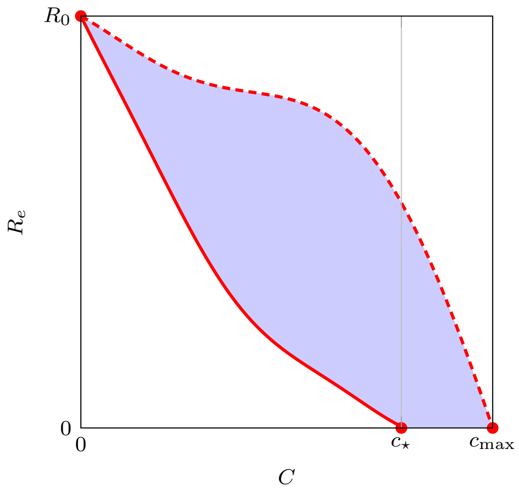
5.3. **Creating a *cordon sanitaire* is not the worst idea.** We say a strategy $\eta \in \Delta$ is a *cordon sanitaire* or *disconnecting* (for the kernel k) if $\eta \neq \mathbb{0}$ and the kernel k restricted to



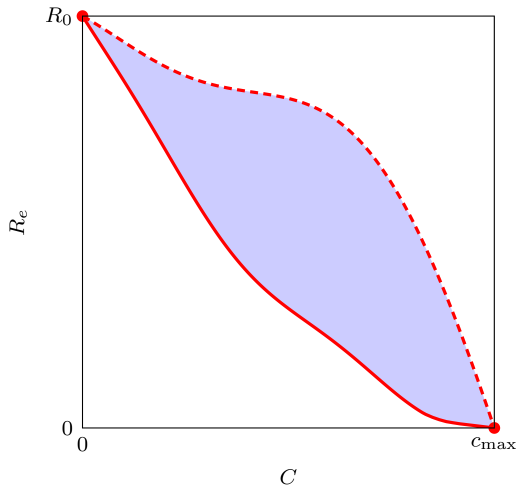
(A) General kernel.



(B) Monatomic kernel.



(C) Irreducible kernel.



(D) Kernel strictly positive almost surely.

FIGURE 5. Generic aspect of the feasible region (light blue), the Pareto frontier (thick red line) and the anti Pareto frontier (dashed red line) for the loss function $R_e[\mathrm{k}]$, with kernel k, and a continuous decreasing cost function C.

the set $\{\eta > 0\}$ is not connected (that is, not irreducible). Let us first give a few elementary comments on disconnecting strategies.

- The strategy $\eta = \mathbb{1}$ is disconnecting if and only if k is not connected.
- Disconnection only depends on fully vaccinated individuals: A strategy η is disconnecting if and only if the strategy $\mathbb{1}_{\{\eta>0\}}$ is disconnecting.
- If $\mathrm{k} > 0$, then there is no disconnecting strategy.
- If $\eta \neq \mathbb{0}$ is a strategy such that $\mathrm{k} = 0$ a.e. on $\{\eta > 0\}^2$, then η is disconnecting.

The next proposition states that if the strategy η is anti-Pareto optimal for a kernel k and non zero, then the kernel k restricted to $\{\eta > 0\}$ is irreducible. Let us remark that in general this implication is not an equivalence.

Proposition 5.3 (A *cordon sanitaire* is never the worst idea)**.** *Suppose that the cost function C is continuous decreasing and consider the loss function $R_e[\mathrm{k}]$, with* k *a finite double norm kernel on Ω such that $R_0[\mathrm{k}] > 0$. Then, a disconnecting strategy is not anti-Pareto optimal.*

In the non-oriented cycle graph from Example 1.1, this property is illustrated in Figure 1 as the disconnecting strategy "one in 4" is not anti-Pareto optimal; see Figure 2

Proof. Let η be a disconnecting strategy, and thus $\eta \neq \mathbb{0}$. Since η is disconnecting, that is, k restricted to $\{\eta > 0\}$ is not irreducible, we deduce there exists $A, B \in \mathscr{F}$ such that $\mu(A) > 0$, $\mu(B) > 0$, $(\mathrm{k}\eta)(B, A) = 0$ and a.e. $A \cup B = \{\eta > 0\}$ and $A \cap B = \emptyset$. We deduce from [5, Equation (29)] where we can replace k by $\mathrm{k}\eta$ that:

$$R_e[\mathrm{k}\eta](\mathbb{1}_A + \mathbb{1}_B) = \max\left(R_e[\mathrm{k}\eta](\mathbb{1}_A), R_e[\mathrm{k}\eta](\mathbb{1}_B)\right). \tag{25}$$

First assume that $R_e[\mathrm{k}\eta](\mathbb{1}_A) \geq R_e[\mathrm{k}\eta](\mathbb{1}_B)$, so that:

$$R_e[\mathrm{k}](\eta) = R_e[\mathrm{k}\eta](\mathbb{1}_{\{\eta>0\}}) = R_e[\mathrm{k}\eta](\mathbb{1}_A + \mathbb{1}_B) = R_e[\mathrm{k}\eta](\mathbb{1}_A).$$

For $\theta \in [0, 1]$, define the strategy $\eta_\theta = \eta\mathbb{1}_A + \theta\eta\mathbb{1}_B$. We deduce that:

$$\begin{aligned} R_e[\mathrm{k}](\eta_\theta) = R_e[\mathrm{k}\eta_\theta](\mathbb{1}_A + \mathbb{1}_B) &= \max(R_e[\mathrm{k}\eta_\theta](\mathbb{1}_A), R_e[\mathrm{k}\eta_\theta](\mathbb{1}_B)) \\ &= \max(R_e[\mathrm{k}\eta](\mathbb{1}_A), \theta R_e[\mathrm{k}\eta](\mathbb{1}_B)) \\ &= R_e[\mathrm{k}\eta](\mathbb{1}_A) \\ &= R_e[\mathrm{k}](\eta), \end{aligned}$$

where we used (25) with η replaced by η_θ for the second equality as $(\mathrm{k}\eta_\theta)(B, A) = 0$, and the homogeneity of the spectral radius in the third. Thus, the map $\theta \mapsto R_e[\mathrm{k}](\eta_\theta)$ is constant on $[0, 1]$. Since $\mu(B) > 0$ and C is decreasing, we get that $\theta \mapsto C(\eta_\theta)$ is decreasing. This implies that η_θ is worse than η for any $\theta \in [0, 1)$, and thus η is not anti-Pareto optimal.

The case $R_e[\mathrm{k}\eta](\mathbb{1}_B) \geq R_e[\mathrm{k}\eta](\mathbb{1}_A)$ is handled similarly. □

Remark 5.4. If the kernel k is irreducible and non-zero, then the upper boundary of the set of outcomes $\mathbf{F}$ is the anti-Pareto frontier, see Figure 5(c) for an instance. We deduce from Proposition 5.3 that if η_0 is a disconnecting strategy, then we have that $R_e[\mathrm{k}](\eta_0)$ is strictly less that $\sup\{R_e[\mathrm{k}](\eta) : C(\eta) = C(\eta_0)\}$.

However, if the kernel k is not irreducible, then the trivial strategy $\mathbb{1}$ is disconnecting. Furthermore, the upper boundary of the set of outcomes $\mathbf{F}$ is not reduced to the anti-Pareto frontier, see Figure 5(a) for instance. In fact, there exists disconnecting strategies that are not anti-Pareto optimal, but whose outcomes lie on the flat parts of the upper boundary of $\mathbf{F}$. In particular, such strategies have the worst loss given their cost. However, it is not difficult to check that they do not disconnect further than the trivial strategy $\mathbb{1}$.

5.4. **Pareto and anti-Pareto frontiers for reducible kernels.** Let us now assume that the kernel k is "truly reducible", in the sense that it has at least two non-zero atoms, and thus $R_0 = R_0[\mathrm{k}] > 0$. We will see in this section how to effectively reduce the study of the global optimization problem to a study of the optimization problem on each non-zero atom. Recall the collection of non-zero atoms $(\Omega_i, i \in I)$ defined in Section 5.1 and the corresponding quasi-irreducible kernels $(\mathrm{k}_i, i \in I)$ in (24). By construction, the kernel k_i has a finite double norm and $R_0[\mathrm{k}_i] > 0$.

We now describe two ways of restricting the problem to an atom. For the kernel k_i and the loss function $R_e[\mathrm{k}_i]$, the atom is still viewed as a part of the larger population Ω. As such, the vaccination strategies that agree on Ω_i but differ on Ω_i^c will have the same loss, but their costs may differ. For $i \in I$ and $\eta \in \Delta$, we set similarly:

$$\eta_i = \eta \mathbb{1}_{\Omega_i}.$$

We consider the loss $R_e[\mathrm{k}_i]$ and the corresponding optimal loss function $R_i^\star$ defined on $[0, c_{\max}]$ and optimal cost function $C_i^\star$ and $C_{i,\star}$. For convenience the functions $C_i^\star$ and $C_{i,\star}$ which are defined on $[0, R_0[\mathrm{k}_i]]$ are extended to $[0, R_0]$ by letting them be equal to 0 on $(R_0[\mathrm{k}_i], R_0]$.

Another point of view is to restrict the kernel and vaccination strategies to the atom, and study it intrisically, in isolation. Quantities and functions defined by this intrinsic approach will be denoted by bold letters. In particular $\mathbf{k}_i : \Omega_i^2 \to \mathbb{R}$ is the kernel k (and k_i) restricted to Ω_i; it is irreducible and non-zero by construction and $R_0[\mathbf{k}_i]$ is a simple positive eigenvalue of the corresponding integral operator. If η is a vaccination strategy, then $\boldsymbol{\eta}_i$ is its restriction to Ω_i. By construction, we have for all $\eta \in \Delta$:

$$R_e[\mathrm{k}_i](\eta) = R_e[\mathrm{k}_i](\eta_i) = R_e[\mathbf{k}_i](\boldsymbol{\eta}_i).$$

If $\boldsymbol{\eta}$ is a $[0,1]$-valued measurable function defined on Ω_i, we define its extension η on Ω (corresponding to no vaccinations outside Ω_i) and its cost by:

$$\eta = \begin{cases} \boldsymbol{\eta} & \text{on } \Omega_i \\ \mathbb{1} & \text{on } \Omega_i^c \end{cases} \qquad \text{and} \quad \boldsymbol{C}_i(\boldsymbol{\eta}) = C(\eta).$$

The optimization problems (17) and (19) may now be stated on each Ω_i for the kernel $\mathbf{k}_i$, the loss $R_e[\mathbf{k}_i]$ and the cost $\boldsymbol{C}_i$: denote by $\boldsymbol{C}_{i,\star}$ and $\boldsymbol{C}_i^\star$ the corresponding optimal cost functions, and extend them to $[0, R_0]$ by letting them be equal to 0 on $(R_0[\mathbf{k}_i], R_0]$. In particular, by construction, $\boldsymbol{C}_{i,\star}$ is equal to $C_{i,\star}$. However, there is no relation in general between $\boldsymbol{C}_i^\star$ and $C_i^\star$. Nevertheless, it is possible to establish such a relation when the cost is extensive. Recall once more that for a vaccination strategy η, the proportion of vaccinated individuals of trait x is given by $1 - \eta(x)$. Thus, two vaccination strategies η and η' target disjoint subsets of the population if $\eta \vee \eta' = \mathbb{1}$.

Definition 5.5 (Extensivity)**.** *Let C be a continuous decreasing cost function with $C(\mathbb{1}) = 0$. The cost C is called* extensive *if vaccinating disjoint subsets of the population is additive:*

$$C(\eta \wedge \eta') = C(\eta) + C(\eta') \quad \textit{for all } \eta, \eta' \in \Delta \textit{ such that } \eta \vee \eta' = \mathbb{1}.$$

If the continuous decreasing cost function C is extensive, then we get for all $\eta \in \Delta$ that:

$$C(\eta) = \sum_{i \in I} C(\eta_i + \mathbb{1}_{\Omega_i^c}) = \sum_{i \in I} \boldsymbol{C}_i(\boldsymbol{\eta}_i), \tag{26}$$

since all the vaccinations $\eta_i + \mathbb{1}_{\Omega_i^c}$ target pairwise disjoint subsets of the population.

Remark 5.6 (Affine costs are extensive). If the cost function takes the form

$$C(\eta) = c_{\max} - \int_\Omega \phi(\eta(x), x)\, \mu(dx)$$

where $\phi : [0,1] \times \Omega \to \mathbb{R}_+$ is measurable and non-decreasing in its first variable, then C is extensive. In particuar, the affine cost functions considered in Proposition 4.1 are extensive.

We are now ready to state the reduction result, which in particular implies that if the cost function is extensive, then the (anti-)Pareto frontier of the full model may be constructed from the family of (anti-)Pareto frontiers of each atom.

Proposition 5.7 (Reduction to atoms)**.** *Let* k *be a kernel with finite double norm on Ω, such that $R_0 = R_0[\mathrm{k}] > 0$. Suppose that the cost function C is continuous decreasing with $C(\mathbb{1}) = 0$.*

(i) ***Decomposition of the loss.*** *For any $\eta \in \Delta$, we have:*

$$R_e[\mathrm{k}](\eta) = \max_{i\in I} R_e[\mathrm{k}](\eta \mathbb{1}_{\Omega_i}) = \max_{i\in I} R_e[\mathrm{k}_i](\eta_i) = \max_{i\in I} R_e[\mathbf{k}_i](\boldsymbol{\eta}_i). \tag{27}$$

(ii) ***Anti-Pareto optimal strategies.*** *For all $\ell \in [0, R_0]$ and $\eta \in \Delta$, the following two properties are equivalent:*

a) The strategy η is anti-Pareto optimal with $R_e[\mathrm{k}](\eta) = \ell$.

b) There exists $j \in \mathrm{argmax}_{i\in I}\, C_i^\star(\ell)$ such that $\eta = 0$ on Ω_j^c and $\eta = \boldsymbol{\eta}_j$ on Ω_j, where $\boldsymbol{\eta}_j$ is anti-Pareto optimal for $\mathbf{k}_j$ on Ω_j and cost function $\boldsymbol{C}_i$ with $R_e[\mathbf{k}_j](\boldsymbol{\eta}_j) = \ell$.

Besides, we have:

$$R_e^\star = \max_{i\in I} R_i^\star \quad \text{on } [0, c_{\max}] \quad \text{and} \quad C^\star = \max_{i\in I} C_i^\star \quad \text{on } [0, R_0]. \tag{28}$$

Furthermore, if the cost function C is extensive, then for all $i \in I$, we have:

$$C_i^\star = \boldsymbol{C}_i^\star + C(\mathbb{1}_{\Omega_i}).$$

(iii) ***Pareto optimal strategies when the cost function is extensive.*** *Suppose that the cost function C is extensive. For all $\ell \in [0, R_0]$ and $\eta \in \Delta$, the following two properties are equivalent:*

a) The strategy η is Pareto optimal with $R_e[\mathrm{k}](\eta) = \ell$.

b) On $(\bigcup_{i\in I} \Omega_i)^c$, $\eta = 1$ and, for all $i \in I$, η restricted to Ω_i, say $\boldsymbol{\eta}_i$, is Pareto optimal for $\mathbf{k}_i$ on Ω_i and cost function $\boldsymbol{C}_i$ with $R_e[\mathbf{k}_i](\boldsymbol{\eta}_i) = \min(\ell, R_0[\mathbf{k}_i])$ (and thus $\boldsymbol{\eta}_i = \mathbb{1}$ if $R_0[\mathbf{k}_i] \leq \ell$).

Besides, we have:

$$C_\star = \sum_{i\in I} \boldsymbol{C}_{i,\star}.$$

Remark 5.8 (Additional consequences). From (21) and the second part of (28), we get that the anti-Pareto frontier is given by:

$$\mathcal{F}^{\mathrm{Anti}} = \left\{ \left(\max_{i\in I} C_i^\star(\ell), \ell \right) \;:\; \ell \in [0, R_0] \right\}.$$

We deduce from Point (ii) that the maximal cost of totally inefficient strategies is given by:

$$c^\star := C^\star(R_0) = \max_{i\in I} \{ C(\mathbb{1}_{\Omega_i}) :\; R_0[\mathrm{k}_i] = R_0[\mathrm{k}] \}.$$

According to [5, Remark 5.1(v)] the number of atoms Ω_i such that $R_0[\mathrm{k}_i] = R_0[\mathrm{k}]$ is equal to the algebraic multiplicity of R_0 for T_k.

As any Pareto optimal strategy is larger than $\mathbb{1}_{(\bigcup_{i\in I} \Omega_i)^c}$ according to Point (iii), we get an upper bound for the minimal cost which ensures that no infection occurs at all:

$$c_\star = C_\star(0) \leq C(\eta) \quad \text{with} \quad \eta = \mathbb{1} - \sum_{i\in I} \mathbb{1}_{\Omega_i}.$$

Remark 5.9. If $R_0[\mathrm{k}] > 0$ and k is not monatomic, then Assumption 7 in [7] (that is any local maximum of the loss function is also a global maximum) may or may not be satisfied for the loss function $R_e = R_e[\mathrm{k}]$; this can happen even in a two homogeneous populations model. In the former case the function $C^\star$ is continuous and the anti-Pareto frontier is connected, whereas in the latter case the function $C^\star$ may have jumps and then the anti-Pareto frontier has more than one connected component.

Proof of Proposition 5.7. Let k be a finite double norm kernel on Ω such that $R_0 = R_0[\mathrm{k}] > 0$. Set $\Omega_0 = \Omega \setminus \cup_{i\in I}\Omega_i$. For $i \in I$ and $\eta \in \Delta$, we set $\eta_i = \eta \mathbb{1}_{\Omega_i}$.

According to (24) and since $R_e[\mathrm{k}_i](\eta) = R_e[\mathrm{k}_i](\eta_i) = R_e[\mathrm{k}](\eta_i)$, we can decompose $R_e[\mathrm{k}]$ according to the quasi-irreducible components $(\mathrm{k}_i, i \in I)$ of k to get that for $\eta \in \Delta$:

$$R_e[\mathrm{k}](\eta) = \max_{i\in I} R_e[\mathrm{k}_i](\eta) = \max_{i\in I} R_e[\mathrm{k}_i](\eta_i) = \max_{i\in I} R_e[\mathrm{k}](\eta_i). \tag{29}$$

Then use that $\mathbf{k}_i$ is the restriction of k_i to Ω_i to get Point (i).

We now prove Point (ii). Equation (29) and the definition of $R_e^\star$ readily implies that $R_e^\star = \max_{i\in I} R_i^\star$, which gives the first part of (28).

We prove that properties a) and b) are equivalent. The case $\ell = 0$ being trivial, we only consider $\ell \in (0, R_0]$. Let η be a strategy such that $R_e[\mathrm{k}](\eta) = \ell$. According to (i), there exists j such that $R_e[\mathrm{k}](\eta) = R_e[\mathrm{k}](\eta_j)$. Since $\ell > 0$, we get that η_j is not equal to $\mathbb{0}$. Hence, we get:

$$C(\eta) \leq \inf_{i\in I} C(\eta_i) \leq C(\eta_j) \leq C_j^\star(\ell) \leq \sup_{i\in I} C_i^\star(\ell) \leq C^\star(\ell),$$

where:

(1) the first and second inequalities become equalities if and only if $\eta_i = \mathbb{0}$ for all $i \neq j$ because C is decreasing;
(2) the third inequality is an equality if and only if $\boldsymbol{\eta}_j$ is anti-Pareto optimal (see Table 1);
(3) the last inequality follows from the fact that $R_e[\mathrm{k}_i](\eta_i) = R_e[\mathrm{k}](\eta_i)$ for all $i \in I$.

Hence, Property a) is equivalent to the following equalities:

$$C_j^\star(\ell) = C(\eta_j) = C(\eta) = C^\star(\ell). \tag{30}$$

which is equivalent to Property b). In particular, it follows from the existence of the anti-Pareto optimal strategy that $\sup_{i\in I} C_i^\star$ is in fact a max.

We now prove that $C_i^\star = \boldsymbol{C}_i^\star + C(\mathbb{1}_{\Omega_i})$ for all $i \in I$ in case C is extensive. Note that the optimal cost $\boldsymbol{C}_i^\star$, defined in terms of the restricted kernel $\mathbf{k}_i$, and which may be viewed as intrinsic on Ω_i, differs from the cost $C_i^\star$, defined on the "extrinsic" kernel k_i defined on the whole space Ω. Let $\ell \in [0, R_0]$. The worst vaccinations on the whole space clearly consist in vaccinating everyone outside Ω_i and vaccinating in the worst possible way inside Ω_i, that is, if η is anti-Pareto optimal for the kernel k_i with loss $\ell \in [0, R_0]$ and cost $C(\eta)$, then $\eta = \boldsymbol{\eta}_i \mathbb{1}_{\Omega_i}$, where $\boldsymbol{\eta}_i$ is anti-Pareto optimal for the kernel $\mathbf{k}_i$ with loss ℓ and cost $\boldsymbol{C}_i^\star(\ell) = \boldsymbol{C}_i(\boldsymbol{\eta}_i)$. Set $\eta' = \eta + \mathbb{1}_{\Omega_i^c}$ and $\eta'' = \mathbb{1}_{\Omega_i}$ so that $\eta = \eta' \vee \eta'' = \mathbb{1}$. By definition of $\boldsymbol{C}_i$, we have $C(\eta') = \boldsymbol{C}_i(\boldsymbol{\eta}_i)$. Since C is extensive, we get:

$$C_i^\star(\ell) = C(\eta) = C(\eta') + C(\eta'') = \boldsymbol{C}_i(\boldsymbol{\eta}_i) + C(\mathbb{1}_{\Omega_i}) = \boldsymbol{C}_i^\star(\ell) + C(\mathbb{1}_{\Omega_i}).$$

Point (iii) follows directly from Point (i) and the the following decomposition of C as an extensive function:

$$C(\eta) = \sum_{i\in I} \boldsymbol{C}_i(\boldsymbol{\eta}_i). \tag{31}$$

□

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