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INFINITE DIMENSIONAL METAPOPULATION SIS MODEL WITH GENERALIZED INCIDENCE RATE

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Abstract. We consider an infinite-dimension SIS model introduced by Delmas, Dronnier and Zitt, with a more general incidence rate, and study its equilibria. Unsurprisingly, there exists at least one endemic equilibrium if and only if the basic reproduction number is larger than 1. When the pathogen transmission exhibits one way propagation, it is possible to observe different possible endemic equilibria. We characterize in a general setting all the equilibria, using a decomposition of the space into atoms, given by the transmission operator. We also prove that the proportion of infected individuals converges to an equilibrium, which is uniquely determined by the support of the initial condition.

We extend those results to infinite-dimensional SIS models with reservoir or with immigration.

1. Introduction

1.1. **Model and relations with existing models.** We consider an inhomogeneous SIS epidemic model, where individuals are either susceptible or infected. The homogeneous model was introduced by Kermack and McKendrick [28], we refer to the monograph of Brauer, Castillo-Chavez et Feng [9] for an analysis of this homogeneous SIS model and some of its variants. Let us recast the model from [28] in the constant population case: let $I(t)$ and $S(t)$ denote respectively the number of the infected and susceptible) individuals at time $t \geqslant 0$, in a population of constant size $N = S(t) + I(t) > 0$. The evolution of the number of infected is given by:

$$I' = k\,\frac{SI}{N} - \gamma I, \tag{1}$$

where $k \geqslant 0$ is the infection rate and $\gamma > 0$ the recovery rate.

The assumption of homogeneity of the population is not always satisfied in practice, see for example: Trauer et al. [54] for a review on tuberculosis, [43] on the impact of health condition, [11] on the number of sexual partners in a sexually transmissible infection, or the review [55] for more possible sources of heterogeneity. The inhomogeneous SIS model from Lajmanovich and Yorke [32] generalizes the Kermack-McKendrick model to a population divided in n sub-groups; the same equation appears also when studying network of communities linked by dispersal, see Mouquet and Loreau [40] and more generally [12]. Later, Thieme [53] and Delmas, Dronnier and Zitt [17] introduced a variant allowing an infinite number (possibly uncountable) of sub-groups or features.

We follow the model given by [17] where the transmission operator can be non-irreducible, see the discussion in Section 1.2 below and allowing furthermore a more general incidence rate, see Section 1.5. The heterogeneity of the population is described as follow: $(\Omega, \mathcal{G}, \mu)$ is a measured space with a non-zero σ-finite measure μ: an element $x \in \Omega$ corresponds to a particular *feature* (or *trait*) of individuals. We assume that individuals with the same feature behave in the same way with respect to the epidemic, and that features stay constant during the whole infection process. We also assume that for a given feature $x \in \Omega$, the size of the population $\mu(\mathrm{d}x)$ of feature x remains constant over time.

Let $u(t, x)$ denote the proportion of individuals with feature $x \in \Omega$ that are infected at time $t \geqslant 0$ among the population of individuals with feature x. Let Δ be the set of measurable functions defined on Ω taking values in $[0, 1]$. The heterogeneous SIS dynamics is given, for an initial condition $h \in \Delta$, by

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the evolution equation on the Banach space L^∞ of measurable bounded real-valued functions defined on Ω by:

$$\begin{cases} u' = F(u), \\ u(0) = h \in \Delta, \end{cases} \tag{2}$$

with

$$F(u) = \varphi(u)\, Tu - \gamma u, \tag{3}$$

where F depends on: a bounded linear *transmission operator* T on L^∞, a bounded real-valued positive *recovery rate* function γ defined on Ω, and a real-valued function φ defined on $\mathbb{R}$ encoding the non-bilinearity of the *incidence rate*. The hypotheses on the parameter (T, γ, φ) are given in Assumptions 1 and 2. Let us stress that the usual *law of mass action* $\varphi = 1 - \mathrm{Id}$, with Id the identity map on $\mathbb{R}$, satisfies the corresponding hypothesis from Assumption 2 summarized in Condition (14).

Remark 1.1 (The kernel model from [17]). Let $k : \Omega^2 \to \mathbb{R}_+$ be a kernel, that is a nonnegative measurable function. The associated kernel operator T_k is defined as follow. For $h \in L^\infty$ and $x \in \Omega$, we define:

$$T_k(h)(x) = \int_\Omega k(x, y) h(x)\, \mu(\mathrm{d}y).$$

The quantity $k(x, y)$ represents the transmission rate from individuals with feature $y \in \Omega$ to those with feature $x \in \Omega$. The heterogeneous SIS model from [17] is then given by (2) and (3) with $T = T_k$ (under some integral hypothesis on the kernel) and the usual law of mass action $\varphi = 1 - \mathrm{Id}$. This in particular encompasses the Lajmanovich and Yorke model.

In epidemiology, *equilibria* are constant solutions of (2), that is, functions $g \in \Delta$ such that:

$$F(g) = 0. \tag{4}$$

They play a significant role in the long-time behavior of the dynamics of an outbreak, see Theorem 3 below. Obviously, the *disease-free equilibrium* (DFE) $g = 0$ is an equilibrium. Any other equilibrium is called *endemic equilibrium* (EE). The *basic reproduction number* denoted R_0 is defined by Heesterbeek and Dietz [27] as "the expected number of secondary cases produced by a typical infected individual during its entire infectious period, in a population consisting of susceptibles only". Following [17] (see also the method of the next-generation operator in Diekmann, Heesterbeek and Metz [20]), the basic reproduction number R_0 for the SIS model (2) with the usual incidence rate associated to $\varphi = 1 - \mathrm{Id}$ is defined as the spectral radius of the operator $TM_{1/\gamma}$, where the operator $M_{1/\gamma}$ is the multiplication by $1/\gamma$. There usually is a threshold behavior for the existence of EE according to the value of R_0: for $R_0 \leqslant 1$ only the DFE exists as an equilibrium, and for $R_0 > 1$ there exists an EE. This is not universal: for example, models with imperfect vaccines or exogenous re-infections might lead to backward bifurcation and produce multiple EE even in the regime $R_0 < 1$, see [24] and more specifically [8] for a SIS model. Nevertheless, we check that threshold behavior holds for the SIS model (2), see Theorem 2 below. A discussion of the uniqueness of EE is given in Section 1.2.

1.2. **A taste of the main results in the finite setting.** Except in the trivial case where the population may be split in subpopulations that do not interact at all, the existence of multiple equilibria is fundamentally linked to *asymmetry* in the transmission dynamics. In this section, we first explain this phenomenon, and a related crucial decomposition of the space, in the simple case where Ω is finite, to give a taste of the general results stated below.

We consider a finite set Ω, let $\mathcal{G}$ be the set of subsets of Ω, and μ a finite measure with support Ω. The transmission operator T is identified with a matrix $K = (K_{x,y})_{x,y \in \Omega}$ where $K_{x,y}$ is the infection rate from individuals with feature y to those with feature x; in particular it takes into account the relative size of the sub-populations. When Ω is a singleton and $\varphi = 1 - \mathrm{Id}$, we recover Equation (1) (with $u = I/N$ and $K = k$). When Ω is finite, we recover the Lajmanovich and Yorke [32] model, and the same framework can be used to describe households models [6] and multi-host and vector-borne diseases [39].

In this finite case, the study of the non-uniqueness for equilibria relies on the properties of the oriented transmission graph $G_K = (\Omega, E_K)$ with the set of edges $E_K = \{(y, x) \in \Omega^2 : K_{x,y} > 0\}$ given by the support of the transmission matrix K. An edge from y to x models the possibility of infection from the sub-population with feature y to the sub-population with feature x; in particular the graph may have self-loops. For transmission graph models see for example [25, 5].

Strongly connected components of G_K will be called *atoms* — the notion will be generalized in the infinite case. An atom is *non-zero* unless it is a singleton with no self loop. Notice the transmission matrix/operator is irreducible if and only if the graph G_K is strongly connected (that is, Ω is an atom), and is said monatomic if there is a unique non-zero atom. For further result on monatomic operators in the general case see [19] and references therein; we refer also to Corollary 4.11 for a characterization of monatomic transmission matrix using the number of EE.

In many examples the transmission graph G_K is symmetric (even though the transmission K might not be symmetric). In this case, all (strongly) connected components behave independently and one can study each connected components separately. Cases where G_K is not strongly connected occur less frequently in the literature; it has been mentioned for example in a multi-type SIR model by [29, 36].

Let us mention two examples of non symmetric transmission graphs.

(i) The West Nile Virus, presented in [7], infects three species, birds (B), humans (H) and mosquitoes (M). It is a vector-borne disease where birds and mosquitoes serve as vectors for a transmission to humans. In this model, mosquitoes infects birds and humans while biting them and mosquitoes get infected by birds while biting them, and we assume there is no infection from humans to mosquitoes, nor between birds and humans. The graph G_K given in Fig. 1a has only one non zero atom $\{B, M\}$ and a zero atom $\{H\}$. In particular K is monatomic.

(ii) In the zoonosis model from [46], a pathogen exists in wild animals (W), is transmitted to domestic animals (D) that transmit themselves the pathogen to humans (H). The graph G_K given in Fig. 1b has three non-zero atoms: $\{W\}, \{D\}$ and $\{H\}$.

In such cases where G_K is not symmetric, the picture is richer: many endemic equilibria may exist, they may be entirely characterized by the atoms contained in their support, and their basins of attraction may be described explicitly.

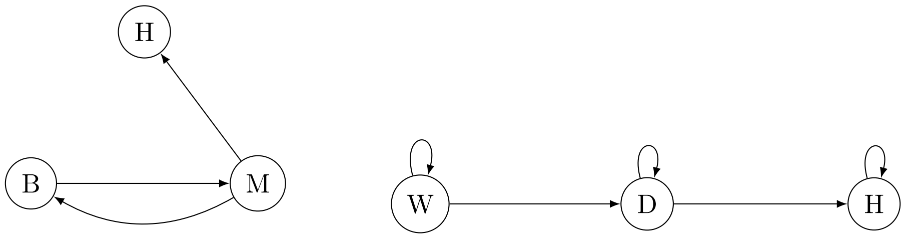
Let us give a few additional definitions to state these results more precisely, before giving the general statements below in Theorem 2 and 3. Define the *future* of a set $A \subset \Omega$ as the set $\mathcal{F}(A)$ of all the vertices in G_K reachable from A by a (possibly empty) path using edges in E_K. For two atoms A and B of G_K, we write $B \preccurlyeq A$ if $B \subset \mathcal{F}(A)$; the relation $\preccurlyeq$ is a partial order. An *antichain* of atoms is a set of atoms which are pairwise unordered. The future of an antichain is the future of the union of its elements.

In the West Nile Virus model (i), the antichains of non-zero atoms are $\varnothing$ and $\{B, M\}$; in the zoonosis model (ii), the antichains of non-zero atoms are: $\varnothing$, $\{H\}$, $\{D\}$ and $\{W\}$.

Finally, an atom is *supercritical* if the basic reproduction of the SIS-model restricted to the atom is larger than 1; in particular a supercritical atom is non-zero and an atom A is trivially supercritical if $K_{x,x}/\gamma(x) > 1$ for all $x \in A$, where γ is the recovery rate function, and the function φ satisfies the regularity Condition (14) below.

Our first main result, Theorem 2, states (in the general possibly infinite setting) that each equilibrium is characterized by a (different) antichain of supercritical atoms, and the DFE is associated to the antichain $\varnothing$. For example, assuming for simplicity that all non-zero atoms are supercritical, we deduce that in the West Nile Virus model (i) there is only one EE and that in the zoonosis model (ii) there are three EE.

Let us mention that a similar result on the existence of multiple EE is obtained in Waters et al. [56] for a waterborne parasites that infect both humans and animals, such as *Giardia* infection in rural Australia. In this model the animals and the humans can be seen as non-zero atoms for the transmission, and the water as an environmental reservoirs. This model does not fit exactly the metapopulation SIS model (2)-(3), nor the model with external disease reservoir presented in Section 1.4 because the reservoir is between the animal population and the human population.



(a) West Nile Virus (b) Zoonosis model from [46]

FIGURE 1. Some examples of transmission graphs G_K

Theorem 2 also states also that the support of an equilibrium is given by the future of its corresponding antichain of supercritical atoms. Furthermore, Corollary 4.2 asserts that for two equilibria g and g', we have $g \leqslant g'$ if and only if $\text{supp}(g) \subset \text{supp}(g')$. This in particular allows to recover the existence of a maximal equilibrium g^*, in the sense that if g is an other equilibrium, then $g \leqslant g^*$.

For example, assuming again for simplicity that all non-zero atoms are supercritical, we deduce that in the zoonosis model (ii), denoting by g_x the equilibrium characterized by the antichain $\{x\}$, we have:

$$\text{supp}(g_H) = \{H\} \subset \text{supp}(g_D) = \{D, H\} \subset \text{supp}(g_W) = \{W, D, H\} \quad \text{and} \quad 0 \neq g_H \lneq g_D \lneq g_W = g^*.$$

Our second main result is a full characterization of basins of attraction of the various equilibria: we show in Theorem 3 that, starting with an initial condition $u(0) = h$, the epidemics converges in long time towards the equilibrium associated to the maximal antichain in $\mathcal{F}(\text{supp}(h))$ the future of the support of the initial condition.

In the example of the West Nile Virus model (i), assuming that all non-zero atoms are supercritical, we deduce that starting with an initial condition where only the human population H is infected, the epidemic converges to the DFE and thus dies out, but starting with an initial condition where the populations H and M (or simply M or B) is infected, the epidemic converges to the unique EE g^*, whose support is $\{B, M, H\}$.

In the example of the zoonosis model (ii), assuming again that all the atoms are supercritical, we deduce from Theorem 3, starting with an initial condition $u(0) = h$, the epidemics converges in long time towards the equilibrium whose support is the support of h.

1.3. **Assumptions and main results.** Recall the SIS model (2)-(3) with parameter (T, γ, φ). We shall consider the following assumptions on the parameters. For $p \in [1, +\infty]$, let L^p denote the usual Lebesque space of measurable function defined on the measured space $(\Omega, \mathcal{G}, \mu)$ endowed with the L^p norm $\|\cdot\|_p$, and L^p_+ the subset of L^p of nonnegative functions.

Assumption 1. *The measure μ is finite and non-zero; the map T is a bounded linear map on L^∞ and there exists $p \in (1, +\infty)$ and a finite constant C_p such that for all $f \in L^\infty$:*

$$\| Tf \|_p \leqslant C_p \, \| f \|_p \, ; \tag{5}$$

the function γ belongs to L^∞ and $\gamma > 0$ a.e.; and the function $\varphi : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, nonnegative on $[0, 1]$ and $\varphi(1) = 0$.

Assumption 2. *Assumption 1 holds; there exists a finite constant C'_p such that for all $f \in L^\infty$:*

$$\| Tf \|_\infty \leqslant C'_p \, \| \gamma f \|_p \, ; \tag{6}$$

and the map φ is decreasing on $[0, 1]$ with $\varphi(0) = 1$.

The next two remarks are related to kernel operators. We also refer to Section 2.6 for further properties of the operator T induced by those two assumptions.

Remark 1.2 (The operator T is a kernel operator). Assume Condition (6) holds. Since γ is bounded, we deduce that there exists a finite constant C''_p such that $\| Tf \|_\infty \leqslant C''_p \| f \|_p$ for all $f \in L^\infty$, and also that Condition (5) holds. According to [49, Theorem 4.2], we deduce that T is a kernel operator (and the kernel is indeed nonnegative by Theorem 1.3 therein).

Remark 1.3 (The SIS model from [17]). Recall the definition of the kernel operator T_k for k a kernel given in Section 1.1. We check that the SIS model from [17], see Assumption 1 therein, satisfies our Assumption 2. In [17], the measure μ is a probability measure on Ω, the function γ is positive and bounded, and the mass-action incidence rate is associated to $\varphi = 1 - \mathrm{Id}$. So the conditions on φ, γ and μ in Assumption 2 are clearly satisfied. Therein, we have $T = T_k$ for $k : \Omega^2 \to \mathbb{R}_+$ a kernel such that:

$$\sup_{x\in\Omega} \int_\Omega \left(\frac{k(x,y)}{\gamma(y)} \right)^q \mu(\mathrm{d}y) < +\infty, \tag{7}$$

for some $q \in (1, +\infty)$. It is then elementary to check that the conditions on the operator $T = T_k$ from Assumption 2 are satisfied with $p \in (1, +\infty)$ given by $1/p + 1/q = 1$.

We now give our main results. Recall that $\Delta = \{f \in L^\infty_+ : 1 - f \in L^\infty_+\}$ is the set of measurable functions taking their value a.e. in $[0, 1]$. Proposition 3.1 below asserts that under Assumption 1, for any initial condition $h \in \Delta$, Equation (2) has a unique global solution in L^∞ given by the semi-flow $(\phi(t,h))_{t\in\mathbb{R}_+}$ and that $\phi(t,h)$ belongs to Δ for all $t \in \mathbb{R}_+$. The following result on long time convergence appears below as Theorem 5.1 (see Section 2.3 below for a precise definition of the convergence involved).

Theorem 1 (Longtime behavior)**.** *Let (T, γ, φ) satisfy Assumption 2. The semiflow ϕ always converges to an equilibrium: for any initial condition $h \in \Delta$, there exists $g \in \Delta$ such that $F(g) = 0$ and*

$$\operatorname*{limess}_{t\to+\infty} \phi(t,h) = g \quad \text{in } L^\infty. \tag{8}$$

Under Assumption 2, we define the basic reproduction number R_0 as the spectral radius $\rho(T_{1/\gamma})$ of the power compact operator $T_{1/\gamma}$ on L^∞ given by $T_{1/\gamma} f = T(f/\gamma)$, see Lemma 2.8. It comes at no surprise that if $R_0 \leqslant 1$, then the zero function $\mathbb{0}$ is the only equilibrium, so that all epidemic disappear in the long run, see Proposition 4.3. However, if $R_0 > 1$, then there exists a maximal endemic equilibrium, say $g^* \neq \mathbb{0}$, see Theorem 4.7. If furthermore T is irreducible (which is equivalent to the existence and uniqueness, up to a scaling factor, of $v \in L^\infty_+ \backslash \{\mathbb{0}\}$ such that $Tv = R_0\, v$ and that v is positive), the maximal equilibrium is the only endemic equilibrium and g in (8) is equal to g^* as soon as the initial condition h is non-zero, see again Proposition 4.3. Those results appear already in [17] in a slightly less general framework for $R_0 \leqslant 1$ or T irreducible (or quasi-irreducible).

The main result of the paper is the description of all the endemic equilibria and their domain of attraction: for any equilibrium g we give all the initial conditions $h \in \Delta$ such that (8) holds. To do so, we shall rely on the decomposition of the state space in atoms associated to the operator T given by Schwartz [50], see also our previous work [19], which is recalled in Section 2.4, and Section 1.2 for the elementary case where Ω is finite. To summarize, a measurable set $A \in \mathcal{G}$ is invariant if the support of the function $T\mathbb{1}_A$ is a subset of A (up to a set of zero measure); a set is admissible if it belongs to the σ-field generated by the invariant sets; the atoms are the minimal admissible sets with positive measure (that is A is an atom if A is admissible with $\mu(A) > 0$ and if $B \subset A$ is admissible then either $\mu(B) = 0$ or $\mu(B) = \mu(A)$). If T is irreducible, then Ω is an atom. For a measurable set A, its future $\mathcal{F}(A)$ is the smallest invariant set containing A (up to a set of zero measure). The set of atoms (identifying atoms which differ by a set of zero measure) can be endowed with an order relation $\preccurlyeq$: $A \preccurlyeq B$ when $\mathcal{F}(A) \subset \mathcal{F}(B)$ (where the inclusion holds up to a set of zero measure). See Section 2.4 for further details. We say that an atom A is supercritical if the spectral radius of the operator $T_{1/\gamma}$ restricted to A, denoted $R_0(A)$, is strictly larger than 1. The number of supercritical atoms is finite; it is positive if and only if $R_0 > 1$, see [19]. A *supercritical antichain* is a finite set of supercritical atoms which are pairwise unordered with respect to $\preccurlyeq$; we define its future as the future of the union of its atoms. For example, with two supercrtical atoms, say A and B, the supercritical antichains are $\varnothing$, $\{A\}$ and $\{B\}$, with also $\{A, B\}$ if A and B are unordered. Notice that $R_0 > 1$ if and only if there exists a non empty supercritical antichain.

We give a complete characterization of equilibria, see Theorem 4.8 for a more complete statement.

Theorem 2 (Equilibria and supercritical antichains are in bijection). *If (T, γ, φ) satisfy Assumption 2, then the set of supercritical antichains and the set of equilibria are in bijection. Furthermore the support of the equilibrium associated to a supercritical antichain is given by its future.*

The empty supercritical antichain corresponds to the DFE $g = \mathbb{0}$. We deduce from this result that if g and g' are two equilibria, then $g \leqslant g'$ if and only if $\mathrm{supp}(g) \subset \mathrm{supp}(g')$ (up to a set of zero measure) and $\mathrm{supp}(h) = \{h > 0\}$ is the support of the function h, see Corollary 4.2.

To complete this theorem we fully describe basins of attraction. To state the result, we denote by T_A the projection of T on a measurable set A, that is, the operator on L^∞ defined by $T_A f = \mathbb{1}_A T(\mathbb{1}_A f)$ for $f \in L^\infty$. Notice that if T satisfies Assumption 2, so does T_A. When this is the case, we say that g is the maximal equilibrium of A when g is the maximal equilibrium of the SIS model with T replaced by T_A. Intuitively, from an initial condition $h \in \Delta$, the epidemic converges to an equilibrium which depends only on the support of the initial condition; it is the same as the one starting from the "worst possible case" where the whole population in $\mathcal{F}(\mathrm{supp}(h)) \subset \Omega$ is infected.

Theorem 3 (Basins of attraction of equilibria). *The limiting equilibrium g of an epidemic with initial condition $h \in \Delta$ from Theorem 1 is the maximal equilibrium of $\mathcal{F}(\mathrm{supp}(h))$.*

This result appears below as Theorem 5.1. It is a full generalization of Theorem 4.13 in [17], which only covers the irreducible case T where, if $R_0 > 1$, the endemic equilibrium g is unique and all epidemic with initial condition $h \neq \mathbb{0}$ converge to g in large time. Proposition 5.4 states that γ times the epidemic converges uniformly to γg. Thus, when $\mathrm{essinf}\, \gamma > 0$, the epidemic converges uniformly to g, see also Remark 4.4 when furthermore $R_0 < 1$. This uniform convergence is no longer true *a priori* when $\mathrm{essinf}\, \gamma = 0$, see Remark 4.5 and Example 4.6.

1.4. **Model with an external disease reservoir.** We consider an infinite-dimensional SIS model with an external disease reservoir, called SISκ model in [42]; it can be seen as an extension of the SIS model (2). An external disease reservoir is a particular case of environmentally transmitted diseases where the population of pathogens in the environment is assumed to be constant over time, see for example [23, 34] and references therein. See also the example of the West Nile Virus, where birds and mosquitoes form a reservoir that is not infected by humans, see [7]. It also encompasses some SIS models with immigration from [10], see Remark 1.4 below.

Recall $I(t) \geqslant 0$ and $S(t) \geqslant 0$ denote respectively the number of infected individuals and suscpetible individuals at time $t \geqslant 0$. According to [42, Eq. (2.1-2)], the corresponding ordinary differential equations model, including infection from the external disease reservoir, is given by:

$$
\begin{cases} S' = \mu_0 N - \mu_0 S - k\, \dfrac{SI}{N} - \kappa S + \gamma I, \\ I' = k\, \dfrac{SI}{N} + \kappa S - (\mu_0 + \gamma) I, \end{cases} \tag{9}
$$

where $N = I + S$ is the total population, μ_0 is the healthy birth rate and the common death rate of the susceptible and infected populations, κS is the rate of disease transmission from the reservoir, with $\kappa > 0$. Notice the total size population N is constant in time.

In an inhomogeneous setting, with the measured space of types $(\Omega, \mathcal{G}, \mu)$ and $\mu(\Omega) \in (0, +\infty)$, the proportion of infected individuals among the individuals with feature x is given by $u(t,x) = I(t,x)/N(x)$, where $I(t,x)$ denotes the number of infected individuals with feature $x \in \Omega$ at time $t \in \mathbb{R}_+$ and $N(x)$ the total size of the population with feature $x \in \Omega$, assumed constant over time. In the inhomogeneous SISκ model inspired by (9), the function $u = (u(t,x))_{t \in \mathbb{R}_+, x \in \Omega}$ is solution in L^∞ of the ODE:

$$
\begin{cases} u' = F_\kappa(u), \\ u(0) = h, \end{cases} \tag{10}
$$

with initial condition $h \in L^\infty$ and:

$$
F_\kappa(u) = \varphi(u)(Tu + \kappa) - \gamma u, \tag{11}
$$

where φ is a continuous function on $\mathbb{R}$ and $\kappa \in L^\infty_+$. The particular case of SISκ model given by (9) corresponds to $\Omega = \{\omega\}$, μ a Dirac mass at ω, T the multiplication operator by k, γ and κ constant functions, and $\varphi = 1 - \mathrm{Id}$.

This model can be related to SIS model with immigration, see the following remark.

Remark 1.4 (SIS model with immigration). For the homogeneous population, we link the SISκ model (9) with the SIS model with immigration of [10, Eq. (1)]. Assume initially that the total population $N(t)$ is not necessarily constant over time t. Let $A \geqslant 0$ be the immigration rate, $p \in [0,1]$ the proportion of infected individuals among the immigrants, and $d > 0$ the death rate among the population. All those parameters are assumed to be constant over time. We assume that the epidemic induces no death (that is $\alpha = 0$ in [10, Eq. (1)]) and that the incidence rate is the standard mass-action. Then, the SIS model with immigration given in [10, Eq.(11)] reduces to:

$$\begin{cases} I' = pA + k\,\dfrac{(N-I)I}{N} - (d+\gamma)I, \\ N' = A - dN. \end{cases} \tag{12}$$

Since $\lim_{t\to\infty} N(t) = A/d$, and since we are interested in the long time equilibrium, it is natural to assume that N start at its equilibrium, that is $N(0) = A/d$, so that the population size is constant over time. In this case, Equation (12) with $u(t) = I(t)/N(0)$ reduces to:

$$u' = (1-u)(ku + pd) - (\gamma + (1-p)d)u. \tag{13}$$

The same arguments applied to an inhomogeneous population would lead to a similar multi/infinite-dimensional ODE with $u(t)$ replaced by a function $u(t,x)$, with $x \in \Omega$ the set of features, and ku replaced by Tu with T the transmission operator, so that (13) becomes:

$$u' = (1-u)(Tu + pd) - (\gamma + (1-p)d)u.$$

This corresponds to the SISκ model (10)-(11) with $\varphi = 1 - \mathrm{Id}$, $\kappa = pd$ and γ replaced by $\gamma + (1-p)d$. In conclusion the SIS model with immigration and the SIS model with an external disease reservoir lead to the same ODE.

In Proposition 6.1 and Corollary 6.2, we prove that the SISκ model with reservoir of (10) can be analyzed using the classical SIS model (2) by adding a new element $\mathbf{r}$ to the set of features Ω corresponding to the reservoir. In particular we provide a full description of the equilibria and their domain of attraction for the SISκ model.

1.5. **Discussion on the incidence rate.** In this section, we discuss different models for the infection rate, and more precisely for the function φ in (3).

In an homogeneous population, Ross [45] considered the so called law of mass action $\beta SI/N$ (which corresponds to $\varphi = 1 - \mathrm{Id}$ in the SIS model): the incidence rate is proportional to the product of the proportion of susceptible individuals and the proportion of infected individuals. According to Wilson and Worcester [57], it corresponds to the assumption that infected individuals are mixing uniformly with the susceptible ones throughout the population, see also Heesterbeek [26] for an historical review. Some epidemic models introduced in the literature replace the law of mass action by various incidence rates, see in particular the survey McCallum, Barlow and Homeo [38]. Concerning the function φ, Assumptions 1 and 2 below reduce to:

$$\varphi \text{ is locally Lipschitz on } \mathbb{R}, \text{ decreasing on } [0,1] \text{ with } \varphi(1) = 0 \text{ and } \varphi(0) = 1. \tag{14}$$

In the examples below from the literature, the set Ω is a singleton and the transmission operator T is thus a constant, which is assumed to be positive; so the condition $\varphi(0) = 1$ is a normalization convention on the (constant) operator T and could be replaced here by the more relevant condition $\varphi(0) > 0$.

(i) London and Yorke [58] consider the incidence rate $\beta SI(1 - cI)$, that is:

$$\varphi(u) = (1-u)(1-au) \quad \text{with} \quad a > 0$$

for measles epidemic (in New York City and Baltimore from 1928 to 1972) in order to eliminate the systematic differences on data observed between years with many cases and years with relatively few cases; however they do not provide a biologic or physical argument for such modification. Notice that by considering u/a instead of u, one can assume that $a \leqslant 1$, in which case Condition (14) holds.

(ii) We recall that in the SIR model, once infected, the individuals recover with a permanent immunity. Rose et al. [44] incorporate in the SIR model (with constant population $N = S + I + R$) a population-level heterogeneity for the infection susceptibility given by the gamma probability distribution; in [44, Section 4] they consider the incidence rate $\beta I S^\alpha$. Using data from the 2009 H1N1 influenza outbreak, they observe that the higher-order models are more consistent with the data than the case $\alpha = 1$. In our setting, this model would correspond to the following function φ with satisfies Condition (14):

$$\varphi(u) = (1-u)^\alpha \quad \text{with} \quad \alpha \geqslant 1.$$

(iii) Capasso and Serio [13] study a SIR model (with constant population $N = S + I + R$) taking into account saturation and "psychological" effects. They consider the incidence rate $g(I)S$. In our setting, this model would correspond to:

$$\varphi(u) = (1-u)\,\frac{g(u)}{u},$$

where the conditions on g translated into our framework correspond to: the function g is defined on $[0,1]$, nonnegative, bounded, differentiable with g' bounded and such that $g(0) = 0$, $g'(0) = 1$ and $g(u) \leqslant u$ on $\mathbb{R}_+$. Under those assumption, the function φ is Lipschitz on $[0,1]$, with $\varphi(1) = 0$ and $\varphi(0) = 1$. However the monotonicity condition on φ on $[0,1]$, which amounts to $g(u) \geqslant u(1-u)\,g'(u)$ on $[0,1]$, is not satisfied in general.

Notice that Condition (14) is indeed satisfied for the following functions g, where $c > 0$: $u/(1+cu)$ in [13, Section 6], $(1-\exp(-cu))/c$ in [30] on SIR model for Covid19 outbreak, and $\log(1+cu)/c$ in Table 1 of the survey [38] on pathogen transmission models. They respectively correspond to:

$$\varphi(u) = \frac{1-u}{1+cu}, \quad \varphi(u) = (1-u)\,\frac{1-\exp(-cu)}{cu} \quad \text{and} \quad \varphi(u) = (1-u)\,\frac{\log(1+cu)}{cu}\cdot$$

(iv) We recall that in the SIRS model, once infected, the individuals recover with a temporary immunity. To exhibit qualitatively different dynamical behaviors, Liu, Lewin and Iwasa [35] introduced a SIRS model (with constant population $N = S + I + R$) where the incidence rate is given by $IH(I,S)$ for some differentiable function H such that $H(I,0) = 0$ and $\partial_S H > 0$ for all $I > 0$. The latter condition reflects the biologically intuitive requirement that the incidence rate be an increasing function of the number of susceptibles. In our setting, this model would correspond to:

$$\varphi(u) = H(u, 1-u),$$

with φ differentiable and $\varphi(1) = 0$. Notice that φ is decreasing on $[0,1]$ if $\partial_S H > \partial_I H$. The authors consider the particular case $\varphi(u) = (1-u)^\alpha\, u^{\beta-1}$ with $\alpha, \beta > 0$. Condition (14) holds for $\beta = 1$ and $\alpha \geqslant 1$, which is already considered in Point (ii).

2. Notations

2.1. **Ordered set.** Let $(E, \leqslant)$ be a (partially) ordered set. Whenever it exists, the *supremum* of $A \subset E$, denoted by $\sup(A)$, is the least upper bound of A: for all $x \in A$, $x \leqslant \sup(A)$ and if for some $z \in E$ one has $x \leqslant z$ for all $x \in A$, then $\sup(A) \leqslant z$. A collection $(x_i)_{i \in \mathcal{I}}$ of elements of E is an *antichain* if for all distinct $i, j \in \mathcal{I}$, the elements x_i and x_j are not comparable for the order relation.

2.2. **Banach space and Banach lattice.** Let $(X, \|\cdot\|)$ be a complex Banach space not reduced to $\{0\}$. An operator T on X is a bounded linear (and thus continuous) map from X to itself. If $Y \subset X$ is a subspace of X such that $T(Y) \subset Y$, we denote $T|_Y$ the restriction of T to the subspace Y, that is an operator on the Banach space $(Y, \|\cdot\|)$. The operator norm of T is given by:

$$\|T\|_X = \sup\{\|Tx\| : x \in X \text{ s.t. } \|x\| = 1\}, \tag{15}$$

its spectrum by $\mathrm{Sp}(T) = \{\lambda \in \mathbb{C} : T - \lambda \mathrm{Id} \text{ has no bounded inverse}\}$, where Id is the identity operator on X. If $\lambda \in \mathbb{C}$ and $x \in X\backslash\{0\}$ satisfy $Tx = \lambda x$, then the element x is an eigenvector of T and λ, which belongs to $\mathrm{Sp}(T)$, is an eigenvalue of T. The spectral radius of T is defined by (see [47, Theorem 18.9]):

$$\rho(T) = \sup\{|\lambda| : \lambda \in \mathrm{Sp}(T)\} = \lim_{n\to\infty} \|T^n\|_X^{1/n}. \tag{16}$$

By convention, we set $T^0 = \mathrm{Id}$. The spectral radius is commutative in the sense that if T and S are two operators on X, we have:

$$\rho(TS) = \rho(ST). \tag{17}$$

We define the spectral bound of the operator T by:

$$s(T) = \sup\{\mathcal{R}\mathrm{e}(\lambda) : \lambda \in \mathrm{Sp}(T)\}. \tag{18}$$

Let $X^\star$ denote the (continuous or topological) dual Banach space of X, that is the set of all the continuous linear forms on X. For $x \in X$, $x^\star \in X^\star$, let $\langle x^\star, x\rangle$ denote the duality product and the norm of $x^\star$ in $X^\star$ is defined by $\|x^\star\| = \sup\{\langle x^\star, x\rangle : \|x\| = 1\}$. For an operator T, the dual operator $T^\star$ on $X^\star$ is defined by $\langle T^\star x^\star, x\rangle = \langle x^\star, Tx\rangle$ for all $x \in X$, $x^\star \in X^\star$. It is well known that $\|T^\star\|_{X^\star} = \|T\|_X$ and $\mathrm{Sp}(T^\star) = \mathrm{Sp}(T)$.

An ordered real Banach space $(X, \|\cdot\|, \leqslant)$ is a real Banach space $(X, \|\cdot\|)$ with an order relation $\leqslant$. For any $x \in X$, we define $|x| = \sup(\{x, -x\})$ the supremum of x and $-x$ whenever it exists. Following [48, Section 2], the ordered Banach space $(X, \|\cdot\|, \leqslant)$ is a *Banach lattice* if:

(1) For any $x, y, z \in X, \lambda \geqslant 0$ such that $x \leqslant y$, we have $x + z \leqslant y + z$ and $\lambda x \leqslant \lambda y$.
(2) For any $x, y \in X$, there exists a supremum of x and y in X.
(3) For any $x, y \in X$ such that $|x| \leqslant |y|$, we have $\|x\| \leqslant \|y\|$.

Let $(X, \|\cdot\|, \leqslant)$ be a real Banach lattice. We denote $X_+ = \{x \in X : x \geqslant 0\}$ the positive cone of X. Recall it is a closed set. We shall also consider the dual cone $X_+^\star = \{x^\star \in X^\star : \langle x^\star, x\rangle \geqslant 0 \text{ for all } x \in X_+\}$. A linear map T on X is *positive* if $T(X_+) \subset X_+$. According to [2, Theorem 4.3] positive linear maps on Banach lattices are bounded (and thus are operators).

If S and T are two operators on X, we write $T \leqslant S$ if the operator $S - T$ is positive. If the operators T, S and $S - T$ are positive, then we have, see [37, Theorem 4.2]:

$$\rho(T) \leqslant \rho(S). \tag{19}$$

Any real Banach lattice X and any operator T on X admits a natural complex extension. The spectrum of T will be identified as the spectrum of its complex extension and denoted by $\mathrm{Sp}(T)$, furthermore by [1, Lemma 6.22], the spectral radius of the complex extension is also given by $\lim_{n\to\infty} \|T^n\|_X^{1/n}$, with $\|\cdot\|_X$ still defined by (15). Moreover, by [1, Corollary 3.23], if T is positive (seen as an operator on the real Banach lattice X), then T and its complex extension have the same norm.

2.3. **Lebesgue spaces and essential limits.** Let $(\Omega, \mathcal{G}, \mu)$ be a measured space with μ a σ-finite measure. For any $\mathcal{A} \subset \mathcal{G}$, we denote by $\sigma(\mathcal{A})$ the σ-field generated by $\mathcal{A}$. If f, g are two real-valued measurable functions defined on Ω, we write $f \leqslant g$ a.e. (resp. $f = g$ a.e.) when $\mu(\{f > g\}) = 0$ (resp. $\mu(\{f \neq g\}) = 0$), and denote $\mathrm{supp}(f) = \{f \neq 0\}$ the support of f. We say that a real-valued measurable function f is nonnegative when $f \geqslant 0$ a.e., we say that f is positive, denoted $f > 0$ a.e., when $\mu(\{f \leqslant 0\}) = 0$, and we say that f is bounded if there exists $M \geqslant 0$ such that $|f| \leqslant M$ a.e.. If $A, B \subset \Omega$ are measurable sets, we write $A \subset B$ a.e. (resp $A = B$ a.e.) when $\mathbb{1}_A \leqslant \mathbb{1}_B$ a.e. (resp. $\mathbb{1}_A = \mathbb{1}_B$ a.e.). Let $L^0(\Omega, \mathcal{G}, \mu)$, simply denoted L^0, be the set of $[-\infty, +\infty]$-valued measurable functions defined on Ω, where functions which are a.e. equal are identified. The elements $\mathbb{1}$ and $\mathbb{0}$ of L^0 denote the functions which are a.e. equal respectively to 1 and to 0. For the sake of clarity, we will omit to write a.e. in the proofs.

Let $(f_t)_{t\in T}$ be a family of measurable functions defined on Ω taking values in $[-\infty, +\infty]$. We recall that $f^* = \operatorname{esssup}_{t\in T} f_t$ is a measurable function such that $f_t \leqslant f^*$ a.e. for all $t \in T$ and if f is measurable function such that $f_t \leqslant f$ a.e. for all $t \in T$ then $f^* \leqslant f$ a.e. (if T is at most countable, then one can take $\operatorname{esssup}_{t\in T} f_t = \sup_{t\in T} f_t$). We now consider $T = \mathbb{R}_+$. Let $(f_t)_{t\in\mathbb{R}_+}$ be a non-decreasing sequence, in the sense that for all $t \leqslant s$ we have $f_t \leqslant f_s$ a.e., then if $(t_n)_{n\in\mathbb{N}}$ is a sequence converging to $+\infty$, we have that the sequence $(f_{t_n})_{n\in\mathbb{N}}$ converges a.e. towards $f^* = \operatorname{esssup}_{t\in\mathbb{R}_+} f_t$, and thus we shall simply write $f^* = \operatorname{limess}_{t\to+\infty} f_t$. We leave to the reader the definition of essinf and the limit of a non-increasing sequence of measurable functions. For the family $(f_t)_{t\in\mathbb{R}_+}$, we consider $f_t^* = \operatorname{esssup}_{s\geqslant t} f_s$ for all $t \in \mathbb{R}_+$, and get that the sequence $(f_t^*)_{t\in\mathbb{R}^+}$ is non-increasing and write $\operatorname{limesssup}_{t\to\infty} f_t = \operatorname{limess}_{t\to\infty} f_t^*$. We define in a similar way $\operatorname{limessinf}_{t\to\infty} f_t$. Notice that if $g_t = f_t$ a.e. for all $t \in \mathbb{R}_+$, then the essential supremum/infimum limits of $(f_t)_{t\in\mathbb{R}^+}$ and $(g_t)_{t\in\mathbb{R}^+}$ are a.e. equal. Therefore, the essential supremum/infimum limits of sequences is well defined on the space L^0. We say the sequence of functions $(f_t)_{t\in\mathbb{R}_+}$ in L^0 essentially converges if $\operatorname{limesssup}_{t\to\infty} f_t = \operatorname{limessinf}_{t\to\infty} f_t$ in L^0, and write $\operatorname{limess}_{t\to+\infty} f_t$ for this common limit (which is an element of L^0). When considering $T = \mathbb{N}$ instead of $T = \mathbb{R}_+$, the analog of the essential convergence is the a.e. convergence, that is the usual convergence in L^0.

For a measurable function f, we write $\mu(f) = \int f \,\mathrm{d}\mu = \int_\Omega f(x)\,\mu(\mathrm{d}x)$ the integral of f with respect to μ when it is well defined. When f is measurable and a.e. finite and nonnegative, we denote $f\mu$ the measure on $(\Omega, \mathcal{G})$ defined by $f\mu(A) = \mu(\mathbb{1}_A f)$ for any measurable set A. For $p \in [1, +\infty]$, the Lebesgue space $L^p(\Omega, \mathcal{G}, \mu)$ is the set of all real-valued measurable functions $f \in L^0$ defined on Ω whose L^p-norm, $\| f \|_p = \mu(|f|^p)^{1/p}$ if $p < +\infty$ and $\| f \|_\infty = \operatorname{esssup}(|f|)$ if $p = +\infty$, is finite. When there is no ambiguity we shall simply write $L^p(\Omega)$, $L^p(\mu)$ or L^p for $L^p(\Omega, \mathcal{G}, \mu)$. The Banach space L^p endowed with the usual order $f \leqslant g$, that is $\mu(\{f > g\}) = 0$, is a Banach lattice. The positive cone L^p_+ is the subset of L^p of nonnegative functions; it is normal (as the norm $\|\cdot\|$ is monotonic, that is, $0 \leqslant f \leqslant g$ implies $\| f \|_p \leqslant \| g \|_p$, see [15, Proposition 19.1]) and reproducing (that is, $L^p_+ - L^p_+ = L^p$). Since the supports of two functions which are a.e. equal are also a.e. equal, we get that the support of $f \in L^p$ is well defined up to the a.e. equality; it will still be denoted by $\operatorname{supp}(f)$. For $p \in [1, +\infty)$, the dual of L^p is L^q where $1/p + 1/q = 1$, with the duality product $\langle g, f\rangle = \int fg \,\mathrm{d}\mu$ for $f \in L^p$ and $g \in L^q$ (for $p = 1$, we use that the measure μ is σ-finite).

For any $f \in L^\infty$, we denote by M_f the multiplication by f, which can be seen as an operator on L^p for $p \in [1, +\infty]$. For $A \in \mathcal{G}$ a measurable set, we denote:

$$M_A = M_{\mathbb{1}_A}. \tag{20}$$

Let T be an operator on L^p. The projection of T on A, denoted T_A, is the operator defined by:

$$T_A = M_A\, T\, M_A, \tag{21}$$

and, if $\mu(A) > 0$, we denote by $T|_A$ the operator T restricted to $L^p(A)$, where the set A is endowed with the trace of $\mathcal{G}$ on A and the measure $\mu|_A(\cdot) = \mu(A \cap \cdot)$.

We now assume that $\mu(\Omega) > 0$, so that $L^p(\Omega)$ is not reduced to a singleton. When there is no ambiguity on the operator T, we simply write $\rho(A)$ for the spectral radius of T_A (and of $T|_A$ when $\mu(A) > 0$). In particular, we have $\rho(\Omega) = \rho(T)$ and $\rho(A) = 0$ if $\mu(A) = 0$. If the operator T is positive, we also have that:

$$A \subset B \implies \rho(A) \leqslant \rho(B).$$

2.4. **Decomposition of positive operators on L^p, with $p \in (1, +\infty)$.** Recall the measure μ is σ-finite and non-zero. We recall the atomic decomposition from Schwartz [50] of a positive operator T on L^p, see also [19]. A measurable set $A \in \mathcal{G}$ is *T-invariant*, or simply invariant when there is no ambiguity, if $M_{A^c} T M_A = 0$, which, see [19, Eq. (7)], is equivalent to:

$$\langle g, Tf\rangle = 0. \tag{22}$$

for all $f \in L^p_+$ and $g \in L^q_+$ such that $\text{supp}(f) \subset A$ a.e. and $\text{supp}(g) \subset A^c$ a.e.. The operator T is *irreducible* if its only invariant sets are a.e. equal to Ω or $\varnothing$. A measurable set A with positive measure is *irreducible* if the operator $T|_A$ on $L^p(A)$ is irreducible. The *future* of a set $A \subset \Omega$, denoted $\mathcal{F}(A)$, is the smallest invariant set that contains A. If $\mathcal{C}$ is an at most countable collection of subsets of Ω, then we denote by $\mathcal{F}(\mathcal{C})$ the future of the union of the elements of $\mathcal{C}$:

$$\mathcal{F}(\mathcal{C}) = \mathcal{F}\left(\bigcup_{A\in\mathcal{C}} A\right) = \bigcup_{A\in\mathcal{C}} \mathcal{F}(A), \tag{23}$$

where the last equality is [19, Lemma 3.13].

A set is *admissible* if it belongs to the σ-field generated by the invariant sets. An *atom* is a minimal admissible set with a positive measure (that is, A is an atom if A is admissible, $\mu(A) > 0$ and if B is an admissible set such that $B \subset A$ a.e. then a.e. $B = \varnothing$ or $B = A$), and we identify two atoms that are a.e. equal. In particular, if the set A is an atom and B is admissible then we have:

$$\mu(A \cap B) > 0 \implies A \subset B \quad \text{a.e.}.$$

According to [19, Theorem 1], a measurable set is admissible and irreducible with positive measure if and only if it is an atom. Since the atoms have positive measure, we get that the set of atoms (up to the a.e. equality), $\mathfrak{A}$, is at most countable. We shall also consider the (at most countable) set of non-zero atoms:

$$\mathfrak{A}^* = \{A \in \mathfrak{A} : \rho(A) > 0\}.$$

The relation $\preccurlyeq$ on $\mathfrak{A}$, defined by $A \preccurlyeq B$ if $\mathcal{F}(A) \subset \mathcal{F}(B)$ a.e. (or equivalently $A \subset \mathcal{F}(B)$ a.e.), is an order relation. We end this section by noticing that antichains of atoms are characterized by their future.

Lemma 2.1 (Antichains with same future)**.** *Let $\mathcal{C}$ and $\mathcal{C}'$ be two antichains of atoms. Then, we have:*

$$\mathcal{F}(\mathcal{C}) = \mathcal{F}(\mathcal{C}') \iff \mathcal{C} = \mathcal{C}'.$$

Proof. Assume that $\mathcal{F}(\mathcal{C}) = \mathcal{F}(\mathcal{C}')$. Consider an atom $A \in \mathcal{C}$. Since $A \subset \mathcal{F}(\mathcal{C}) = \mathcal{F}(\mathcal{C}')$, there exists $A' \in \mathcal{C}'$ such that we have $\mu(A \cap \mathcal{F}(A')) > 0$, which implies $A \preccurlyeq A'$ as A is an atom. Conversely there exists $B \in \mathcal{C}$ such that $A' \preccurlyeq B$, and by transitivity $A \preccurlyeq B$. Since $\mathcal{C}$ is an antichain, we obtain $A = B$ and thus $A = A'$ is an element of $\mathcal{C}'$. The reverse implication is trivial by (23). ☐

2.5. **Power compact operators on L^p.** A linear map T on a Banach space is *compact* if the image of the unit ball is relatively compact; it is then bounded. An operator T on a Banach space is *power compact* if there exists $k \in \mathbb{N}^*$ such that T^k is compact. We recall some well-known properties of power compact operators, see [21, 31] for instance.

Lemma 2.2 (Spectrum of power compact operators)**.** *Let T be an operator on a Banach space.*

(i) The operator $T^\star$ is power compact if and only if the operator T is power compact.
(ii) If T is power compact, then the set $\text{Sp}(T)$ *is at most countable and has no accumulation points except possibly* 0 *(it is thus totally disconnected).*

It is well known that the spectral radius (and more generally the spectra) is a continuous function on the set of compact operators with respect to the operator norm, see [41, Theorem 11]. We shall however use a weaker result from Anselone [3]. We say a family $\mathcal{V}$ of operators on X is *collectively compact* if $\bigcup_{V\in\mathcal{V}} V(B)$ is relatively compact, where B is the unit ball of X. The following result is a direct consequence of Proposition 4.1 and Theorem 4.16 in [3].

Lemma 2.3 (Collectively compact operators)**.** *Let I be an interval of $\mathbb{R}$ and $(V_t)_{t\in I}$ be a family of collectively compact operators on a Banach space X. If $t \in I$ is such that* $\lim_{s\to t} \| (V_s - V_t)x \| = 0$ *for all $x \in X$, then we have* $\lim_{s\to t} \rho(V_s) = \rho(V_t)$.

We give a result on compact operators in Lebesgue space. Recall that μ is a non-zero σ-finite measure on Ω, and that L^p denote $L^p(\Omega, \mathcal{G}, \mu)$.

Lemma 2.4 (On compactness)**.** *Let $p \in (1, +\infty)$. A positive operator on L^p which is dominated by a compact operator is compact.*

Proof. Notice that L^p has an order continuous norm for $p \in [1, \infty)$, see [2, Definition 4.7], that is, according to [2, Theorem 4.9], if $(f_n)_{n\in\mathbb{N}}$ is a non-decreasing sequence of L^p_+ such that $\sup_{n\in\mathbb{N}} f_n = f \in L^p$, then $\lim_{n\to\infty} \| f_n - f \|_p = 0$. In particular, when $p \in (1, +\infty)$, the dual of L^p, that is isomorphic to L^q with $1/p + 1/q = 1$, also has an order continuous norm. The lemma is then a direct consequence of [2, Theorem 5.20]. □

We recall in our framework some classical results, see [19, Theorem 6.2 and Lemma 6.5].

Theorem 2.5. *Let T be a positive power compact operator on L^p with $p \in (1, +\infty)$.*

- *(i)* ***Krein-Rutman.*** *If $\rho(T)$ is positive then $\rho(T)$ is an eigenvalue of T, and there exists a corresponding nonnegative right eigenfunction denoted v_T.*
- *(ii)* ***de Pagter.*** *If T is irreducible then $\rho(T)$ is positive unless $T = 0$ and $\dim(L^p) = 1$, that is, if A is measurable then either $\mu(A) = 0$ or $\mu(A^c) = 0$.*
- *(iii)* ***Schwartz.*** *We have that for any admissible set A:*

$$\rho(A) = \max_{B\in\mathfrak{A}^*,\, B\subset A} \rho(B). \tag{24}$$

Following [18, Lemma 4.2], we now state a technical result based on Collatz-Wielandt inequality and the Krein-Rutman theorem, giving a bound on the spectral radius given a strict supersolution of the eigenvalue equation.

Lemma 2.6 (Supersolutions and spectral radius)**.** *Let S be a positive operator on L^p with $p \in (1, +\infty)$. If there exists $\lambda > 0$ and a non-null nonnegative function $v \in L^p_+$ such that $Sv \geqslant \lambda v$, then:*

$$\rho(S) \geqslant \rho(S_A) \geqslant \lambda,$$

where $A = \mathrm{supp}(v)$. Furthermore, if S is power compact, then we have:

- *(i) If $Sv = \lambda v$ on A, then $\rho(S_A) = \lambda$.*
- *(ii) If $Sv - \lambda v$ is positive on A, then $\rho(S_A) > \lambda$.*

Proof. We first note that $\rho(S) \geqslant \rho(S_A)$ by (19). Multiplying the inequality $Sv \geqslant \lambda v$ by $\mathbb{1}_A$ yields that $S_A v \geqslant \lambda v$. The fact that $\rho(S_A) \geqslant \lambda$, and thus $\rho(S) \geqslant \lambda$, is then a direct consequence of the Collatz-Wielandt inequality [22, Propositions 2.1 and 2.2].

We now assume that S is power compact. Since $(S_A)^\star = (S^\star)_A$, we shall denote them simply by $S^\star_A$. Since $S_A \leqslant S$, we deduce from Lemma 2.4 that the operator S_A, and thus $S^\star_A$, is power compact. We apply the Krein-Rutman theorem (Theorem 2.5 (i)) to the power compact operator $S^\star_A$, which has a positive spectral radius $\rho(S^\star_A) = \rho(S_A) > 0$: there exists $w \in L^q_+$ a non-null eigenfunction of $S^\star_A$ related to the eigenvalue $\rho(S_A)$. Since $\rho(S_A)w = S^\star_A w = \mathbb{1}_A S^\star(\mathbb{1}_A w)$, we deduce that $\mathrm{supp}(w) \subset A$ and thus $\langle w, v\rangle > 0$.

We now consider the following duality product:

$$\langle w, S_A v - \lambda v\rangle = \langle S^\star_A w, v\rangle - \lambda \langle w, v\rangle = (\rho(S_A) - \lambda) \langle w, v\rangle . \tag{25}$$

Under the hypothesis that $Sv = \lambda v$ on $A = \mathrm{supp}(v)$, the left hand side of (25) is zero, and thus $\rho(S_A) = \lambda$. This gives Point (i).

To prove Point (ii), since $S_A v - \lambda v$ is by hypothesis positive on A, and $w\mathbb{1}_A = w$ is nonnegative and not identically equal to zero, we get $\langle w, S_A v - \lambda v\rangle > 0$. Then use (25) again to get that $\rho(S_A) > \lambda$. □

2.6. **Operators related to the SIS model.** Let $(\Omega, \mathcal{G}, \mu)$ be a measured space with finite non-zero measure μ, that is, $\mu(\Omega) \in (0, +\infty)$. Recall we simply write L^p for $L^p(\Omega, \mathcal{G}, \mu)$. Observe that $L^\infty \subset L^p$ for $p \in (1, +\infty)$ and that the identity map ι from L^∞ to L^p is a bounded injection.

Lemma 2.7 (On compactness)**.** *Let $p \in (1, +\infty)$. Let T be an operator from L^p to L^∞. The linear map ιT is a compact operator on L^p.*

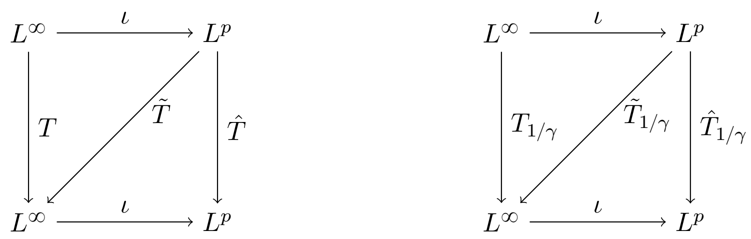


FIGURE 2. Operators related to T in the SIS model (T, γ, φ).

Proof. Since L^p is reflexive, see [21, Corollary IV.8.2], we get by [21, Corollary VI.4.3] that T and ι are weakly compact. By [2, Theorem 5.85], as the space L^∞ is an AM-space, it satisfies the Dunford-Pettis property, thus, by [2, Theorem 5.87], the operator ιT on L^p is then compact. □

Let $p \in (1, +\infty)$ and $\gamma \in L^\infty$ such that $\gamma > 0$ a.e.. Let T be a positive operator on L^∞ such that (5) holds for all $f \in L^\infty$. In particular, as L^∞ is dense in L^p (for the L^p-norm), we can extend T by density into a bounded linear map $\hat{T}$ on L^p. Recall that $M_{1/\gamma}$ denotes the multiplication by $1/\gamma$. Notice that under Assumption 2, see (6), the linear map $TM_{1/\gamma}$, denoted by $T_{1/\gamma}$, can be seen as a bounded linear map on L^∞, and it can also be extended by density into a bounded linear map $\hat{T}_{1/\gamma}$ on L^p, see also Fig. 2.

Lemma 2.8 (Properties of operators related to T)**.** *Let (T, γ, φ) satisfy Assumption 1. Then $\hat{T}$ is an operator on L^p. If Assumption 2 holds, then we have:*

(a) $\hat{T}$ is a compact operator on L^p;
(b) T^2 is a compact operator on L^∞;
(c) $\hat{T}_{1/\gamma}$ is a compact operator on L^p;
(d) $T_{1/\gamma}$ is an operator on L^∞ and $T^2_{1/\gamma}$ is compact;
(e) $\mathrm{Sp}(T) = \mathrm{Sp}(\hat{T})$ and $\mathrm{Sp}(T_{1/\gamma}) = \mathrm{Sp}(\hat{T}_{1/\gamma})$.

Proof. Suppose Assumption 2 holds. We deduce from (6) that there exists a finite constant C such that $\| Tf \|_\infty \leqslant C \| f \|_p$ for all $f \in L^\infty$. By density, we can extend T into an operator $\tilde{T}$ from L^p to L^∞. This gives that $\hat{T} = \iota \tilde{T}$ and Point (a) on the compactness property of $\hat{T}$ is a consequence of Lemma 2.7. As $T = \tilde{T} \iota$ and thus $T^2 = \tilde{T} \hat{T} \iota$, we also get that T^2 is compact, that is Point (b). Using (6), we can also extend $TM_{1/\gamma}$ into an operator $\tilde{T}_{1/\gamma}$ from L^p to L^∞. Arguing as above gives Points (c) and (d).

We now prove Point (e). Two complex Banach spaces $(E, \|\cdot\|)$ and $(E', \|\cdot\|')$ are compatible if $(E'', \|\cdot\| + \|\cdot\|')$, with $E'' = E \cap E'$, is a Banach space, and E'' is dense in E and in E'. Given two compatible spaces E and E', two operators A on E and A' on E' are said to be consistent if $A(E'') \subset E''$, $A'(E'') \subset E''$ and $Ax = A'x$ for all $x \in E''$. If furthermore A and A' are compact, then [14, Theorem 4.2.15] gives that $\mathrm{Sp}(A) = \mathrm{Sp}(A')$. The proof therein relies on the spectrum to be at most countable and with no accumulation points except possibly 0 and that the spectral projections have finite rank, see Theorems 5 and 6 p.579 in [21]. Since this also holds for power compact operators, the results can be extended to A and A' being power compact operators.

As μ is a finite measure, the spaces L^p and L^∞ are pairwise compatible. Notice also the operators T and $\hat{T}$, as well as $T_{1/\gamma}$ and $\hat{T}_{1/\gamma}$, are consistent. Then the equalities follow. □

3. EQUILIBRIA AND RESTRICTION

We consider the SIS model (2)-(3) on $L^\infty = L^\infty(\Omega, \mathcal{G}, \mu)$ with parameter (T, γ, φ) such that Assumption 1 holds. In particular the measure μ is finite and non-zero. We consider the following subset of L^∞:

$$\Delta = \{f \in L^\infty : \mathbb{0} \leqslant f \leqslant \mathbb{1}\}.$$

Recall that g is an equilibrium if g belongs to Δ and solves (4), that is:

$$F(g) = 0.$$

In particular, the function $\mathbb{0}$ is an equilibrium. We say that $g^* \in L^\infty$ is the *maximal equilibrium* if g^* is an equilibrium and all other equilibrium $g \in L^\infty$ are such that $g \leqslant g^*$.

The existence result of the semi-flow and the maximal equilibrium follows [17, Propositions 2.7 and 2.15] with slightly more general hypothesis on T and is obtained similarly, see a proof in Section 8 for completeness. We shall refer to this section for notations and definitions/properties of the semi-flow.

Proposition 3.1 (Existence of a global solution and of the maximal equilibrium). *Let (T, γ, φ) be parameters of the SIS model satisfying Assumption 1. The following properties hold.*

(i) Equation (2) in L^∞ with initial condition $h \in \Delta$ has a unique global solution given by the semi-flow $\phi(\cdot, h) = (\phi(t, h))_{t\in\mathbb{R}_+}$. The semi-flow belongs to $\mathcal{C}^1(\mathbb{R}_+)$.
(ii) For all $t \geqslant 0$ and $h \in \Delta$, we have $\phi(t, h) \in \Delta$.
(iii) The sequence $(\phi(t, \mathbb{1}))_{t\in\mathbb{R}_+}$ is non-increasing and converges essentially to a limit, g^*, which is the maximal equilibrium:

$$\operatorname*{limess}_{t\to+\infty} \phi(t, \mathbb{1}) = g^*.$$

The convergence in Point (iii) is not uniform in general, see Remarks 4.4 and 4.5 and Example 4.6.

The proof of the monotonicity of the maximal equilibrium in the parameters is given in Section 8 for consistency of the arguments.

Lemma 3.2 (Monotonicity of the maximal equilibrium). *For $i = 1, 2$, let $(T_i, \gamma_i, \varphi_i)$ be parameters of the SIS model satisfying Assumption 1 and denote g_i^* the corresponding maximal equilibrium. If $T_1 \geqslant T_2$, $\varphi_1 \geqslant \varphi_2$ and $\gamma_1 \leqslant \gamma_2$, then we have $g_1^* \geqslant g_2^*$.*

Let A be a measurable set. Since (T, γ, φ) satisfy Assumption 1, so does (T_A, γ, φ), where $T_A = T_A = M_A T M_A$ is the projection of T on A. We shall now focus on this restricted (T_A, γ, φ)-SIS model. We set for $u \in L^\infty$:

$$F_A(u) = \varphi(u) T_A(u) - \gamma u, \tag{26}$$

and call $g \in \Delta$ an equilibrium of A if $F_A(g) = 0$. In this case, notice that $\operatorname{supp}(g) \subset A$ a.e.. We also denote ϕ_A the corresponding semi-flow and g_A^* the corresponding maximal equilibrium of A given by Proposition 3.1.

Lemma 3.3 (Maximal equilibria). *Let (T, γ, φ) satisfy Assumption 1. Let $A \subset B$ a.e. be measurable sets. We have $g_A^* \leqslant g_B^*$.*

Proof. Apply Lemma 3.2 with $T_1 = T_B \geqslant T_A = T_2$, $\varphi_1 = \varphi_2 = \varphi$ and $\gamma_1 = \gamma_2 = \gamma$. □

We now provide results on equilibria and semi-flows associated to T and T_A.

Lemma 3.4 (Equilibrium and restriction). *Let (T, γ, φ) satisfy Assumption 1. Let A be a measurable set and $g \in \Delta$.*

(i) If g is an equilibrium and $\operatorname{supp}(g) \subset A$ a.e., then g is an equilibrium of A.
(ii) If A is invariant and g is an equilibrium of A, then g is an equilibrium (and $\operatorname{supp}(g) \subset A$ a.e.).
(iii) If A is invariant and g is an equilibrium, then $\mathbb{1}_{A^c} g$ is an equilibrium of A^c and of $A^c \cap \operatorname{supp}(g)$.

Proof. If $\operatorname{supp}(g) \subset A$ and g is an equilibrium, then we get that g is an equilibrium of A as:

$$F_A(g) = \varphi(g)\mathbb{1}_A T(\mathbb{1}_A g) - \gamma g = \mathbb{1}_A\big(\varphi(g)T(g) - \gamma g\big) = 0.$$

Let A be an invariant set and $h \in \Delta$. Since $\mathbb{1}_{A^c} T(\mathbb{1}_A h) = 0$, we deduce that:

$$F_{A^c}(\mathbb{1}_{A^c} h) = \mathbb{1}_{A^c} F(h). \tag{27}$$

If furthermore $\operatorname{supp}(h) \subset A$, then we have:

$$F_A(h) = F(h) - \varphi(h)\mathbb{1}_{A^c} T(\mathbb{1}_A h) = F(h). \tag{28}$$

If g is an equilibrium of A, then we get $\text{supp}(g) \subset A$ and, since A is invariant, we deduce from (28) that g is an equilibrium. If g is an equilibrium, we deduce from (27) that $\mathbb{1}_{A^c} g$ is an equilibrium of A^c. Then use (i) with T replaced by T_{A^c} and A by $\text{supp}(g)$ to deduce that $\mathbb{1}_{A^c} g$ is also an equilibrium of $A^c \cap \text{supp}(g)$. □

Lemma 3.5 (Semi-flow and restriction). *Let (T, γ, φ) satisfy Assumption 1. Let A be a measurable set and $h \in \Delta$. The following properties hold:*

(i) $\text{limess}_{t\to+\infty}\, \phi_A(t, \mathbb{1}_A) = g_A^*$.
(ii) If A is invariant, then we have $\mathbb{1}_{A^c}\, \phi(\cdot, h) = \phi_{A^c}(\cdot, \mathbb{1}_{A^c} h)$.
(iii) If A is invariant and $\text{supp}(h) \subset A$ *a.e., then we have $\phi(\cdot, h) = \phi_A(\cdot, h)$.*

Proof. As g_A^* is an equilibrium of A, we get $\text{supp}(g_A^*) \subset A$ and thus we have $g_A^* \leqslant \mathbb{1}_A \leqslant \mathbb{1}$. By the monotonicity of the semi-flow, see Lemma 8.5 (i), we have $g_A^* = \phi_A(t, g_A^*) \leqslant \phi_A(t, \mathbb{1}_A) \leqslant \phi_A(t, \mathbb{1})$ for all $t \in \mathbb{R}_+$. Then Proposition 3.1 (iii) gives Point (i).

Let A be invariant. For simplicity, we write ϕ instead of $\phi(t, h)$ and ϕ' for its derivative. By definition of the semi-flow ϕ, we deduce from (27) that:

$$\mathbb{1}_{A^c}\phi' = \mathbb{1}_{A^c} F(\phi) = F_{A^c}(\mathbb{1}_{A^c}\phi).$$

The map $t \mapsto \mathbb{1}_{A^c}\phi(t, h)$ is thus a solution of Equation (2), with F replaced by F_{A^c} and initial condition $\mathbb{1}_{A^c} h$. By the uniqueness of the semi-flow, we get $\mathbb{1}_{A^c}\phi(\cdot, h) = \phi_{A^c}(\cdot, \mathbb{1}_{A^c} h)$, that is Point (ii).

We now assume that $\text{supp}(h) \subset A$. By Point (ii), we have $\text{supp}(\phi) \subset A$. We deduce from (28) that:

$$\phi' = F(\phi) = F_A(\phi).$$

Then, the same argument as above yields Point (iii). □

4. Characterization of equilibria

In this section, we assume that Assumption 2 holds for the SIS model (T, γ, φ). In particular the map φ restricted to $[0, 1]$ is a decreasing bijection onto $[0, 1]$. Recall the operators related to T defined in Section 2.6 and their properties.

4.1. Equilibria, supports and spectral radius.

For any $g \in \Delta$, let L_g denote the compact operator on L^p defined by:

$$L_g = M_{\varphi(g)}\, \hat{T}_{1/\gamma}. \tag{29}$$

This operator is associated to the linearization $M_{\varphi(g)}T - \gamma$ of the dynamics near g in the same way as $\hat{T}_{1/\gamma}$ is associated to the linearization $T - \gamma$ of the dynamics near 0 (as $\varphi(0) = 1$). Notice that when (T, γ, φ) satisfy Assumption 2, then $(M_{\varphi(g)}T, \gamma, \varphi)$ also does. It is immediate to check that for $g \in \Delta$:

$$g \text{ is an equilibrium} \iff L_g(\gamma g) = \gamma g. \tag{30}$$

Lemma 4.1 (Equilibria as nonnegative eigenfunctions). *Let (T, γ, φ) satisfy Assumption 2. Let $g \in \Delta$ be an equilibrium. We have the following properties.*

(i) $\varphi(g) > 0$ a.e..
(ii) The operators $\hat{T}$, $\hat{T}_{1/\gamma}$ and L_g have the same invariant sets, irreducible sets, atoms and non-zero atoms.
(iii) The set $\text{supp}(g)$ *is invariant.*

If $g \neq \mathbb{0}$, then the following additional properties hold. Let $h \in \Delta$ be an equilibrium. Set $A = \text{supp}(g)$.

(iv) $\rho(L_g) \geqslant \rho((L_g)_A) = 1$.
(v) If $h \leqslant g$, then either $\rho((L_h)_A) = 1$ and $h = g$, or $\rho((L_h)_A) > 1$ and $\text{supp}(h) \neq \text{supp}(g)$ *a.e..*
(vi) $h \leqslant g \iff$ $\text{supp}(h) \subset \text{supp}(g)$ *a.e..*

As a consequence of Point (vi) we directly get the following corollary.

Corollary 4.2 (Equilibria and their support). *Let (T, γ, φ) that satisfy Assumption 2. Two equilibria with the same support are equal.*

Proof of Lemma 4.1. Since $\varphi(g)T(g) = \gamma g$, $\varphi(g)$ does not vanish on $\text{supp}(g)$. On the complement set, $g = 0$ so $\varphi(g) = 1$ does not vanish either. Since φ is nonnegative, we get Point (i).

Point (ii) is a direct consequence of the characterization of invariant sets given by (22) as $1/\gamma$ and $\varphi(g)$ are positive by Point (i) and Thereom 2.5 (ii) on non-zero atoms. Point (iii) is a direct consequence of (30), [19, Lemma 3.6] (which state that the support of the eigenfunction γg is L_g-invariant) and Point (ii).

Point (iv) follows directly from Lemma 2.6 with $S = L_g$, $v = \gamma g$ and $\lambda = 1$.

The proof of Point (v) follows similar lines. Let $h \leqslant g$ be two equilibria with $h \neq \mathbb{0}$. The two eigenvalue equations written for h and g yield $(L_g - L_h)(\gamma g) + L_h(\gamma(g-h)) = \gamma(g-h)$, which we rewrite as:

$$L_h(\gamma(g-h)) = \gamma(g-h) + (\varphi(h) - \varphi(g))Tg.$$

On the right hand side, $\varphi(h) - \varphi(g)$ and Tg are nonnegative, and by strict monotonicity of φ they are both positive on $B = \text{supp}(g-h) \subset \text{supp}(g)$. If B is empty, then $g = h$ and we are back to Point (iv). If not, we apply Lemma 2.6 (ii) to $S = L_h$, $\lambda = 1$ and $v = \gamma(g-h) \neq \mathbb{0}$ which is non-negative: $\rho((L_h)_B) > 1$. If B was a subset of $\text{supp}(h)$ this would imply $\rho((L_h)_{\text{supp}(h)}) > 1$, a contradiction with Point (iv). So B is not a subset of $\text{supp}(h)$, or in other words $\text{supp}(h) \neq \text{supp}(g)$.

Finally let us prove Point (vi). Clearly $h \leqslant g$ implies that $\text{supp}(h) \subset \text{supp}(g)$. To prove the reverse implication, let us assume that $\text{supp}(h) \subset \text{supp}(g)$. Since $F(\max(g,h)) \geqslant 0$, by Lemmas 8.5 (ii) and 8.6, the semi-flow starting from $\max(g,h)$ is non-decreasing and converges to an equilibrium $\tilde{g}$, which therefore satisfies $\max(g,h) \leqslant \tilde{g}$. Since $\text{supp}(\max(g,h)) = \text{supp}(g)$ is invariant by Point (iii), we deduce from Lemma 3.5 (iii) that $\text{supp}(\tilde{g}) \subset \text{supp}(g)$ and thus $\text{supp}(\tilde{g}) = \text{supp}(g)$. Since $g \leqslant \tilde{g}$, by the previous point, the functions g and $\tilde{g}$ must be equal, so $g = \max(g,h)$, or in other words $h \leqslant g$. □

4.2. **Maximum equilibria and critical vaccination.** Recall the notations of Section 2.6. Let A be a measurable set. Notice the linear map $(\hat{T}_{1/\gamma})_A = \widehat{(T_A)}_{1/\gamma}$ is an operator on L^p. Following [17], we then define the *basic reproduction number* of A as the spectral radius of this operator $R_0(A) = \rho((\hat{T}_{1/\gamma})_A)$, and simply write R_0 for $R_0(\Omega)$. Notice that, by (19), the map $A \mapsto R_0(A)$ is non-decreasing, that is, for any A, B measurable sets with $A \subset B$ a.e., we have $R_0(A) \leqslant R_0(B)$.

The following result generalizes [17, Theorems 4.7 and 4.13], and is proved similarly, see Section 9 for details.

Proposition 4.3. *Let (T, γ, φ) satisfy Assumption 2. Then we have the following properties.*

(i) If $R_0 \leqslant 1$, then we have $g^* = \mathbb{0}$, and for all $h \in \Delta$:

$$\operatorname*{limess}_{t\to+\infty} \phi(t,h) = \mathbb{0}.$$

(ii) If $R_0 > 1$, then the maximal equilibrium g^* is non-null, (that is, $\mu(\text{supp}(g^*)) > 0$).

(iii) If $R_0 > 1$ and T is quasi-irreducible, that is, $T = T_A$ with A an irreducible set, then we have $\text{supp}(g^*) = A$ a.e. and g^* is the unique non-null equilibrium.

(iv) If $R_0 > 1$, T is quasi-irreducible, that is, $T = T_A$ with A an irreducible set, and $h \in \Delta$, then we have $\text{limess}_{t\to+\infty} \phi(t,h) = \mathbb{0}$ if $\text{supp}(h) \cap A = \varnothing$ a.e. and:

$$\operatorname*{limess}_{t\to+\infty} \phi(t,h) = g^* \quad \text{if} \quad \mu(\text{supp}(h) \cap A) > 0.$$

In the next remarks and examples, we explore the uniformity of the convergence in Point (i) and Proposition 3.1 (iii).

Remark 4.4 (Exponential rate of convergence to $\mathbb{0}$ when $R_0 < 1$ and $\text{essinf}\,\gamma > 0$). Assume (T, γ, φ) satisfies Assumption 2, $R_0 < 1$ and $\text{essinf}\,\gamma > 0$. By Lemma 9.2 we also have $s(T-\gamma) < 0$. Then, mimicking the proof of [17, Theorem 4.6] and using that $1-\varphi \geqslant 0$, we get that for all $c \in (0, -s(T-\gamma))$, there exists $\theta \in \mathbb{R}_+$ such that, for all $h \in \Delta$, $t \geqslant 0$, we have:

$$\| \phi(t,h) \|_\infty \leqslant \theta \, \| h \|_\infty \, \mathrm{e}^{-ct}.$$

Remark 4.5 (Non uniform convergence when $R_0 \leqslant 1$ and $\operatorname{essinf} \gamma = 0$). We assume that $R_0 \leqslant 1$ and $\operatorname{essinf} \gamma = 0$. Consider the function v defined by $v(t) = \exp(-t\gamma) \in \Delta$ for $t \geqslant 0$. As $v'(t) - F(v(t)) = -\varphi(v(t))\, Tv(t) \leqslant 0$, we deduce from Lemma 8.2 and Proposition 7.2 that $\phi(t, \mathbb{1}) \geqslant v(t)$ for all $t \geqslant 0$. We obtain $1 \geqslant \| \phi(t, \mathbb{1}) \|_\infty \geqslant \| v(t) \|_\infty = 1$ as $\operatorname{essinf} \gamma = 0$. Thus the semi-flow $(\phi(t, \mathbb{1}))_{t \in \mathbb{R}_+}$ does not converge to $g^* = \mathbb{0}$ in L^∞.

Notice the same conclusion holds (with the same arguments) if $R_0 \leqslant 1$ is replaced by the more general condition $\operatorname{esssup} g^* < 1$.

Example 4.6 (A uniform convergence when $\operatorname{essinf} \gamma = 0$). If $\operatorname{esssup} g^* = 1$ and $\operatorname{essinf} \gamma = 0$ it is possible for the semi-flow $(\phi(t, \mathbb{1}))_{t \in \mathbb{R}_+}$ to converge to g^* in L^∞. Consider the particular case: $\Omega = (0, 1]$ with $\mu = \nu + \delta_1$, where ν is the Lebesgue measure and δ_1 the Dirac mass at 1; $Tf = f(1)\mathbb{1}$ for all $f \in L^p$; $\gamma(x) = x/2$ and $\varphi(r) = 1 - r$. In this case, $\{1\}$ is the only atom, and $R_0(\{1\}) = 2$. We get $g^*(x) = 1/(1 + x)$ and thus $\operatorname{esssup} g^* = 1$. (Notice that $g^*(0+) = \operatorname{esssup} g^*$ and $\gamma(0+) = \operatorname{essinf} \gamma$.) Elementary calculus give that, for $t \geqslant 0$, $\phi(t, \mathbb{1})(1) = 1/(2 - \mathrm{e}^{-t/2})$ and for $x \in (0, 1)$:

$$\phi(t, \mathbb{1})(x) = \frac{2 + (x - 1)\mathrm{e}^{-(x+1)t/2}}{x + 1}\, \phi(t, \mathbb{1})(1),$$

so that $\| \phi(t, \mathbb{1}) - g^* \|_\infty \leqslant \mathrm{e}^{-t/2}$. So the semi-flow $(\phi(t, \mathbb{1}))_{t \in \mathbb{R}_+}$ converges to g^* in L^∞.

We now focus on critical vaccination. Let φ_0 defined by $\varphi_0(r) = 1 - r$ and T_k the kernel operator form Remark 1.3 with k and γ satisfying (7) so that $(\mathrm{T}_k, \gamma, \varphi_0)$ satisfies Assumption 2. Let $\eta \in \Delta$ seen as a perfect vaccination strategy: the SIS model $(\mathrm{T}_k M_\eta, \gamma, \varphi_0)$ (which indeed satisfies Assumption 2) corresponds to the initial SIS model, where for $x \in \Omega$, a proportion $1 - \eta(x)$ of the population is vaccinated and thus does not spread the disease, see [16] and references therein. In this setting vaccinating the population amounts to replace the measure μ by $\eta\mu$.

Motivated by this example, we shall consider the effective reproduction number defined by:

$$R_e(\eta) = \rho(\hat{T}_{1/\gamma} M_\eta),$$

for $\eta \in \Delta$ (notice that $(TM_\eta, \gamma, \varphi)$ satisfies Assumption 2 and $\hat{T}_{1/\gamma} M_\eta = (\widehat{TM_\eta})_{1/\gamma}$). Following [16], we shall be interested in critical vaccination η for which $R_e(\eta) = 1$. It is observed in [18] that for the SIS model $(\mathrm{T}_k, \gamma, \varphi_0)$, the vaccination strategy $\eta = \varphi_0(g^*)$ is critical. We generalize this result (with a shorter proof based on the fact that $R_e(\varphi(g)) = \rho(L_g)$, see (29)) for more general operators T and functions φ.

Theorem 4.7 (Equilibria and critical vaccination)**.** *Let (T, γ, φ) that satisfy Assumption 2. Let $h \in \Delta$ be an equilibrium. Then we have $h = g^* \iff R_e(\varphi(h)) \leqslant 1$. If furthermore $R_0 > 1$, then we have:*

$$h = g^* \iff R_e(\varphi(h)) = 1.$$

Proof. First, remark that $\rho(L_g) = R_e(\varphi(g))$, where L_g is defined by (29).

If $R_0 \leqslant 1$, then by Proposition 4.3, we have $g^* = \mathbb{0}$. As $R_e(\varphi(0)) = R_0 \leqslant 1$ by Assumption 2, we directly get the equivalence $h = g^* \iff R_e(\varphi(h)) \leqslant 1$.

If $R_0 > 1$, then we have $g^* \neq \mathbb{0}$ by Proposition 4.3. According to Lemma 4.1 (iv), if $h \neq \mathbb{0}$, we have $R_e(\varphi(h)) = \rho(L_h) \geqslant 1$, and by (19) and (v) that if $h \neq g^*$ then $\rho(L_h) \geqslant \rho\big((L_h)_A\big) > 1$ with $A = \operatorname{supp}(g^*)$.

To complete the proof, that is, $R_e(\varphi(g^*)) = 1$, we shall assume that $\rho(L_h) > 1$ and show that $h \neq g^*$. Informally the idea is to follow the unstable direction at the equilibrium h to construct a trajectory leading to another equilibrium. Since L_h is a positive compact operator, thanks to the Krein-Rutman theorem (Theorem 2.5 (i)), we can consider an eigenvector $u \in L^p_+ \backslash \{\mathbb{0}\}$ of L_h related to $\rho(L_h)$. Since the set $A = \operatorname{supp}(h)$ is invariant by Lemma 4.1 (iii), we have:

$$\rho(L_h)u = L_h u = (L_h)_A u + L_h(\mathbb{1}_{A^c} u). \tag{31}$$

If $u\mathbb{1}_{A^c}$ was equal to $\mathbb{0}$, $\rho(L_h) > 1$ would be an eigenvalue of $(L_h)_A$, contradicting Lemma 4.1 (iv). As A is invariant and $u\mathbb{1}_{A^c} \neq \mathbb{0}$, multiplying (31) by $\mathbb{1}_{A^c}$ gives $\rho(L_h)u\mathbb{1}_{A^c} = (L_h)_{A^c}(u\mathbb{1}_{A^c})$, showing that $u\mathbb{1}_{A^c}$ is an eigenvector of $(L_h)_{A^c}$, so $\rho((L_h)_{A^c}) = \rho(L_h) > 1$.

Since $(L_h)_{A^c} = M_{\varphi(h)}(\hat{T}_{1/\gamma})_{A^c} = \widehat{(T_{A^c})}_{1/\gamma}$ (as $h = 0$ on A^c and $\varphi(0) = 1$), we may apply Proposition 9.3 with $T = T_{A^c}$: there exists a $\lambda > 0$ and $w \in L^\infty_+\backslash\{\mathbb{0}\}$ such that $T_{A^c}w - \gamma w = \lambda w$. Without loss of generality, we can assume that $\| w \|_\infty$ is small enough to ensure that $w \in \Delta$ and, as φ is continuous with $\varphi(0) = 1$, that $\varphi(\| w \|_\infty) \geqslant 1 - \delta$ with $\delta > 0$ small enough so that $\delta(\lambda + \| \gamma \|_\infty) \leqslant \lambda$. Note that $\text{supp}(w) \subset A^c$. Since $h+w = h\mathbb{1}_A + w\mathbb{1}_{A^c}$, and $A = \text{supp}(h)$ is invariant, we get $\varphi(h+w)Th = \varphi(h)Th$. Since h is an equilibrium, we obtain that:

$$\begin{aligned} F(h + w) = \varphi(h + w)T(h + w) - \gamma(h + w) &= \varphi(h + w)Tw - \gamma w \\ &\geqslant (1 - \delta)T_{A^c}w - \gamma w \\ &\geqslant (\lambda - \delta(\lambda + \| \gamma \|_\infty))\, w \\ &\geqslant 0. \end{aligned}$$

By Lemmas 8.5 (ii) and 8.6, this implies that the trajectory starting from $h+w$ converges monotonously to an equilibrium g. Since $h \leqslant h + w \leqslant g \leqslant g^*$ and $w \neq \mathbb{0}$, we get that $h \neq g^*$ as claimed. □

4.3. **Equilibria and antichains of atoms.** We now focus on the characterization of equilibria. We recall from (23) that the the future of an antichain $\mathcal{C}$ of atoms (which is at most countable) is given by $\mathcal{F}(\mathcal{C}) = \mathcal{F}\left(\bigcup_{A\in\mathcal{C}} A\right) = \bigcup_{A\in\mathcal{C}} \mathcal{F}(A)$. The set of *supercritical* atoms:

$$\mathfrak{A}^{\text{sup}} = \{A \in \mathfrak{A} :\ R_0(A) > 1\}$$

is finite by [19, Lemma 6.5]. We say an antichain $\mathcal{C}$ of atoms is supercritical if all its elements are supercritical atoms, that is, $\mathcal{C} \subset \mathfrak{A}^{\text{sup}}$. We denote by $\mathfrak{S}$ the (finite) set of supercritical antichains. For a set A, let $\mathcal{C}_A$ denote the (possibly empty) supercritical antichain given by the maximal elements of $\{B \in \mathfrak{A}^{\text{sup}} :\ B \subset A \text{ a.e.}\}$. Notice that when A is admissible, we get by (24) that $\mathcal{C}_A$ is non-empty if and only if $R_0(A) > 1$. For $h \in \Delta$, we simply write $\mathcal{C}_h$ for $\mathcal{C}_{\text{supp}(h)}$.

The following theorem generalizes the uniqueness result of Proposition 4.3 when the operator T is not necessarily quasi-irreducible. Recall that, by Lemma 4.1 (vi), equilibria are characterized by their support.

Theorem 4.8 (Equilibria and supercritical antichains are in bijection)**.** *Let (T, γ, φ) satisfy Assumption 2. The set of the equilibria and the set of supercritical antichains are in bijection through the equivalent relations:*

$$\text{supp}(g) = \mathcal{F}(\mathcal{C}) \iff \mathcal{C}_g = \mathcal{C}, \tag{32}$$

where $g \in \Delta$ is an equilibrium and $\mathcal{C} \in \mathfrak{S}$ a supercritical antichain. Furthermore, if $g \neq \mathbb{0}$, then the equilibrium g is the maximal equilibrium of $\mathcal{F}(\mathcal{C}_g)$.

We divide the proof in two lemmas.

Lemma 4.9 (Support of an equilibrium and related supercritical antichain)**.** *If $g \in \Delta$ is an equilibrium, then we have* $\text{supp}(g) = \mathcal{F}(\mathcal{C}_g)$ *a.e.. In particular if g and h are two equilibria, we get:*

$$\mathcal{C}_g = \mathcal{C}_h \iff g = h.$$

Proof. As the set $\text{supp}(g)$ is invariant by Lemma 4.1 (iii) and as every element of $\mathcal{C}_g$ is included in $\text{supp}(g)$, we have $\mathcal{F}(\mathcal{C}_g) \subset \text{supp}(g)$.

By construction of $\mathcal{C}_g$, every atom $B \subset \text{supp}(g)$ with $R_0(B) > 1$ is included in $\mathcal{F}(\mathcal{C}_g)$. This implies by (24) that, with $A = \text{supp}(g) \cap \mathcal{F}(\mathcal{C}_g)^c$ an invariant (by Lemma 4.1 (iii)) and thus admissible set:

$$R_0(A) = \max_{B\subset A,\, B\in\mathfrak{A}} R_0(B) \leqslant 1.$$

Then Lemma 3.4 (iii) gives that $\mathbb{1}_{\mathcal{F}(\mathcal{C}_g)^c}g$ is an equilibrium of $\mathcal{F}(\mathcal{C}_g)^c$ and of A. Then Proposition 4.3 (i) (with the SIS model (T_A, γ, φ)) implies that $\mathbb{1}_{\mathcal{F}(\mathcal{C}_g)^c}\, g = \mathbb{0}$, that is $\text{supp}(g) \subset \mathcal{F}(\mathcal{C}_g)$. Thus, we get $\text{supp}(g) = \mathcal{F}(\mathcal{C}_g)$. The second part of the lemma is then a direct consequence of Corollary 4.2 and Lemma 2.1. □

Lemma 4.10. *For any supercritical antichain $\mathcal{C}$, there exists an equilibrium $g \in \Delta$ such that $\mathcal{C} = \mathcal{C}_g$. If $\mathcal{C}$ is non empty, then $g \neq \mathbb{0}$ is the maximal equilibrium of $\mathcal{F}(\mathcal{C})$.*

Proof. If $\mathcal{C}$ is empty, then taking the equilibrium $g = \mathbb{0}$, we get $\mathcal{C} = \mathcal{C}_g$. We assume now that $\mathcal{C}$ is not empty. Let g be the maximal equilibrium on $\mathcal{F}(\mathcal{C})$. It is also an equilibrium by Lemma 3.4 (ii) and $\text{supp}(g) \subset \mathcal{F}(\mathcal{C})$. For any $A \in \mathcal{C}$, we have $g^*_A \leqslant g$ by Lemma 3.3, and g^*_A is positive on A by Proposition 4.3 (iii) since A is a supercritical atom and thus an irreducible set with $R_0(A) > 1$. This implies that $A \subset \text{supp}(g)$ and thus $\mathcal{F}(A) \subset \text{supp}(g)$ as $\text{supp}(g)$ is invariant. Then use (23) to get $\mathcal{F}(\mathcal{C}) \subset \text{supp}(g)$, so that $\mathcal{F}(\mathcal{C}) = \text{supp}(g)$, and thus $\mathcal{F}(\mathcal{C}) = \mathcal{F}(\mathcal{C}_g)$ by Lemma 4.9. The two antichains $\mathcal{C}$ and $\mathcal{C}_g$ have the same future and are thus equal by Lemma 2.1. The proof is then complete. □

Let $\mathcal{E} \subset \Delta$ denote the set of equilibria of the SIS model (T, γ, φ).

Proof of Theorem 4.8. The map $g \mapsto \mathcal{C}_g$ from $\mathcal{E}$ to $\mathfrak{S}$ is one-to-one by Lemma 4.9 and onto by Lemma 4.10. The equivalence given by (32) is a direct consequence of Lemmas 2.1 and 4.9. Use the last part of Lemma 4.10 to get the last part of the theorem. □

4.4. **Monatomicity and order relation via equilibria.** Let (T, γ, φ) that satisfy Assumption 2. Consider the SIS model $(T, \lambda\gamma, \varphi)$ with recovery rate γ multiplied by a real parameter $\lambda > 0$. The reproduction number of a measurable set A for this model is $\rho((\hat{T}_{1/\lambda\gamma})_A) = \rho((\hat{T}_{1/\gamma})_A\, M_{1/\lambda}) = R_0(A)/\lambda$. We deduce from Theorem 4.8 that the number of equilibria of the SIS model $(T, \lambda\gamma, \varphi)$ is decreasing with λ.

We say that the operator T on L^p for $p \in (1, +\infty)$ is *monatomic* if it has exactly one non-zero atom, that is, $\text{card}(\mathfrak{A}^*) = 1$. Monatomicity is a natural extension of (quasi-)irreducibility, see [19, Remark 1.2] and references therein. We complete the characterization of monatomic operator T given in [19, Theorem 2] using the number of equilibria of the SIS models $(T, \lambda\gamma, \varphi)$.

Corollary 4.11 (Criterium of monatomicity)**.** *Let (T, γ, φ) satisfy Assumption 2. The operator T is monatomic if and only if the two following properties hold:*

(i) For $\lambda > 0$, the SIS model $(T, \lambda\gamma, \varphi)$ has at most one non-null equilibrium.
(ii) There exists $\lambda > 0$ such that the SIS model $(T, \lambda\gamma, \varphi)$ has a non-null equilibrium.

Proof. By Theorem 4.8, we deduce that if $\mathcal{C}$ is a finite antichain of non-zero atoms then, for all $\lambda \in (0, \min_{A \in \mathcal{C}} R_0(A))$ there exists an equilibrium g^λ for the model $(T, \lambda\gamma, \varphi)$ such that $\mathcal{F}(\mathcal{C}) = \text{supp}(g^\lambda)$.

Then, Point (i) means that the number of finite antichains of non-zero atoms is at most two, and Point (ii) that it is at least two: so the two points are equivalent to the number of antichains being exactly two (one being empty), that is T is monatomic. □

5. Convergence and attraction domains

In this section, we are interested in the behavior of the semi-flow $(\phi(t,h))_{t \in \mathbb{R}_+}$ of Equation (2) when t goes to infinity for an initial condition $h \in \Delta$. If T is a quasi-irreducible kernel positive operator, then according to Proposition 4.3 (iv), see also [17, Theorem 4.13] when T is an irreducible kernel operator, the semi-flow $(\phi(t,h))_{t \in \mathbb{R}_+}$ converges essentially to g^* if $\mu(\text{supp}(h) \cap \text{supp}(g^*)) > 0$ and $\mathbb{0}$ otherwise. We generalize this result to general operators, see Section 10 for a proof.

Theorem 5.1 (Convergence to an equilibrium)**.** *Let (T, γ, φ) satisfy Assumption 2. The semi-flow $(\phi(t,h))_{t \in \mathbb{R}_+}$ with initial condition $h \in \Delta$ converges essentialy to a limit, say $g \in \Delta$; and g is an equilibrium and more precisely the maximal equilibrium of the set $\mathcal{F}(\text{supp}(h))$:*

$$\operatorname*{limess}_{t \to \infty} \phi(t,h) = g \quad \textit{and} \quad \mathcal{C}_g = \mathcal{C}_{\mathcal{F}(\text{supp}(h))}.$$

We derive directly the next corollary, where the maximal equilibrium g^* is possibly equal to $\mathbb{0}$.

Corollary 5.2 (Attraction domain of the maximum equilibrium)**.** *Let (T, γ, φ) satisfy Assumption 2. The semi-flow with initial condition $h \in \Delta$ converges essentially to the maximal equilibrium g^*, that is, $\text{limess}_{t \to \infty} \phi(t,h) = g^*$, if and only if $\mathcal{F}(\text{supp}(h))$ contains all the supercritical atoms.*

In particular, we recover Proposition 4.3 as $R_0 \leqslant 1$ means there is no supercritical atom, and $R_0 > 1$ and T quasi-irreducible means there is only one supercritical atom. In the previous corollary, it may however happen that none of the supercritical atoms is included in $\text{supp}(h)$, see the next example.

Example 5.3. Let $\Omega = \{a, b\}$ with the counting measure, and consider the SIS model (T, γ, φ) with T identified with the matrix $\begin{pmatrix} 1 & 0 \\ \star & 2 \end{pmatrix}$ (with $\star > 0$), $\gamma = \mathbb{1}$, $\varphi(r) = 1 - r$. Notice that $\{a\}$ and $\{b\}$ are non-zero atoms, the former being critical with $\mathcal{F}(\{a\}) = \Omega$ and the latter being supercritical and invariant. Thus, there exists only two equilibria: $\mathbb{0} = (0, 0)$ and $g^* = (0, 1/2)$. If $h \neq \mathbb{0}$, then we have $\lim_{t\to\infty} \phi(t, h) = g^*$. For $h = (0, 1)$, we have $\text{supp}(h) = \text{supp}(g^*)$, but for $h = (1, 0)$ we have $\text{supp}(h) \cap \text{supp}(g^*) = \varnothing$.

Proposition 5.4 (Uniform convergence to an equilibrium)**.** *Let (T, γ, φ) satisfy Assumption 2. For $h \in \Delta$ and g the maximal equilibrium of $\mathcal{F}(\text{supp}(h))$, we have that:*

$$\lim_{t\to\infty} \left\| \left(\phi(t, h) - g\right) \gamma \right\|_\infty = 0. \tag{33}$$

In particular, when $\text{essinf}\, \gamma > 0$, the convergence given by Theorem 5.1 is uniform. Notice we have a stronger result if furthermore $R_0 < 1$ (and thus $g = \mathbb{0}$), see Remark 4.4.

Proof. We start with a preliminary result. Set M the norm of the operator $\tilde{T}$ from L^p to L^∞, which coincide with T on L^∞, see the proof of Lemma 2.8; it is finite by Assumption 2. Let $m > 0$. Let $h_1 \geqslant h_2$ be elements of Δ, and for $t \in \mathbb{R}_+$ set $f(t) = \phi(t, h_1) - \phi(t, h_2)$. We claim that:

$$\limsup_{t\to\infty} \left\| f(t)\mathbb{1}_{\{\gamma \geqslant m\}} \right\|_\infty \leqslant \frac{M}{m} \limsup_{t\to\infty} \| f(t) \|_p . \tag{34}$$

Indeed, by monotonicity of the semi-flow (see Lemma 8.5), we have $\phi(t, h_1) \geqslant \phi(t, h_2)$ and thus $f(t) \geqslant \mathbb{0}$. We also have, as φ is decreasing on $[0, 1]$ and T is positive that for $t \in \mathbb{R}_+$:

$$\begin{aligned} f'(t) &= \varphi(\phi(t, h_1))T\phi(t, h_1) - \varphi(\phi(t, h_2))T\phi(t, h_2) - \gamma f(t) \\ &\leqslant \varphi(\phi(t, h_1))Tf(t) - \gamma f(t) \\ &\leqslant Tf(t) - \gamma f(t). \end{aligned}$$

On $\{\gamma \geqslant m\}$, we get for $v(t) = \mathrm{e}^{mt} f(t)$ that for all $t \geqslant 0$:

$$v'(t) \leqslant (m - \gamma)v(t) + Tv(t) \leqslant \left\| \tilde{T}\, \iota v(t) \right\|_\infty \leqslant M \| v(t) \|_p . \tag{35}$$

By (39) and (40) on the Bochner integral, we deduce that $v(t) \leqslant v(0) + M \int_0^t \| v(s) \|_p \, \mathrm{d}s$ on $\{\gamma \geqslant m\}$. Since f is nonnegative, we get:

$$\left\| f(t)\mathbb{1}_{\{\gamma\geqslant m\}} \right\|_\infty \leqslant \mathrm{e}^{-mt} \| h_1 - h_2 \|_\infty + M \int_0^t \mathrm{e}^{-m(t-s)} \| f(s) \|_p \, \mathrm{d}t \leqslant 2\mathrm{e}^{-mt} + M \int_0^t \mathrm{e}^{-ms} \| f(t - s) \|_p \, \mathrm{d}t.$$

This gives (34).

Let $h \in \Delta$ and g be the maximal equilibrium of $A = \mathcal{F}(\text{supp}(h))$. By Lemma 9.1, we have $\text{supp}(\phi(1, h)) = A$. Let $h_1 = \max(\phi(1, h), g)$ and $h_2 = \min(\phi(1, h), g)$. We thus have $h_1 \leqslant h_2$ and $\text{supp}(g) = \text{supp}(h_2) \subset \text{supp}(h_1) = A$. By Theorem 5.1 (and using that g is the maximal equilibrium on the invariant set $\mathcal{F}(\text{supp}(h_i))$ for $i = 1, 2$), we get that $\text{limess}_{t\to\infty}\, \phi(t, h_i) = g$ for $i = 1, 2$. With $f(t) = \phi(t, h_1) - \phi(t, h_2)$, we get by the dominated convergence theorem that $\lim_{t\to\infty} \| f(t) \|_p = 0$, and by (34) that $\lim_{t\to\infty} \left\| f(t)\mathbb{1}_{\{\gamma\geqslant m\}} \right\|_\infty = 0$ for all $m > 0$ and thus $\lim_{t\to\infty} \| f(t)\, \gamma \|_\infty = 0$ as $\| f \|_\infty \leqslant 1$. Use the monotonicity of the semi-flow to get $\phi(t, h_1) \geqslant \phi(t + 1, h) \geqslant \phi(t, h_2)$ and deduce that (33) holds. □

6. SIS model with an external disease reservoir

In this section, we consider the infinite-dimensional inhomogeneous SIS model with an external disease reservoir, called SISκ model, presented in Section 1.4. The function $u = (u(t, x))_{t\in\mathbb{R}_+, x\in\Omega}$, where $u(t, x)$ is the proportion of infected population among the population with feature x, is solution in L^∞ of the ODE:

$$\begin{cases} u' = F_\kappa(u), \\ u(0) = h, \end{cases} \tag{36}$$

with initial condition $h \in L^\infty$ and:

$$F_\kappa(u) = \varphi(u)(Tu + \kappa) - \gamma u, \tag{37}$$

where φ is a continuous function on $\mathbb{R}$ and $\kappa \in L^\infty_+$. To study solutions of (36) and the corresponding equilibria, that is functions $g \in \Delta$ such that $F_\kappa(g) = 0$, we shall use the formalism of Section 3 by adding a sub-population corresponding to the reservoir with type $\mathbf{r}$. Notice the case $\varphi([0,1]) = \{0\}$ (which is not possible under Assumption 2) is trivial, and thus we shall assume there exists $a \in (0,1)$ such that $\varphi(a) > 0$.

We set $\Omega_\mathbf{r} = \Omega \sqcup \{\mathbf{r}\}$ (assuming without loss of generality that $\mathbf{r} \notin \Omega$), $\mathcal{G}_\mathbf{r} = \sigma(\mathcal{G}, \{\mathbf{r}\})$ and $\mu_\mathbf{r}$ a measure on $(\Omega_\mathbf{r}, \mathcal{G}_\mathbf{r})$ which coincides with μ on Ω and with positive finite weight on $\mathbf{r}$. For a function $f_\mathbf{r}$ defined on $\Omega_\mathbf{r}$, we simply write f for its restriction to Ω (and similarly, for a function f defined on Ω, we write $f_\mathbf{r}$ for a function defined on $\Omega_\mathbf{r}$ which coincides with f on Ω, the value of $f_\mathbf{r}$ on $\mathbf{r}$ being given when needed). We simply write $L^p_\mathbf{r}$ for $L^p(\Omega_\mathbf{r}, \mathcal{G}_\mathbf{r}, \mu_\mathbf{r})$, where $p \in [1, +\infty]$. We define the positive operator $T_\mathbf{r}$ on $L^\infty_\mathbf{r}$, as an extension of T on $\Omega_\mathbf{r}$, by:

$$T_\mathbf{r} f_\mathbf{r}(x) = \mathbb{1}_\Omega(x)\, Tf(x) + f_\mathbf{r}(\mathbf{r})\, \alpha_\mathbf{r}(x),$$

with $\alpha_\mathbf{r} \in (L^\infty_\mathbf{r})_+$ such that $\alpha = \kappa/a$ on Ω, with $a \in (0,1)$, and $\alpha_\mathbf{r}(\mathbf{r}) = b > 0$. For $p \in (1, +\infty)$, we define similarly the operator $\hat{T}_\mathbf{r}$ on $L^p_\mathbf{r}$ based on the operator $\hat{T}$ on L^p, see Lemma 2.8. We also define the function $\gamma_\mathbf{r}$ (which coincides with γ on Ω by definition) such that $\gamma_\mathbf{r}(\mathbf{r}) = b\varphi(a)$ is assumed to be positive (that is, $\varphi(a) > 0$). Let $\Delta_\mathbf{r} = \{f_\mathbf{r} \in (L^\infty_\mathbf{r})_+ : 1 - f_\mathbf{r} \in (L^\infty_\mathbf{r})_+\}$ be the analogue of Δ for $\Omega_\mathbf{r}$. It is elementary to check the following result.

Proposition 6.1 (Solution to the SISκ model)**.** *Let (T, γ, φ) satisfy Assumption 1. Assume furthermore there exists $a \in (0,1)$ such that $\varphi(a) > 0$, and let $\kappa \in L^\infty_+$. A function $(u(t,x))_{t\in\mathbb{R}_+, x\in\Omega}$ is a solution to* (36) *related to the SISκ model with initial condition $h \in \Delta$ if and only if the function $(u_\mathbf{r}(t,x))_{t\in\mathbb{R}_+, x\in\Omega_\mathbf{r}}$, where $u_\mathbf{r}(t, \mathbf{r}) = a$ for all $t \in \mathbb{R}_+$, is a solution to* (2) *related to the SIS model with parameter $(T_\mathbf{r}, \gamma_\mathbf{r}, \varphi)$ on $\Omega_\mathbf{r}$ and with initial condition $h_\mathbf{r} \in \Delta_\mathbf{r}$ such that $h_\mathbf{r}(\mathbf{r}) = a$.*

We shall consider the supercritical atoms out of the individuals infected by the reservoir:

$$\mathfrak{A}^{\sup}_\mathbf{r} = \{A \in \mathfrak{A}^{\sup} : A \cap \mathcal{F}(\mathrm{supp}(\kappa)) = \varnothing \text{ a.e.}\}.$$

Based on Theorems 4.8 and 5.1 for the $(T_\mathbf{r}, \gamma_\mathbf{r}, \varphi)$ SIS model we can give a representation of the equilibria of SISκ model, that is, of the solutions to $F_\kappa(g) = 0$ in Δ, prove that the equilibria are characterized by their support, and explicit their attraction domain. For a function $h \in L^\infty_+$, we shall denote $\mathcal{C}_{\mathbf{r},h}$ the antichain given by the maximal elements of $\{B \in \mathfrak{A}^{\sup}_\mathbf{r} : B \subset \mathcal{F}(\mathrm{supp}(h)) \text{ a.e.}\}$. Notice that Assumption 2 implies that φ is positive on $[0,1)$.

Corollary 6.2 (Equilibria of the SISκ model)**.** *Let (T, γ, φ) satisfy Assumption 2 and $\kappa \in L^\infty_+$. The set of equilibria and the set of antichains in $\mathfrak{A}^{\sup}_\mathbf{r}$ are in bijection through the equivalent relations:*

$$\mathrm{supp}(g) = \mathcal{F}(\mathcal{C}) \cup \mathcal{F}(\mathrm{supp}(\kappa)) \quad a.e. \quad \Longleftrightarrow \quad \mathcal{C}_{\mathbf{r},g} = \mathcal{C},$$

where $\mathcal{C} \subset \mathfrak{A}^{\sup}_\mathbf{r}$ is an antichain and $g \in \Delta$ an equilibrium, that is, $F_\kappa(g) = 0$.

Furthermore, the semi-flow $(\phi(t,h))_{t\in\mathbb{R}_+}$ solution of (36) *with initial condition $h \in \Delta$ is well defined and it converges a.e. to a limit, say $g \in \Delta$; and g is an equilibrium and more precisely:*

$$\operatorname*{limess}_{t\to\infty} \phi(t,h) = g \quad and \quad \mathcal{C}_{\mathbf{r},g} = \mathcal{C}_{\mathbf{r},h}.$$

Remark 6.3. We deduce the following properties under the hypothesis of Corollary 6.2.

(1) Notice that $\mathfrak{A}^{\sup}_\mathbf{r}$ is empty if and only if there exists a unique equilibrium $g \in \Delta$ for the SISκ model. In this case, we have $\mathrm{supp}(g) = \mathcal{F}(\mathrm{supp}(\kappa))$ a.e. and $\mathrm{limess}_{t\to\infty}\, \phi(t,h) = g$ for all $h \in \Delta$.
(2) If $\mathrm{essinf}\, \kappa > 0$, then $\mathfrak{A}^{\sup}_\mathbf{r}$ is empty.
(3) If g is an equilibrium for the SISκ model, then g is positive on $\mathcal{F}(\mathrm{supp}(\kappa))$ and thus on $\mathrm{supp}(\kappa)$.

Proof of Corollary 6.2. If $A \in \mathcal{G}$ is invariant for $\hat{T}$, then A seen as an element of $\mathcal{G}_\mathbf{r}$ is also invariant for $\hat{T}_\mathbf{r}$. Furthermore, the reservoir $\{\mathbf{r}\}$ is an atom of $\mu_\mathbf{r}$ and, as Ω is $\hat{T}_\mathbf{r}$-invariant, we get that $\{\mathbf{r}\}$ is a $\hat{T}_\mathbf{r}$-atom. We deduce that a set $B \in \mathcal{G}_\mathbf{r}$ is admissible for $\hat{T}_\mathbf{r}$ if and only if $B \cap \Omega$ is admissible for $\hat{T}$.

In particular a set A is an atom of $\hat{T}_{\mathbf{r}}$ if and only if either $A = \{\mathbf{r}\}$ or $A \subset \Omega$ and A is an atom of $\hat{T}$. We denote by $\mathcal{F}_{\mathbf{r}}(A)$ the future of a set $A \subset \Omega_{\mathbf{r}}$ with respect to $\hat{T}_{\mathbf{r}}$. Notice that $\mathcal{F}_{\mathbf{r}}(A) = \mathcal{F}(A)$ for any $A \subset \Omega$ and that the future of the reservoir is:

$$\mathcal{F}_{\mathbf{r}}(\{\mathbf{r}\}) = \{\mathbf{r}\} \cup \mathcal{F}(\mathrm{supp}(\kappa)). \tag{38}$$

Recall the basic reproduction number $R_0(A)$ of a measurable set for the SIS model (T, γ, φ) given in Section 4.2. We simply write $R_{\mathbf{r}}(A)$ when considering the basic reproduction number of $A \in \mathcal{G}_{\mathbf{r}}$ for the SIS model $(T_{\mathbf{r}}, \gamma_{\mathbf{r}}, \varphi)$. Since $(T_{\mathbf{r}})_A = T_A$ for $A \subset \Omega$ measurable, we deduce that $R_{\mathbf{r}}(A) = R_0(A)$ for any $A \subset \Omega$ measurable. We also have:

$$R_{\mathbf{r}}(\{\mathbf{r}\}) = \frac{\alpha_{\mathbf{r}}(\mathbf{r})}{\gamma_{\mathbf{r}}(\mathbf{r})} = \frac{1}{\varphi(a)}.$$

Note that under Assumption 2, we have $\varphi((0, 1)) = (0, 1)$ and thus the atom $\{\mathbf{r}\}$ is super-critical for the $(T_{\mathbf{r}}, \gamma_{\mathbf{r}}, \varphi)$ SIS model.

We deduce from Proposition 6.1. that a function $g \in \Delta$ is an equilibrium for the SISκ model if and only if $g_{\mathbf{r}} \in \Delta_{\mathbf{r}}$, such that $g_{\mathbf{r}}(\mathbf{r}) > 0$ is an equilibrium for the $(T_{\mathbf{r}}, \gamma_{\mathbf{r}}, \varphi)$ SIS model on $\Omega_{\mathbf{r}}$. Notice that necessarily $g_{\mathbf{r}}(\mathbf{r}) = a$. The equilibria of the $(T_{\mathbf{r}}, \gamma_{\mathbf{r}}, \varphi)$ SIS model whose support contains the reservoir $\{\mathbf{r}\}$ are according to Theorems 4.8 in bijection will all the supercritical antichains containing the atom $\{\mathbf{r}\}$. Those supercritical antichains are exactly the antichains of $\mathfrak{A}^{\mathrm{sup}}_{\mathbf{r}}$ with the atom $\{\mathbf{r}\}$ added to them. Then, use (38) and Theorems 4.8 and 5.1 to conclude. □

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7. A short reminder on integration, derivation and ODE in Banach space

This section is devoted to the definition and properties of the Bochner integral, the differentiation and differential equations in Banach spaces. We consider $(X, \|\cdot\|)$ a real Banach space.

7.1. Integration in Banach spaces.

We give a short summary on the Bochner integral, and refer to [4] and [51] for a more detailed presentation. We consider the Borel σ-field on $\mathbb{R}$ and write $\nu(\mathrm{d}t) = \mathrm{d}t$ for the Lebesgue measure. Let I be an interval of $\mathbb{R}$. A function $f : I \mapsto X$ is *simple* if $f = \sum_{k=1}^n a_k \mathbb{1}_{A_k}$ where $n \in \mathbb{N}$, the a_k's belong to X and the A_k's are Borel subsets of I with finite Lebesgue measure. We define its Bochner integral as:

$$\int_I f \,\mathrm{d}\nu = \sum_{k=1}^n a_k\, \nu(A_k).$$

Notice the integral belongs to X and does not depend on the representation of the simple function f. A function $f : I \mapsto X$ is Bochner measurable (simply called measurable in [4]) if there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$, with $f_n : I \mapsto X$, such that ν-a.e. $\lim_{n\to\infty} f_n = f$ (that is, $\lim_{n\to\infty} \| f(t) - f_n(t) \| = 0$ $\mathrm{d}t$-a.e. on I); it is furthermore Bochner integrable if one can find such approximating sequence $(f_n)_{n\in\mathbb{N}}$ so that $\lim_{n\to\infty} \int_I \| f(t) - f_n(t) \| \,\mathrm{d}t = 0$. In this case the *Bochner integral* of f is defined as:

$$\int_I f \,\mathrm{d}\nu = \lim_{n\to\infty} \int_I f_n \,\mathrm{d}\nu,$$

where the limit holds in the Banach space. The Bochner integrable $\int_I f \,\mathrm{d}\nu$ does not depend on the approximating sequence $(f_n)_{n\in\mathbb{N}}$; we shall also denote it by $\int_I f(t)\,\mathrm{d}t$. Thanks to [4, Corollary 1.1.2] X-valued continuous function are Bochner measurable and a.e. limits of Bochner measurable function are Bochner measurable. According to [4, Corollary 1.1.2], a function $f : I \mapsto X$ is Bochner integrable if and only if it is Bochner measurable and $\| f \| : I \mapsto \mathbb{R}_+$ is integrable; in this case we have:

$$\left\| \int_I f \,\mathrm{d}\nu \right\| \leqslant \int_I \| f \| \,\mathrm{d}\nu. \tag{39}$$

When $\bar{I} = [a, b]$ with $-\infty \leqslant a < b \leqslant +\infty$ and f is Bochner integrable on I, we simply write the Bochner integral as:

$$\int_I f \,\mathrm{d}\nu = \int_a^b f \mathrm{d}\nu.$$

The Bochner integral enjoy many properties as the usual Lebesgue integral, as such we shall use the dominated convergence, see [4, Theorem 1.1.8], which we recall.

Theorem 7.1 (Dominated convergence theorem)**.** *Let I be a non-empty interval of $\mathbb{R}$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of Bochner-integrable functions defined on I which converges ν-a.e. to $f : I \to X$. Assume* *there exists a Lebesgue-integrable function* $g : I \to \mathbb{R}_+$ *such that* $\| f_n \| \leqslant g$ ν*-a.e. for all* $n \in \mathbb{N}$*. Then the function* f *is Bochner-integrable and:*

$$\int_I f \, \mathrm{d}\nu = \lim_{n\to\infty} \int_I f_n \, \mathrm{d}\nu \quad \text{in } X.$$

7.2. **Differential equations in Banach spaces.** We now consider the derivation of functions on a Banach space. Let I be an interval of $\mathbb{R}$ with non-empty interior. We say that a function $f : I \mapsto X$ is *differentiable* at a point $t \in I$ if the following limit, $f'(t)$, exists in $(X, \| \cdot \|)$:

$$f'(t) = \lim_{\substack{s\to 0\\ t+s\in I}} \frac{f(t+s) - f(t)}{s}.$$

Notice that if f is differentiable at t, then it is continuous at t. We say that f is differentiable on I if it is differentiable at any point of I, and that f belongs to $\mathcal{C}^1(I)$ if it is differentiable on I and f' is continuous on I. We have the following fundamental theorem of calculus, see [4, Proposition 1.2.2] and [51, Corollary 3.1.7]. Assume $I = [a, b]$ with $-\infty < a < b < +\infty$ and that $f : I \to X$ belongs to $\mathcal{C}^1(I)$, then f' is Bochner-integrable on I and we have:

$$f(b) - f(a) = \int_a^b f' \, \mathrm{d}\nu. \tag{40}$$

We now recall some results on differential equations in Banach spaces. Let $F : X \to X$ be locally-Lipschitz, that is, for all $x \in X$, there exists $\eta > 0$ and C finite such that for all $y \in X$, we have:

$$\| x - y \| \leqslant \eta \quad \Longrightarrow \quad \| F(x) - F(y) \| \leqslant C \| x - y \| .$$

The Picard-Lindelöf theorem, see [33, Corollaries IV 1.6-8], ensures the existence of (u, τ), with $\tau \in (0, +\infty]$ and $u \in \mathcal{C}^1([0, \tau))$ taking values in X, that is a solution to the Cauchy problem:

$$\begin{cases} u' = F(u), \\ u(0) = x, \end{cases} \tag{41}$$

where the first equality in (41) holds in $[0, \tau)$ and $x \in X$ is the so-called initial condition, and furthermore the solution (u, τ) is unique and maximal (that is, if (u', τ') is another solution to (41), then $\tau' \leqslant \tau$ and $u' = u$ on $[0, \tau')$). We say the solution is *global* if $\tau = +\infty$.

We end this section with a comparison theorem. Let $(X, \| \cdot \|, \leqslant)$ be a real Banach lattice. Let $D_1, D_2 \subset X$. A map $F : X \mapsto X$ is *cooperative* on $D_1 \times D_2$ if for any $(x, y) \in D_1 \times D_2$ with $x \leqslant y$ and any $\nu \in X^\star_+$, we have:

$$\langle \nu, x - y \rangle = 0 \quad \Longrightarrow \quad \langle \nu, F(x) - F(y) \rangle \leqslant 0. \tag{42}$$

We recall [17, Theorem 2.4].

Proposition 7.2 (Comparison)**.** *Assume that* X_+ *has non-empty interior. Let* $F : X \to X$ *be locally-Lipschitz,* $D_1, D_2 \subset X$ *and* $\tau \in (0, +\infty]$*. Let* $u : [0, \tau) \to D_1$ *and* $v : [0, \tau) \to D_2$ *be two* $\mathcal{C}^1$ *maps. Suppose that* F *is cooperative on* $D_1 \times X$ *or on* $X \times D_2$*, that* $u(0) \leqslant v(0)$*, and that* $u'(t) - F(u(t)) \leqslant v'(t) - F(v(t))$ *for all* $t \in [0, \tau)$*. Then, we have* $u(t) \leqslant v(t)$ *for all* $t \in [0, \tau)$*.*

8. Existence and regularity of the semi-flow for the SIS model

We prove Proposition 3.1 and Lemma 3.2 in this section assuming that (T, γ, φ) satisfies Assumption 1. The results and proofs are very close to those in [17].

8.1. **Existence of the semi-flow.** Recall that Assumption 1 holds with the function F defined in (3) by $F(h) = \varphi(h)Th - \gamma h$ for $h \in L^\infty$. Recall also the set $\Delta = \{f \in L^\infty : \mathbb{0} \leqslant f \leqslant \mathbb{1}\}$.

Lemma 8.1 (Regularity of F)**.** *Let F be the function from L^∞ to L^∞ defined by* (3)*.*

(i) The function F is locally-Lipschitz on $(L^\infty, \| \cdot \|_\infty)$.

(ii) There exists a finite constant C_p such that for $g, h \in \Delta$, we have:

$$\| F(g) - F(h) \|_p \leqslant C_p \, \| g - h \|_p .$$

(iii) Let $(h_n)_{n\in\mathbb{N}}$ be a monotonous sequence of elements of Δ. Then the sequence $(F(h_n))_{n\in\mathbb{N}}$ converges a.e. to $F(h)$, where h is the a.e. limit of $(h_n)_{n\in\mathbb{N}}$.

Proof. Since φ is locally Lipschitz, we denote by K_r the corresponding (finite) Lipschitz constant of φ on $[-r, r]$ and $M_r = \sup_{[-r,r]} |\varphi| \leqslant |\varphi(0)| + rK_r$.

We prove Point (i). Let $r > 0$ and $u, v \in L^\infty$ with $\| u \|_\infty \leqslant r$ and $\| v \|_\infty \leqslant r$. We have:

$$\begin{aligned} \| F(u) - F(v) \|_\infty &= \| \varphi(u)Tu - \varphi(v)Tv - \gamma(u - v) \|_\infty \\ &\leqslant \| \varphi(u) \|_\infty \| T(u - v) \|_\infty + \| \varphi(u) - \varphi(v) \|_\infty \left\| T'v \right\|_\infty + \| \gamma \|_\infty \, \| u - v \|_\infty \\ &\leqslant (M_r \| T \|_{L^\infty} + K_r r \left\| T' \right\|_{L^\infty} + \| \gamma \|_\infty) \| u - v \|_\infty . \end{aligned}$$

This concludes the proof of Point (i).

We prove Point (ii) in a similar way. Let $u, v \in \Delta$. We have:

$$\begin{aligned} \| F(u) - F(v) \|_p &= \| \varphi(u)Tu - \varphi(v)Tv - \gamma(u - v) \|_p \\ &\leqslant \| \varphi(u) \|_\infty \| T(u - v) \|_p + \| \varphi(u) - \varphi(v) \|_p \, \| Tv \|_\infty + \| \gamma \|_\infty \, \| u - v \|_{\gamma^p, p} \\ &\leqslant \left(M_1 \left\| \hat{T} \right\|_{L^p} + K_1 \| T \|_{L^\infty} + \| \gamma \|_\infty \right) \| u - v \|_p . \end{aligned}$$

This concludes the proof of Point (ii).

For simplicity, we assume that $(h_n)_{n\in\mathbb{N}}$ is non-decreasing. Thus it converges a.e. (that is, in L^0) to a limit, say h, and this limit belongs to Δ. Since T is positive, we also get that the sequence $(Th_n)_{n\in\mathbb{N}}$ is non-decreasing and bounded by $T\mathbb{1} \in L^\infty$, thus it converges a.e. (that is, in L^0) to a limit, say $w \in L^\infty$. On the other hand, by dominated convergence, we also get that $(h_n)_{n\in\mathbb{N}}$ converges to h in L^p, and thus, as $\hat{T}$ is bounded on L^p, we get that $(\hat{T}h_n)_{n\in\mathbb{N}}$ converges to $\hat{T}h$ in L^p. We thus deduce that $w = \hat{T}h = Th$. Then use that φ is continuous, to deduce that $(F(h_n))_{n\in\mathbb{N}}$ converges a.e. to $F(h)$. This gives Point (iii). ◻

We now prove that F is cooperative, see (42), using that φ is non-negative on $[0, 1]$.

Lemma 8.2 (F is cooperative)**.** *The map F is cooperative on $\Delta \times L^\infty$ and on $L^\infty \times \Delta$.*

Proof. We first prove that F is cooperative on $\Delta \times L^\infty$. Let $u \in \Delta$ and $v \in L^\infty$ with $u \leqslant v$. Let $\nu \in L_+^{\infty,\star}$ such that $\langle \nu, u - v\rangle = 0$. Since $v - u \geqslant 0$, we deduce that for any $h \in L^\infty$:

$$\langle \nu, (u - v)h\rangle = 0 \tag{43}$$

(see [17, Lemma 2.6] for a proof in a very similar setting). Then, using (43) with $h = \gamma$, we get:

$$\langle \nu, F(u) - F(v)\rangle = \langle \nu, \varphi(u)Tu - \varphi(v)Tv - \gamma(u - v)\rangle = \langle \nu, \varphi(u)Tu - \varphi(v)Tv\rangle.$$

For $s, t \in \mathbb{R}_+$, we set $\Phi(s, t) = (\varphi(s) - \varphi(t))/(s - t)$ if $s \neq t$ and $\Phi(s, s) = 0$. We have:

$$\varphi(u)Tu - \varphi(v)Tv = \varphi(u)T(u - v) + (u - v)h \quad \text{with} \quad h = \Phi(u, v)\, Tv.$$

As φ is locally-Lipschitz by Assumption 1 and as u, v and Tv belongs to L^∞, we deduce that $h \in L^\infty$. We deduce from (43) that:

$$\langle \nu, F(u) - F(v)\rangle = \langle \nu, \varphi(u)T(u - v)\rangle. \tag{44}$$

As we have $\varphi \geqslant 0$ on $[0, 1]$ by Assumption 1 and $u \leqslant v$, we have $\varphi(u)T(u - v) \leqslant 0$. Thus, as ν is a positive linear form on L^∞, we have $\langle \nu, F(u) - F(v)\rangle \leqslant 0$. Therefore the map F is cooperative on $\Delta \times L^\infty$.

If $(u, v) \in L^\infty \times \Delta$ satisfy $u \leqslant v$, then using similar computations with $h = \Phi(v, u)Tu$, one get instead of (44) that $\langle \nu, F(u) - F(v)\rangle = \langle \nu, \varphi(v)T(u - v)\rangle$. Similar arguments yields then that F is also cooperative on $L^\infty \times \Delta$. □

Mimicking the proof of [17, Proposition 2.7 (i)] (which in particular uses that $\varphi(1) = 0$), we get that any solution of (2) with an initial condition in Δ remains in Δ.

Lemma 8.3. *The domain Δ is forward invariant for the differential equation $u' = F(u)$ in L^∞.*

We then conclude on the existence of global solutions in Δ.

Lemma 8.4 (Maximal solutions are global)**.** *Any maximal solution of $u' = F(u)$ in L^∞ with initial condition $u(0) = h \in \Delta$ is global.*

Proof. The bounded open set $U = \{f \in L^\infty : \; \| f \|_\infty < 2\}$ of L^∞ contains Δ, and the map F is Lipschitz on U by Lemma 8.1 (i). As the set Δ is forward invariant by Lemma 8.3, one can apply [33, Corollary IV 1.8] to conclude that any maximal solution to $u' = F(u)$ with initial condition in Δ is global. □

Under Assumption 1, using Picard-Lindelöf theorem [33, Corollaries IV 1.6-8] and Lemma 8.4, which ensure the existence and uniqueness of maximal solution to $u' = F(u)$ in L^∞ with initial condition in Δ, we can define the semi-flow $\phi : \; \mathbb{R}_+ \times\Delta \to \Delta$, where the L^∞-valued function $\phi(\cdot, h) = (\phi(t, h))_{t\in\mathbb{R}_+}$ is the global solution to (2) with initial condition $u_0 = h \in \Delta$. Notice that $\phi(\cdot, h)$ belongs to $\mathcal{C}^1(\mathbb{R}_+)$ and statisfies the semi-group property:

$$\phi(t + s, h) = \phi(t, \phi(s, h)) \quad \text{for all} \quad s, t \in \mathbb{R}_+ \quad \text{and} \quad h \in \Delta. \tag{45}$$

8.2. **Properties of the semi-flow.** We now establish the following properties of the semi-flow. We stress in the next lemmas that Assumption 1 holds.

Lemma 8.5 (Properties of the semi-flow)**.** *Let (T, γ, φ) satisfy Assumption 1.*

(i) If $h_1 \leqslant h_2$ belong to Δ, then we have $\phi(t, h_1) \leqslant \phi(t, h_2)$ for all $t \in \mathbb{R}_+$.
(ii) Let $h \in \Delta$. The function $t \mapsto \phi(t, h)$ from $\mathbb{R}^+$ to L^∞ is non-decreasing (resp. non-increasing) if and only if we have $F(h) \geqslant \mathbb{0}$ (resp. $F(h) \leqslant \mathbb{0}$) in L^∞.
(iii) Let $t \in \mathbb{R}_+$. The function $h \mapsto \phi(t, h)$ defined on Δ is Lipschitz with respect to $\| \cdot \|_\infty$.
(iv) Let $t \in \mathbb{R}_+$. The function $h \mapsto \phi(t, h)$ defined on Δ is continuous with respect to the a.e. convergence, and, more generally, if $(h_r)_{r\in\mathbb{R}_+}$ is a sequence of elements of Δ such that $h = \operatorname{limess}_{r\to+\infty} h_r$ exists (and thus belongs to Δ), then $\operatorname{limess}_{r\to+\infty} \phi(t, h_r) = \phi(t, h)$.

Proof. For all the Points but (iv), the proof mimic respectively the proofs of Propositions 2.8, 2.10 and 2.11 (ii) from [17]. Following the proof of Proposition 2.11 (iii) in [17], we see that to get Point (iv), it is enough to check the following claim: if $(h_n)_{n\in\mathbb{N}}$ is a monotonous sequence of elements of Δ, which thus converges a.e. to a limit, say $h \in \Delta$, then $(\phi(t, h_n))_{n\in\mathbb{N}}$ converges also a.e. to $\phi(t, h)$ for all $t \geqslant 0$.

For simplicity, we assume that the sequence $(h_n)_{n\in\mathbb{N}}$ is non-decreasing. From Point (i), we get that the sequence $(\phi(s, h_n))_{n\in\mathbb{N}}$ is also non-decreasing and thus converges a.e. to a limit say $f_s \in \Delta$ for all $s \in \mathbb{R}_+$. We deduce from Lemma 8.1 (iii) that $(F(\phi(s, h_n)))_{n\in\mathbb{N}}$ converges a.e. towards $F(f_s)$. Since F is bounded on Δ (as it is locally Lipschitz on L^∞) we deduce that the convergence also holds in L^p. Since the identity map from $(L^\infty, \| \cdot \|_\infty)$ to $(L^p, \| \cdot \|_p)$ is continuous, we deduce that a solution to (2) in L^∞ is also a solution to (2) in L^p. By the fundamental Theorem of calculus, see (40), we get that for all $s \geqslant 0$:

$$\phi(s, h_n) = h_n + \int_0^s F(\phi(r, h_n))\, \mathrm{d}r \quad \text{and} \quad \phi(s, h) = h + \int_0^s F(\phi(r, h))\, \mathrm{d}r \quad \text{hold in } L^p.$$

By the dominated comvergence Theorem 7.1 (with $X = L^p$ and g the constant function on $\mathbb{R}_+$ equal to 1), we deduce that $(f_s)_{s\in\mathbb{R}_+}$ is Bochner integrable on bounded intervals of $\mathbb{R}_+$ and for all $s \geqslant 0$:

$$f_s = h + \int_0^s F(f_r)\, \mathrm{d}r \quad \text{holds in } L^p.$$

We deduce from (39) and Lemma 8.1 (ii) that for all $s \geqslant 0$:

$$\| f_s - \phi(s,h) \|_p \leqslant C_p \int_0^s \| f_r - \phi(r,h) \|_p \, \mathrm{d}r.$$

Since f_r and $\phi(r,h)$ belong to Δ, we get that $\| f_r - \phi(r,h) \|_p \leqslant 2$, so that $r \mapsto \| f_r - \phi(r,h) \|_p$ is locally ν-integrable. We deduce from the Grönwall's inequality that $\| f_s - \phi(s,h) \|_p = 0$ for all $s \geqslant 0$. This gives that $f_s = \phi(s,h)$ for all $s \geqslant 0$, which proves the claim. ☐

Following [17, Proposition 2.13], we prove that the limit of the semi-flow is an equilibrium.

Lemma 8.6 (Limits of the semi-flow are equilibria)**.** *Let (T, γ, φ) satisfy Assumption 1. Let $h \in \Delta$. If* $\mathrm{limess}_{t\to+\infty}\, \phi(t,h)$ *exists, then it belongs to Δ and is an equilibrium.*

Proof. Let $h^* = \mathrm{limess}_{t\to+\infty}\, \phi(t,h)$. By Lemma 8.5 (iv) and by (45), we have for all $s \in \mathbb{R}_+$ that:

$$\phi(s,h^*) = \phi(s, \operatorname*{limess}_{t\to+\infty} \phi(t,h)) = \operatorname*{limess}_{t\to+\infty} \phi(s, \phi(t,h)) = \operatorname*{limess}_{t\to+\infty} \phi(t,h) = h^*.$$

Then use Lemma 8.5 (ii) to get that $F(h^*) = 0$. ☐

8.3. Proof of Proposition 3.1 and Lemma 3.2.

Proof of Proposition 3.1. The solution to Equation (2) in L^∞ with initial condition in Δ is given by the semi-flow ϕ, see Section 8.1 and Lemma 8.4 therein. This gives Point (i). Point (ii) is Lemma 8.3.

Since $F(\mathbb{1}) = \varphi(1)T(\mathbb{1}) - \gamma = -\gamma \leqslant 0$ by Assumption 1, we get by Lemma 8.5 (ii) that the semi-flow $t \mapsto \phi(t,\mathbb{1})$ is non-increasing. This implies that $g^* = \mathrm{limess}_{t\to+\infty}\, \phi(t,\mathbb{1})$ exists. By Lemma 8.6, we get that g^* is an equilibrium. Let $h \in \Delta$ be an equilibrium. We have $h \leqslant \mathbb{1}$, thus by Lemma 8.5 (i) we have $h = \phi(t,h) \leqslant \phi(t,\mathbb{1})$ for all $t \geqslant 0$. Taking the essential limit, we get $h \leqslant g^*$. This gives Point (iii). ☐

Proof of Lemma 3.2. For $h \in L^\infty$ and $i = 1,2$, let $F_i(h) = \varphi_i(h)T_i(h) - \gamma_i h$ and let ϕ_i be the semi-flow of Equation (2) with the parameters $(T_i, \gamma_i, \varphi_i)$. By assumption, we have $F_1(g_2^*) \geqslant F_2(g_2^*) = 0$. Thus, by Lemma 8.5 (ii), the semi-flow $t \mapsto \phi_1(t, g_2^*)$ is non-decreasing. By Lemma 8.6, since the essential limit $g = \mathrm{limess}_{t\to+\infty}\, \phi_1(t, g_2^*)$ exists, it belongs to Δ and is an equilibrium (for the parameters $(T_1, \gamma_1, \varphi_1)$). As g_1^* is the maximal equilibrium, we have $g_1^* \geqslant g$, and thus $g_1^* \geqslant g_2^*$ as the semi-flow $t \mapsto \phi_1(t, g_2^*)$ is non-decreasing. ☐

9. Proof of Proposition 4.3

9.1. On the support of the semi-flow.

The following lemma is a generalization of [17, Lemma 4.10] where T is assumed to be irreducible.

Lemma 9.1 (Support of the semi-flow)**.** *Let (T, γ, φ) that satisfies Assumption 1 with $\varphi(0) > 0$. Let $h \in \Delta$. We have* $\mathrm{supp}(\phi(t,h)) = \mathcal{F}(\mathrm{supp}(h))$ *a.e. for all $t > 0$.*

Proof. Since $\varphi(0) > 0$, there exists $a, \eta \in (0,1)$ such that $a - \varphi(r) < 0$ for all $r \in [0,\eta]$. Notice the operator $Q = aT - \gamma + \| \gamma \|_\infty$ on L^∞ is positive and that the invariant sets for Q and T are the same. Set $f = \eta h/2$ and $A = \mathrm{supp}(h) = \mathrm{supp}(f)$. There exists $c > 0$ small enough such that for all $t \in [0,c]$:

$$\exp(at \| T \|_{L^\infty}) < 2. \tag{46}$$

Set for $t \geqslant 0$:

$$u(t) = \mathrm{e}^{t(aT-\gamma)} f = \mathrm{e}^{-\| \gamma \|_\infty t}\, \mathrm{e}^{tQ} f. \tag{47}$$

By [19, Corollary 5.7] applied to the operator Q, we get that $\mathrm{supp}(u(t)) = \mathrm{supp}(u(t)\mathrm{e}^{\| \gamma \|_\infty t}) = \mathcal{F}(A)$ for $t > 0$. Differentiating (47) leads to:

$$u'(t) - F(u(t)) = (a - \varphi(u(t)))Tu(t). \tag{48}$$

We deduce from (46) and (47) that $\| u(t) \|_\infty < \eta$ for $t \in [0,c]$, and thus, by definition of a and η, that $u'(t) - F(u(t)) \leqslant 0$ for $t \in [0,c]$. Then, since $u(0) = f \leqslant h$, Theorem 7.2 implies that $u(t) \leqslant \phi(t,h)$ for $t \in [0,c]$, and thus $\mathcal{F}(A) \subset \mathrm{supp}(\phi(t,h))$ for $t \in (0,c]$. Then, use the semi-flow equation (45) to propagate the result to all $t > 0$.

We deduce from Lemma 3.5 (iii) that $\mathrm{supp}(\phi(t,h)) = \mathrm{supp}(\phi_{\mathcal{F}(A)}(t,h)) \subset \mathcal{F}(A)$ for all $t \geqslant 0$. This gives that $\mathrm{supp}(\phi(t,h)) = \mathcal{F}(A)$ for all $t > 0$. □

9.2. **Preliminary results on the spectral radius and bound.** We refer to [52, Section 3] for results on the spectral bound on Banach lattices defined by (18).

Lemma 9.2 (Spectral radius and spectral bound). *Let (T, γ, φ) that satisfy Assumption 2. Let $\delta : \Omega \to \mathbb{R}$ be a measurable positive function with $\delta \geqslant \gamma$ and* $\operatorname{essinf} \delta > 0$. *Then the quantities $s(T - \delta)$, $s(\hat{T} - \delta)$, $\rho(T_{1/\delta}) - 1$ and $\rho(\hat{T}_{1/\delta}) - 1$ have the same sign.*

Proof. Notice that (T, δ, φ) also satisfies Assumption 2. By Lemma 2.8 (e) with γ replaced by δ, we have that $\rho(\hat{T}_{1/\delta}) = \rho(T_{1/\delta})$. So it is enough to prove that $s(\hat{T} - \delta)$ and $\rho(\hat{T}_{1/\delta}) - 1$ have the same sign, and that $s(T - \delta)$ and $\rho(T_{1/\delta}) - 1$ have the same sign. This can be done by mimicking the proof of [17, Proposition 4.1] based on [52], noticing that the cone L^p_+ is normal and reproducing for $p \in [1, +\infty]$, and that, by Lemma 2.8, the operators $\hat{T}_{1/\delta}$ and $T_{1/\delta}$ are respectively compact on L^p and power compact on L^∞, and that the linear maps $\hat{T} - \delta$ and $T - \delta$ are operators respectively on L^p and L^∞. □

Adapting the proof of [17, Proposition 4.2] on kernel operators, we provide a weaker link between $\rho(\hat{T}_{1/\gamma}) - 1$ and $s(\hat{T} - \gamma)$ without the condition $\operatorname{essinf} \gamma > 0$.

Proposition 9.3 (Positive spectral bound and Krein-Rutman theorem). *Let (T, γ, φ) that satisfy Assumption 2. Then the following assertions are equivalent:*

(i) $s(T - \gamma) > 0$ or equivalently $s(\hat{T} - \gamma) > 0$.
(ii) $\rho(T_{1/\gamma}) > 1$ or equivalently $\rho(\hat{T}_{1/\gamma}) > 1$.
(iii) There exists $\lambda > 0$ and $w \in L^\infty_+ \backslash \{\mathbb{0}\}$ such that we have $Tw - \gamma w = \lambda w$.

Proof. Recall Assumption 2 holds and $p \in (1, +\infty)$. Since $s(A - (\gamma + \varepsilon)) = s(A - \gamma) - \varepsilon$ for $\varepsilon \in \mathbb{R}$ and A equal to T or $\hat{T}$, we deduce from Lemma 9.2 that the two conditions in Point (i) are equivalent. We also deduce from Lemma 2.8 (e) that the two conditions in Point (ii) are equivalent. So, we shall only consider the second ones. It is immediate that Point (iii) implies Point (i) as $L^\infty \subset L^p$ and T and $\hat{T}$ coincide on L^∞.

We assume Point (i) and prove Point (ii). For any $a \in \mathbb{R}_+$, we denote $\psi(a) = \rho(V_a)$ with $V_a = \hat{T}_{1/(\gamma+a)}$. Notice that $V_a = \hat{T} M_{1/(\gamma+a)}$ for $a > 0$. By Assumption 2, the operator V_a on L^p is positive and that $V_a \geqslant V_b$ for $0 \leqslant a \leqslant b$. Thus the map ψ is non-increasing on $\mathbb{R}_+$ by (19). By Point (i), there exists $\varepsilon > 0$ such that $s(\hat{T} - (\gamma + \varepsilon)) = s(\hat{T} - \gamma) - \varepsilon > 0$, therefore we have $\psi(\varepsilon) > 1$ by Lemma 9.2 applied to $\delta = \gamma + \varepsilon$. We thus get $\psi(0) = \rho(\hat{T}_{1/\gamma}) > 1$, that is Point (ii).

We assume Point (ii) and prove Point (iii). By Point (ii), we have $\psi(0) > 1$. As for all $a > 0$, we have $\psi(a) \leqslant \| V_a \|_{L^p} \leqslant a^{-1} \left\| \hat{T} \right\|_{L^p}$. We deduce that $\lim_{a\to\infty} \psi(a) = 0$.

We now prove that ψ is continuous on $\mathbb{R}_+$. Let B denote the unit ball of L^p. Notice that $M_{1/(\gamma+a)}(B) \subset M_{1/\gamma}(B)$ for $a \in \mathbb{R}_+$ and thus $\bigcup_{a\in\mathbb{R}_+} V_a(B) = V_0(B) = \hat{T}_{1/\gamma}(B)$ is relatively compact in L^p, and thus the family $(V_a)_{a\in\mathbb{R}_+}$ is collectively compact. Thanks to Lemma 2.3, the continuity of ψ holds if $\lim_{|a-b|\to 0} \| (V_a - V_b) f \|_p = 0$ for any $f \in L^p$. This is indeed the case as, for $f \in L^p$, we have:

$$\| (V_a - V_b) f \|_p = \left\| \hat{T}_{1/\gamma} \left(\frac{(b-a)\gamma}{(\gamma+a)(\gamma+b)} f \right) \right\|_p \leqslant \left\| \hat{T}_{1/\gamma} \right\|_{L^p} \left\| \frac{(b-a)\gamma}{(\gamma+a)(\gamma+b)} f \right\|_p,$$

and the right members goes to 0 as $|a - b|$ goes to 0 using $|b - a|\gamma/(\gamma + a)(\gamma + b) \leqslant 1$ and dominated convergence. In conclusion, the function ψ is continuous on $\mathbb{R}_+$.

Since $\psi(0) > 1$ and $\lim_{a\to\infty} \psi(a) = 0$, we deduce from the continuity of ψ, that there exists $\lambda > 0$ such that $\psi(\lambda) = 1$. Thus by the Krein-Rutman Theorem 2.5 (i) applied to the positive compact operator V_λ on L^p, there exists $v \in L^p_+ \backslash \{\mathbb{0}\}$ such that $V_\lambda v = v$. Thanks to (6), we have $\| v \|_\infty = \| V_\lambda v \|_\infty \leqslant \| T(v/\gamma) \|_\infty \leqslant C'_p \| v \|_p$, we deduce that v belongs also to L^∞. Setting $w = v/(\gamma + \lambda) \in L^\infty_+ \backslash \{\mathbb{0}\}$, we get that $\hat{T}w - \gamma w = \lambda w$. As $\hat{T}$ and T coincide on L^∞, we get Point (iii). □

9.3. **Proof of Proposition 4.3 (i).** Let g be a non-zero equilibrium. By Assumption 2, $\varphi(g) < 1$ on $\text{supp}(g)$ (as $\varphi(r) < 1$ for $r \in (0,1]$). Recall the operator $S = \hat{T}_{1/\gamma}$ is compact on L^p, see Lemma 2.8 (c). Since $g \in \Delta$ is an equilibrium, we obtain that:

$$S(\gamma g) = \frac{\gamma g}{\varphi(g)} > \gamma g \quad \text{on} \quad \text{supp}(g).$$

We deduce from Lemma 2.6 (ii), with $\lambda = 1$, and (19) that $R_0 = \rho(S) \geqslant \rho\left(S_{\text{supp}(g)}\right) > 1$. In other words, $R_0 \leqslant 1$ implies that $\mathbb{0}$ is the only equilibrium. The last part of Point (i) is a consequence of Proposition 3.1 (iii) and the monotonicity of the semi-flow from Lemma 8.5 (i).

9.4. **Proof of Proposition 4.3 (ii).** We assume that we have $R_0 > 1$. Similarly to [17, Section 4.4], we will prove that there exists a non-zero initial condition $w \in \Delta$ such that the semi-flow $\phi(., w)$ is non-decreasing. As $R_0 > 1$, there exists $a \in (0,1)$ such that $\rho\left(a\hat{T}_{1/\gamma}\right) > 1$. Thus, by Proposition 9.3 (with (aT, γ, φ)), there exists $\lambda > 0$ and $w \in L^\infty_+ \backslash \{\mathbb{0}\}$ such that $aTw - \gamma w = \lambda w$. By Assumption 2, the map φ is continuous with $\varphi(0) = 1$; thus there exists $\eta \in (0,1)$ such that for all $r \in [0,\eta]$, we have $\varphi(r) \geqslant a$. Without loss of generality, we assume that $\| w \|_\infty \leqslant \eta$, and thus $w \in \Delta$. We deduce that:

$$F(w) = \varphi(w)Tw - \gamma w \geqslant aTw - \gamma w = \lambda w \geqslant 0.$$

By Lemma 8.5 (ii), the semi-flow $t \mapsto \phi(t, w)$ is thus non-decreasing on $\mathbb{R}_+$ and its essential limit, say g, exists and belongs to Δ. It is an equilibrium by Lemma 8.6. Let g^* denote the maximal equilibrium. As we have $g^* \geqslant g \geqslant w$ and $\mu(\text{supp}(w)) > 0$, we deduce that $\mu(\text{supp}(g^*)) > 0$.

9.5. **Proof of Proposition 4.3 (iii).** We now assume that $T = T_A$ with A an irreducible set. Notice the set A is invariant and thus admissible; and it has positive measure as $R_0 > 0$. It is thus a (non-zero) atom by [19, Theorem 1]. Let g be a non-zero equilibrium. We have $\text{supp}(g) \subset A$ by (4). Since $\text{supp}(g)$ is invariant, see Lemma 4.1 (iii), and A is an atom, we deduce that $\text{supp}(g) = A$. Then use that the support of g characterizes g, see Corollary 4.2 to deduce that g is the only non-zero equilibrium.

9.6. **Proof of Proposition 4.3 (iv).** By Point (iii), we have $\text{supp}(g^*) = A$. On A^c, we have $\phi(t,h)' = -\gamma\phi(t,h)$, so that $\text{limess}_{t\to\infty} \phi(t,h)\mathbb{1}_{A^c} = \mathbb{0}$. So it is enough to prove the result when $\text{supp}(h) \subset \text{supp}(g^*) = A$. As $\text{supp}(h)$ is a non-empty set included in the invariant and irreducible set A, its future is equal to A. Thus, by Lemma 9.1 and considering $\phi(1,h)$ instead of h, one can assume without loss of generality that $\text{supp}(h) = \text{supp}(g^*) = A$.

For $\varepsilon \in (0,1]$, we consider the operator $U_\varepsilon = \varphi(\varepsilon)M_{\{h\geqslant\varepsilon\}}\hat{T}_{1/\gamma}$ on $L^p(\mu)$, and set $U_0 = M_{\{h>0\}}\hat{T}_{1/\gamma}$. Let B be the unit ball in L^p. Since $\hat{T}_{1/\gamma}$ is compact and $\lim_{\varepsilon\to 0} \varphi(\varepsilon)\mathbb{1}_{\{h\geqslant\varepsilon\}} = \mathbb{1}_{\{h>0\}}$ a.e., we deduce that $\bigcup_{\varepsilon\in[0,1]} U_\varepsilon(B)$ is relatively compact. Thus the family of operators $(U_\varepsilon)_{\varepsilon\in[0,1]}$ is collectively compact. By dominated convergence, we also get that $\lim_{\varepsilon\to 0} \| (U_\varepsilon - U_0)f \|_p = 0$. We deduce from Lemma 2.3 that the map $\varepsilon \mapsto \rho(U_\varepsilon)$ is continuous at 0. Thus, there exists $\varepsilon \in (0,1)$ such that $\rho(U_\varepsilon) > 1$. Notice that $(\varphi(\varepsilon)\mathbb{1}_{\{h\geqslant\varepsilon\}}T, \gamma, \varphi)$ satisfies Assumption 2. By Proposition 9.3 with T replaced by $\varphi(\varepsilon)\mathbb{1}_{\{h\geqslant\varepsilon\}}T$, there exists $\lambda > 0$ and $w \in L^\infty_+\backslash\{\mathbb{0}\}$ such that:

$$\varphi(\varepsilon)\mathbb{1}_{\{h\geqslant\varepsilon\}}Tw - \gamma w = \lambda w. \tag{49}$$

Without loss of generality, we can assume that $\| w \|_\infty \leqslant \varepsilon$. We also have:

$$\text{supp}(w) \subset \{h \geqslant \varepsilon\} \subset A. \tag{50}$$

Using (49) we get that:

$$F(w) = \varphi(w)Tw - \gamma w \geqslant \varphi(\varepsilon)Tw - \gamma w \geqslant \lambda w \geqslant 0. \tag{51}$$

We deduce that the map $t \mapsto \phi(t,w)$ is non-decreasing by Lemma 8.5 (ii) and, as g^* is the only non-zero equilibrium, that $\text{limess}_{t\to\infty} \phi(t,w) = g^*$ by Lemma 8.6. As the semi-flow is monotone by Lemma 8.5 (i), we deduce that $\phi(t,w) \leqslant \phi(t,h) \leqslant \phi(t,\mathbb{1})$ for all $t \in \mathbb{R}_+$. Then use Proposition 3.1 (iii) to conclude that $\text{limess}_{t\to\infty} \phi(t,h) = g^*$.

10. Proof of Theorem 5.1

Let (T, γ, φ) that satisfy Assumption 2. We keep notations from Section 4; so $\mathcal{C}_A$ is the supercritical antichain given by the maximal elements of the supercritical atoms included in the set A.

Proof of Theorem 5.1. Let $A = \mathcal{F}(\text{supp}(h))$ and g_A^* be the maximal equilibrium on A (notice that $A = \varnothing$ if $h = \mathbb{0}$). Since A is invariant, g_A^* is also an equilibrium by Lemma 3.4 (ii).

We will, as in the proof of Proposition 4.3 (iv), prove the existence of $w \in \Delta$ with $w \leqslant h$ such that the semi-flow $(\phi(t, w))_{t \in \mathbb{R}_+}$ is non-decreasing and converges essentially to g_A^*. By Lemma 9.1 and considering $\phi(1, h)$ instead of h, one can assume without loss of generality that $\text{supp}(h) = A$. If $\mathcal{C}_A$ is empty, we get that $g_A^* = \mathbb{0}$ and by Lemma 3.5 (i) and (iii), we have $\text{limess}_{t \to +\infty} \phi(t, \mathbb{1}_A) = \mathbb{0}$, and by monotonicity of the semi-flow that $\text{limess}_{t \to +\infty} \phi(t, h) = \mathbb{0}$, which proves Theorem 5.1 in this case.

We now assume that $\mathcal{C}_A$ is not empty. Let $B \in \mathcal{C}_A$. By considering T_B instead of T, mimicking the proof of Proposition 4.3 (iv), see (50) and (51), we deduce that there exists a function $w_B \in \Delta$ such that $\text{supp}(w_B) \subset B$, $w_B \leqslant h$, $F(w_B) \geqslant 0$ and $\text{limess}_{t \to \infty} \phi_B(t, w_B) = g_B^*$. Since $\mathcal{C}_A$ is an antichain of atoms, we deduce that $\mathcal{F}(\text{supp}(w_B)) \cap \text{supp}(w_{B'}) \subset \mathcal{F}(B) \cap B' = \varnothing$ for all $B', B \in \mathcal{C}_A$ such that $B \neq B'$. Set $w = \sum_{B \in \mathcal{C}_A} w_B \leqslant h$. We have:

$$F(w) = \sum_{B \in \mathcal{C}_A} F(w_B) + \mathbb{1}_{\mathcal{F}(B)} \left(\varphi(w) - \varphi(w_B) \right) T w_B = \sum_{B \in \mathcal{C}_A} F(w_B) \geqslant 0,$$

where for the first equality we used that $\mathcal{F}(B)$ is invariant and $\text{supp}(w_B) \subset B \subset \mathcal{F}(B)$, and for the second that $\varphi(w) - \varphi(w_B) = 0$ on $\mathcal{F}(B)$ for $B \in \mathcal{C}_A$. Arguing as in the end of the proof of Proposition 4.3 (iv), we deduce that the semi-flow $(\phi(t, w))_{t \in \mathbb{R}_+}$ is non-decreasing and that $\text{limess}_{t \to \infty} \phi(t, w) = g \in \Delta$ with g an equilibrium.

We now prove the equality $\mathcal{C}_A = \mathcal{C}_g = \mathcal{C}_{g_A^*}$. Since $w \leqslant g$, we deduce that $\mathcal{C}_A \subset \mathcal{C}_g$. Since $\text{supp}(w) \subset A$ and A is invariant, we have $\text{supp}(g) \subset A$ by Lemma 9.1. In particular, g is an equilibrium of A by Lemma 3.4 (i), thus we get $g \leqslant g_A^*$ and deduce that $\mathcal{C}_g \subset \mathcal{C}_{g_A^*} \subset \mathcal{C}_A$ as $\text{supp}(g_A^*) \subset A$. This proves the claim and that $g = g_A^*$ by Corollary 4.2.

By monotonicity of the semi-flow, we have that $\phi(t, w) \leqslant \phi(t, h) \leqslant \phi(t, \mathbb{1}_A)$ for all $t \geqslant 0$. The left term converges essentially to $g = g_A^*$, and the right term to g_A^* by Lemma 3.5. This gives $\text{limess}_{t \to \infty} \phi(t, h) = g_A^*$. This ends the proof. □

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