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Some properties of the exit measure for super Brownian motion

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Abstract. We consider the exit measure of super Brownian motion with a stable branching mechanism of a smooth domain D of $\mathbb{R}^d$. We derive lower bounds for the hitting probability of small balls for the exit measure and upper bounds in the critical dimension. This completes results given by Sheu [22] and generalizes the results of Abraham and Le Gall [2]. Because of the links between exits measure and partial differential equations, those results imply bounds on solutions of elliptic semi-linear PDE. We also give the Hausdorff dimension of the support of the exit measure and show it is totally disconnected in high dimension. Eventually we prove the exit measure is singular with respect to the surface measure on ∂D in the critical dimension. Our main tool is the subordinated Brownian snake introduced by Bertoin, Le Gall and Le Jan [4].

1. Introduction

1.1. *Presentation of the results*

A superprocess $X = (X_t, \mathbb{P}^X_\nu)$ on $\mathbb{R}^d$ is a Markov process taking values in the space of finite measures on $\mathbb{R}^d$, M_f, which describes the evolution of a cloud of branching particles. We refer to Dynkin [10] and Dawson [6] for a detailed introduction to the subject. We consider here the α-super Brownian motion X, where $\alpha \in (1, 2]$. We introduce the notation $(\nu, f) = \int f(y)\nu(dy)$, where the measure $\nu \in M_f$ and the function f is an element of $\mathcal{B}(\mathbb{R}^d)$, the set of measurable functions on $\mathbb{R}^d$ taking values in $\mathbb{R}$. The law of X is characterized by its Laplace transform:

- $X_0 = \nu \qquad \mathbb{P}^X_\nu$-a.s.
- Let γ be a Brownian motion in $\mathbb{R}^d$ starting at x. We denote by P_x its law. For every nonnegative bounded function $f \in \mathcal{B}(\mathbb{R}^d)$, and for every $t \geq s \geq 0$,

$$\mathbb{E}^X_\nu \left[e^{-(X_t, f)} \mid \sigma(X_u, 0 \leq u \leq s) \right] = e^{-\left(X_s, v(t-s,\cdot)\right)}$$

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where v is the unique nonnegative measurable solution of the integral equation:

$$v(t,x) + \mathrm{E}_x\left[\int_0^t ds\, v(s,\gamma_{t-s})^\alpha\right] = \mathrm{E}_x\big[f(\gamma_t)\big].$$

Let D be a bounded domain (i.e. open connected subset) of $\mathbb{R}^d$. The goal of this paper is to study the exit measure of D for the α-super Brownian motion. This is a measure on ∂D introduced by Dynkin in [11] which describes the position of the cloud of particles at their first exit time from D. This exit measure is related to the semi-linear partial differential equation $\frac{1}{2}\Delta u = u^\alpha$ in D.

More precisely, the law of the exit measure is characterized by: for every $\nu \in M_f$, such that supp $\nu \subset D$, for every nonnegative bounded function $f \in \mathcal{B}(\mathbb{R}^d)$,

$$\mathbb{E}_\nu^X\left[\mathrm{e}^{-(X_D,f)}\right] = \mathrm{e}^{-(\nu,v)},$$

where v is the unique nonnegative measurable solution of the integral equation

$$v(x) + \mathrm{E}_x\left[\int_0^{\kappa_D} ds\; v(\gamma_s)^\alpha\right] = \mathrm{E}_x[f(\gamma_{\kappa_D})], \quad x \in D. \tag{1}$$

The stopping time $\kappa_D = \inf\{s > 0; \gamma_s \notin D\}$, with the convention $\inf \emptyset = +\infty$, is the first exit time of D for γ. The function v solves $\frac{1}{2}\,\Delta u = u^\alpha$ in D.

From now on, we assume D is regular. If f is continuous, then v is continuous in $\bar{D}$ and is the unique nonnegative solution of the Dirichlet problem:

$$\begin{cases} \frac{1}{2}\Delta u = u^\alpha & \text{in } D, \\ u_{|\partial D} = f. \end{cases}$$

We will be mainly interested in solution of $\frac{1}{2}\,\Delta u = u^\alpha$ in D with boundary conditions which blow up. Let $y_0 \in \partial D$ be fixed. The set $B_{\partial D}(y_0,\varepsilon) = \{y \in \partial D; |y - y_0| < \varepsilon\}$ is a ball on the boundary of D. We will also write B_ε when there is no confusion. We write δ_x for the Dirac mass at point $x \in \mathbb{R}^d$. From [14] (see also [12] theorem 1.4 and remark 4.3), the function

$$u_\varepsilon(x) = -\log \mathbb{P}_{\delta_x}^X[X_D(B_\varepsilon) = 0], \quad x \in D, \tag{2}$$

is the minimal nonnegative solution of:

$$\begin{cases} \frac{1}{2}\,\Delta u = u^\alpha & \text{in} \quad D, \\ \lim_{x\to y, x\in D} u(x) = \infty & \text{where} \quad y \in B_\varepsilon. \end{cases}$$

Let $\mathcal{R}_D$ be the range of the α-super Brownian motion in D (it can be viewed as the range of the α-superprocess where the underlying Brownian motion is replaced by a Brownian motion killed out of D). From [15] theorem 2.5 (see also [12] theorem 2.1 and remark 4.3) the function $v_\varepsilon(x) = -\log \mathbb{P}_{\delta_x}^X[\mathcal{R}_D \cap \overline{B_\varepsilon} = \emptyset]$ is the maximal solution of:

$$\begin{cases} \frac{1}{2}\,\Delta u = u^\alpha & \text{in} \quad D, \\ \lim_{x\to y, x\in D} u(x) = 0 & \text{where} \quad y \in \partial D \backslash \overline{B_\varepsilon}, \end{cases} \tag{3}$$

and $\overline{B_\varepsilon}$ denotes the closure of B_ε. There is a natural way to build $\mathcal{R}_D$ and X_D on the same probability space (see [12]). Let $(F_n, n \geq 1)$ be an increasing sequence of closed sets such that $F_n \subset \bar{D}\backslash\overline{B_\varepsilon}$ and $\bigcup_{n\geq 1} F_n = \bar{D}\backslash\overline{B_\varepsilon}$. Since $\mathcal{R}_D$ is a.s. a closed subset of $\overline{D}$, we have a.s.

$$\left\{\mathcal{R}_D \subset \bar{D}\backslash\overline{B_\varepsilon}\right\} = \bigcup_{n\geq 1} \{\mathcal{R}_D \subset F_n\}.$$

By lemma 2.1 of [12] with $Q = \mathbb{R} \times D$, we have the following inclusion:

$$\{\mathcal{R}_D \subset F_n\} \subset \left\{X_D(F_n^c) = 0\right\}.$$

Since $\left\{X_D(F_n^c) = 0\right\} \subset \{X_D(B_\varepsilon) = 0\}$, we deduce the inclusion

$$\left\{\mathcal{R}_D \subset \bar{D}\backslash\overline{B_\varepsilon}\right\} \subset \{X_D(B_\varepsilon) = 0\}.$$

As a consequence we have $u_\varepsilon \leq v_\varepsilon$ in D, and we deduce that u_ε is the minimal nonnegative solution of

$$\begin{cases} \frac{1}{2}\,\Delta u = u^\alpha & \text{in} \quad D \\ \lim_{x\to y, x\in D} u(x) = 0 & \text{where} \quad y \in \partial D\backslash\overline{B_\varepsilon} \\ \lim_{x\to y, x\in D} u(x) = \infty & \text{where} \quad y \in B_\varepsilon. \end{cases} \tag{4}$$

From now on we assume that D is of class C^2. In particular D enjoys the uniform outer ball condition. Using the exit measure X_D we prove the following results on the semi-linear PDE.

Theorem 1.1. *For $\varepsilon > 0$, small enough, the function u_ε is the unique nonnegative measurable solution of (4).*

In particular u_ε is the maximal nonnegative solution of (3). We can describe the behavior of u_ε as a function of ε. The critical dimension is $d_c = (\alpha + 1)/(\alpha - 1)$. Let us introduce the function $\varphi_d(\varepsilon)$ defined on $(0, \infty)$ by:

$$\varphi_d(\varepsilon) = \begin{cases} 1 & \text{if } d < d_c \\ \left[\log(1/\varepsilon)\right]^{-1/(\alpha-1)} & \text{if } d = d_c \\ \varepsilon^{d-d_c} & \text{if } d > d_c. \end{cases}$$

We first give a lower bound of u_ε.

Theorem 1.2. *Let K be a compact subset of D. There exist positive constants c_d and ε_0, such that for every $\varepsilon \in (0, \varepsilon_0]$, $x \in K$, we have*

$$c_d\varphi_d(\varepsilon) \leq u_\varepsilon(x).$$

An upper bound for v_ε and thus for u_ε was given by Sheu (see lemma 4.2 and the following remark in [22]) for $d \neq d_c$. Let K be a compact subset of D. He proved there exist positive constants C_d and ε_0, such that for every $\varepsilon \in (0, \varepsilon_0]$, $x \in K$, we have

$$u_\varepsilon(x) \le C_d \varphi_d(\varepsilon).$$

The critical dimension is more delicate. It was proved by Abraham and Le Gall in [2] for the particular case $\alpha = 2$. For a general $\alpha \in (1, 2]$, we get:

Theorem 1.3. *Let $d = d_c$. Let K be a compact subset of D. There exist positive constants C and ε_0, such that for every $\varepsilon \in (0, \varepsilon_0]$, $x \in K$, we have*

$$u_\varepsilon(x) \le C \left[\log(1/\varepsilon)\right]^{-1}.$$

However, the proof of this theorem suggests that the upper bound should be $\varphi_{d_c}(\varepsilon) = \left[\log(1/\varepsilon)\right]^{-1/(\alpha-1)}$. (This is explained in Remark 6.2).

As an immediate corollary of the above results, we get that if K is a compact subset of D, then there exist positive constants C_d, ε_0 such that any solution of (3) with $\varepsilon \in (0, \varepsilon_0]$ is bounded from above by $C_d\varphi(\varepsilon)$ if $d \neq d_c$ or by $C_{d_c}\left[\log(1/\varepsilon)\right]^{-1}$ if $d = d_c$.

We give now results on the exit measure X_D. Sheu [22] proved that if $d > d_c$ (resp. $d < d_c$) then a.s. the exit measure, X_D, is singular (resp. absolutely continuous) with respect to the surface measure on D. As a consequence of the above theorem, we get:

Corollary 1.4. *Let $\nu \in M_f$ with its support in D. $\mathbb{P}_\nu^X$-a.s., the measure X_D is singular (resp. absolutely continuous) with respect to the Lebesgue measure on ∂D if and only if $d \ge d_c$ (resp. $d < d_c$).*

Proof. The case $d \neq d_c$ is from [22] theorems 3.3 and 4.3. Let us consider the critical case. Let $y_0 \in \partial D$ and $B_\varepsilon = B_{\partial D}(y_0, \varepsilon)$. From the definition of X_D, we have for $\nu \in M_f$ with its support in D, and $x \in D$,

$$\mathbb{E}_\nu^X\left[\mathrm{e}^{-\lambda X_D(B_\varepsilon)}\right] = \mathrm{e}^{-(\nu, u^\lambda)} \quad \text{and} \quad \mathbb{E}_{\delta_x}^X\left[\mathrm{e}^{-\lambda X_D(B_\varepsilon)}\right] = \mathrm{e}^{-u^\lambda(x)},$$

where u^λ is the solution of (1) with $f = \lambda \mathbf{1}_{B_\varepsilon}$. Letting λ go to infinity in the latter equation, we deduce that $u^\lambda(x)$ converges to $-\log \mathbb{P}_{\delta_x}^X[X_D(B_\varepsilon) = 0] = u_\varepsilon(x)$. Therefore, by letting λ go to infinity in the former equality, we get

$$\mathbb{P}_\nu^X[X_D(B_\varepsilon) > 0] = 1 - \mathrm{e}^{-(\nu, u_\varepsilon)}.$$

This can also be seen as a consequence of the well known cluster representation of the superprocesses. Thanks to theorem 1.3, taking the limit as ε goes to 0, we get $\mathbb{P}_\nu^X[y_0 \in \mathrm{supp}\ X_D] = 0$ for every $y_0 \in \partial D$. By integrating with respect to $\theta(dy_0)$, the Lebesgue measure on ∂D, we get

$$\mathbb{E}_\nu^X\left[\theta(\mathrm{supp}\ X_D)\right] = 0,$$

which gives the result. □

If $A \in \mathcal{B}(\mathbb{R}^d)$, we denote by dim A its Hausdorff dimension. An upper bound of the Hausdorff dimension of the support of the exit measure was given in [22]. We complete this result with the following theorem.

Theorem 1.5. *Let $\nu \in M_f$ with its support in D. $\mathbb{P}_\nu^X$-a.s. on $\{X_D \neq 0\}$, we have*

$$\dim \operatorname{supp} X_D = \frac{2}{\alpha - 1} \wedge (d-1).$$

Once we have the result on the hitting probability of small balls of the boundary of ∂D, we can derive a result on the connected components of X_D (see [1] for more result in the particular case $\alpha = 2$).

Theorem 1.6. *If $d > 2d_c - 1$, then $\mathbb{P}_\nu^X$-a.s. the support of X_D is totally disconnected.*

1.2. Description of the proofs

The main tool used here to study α-super Brownian motion is the Brownian snake, a path valued Markov process, introduced by Le Gall. Unfortunately, this process gives only a representation of superprocesses for $\alpha = 2$. To treat general α's, we used a subordination method described in [4]. The intuitive idea of subordination is to consider (in the particles system picture) Brownian particles and to "freeze" them from time to time. While these particles are motionless, the branching mechanism still goes on. Thus, we can consider the particle paths as Brownian paths along which some masses are added (these masses correspond to the "freezing times"). When there is a large mass, a lot of branching occur at this point (a large mass corresponds to a large interval of time during which the particle is motionless). Hence, the paths of the range are still Brownian paths but the branching mechanism has changed. To get the desired superprocess, the "freezing times" are given by jumps of a subordinator. The construction of the Brownian snake and the subordination procedure will be developed in section 2.

The proof of the lower bound of u_ε (theorem 1.2) in section 3 uses the integral equation (1) and bounds on the Poisson kernel and Green function in D.

Section 4 is devoted to some technical lemmas on the typical behavior of the snake paths near the end points. They are generalizations of results from [21] and [2] where $\alpha = 2$. The proofs are more involved because of the time change. We have to look not only at the spatial motion but also at the time change. For a first reading, this section may be skipped, but for the notations and remarks of subsection 4.1.

We then use those lemmas to prove uniqueness (theorem 1.1) in section 5. This proof relies heavily on the snake construction.

The proof of the upper bound of u_ε in the critical dimension (theorem 1.3) in section 6 is based on bounds of the Poisson kernel. For these bounds to be accurate, we need to control the behavior of the snake paths near their end point, this is the aim of the lemmas of section 4. We follow the proof of theorem 4.1 in [2], but the arguments are more delicate because of the subordination method.

The upper bound in theorem 1.5 is due to Sheu [22]. The lower bound of the Hausdorff dimension of the exit measure is proved in section 7. And theorem 1.6

on connected component is proved in section 8. Those results are the elliptic counterpart of section 5.2 and theorem 2.4 in [8].

Eventually the appendix deals with the law of the time reversal of stable subordinators. Some notations are recalled at the end of the appendix.

All the theorems were known for $\alpha = 2$. From now on we assume that $\alpha \in (1, 2)$. We denote by c a generic non trivial constant whose value may vary from line to line.

2. The subordination approach to superprocesses

2.1. The Brownian snake

Let E be a Polish space and (β_t) a càdlàg Markov process with values in E. Let $\mathcal{W}$ be the set of all killed paths in E. By definition, a killed path in E is a càdlàg mapping $w : [0, \zeta) \longrightarrow E$ where $\zeta = \zeta_w > 0$ is called the lifetime of the path. By convention, we also agree that every point $z \in E$ is a killed path of lifetime 0. Let us fix $z \in E$ and let us denote by $\mathcal{W}_z$ the subset of $\mathcal{W}$ of all killed paths w with initial point $w(0) = z$ (in particular, $z \in \mathcal{W}_z$).

From proposition 5 of [4], we know that there exists a continuous strong Markov process in $\mathcal{W}_z$, denoted by $W = (W_s, s \geq 0)$, whose law is described by:

- The lifetime process $(\zeta_t, t \geq 0)$ is a one-dimensional reflecting Brownian motion in $\mathbb{R}^+$.
- Conditionally on $(\zeta_t, t \geq 0)$, the process $(W_t, t \geq 0)$ is still a Markov process whose transition kernels are described by: let $0 < s < s'$ and $m_{s,s'} = \inf_{u\in[s,s']} \zeta_u$.
  - $W_s(u) = W_{s'}(u)$ for every $u \in [0, m_{s,s'}]$.
  - The processes $\big(W_s(u + m_{s,s'}), u \geq 0\big)$ and $\big(W_{s'}(u + m_{s,s'}), u \geq 0\big)$ are conditionally on $\beta_0 = W_s(m_{s,s'})$, independent and distributed as the Markov process β killed respectively at times $\zeta_s - m_{s,s'}$ and $\zeta_{s'} - m_{s,s'}$.

Remark 2.1. Notice this description is not really complete. Indeed, if β is not continuous, the quantity $W_s(m_{s,s'})$ is not defined when $m_{s,s'}$ is equal to ζ_s or ζ_s'. However this description gives the right intuition. Furthermore under rather general conditions on β, a.s. $W_s(t)$ has a limit as t goes to the lifetime ζ_s. In this case $W_s(m_{s,s'})$ will be well defined. We refer to [4] for a precise description.

2.2. The subordination method

Our main goal in this section is to recall from [4] how superprocesses with a general branching mechanism can be constructed using the Brownian snake and a subordination method. As mentioned in the introduction, the idea is to use the previous construction using for β a "freezed" Brownian motion.

Let $S = (S_t, t \geq 0)$ be a ρ-stable subordinator, where $\rho = \alpha - 1$, whose Laplace transform is: for $\lambda \geq 0$, $\bar{\mathrm{E}}\left[\mathrm{e}^{-\lambda S_t}\right] = \mathrm{e}^{-c^*_\rho t \lambda^\rho}$. We set $c^*_\rho = 2^{-\rho}/\,\Gamma(1+\rho)$, this choice will be explained in section 2.4. We denote by ξ the associated residual lifetime process defined by $\xi_t = \inf\{S_s - t;\, S_s > t\}$, and by L the right continuous inverse of S, $L_t = \inf\{s;\, S_s > t\}$, so that L can be viewed as the local time at 0 of the Markov process ξ. Let $\gamma = (\gamma_t, t \geq 0)$ be an independent Brownian motion in $\mathbb{R}^d$. We would like here to take $\beta = \gamma \circ L$, which is a Brownian motion freezed at random times. However, this is not a Markov process and so the previous construction does not apply. For this reason, we will consider the process $\bar{\xi}_t = \left(\xi_t, L_t, \gamma_{L_t}\right)$ which is a Markov process with values in $E = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^d$. The second component will give the time change and only the third component will really give the spatial motion.

Let $\bar{\mathrm{P}}_z$ be the law of $\bar{\xi}$ started at $z \in E$. For simplicity we write $\Gamma_t = \gamma_{L_t}$, and $\bar{\mathrm{P}}_x = \bar{\mathrm{P}}_z$ when $z = (0, 0, x)$. We denote by $W = (W_s, s \geq 0)$ the Brownian snake with spatial motion $\bar{\xi}$. Denote by $\mathbb{E}_w$ the probability measure under which W starts at w, and by $\mathbb{E}^*_w$ the probability under which W starts at w and is killed when ζ reaches zero. We introduce an obvious notation for the coordinates of a path $w \in \mathcal{W}$:

$$w(t) = (\xi_t(w), L_t(w), \Gamma_t(w)) \quad \text{for} \quad 0 \leq t < \zeta_w.$$

We set $\hat{w} = \lim_{t\uparrow\zeta_w} \Gamma_t(w)$ (resp $\hat{L}(w) = \lim_{t\uparrow\zeta_w} L_t(w)$) if the limit exists, $\hat{w} = \partial$ (resp. $\hat{L}(w) = \partial'$) otherwise, where ∂ (resp. ∂') is a cemetery point added to $\mathbb{R}^d$ (resp $\mathbb{R}$). Some continuity properties hold for the process W (see [4] lemma 10 and [8] lemma 5.3). Fix $w_0 \in \mathcal{W}_z$, such that the functions $t \mapsto L_t(w_0)$ and $t \mapsto \Gamma_t(w_0)$ are continuous on $[0, \zeta_{w_0})$ and have a continuous extension on $[0, \zeta_{w_0}]$. Then $\mathbb{E}_{w_0}$-a.s. the mappings $s \mapsto \left(L_{t\wedge\zeta_s}(W_s), t \geq 0\right)$ and $s \mapsto \left(\Gamma_{t\wedge\zeta_s}(W_s), t \geq 0\right)$ are continuous with respect to the uniform topology on the set of continuous functions defined on $\mathbb{R}^+$. In particular, the processes $\hat{W}_s$ and $\hat{L}(W_s)$ are well defined and continuous $\mathbb{E}_{w_0}$-a.s.

It is clear that the trivial path $z \in E$ is a regular recurrent point for W. We denote by $\mathbb{N}_z$ the associated excursion measure (see [5]). The law under $\mathbb{N}_z$ of $(\zeta_s, s \geq 0)$ is the Itô measure of positive excursions of linear Brownian motion. We assume that $\mathbb{N}_z$ is normalized so that

$$\mathbb{N}_z\left[\sup_{s\geq 0} \zeta_s > \varepsilon\right] = \frac{1}{2\varepsilon}\,.$$

We also set $\sigma = \inf\{s > 0, \zeta_s = 0\}$, which represents the duration of the excursion. Then for any nonnegative measurable function G on $\mathcal{W}_z$, we have:

$$\mathbb{N}_z \int_0^\sigma G(W_s)\, ds = \int_0^\infty ds\, \bar{\mathrm{E}}_z\left[G\left(\left(\bar{\xi}_t, 0 \leq t < s\right)\right)\right]. \tag{5}$$

For simplicity we write $\mathbb{N}_x = \mathbb{N}_z$ when $z = (0, 0, x)$. The continuity properties mentioned above under $\mathbb{E}_{w_0}$ also hold under $\mathbb{N}_z$.

Let $C(\mathbb{R}^+, \mathcal{W})$ denote the set of continuous functions from $\mathbb{R}^+$ to $\mathcal{W}$. Let $w \in \mathcal{W}_z$. We now recall the excursion decomposition of the Brownian snake under $\mathbb{E}^*_w$. We define the minimum process for the lifetime $\tilde{\zeta}_s = \inf\{\zeta_u, u \in [0, s]\}$. Let $(\alpha_i, \beta_i), i \in I$ be the excursion intervals of $\zeta - \tilde{\zeta}$ above 0 before time σ. For every $i \in I$, we set $W^i_s(t) = W_{s+\alpha_i}(t + \zeta_{\alpha_i})$, for $0 \leq t < \zeta_{\alpha_i+s} - \zeta_{\alpha_i}$, and $s \in (0, \beta_i - \alpha_i)$. Although the process $\bar{\xi}$ is not continuous, proposition 2.5 of [20] still holds.

Proposition 2.2. *The random measure $\sum_{i\in I}\delta_{(\alpha_i,W^i)}$ is under $\mathbb{E}^*_w$ a Poisson point measure on* $[0,\zeta_w]\times C(\mathbb{R}^+,\mathcal{W})$ *with intensity*

$$2dt\mathbb{N}_{w(t)}(d\mathrm{W}).$$

2.3. Exit measures

Let Q be an open subset of E with $z\in Q$ (or $w_0(0)\in Q$). As in [4], we can define the exit local time from Q, denoted by $\left(L^Q_s, s\geq 0\right)$. $\mathbb{N}_z$-a.e. (or $\mathbb{E}_{w_0}$-a.s.), the exit local time L^Q is a continuous increasing process given by the approximation: for every $s\geq 0$,

$$L^Q_s=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^s \mathbf{1}_{\{\tau_Q(W_u)<\zeta_u<\tau_Q(W_u)+\varepsilon\}}du,$$

where $\tau_Q(w)=\inf\{r>0; w(r)\notin Q\}$ is the exit time of Q for w. We then define under the excursion measure $\mathbb{N}_z$ the exit measure of Q for the snake, $Y_Q(\mathrm{W})$, by the formula: for every bounded nonnegative function $\varphi\in\mathcal{B}(\mathbb{R}^d)$,

$$\left(Y_Q,\varphi\right)=\int_0^\sigma \varphi(\hat{W}_s)dL^Q_s.$$

We write Y_Q for $Y_Q(W)$ when there is no confusion.

Intuitively, this measure describes the particles frozen when they first leave Q. There are mainly two kinds of interesting domains Q: first when $Q=\mathbb{R}^+\times[0,t)\times\mathbb{R}^d$ and $Q=\mathbb{R}^+\times\mathbb{R}^+\times D$. The former was used in [8] to get path properties of α-super Brownian motion. The latter will be useful here to get properties of the exit measure X_D.

The first moment of the random measure can be derived by taking the limit in (5) (see [20] proposition 3.3 for details). We have for every nonnegative measurable function G on $\mathcal{W}_z$

$$\mathbb{N}_z\int_0^\sigma G(W_s)\,dL^Q_s=\bar{\mathrm{E}}^Q_z[G], \tag{6}$$

where $\bar{\mathrm{P}}^Q_z$ is the sub-probability on $\mathcal{W}_z$ defined as the law of $\bar{\xi}$ stopped at time τ_Q under $\bar{\mathrm{P}}_z(\cdot\cap\left\{\tau_Q<\infty\right\})$.

We apply the construction of the exit measure with $Q=Q_D=\mathbb{R}^+\times\mathbb{R}^+\times D$, where D is a domain of $\mathbb{R}^d$. For convenience, we write $Y_D=Y_{Q_D}$, $\tau_D=\tau_{Q_D}$, $\bar{\mathrm{P}}^D_z=\bar{\mathrm{P}}^{Q_D}_z$ and also $\bar{\mathrm{P}}^D_x$ for $\bar{\mathrm{P}}^D_z$ when $z=(0,0,x)$.

Informally, in this case, the exit local time L^D increases only when the snake path W_s hits $\mathbb{R}^+\times\mathbb{R}^+\times\overline{D}$ for the first time at its lifetime ζ_s. Thus, the support of the exit measure Y_D is given by the end points $\hat{W}_s$ of such paths W_s. In the branching particles representation, Y_D describes the particles frozen at their first exit time of D.

Let φ be a nonnegative bounded measurable function on ∂D. Thanks to proposition 6 of [4] the function

$$u(z)=\mathbb{N}_z\left[1-\mathrm{e}^{-(Y_D,\varphi)}\right],\quad z\in\mathbb{R}^+\times\mathbb{R}^+\times D,$$

satisfies

$$u(z) = \bar{\mathrm{E}}_z\left[\varphi(\Gamma_{\tau_D})\right] - 2\bar{\mathrm{E}}_z\left[\int_0^{\tau_D} ds\; u(\bar{\xi}_s)^2\right]. \tag{7}$$

By arguing as in [20], theorem 4.1, we easily get a "Palm measure formula" for the random measure Y_D.

Proposition 2.3. *For every nonnegative measurable function F on $\mathbb{R}^d \times M_f$, for every $t > 0$ and $z \in \mathbb{R}^+ \times \mathbb{R}^+ \times D$, we have*

$$\mathbb{N}_z\left[\int Y_D(dy)F\left(y, Y_D\right)\right] = \int \bar{\mathrm{P}}_z^D(dw)\mathbb{E}\left[F\left(\hat{w}, \int \mathcal{N}_w(d\mathrm{W})Y_D(\mathrm{W})\right)\right],$$

where for every $w \in \mathcal{W}_z$, $\mathcal{N}_w(d\mathrm{W})$ denotes under $\mathbb{E}$, a Poisson measure on $C\left(\mathbb{R}^+, \mathcal{W}\right)$ with intensity

$$4\int_0^{\zeta_w} du\; \mathbb{N}_{w(u)}[d\mathrm{W}].$$

Intuitively, we pick a point on ∂D according to the exit measure. It is a typical point of the support of the exit measure. This point is the end point of a path distributed as the Markov process $\bar{\xi}$ (a "freezed" Brownian motion) stopped when it leaves D. The distribution of the snakes that branch from this path is described by a Poisson point process. The branching points are uniformly distributed along the path taking into account of the freezing times.

2.4. Relationship between the snake and the α-super Brownian motion

We introduced the process Y_D because its distribution under the excursion measure $\mathbb{N}_x$ is the canonical measure of the α-super Brownian motion started at δ_x.

Proposition 2.4. *Let $\nu \in M_f$, such that* supp $\nu \subset D$*, and let $\sum_{i\in I}\delta_{W^i}$ be a Poisson measure on $C(\mathbb{R}^+, \mathcal{W})$ with intensity $\int \nu(dx)\mathbb{N}_x[d\mathrm{W}]$. The random measure*

$$\sum_{i\in I} Y_D(W^i)$$

has the same distribution as X_D under $\mathbb{P}_\nu^X$.

Let $f \in \mathcal{B}(\mathbb{R}^d)$ be bounded and nonnegative. For $z = (k, l, x) \in Q_D$, we set $u(z) = \mathbb{N}_z[1 - \mathrm{e}^{-(Y_D, f)}]$ and $v(x) = u(0, 0, x)$. To prove the proposition, it is enough to check that the nonnegative function v solves (1). From (7), we see we need to express $u(k, l, x)$ in term of $v(x)$. The proof is then similar to the proof of theorem 8 in [4] and is not reproduced here. In particular it involves some integral of the Lévy measure of the Subordinator S. In order for the constant in front of $v(\gamma_s)^\alpha$ in (1) to be equal to 1, the computations yield the exact value of the constant c_ρ^*.

3. Lower bound of the hitting probability of small balls for X_D and Y_D

Thanks to the snake representation of the α-super Brownian motion (proposition 2.4), theorem 1.2 on the lower bound of the hitting probability of the exit measure X_D is equivalent to the following proposition.

Proposition 3.1. *Let K be a compact subset of D. There exists a constant c_K, such that for every $x \in K$, for every $y \in \partial D$, $\varepsilon \in (0, 1/2)$,*

$$\mathbb{N}_x \left[Y_D \left(B_{\partial D}(y, \varepsilon) \right) > 0 \right] \geq c_K \varphi_d(\varepsilon).$$

Its proof relies on well known bounds on the Poisson kernel and on the Green function in D. At the end of this section we complete this proposition by describing the behavior of the hitting measure $\mathbb{N}_x \left[Y_D \left(B_{\partial D}(y, \varepsilon) \right) > 0 \right]$ when x is close to y (lemma 3.2).

We first recall that (1) can be rewritten as

$$v(x) + \int_D dy\; G_D(x, y) v(y)^{1+\rho} = \int_{\partial D} P_D(x, z) f(z) \theta(dz), \tag{8}$$

where θ is the surface measure on ∂D, P_D is the Poisson kernel in D and G_D the Green function of D. We then give some useful bounds for the Poisson kernel and the Green function. There exist positive constants $c(D)$ and $C(D)$ (see [17] formula (3.19)) such that for every $(x, y) \in D \times \partial D$,

$$c(D) d(x, \partial D) \left| x - y \right|^{-d} \leq P_D(x, y) \leq C(D) d(x, \partial D) \left| x - y \right|^{-d}, \tag{9}$$

where $d(x, \partial D) = \inf\{|x - y|;\, y \in \partial D\}$. There exists a positive constant $C(D)$ (see [25] theorem 3 with $q = 0$) such that for every $(x, y') \in D \times D$,

$$G_D(x, y') \leq C(D) \left| x - y' \right|^{1-d} d(y', \partial D). \tag{10}$$

Proof of Proposition 3.1. Let $a > 0$. Let $x \in K$, $y \in \partial D$, $\varepsilon \in (0, 1/2)$. We set $h_d(\varepsilon) = \varepsilon^{-d+1} \varphi_d(\varepsilon)$. We have:

$$\mathbb{N}_x \left[Y_D \left(B_{\partial D}(y, \varepsilon) \right) > 0 \right] \geq= \mathbb{N}_x \left[1 - \exp\left[-a h_d(\varepsilon) Y_D \left(B_{\partial D}(y, \varepsilon) \right) \right] \right] =: v_\varepsilon(x),$$

where, thanks to proposition 2.4, the function v_ε is the only nonnegative solution of (8) with $f = a h_d(\varepsilon) \mathbf{1}_{B_{\partial D}(y, \varepsilon)}$. As

$$v_\varepsilon(x) \leq a h_d(\varepsilon) \int_{B_{\partial D}(y, \varepsilon)} P_D(x, z) \theta(dz),$$

we deduce from (8) that

$$\begin{aligned} v_\varepsilon(x) \geq\; & a h_d(\varepsilon) \int_{B_{\partial D}(y, \varepsilon)} P_D(x, z) \theta(dz) \\ & - \left[a h_d(\varepsilon) \right]^{1+\rho} \int_D dy\; G_D(x, y) \left[\int_{B_{\partial D}(y, \varepsilon)} P_D(y, z) \theta(dz) \right]^{1+\rho}. \end{aligned} \tag{11}$$

We now bound the second term of the right-hand side, which we denote by I. We decompose the integration over D in an integration over $D \cap B(y, 2\varepsilon)^c$ (denoted by I_1) and over $D \cap B(y, 2\varepsilon)$ (denoted by I_2), where $B(x, r)$ is the ball in $\mathbb{R}^d$ centered at x with radius r. We easily get an upper bound on I_1. We have for $\varepsilon > 0$ small enough,

$$\begin{aligned}
I_1 &= \int_{D \cap B(y,2\varepsilon)^c} dy' \; G_D(x, y') \left[\int_{B_{\partial D}(y,\varepsilon)} P_D(y', z)\theta(dz) \right]^{1+\rho} \\
&\le c \int_{D \cap B(y,2\varepsilon)^c} dy' \; \left|x - y'\right|^{1-d} d(y', \partial D)^{2+\rho} \sup_{z \in B(y,\varepsilon)} \left|y' - z\right|^{-d(1+\rho)} \\
&\quad \times \left[\int_{B_{\partial D}(y,\varepsilon)} \theta(dz') \right]^{1+\rho} \\
&\le c\varepsilon^{(d-1)(1+\rho)} \left[c + \int_{\text{diam } D \ge r \ge 2\varepsilon} r^{d-1} r^{2+\rho} r^{-d(1+\rho)} dr \right] \\
&\le c\varepsilon^{d-1} h_d(\varepsilon)^{-\rho}.
\end{aligned}$$

We use the notation diam $D = \sup\{\left|z - z'\right|; (z, z') \in D^2\}$. We also have for $\varepsilon > 0$ small enough,

$$\begin{aligned}
I_2 &= \int_{D \cap B(y,2\varepsilon)} dy' \; G_D(x, y') \left[\int_{B_{\partial D}(y,\varepsilon)} P_D(y', z)\theta(dz) \right]^{1+\rho} \\
&\le c \int_{D \cap B(y,2\varepsilon)} dy' \left[\int_{B_{\partial D}(y,\varepsilon)} d(y', \partial D)^{1+\frac{1}{[1+\rho]}} \left|y' - z\right|^{-d} \theta(dz) \right]^{1+\rho} \\
&\le c \int_{D \cap B(y,2\varepsilon)} dy' \left[\int_{B_{\partial D}(y,\varepsilon)} \left|y' - z\right|^{-d+1+\frac{1}{[1+\rho]}} \theta(dz) \right]^{1+\rho} \\
&\le c \int_{D \cap B(y,2\varepsilon)} dy' \left[\varepsilon^{1/[1+\rho]} \right]^{1+\rho} \\
&= c\varepsilon^{d+1}.
\end{aligned}$$

Combining those results together, we get that there exists a positive constant c_1' such that for every $(x, y) \in K \times \partial D$, $\varepsilon \in (0, 1/2)$,

$$I \le c_1' [a h_d(\varepsilon)]^{1+\rho} \varepsilon^{d-1} h_d(\varepsilon)^{-\rho}.$$

On the other hand, there exists a constant c_2' such that for every $(x, y) \in K \times \partial D$, $\varepsilon \in (0, 1/2)$:

$$\int_{B_{\partial D}(y,\varepsilon)} P_D(x, z)\theta(dz) \ge c_2' \varepsilon^{d-1}.$$

Plugging the previous inequalities into (11), we get

$$v_\varepsilon(x) \ge a\varphi_d(\varepsilon) \left[c_2' - c_1' a^\rho \right].$$

Since the constant a is arbitrary, we can take $a = (c'_2/2c'_1)^{1/\rho}$ to get

$$\mathbb{N}_x \left[Y_D \left(B_{\partial D}(y, \varepsilon) \right) > 0 \right] \geq v_\varepsilon(x) \geq \frac{1}{2} c'_2 a \varphi_d(\varepsilon). \qquad \square$$

We can also derive another bound when the starting point x is near the boundary using similar techniques.

Lemma 3.2. *Let $A > a > 0$. There exist two constants $c(A, a) > 0$ and $\varepsilon(D) > 0$, such that for every $y_0 \in \partial D$, $\varepsilon \in (0, \varepsilon(D))$, $y \in \overline{B_{\partial D}(y_0, \varepsilon)}$, $\eta \in (0, \varepsilon)$, $x \in D$ with $d(x, y) < A\eta$ and $d(x, \partial D) > a\eta$, we have*

$$\mathbb{N}_x \left[Y_D(B_{\partial D}(y_0, \varepsilon) \cap B_{\partial D}(y, \eta)) > 0 \right] \geq c(A, a)\eta^{-2/\rho}.$$

Proof. We use the same techniques as in the proof of the previous proposition. We replace the upper bound of the Green function by the following : there exists a constant c such that, for every $(x, y) \in D \times D$,

$$G_D(x, y) \leq c|y - x|^{2-d} \quad \text{if} \quad d \geq 3.$$

For $d = 2$, we bound $G_D(x, y)$ by the Green function of $\mathbb{R}^2 \backslash B$, where B is a ball outside D tangent to D in y_0. Since D is bounded of class C^2, the “uniform exterior sphere” condition holds, that is the radius of B can be chosen independently of y_0. $\square$

4. Some technical lemmas

4.1. Remarks and notations

In this section we look at the behavior of the path $W_s = (\xi_t(W_s), L_t(W_s), \Gamma_t(W_s); t \in [0, \zeta_s])$ near its end point. This behavior is crucial for the proof of the uniqueness theorem in the next section and for giving an estimate of the hitting probability of small balls for the exit measure in the critical dimension (proposition 6.1). We will write $\hat{W}_s = \Gamma_{\zeta_s}(W_s)$ for the spatial end point and $\hat{L}_s = L_{\zeta_s}(W_s)$ for the time change end point of the path W_s. Lemma 4.2 and lemma 4.4 are devoted to the upper bounds of the probability of unusually large value of the continuous processes $\hat{L}_s$ and $\hat{W}_s$ from a constant piecewise approximation. Eventually, we then deduce some uniform behavior of the spatial motion $\gamma(W_s) = \Gamma \circ L^{-1}(W_s)$ near the end point $\hat{W}_s = \gamma_{\hat{L}_s}(W_s)$ (lemma 4.5 and lemma 4.6) and of the inverse of the time change $S(W_s) = L^{-1}(W_s)$ near the end point $\hat{L}_s$ (lemma 4.7). These estimates would be easy to obtain if we considered a typical path. But we want to apply them to the first path hitting B_ε. Consequently, we must get uniform estimates which are true for all the paths of the snake.

More precisely, we show in lemma 4.5 that outside a set of small probability, the spatial motion is not abnormally fast near its end point. We show in lemma 4.6, that if the end point is on ∂D, then the path does not spend too much time near ∂D (for example, the path does not approach ∂D tangentially). Eventually in lemma

4.7 we prove the time change increments are not abnormally small near the end point. This means that the intensity of the branching mechanism is not too low.

Let us introduce some notations. For $w \in \mathcal{W}$, we define $\kappa_D(w) = L_{\tau_D}(w)$ if $\tau_D(w) < \infty$, $\kappa_D(w) = \infty$ otherwise. We extend this definition to the process $\bar{\xi}$. With the notations of section 2.2, under $\bar{\mathrm{P}}_x$, $x \in D$, κ_D is the exit time of D for γ, whereas τ_D is the exit time of D for $\Gamma = \gamma_L$. Notice that $\bar{\mathrm{P}}_x$-a.s. we have $S_{\kappa_D -} = \tau_D$. We define for $w \in \mathcal{W}$ such that $\hat{L}(w)$ is finite, the inverse of the time process L: $S_t(w) = \inf\{u \geq 0, L_u(w) > t\}$ and for the spatial motion, we also set $\gamma_t(w) = \Gamma_{S_t}(w)$ for $t \in [0, \hat{L}(w))$.

We write $\hat{L}_s$ for $\hat{L}(W_s)$, and we set $\hat{L}_s = \hat{L}_0$ for $s \geq \sigma$.

Remark 4.1. In particular, for a given $s > 0$, if $s < \sigma$, then the path $\gamma(W_s)$ is under $\mathbb{N}_x$ distributed as a Brownian motion started at point x (and killed at time $\hat{L}_s$). Similarly, the path $S(W_s)$ is distributed as the subordinator introduced in section 2.2 (killed at time $\hat{L}_s$). The notations are consistent with those from section 2.2.

4.2. Lemmas and proofs

Lemma 4.2. *Let $\theta > 0$. There exist a constant $C(\theta)$ such that for every stopping time τ with respect to the filtration generated by ζ, for every $a > 0, c > \theta, x \in \mathbb{R}^d$, on $\{\tau < \infty\}$,*

$$\mathbb{N}_x \left[\sup_{u \in [\tau, \tau + a]} \left| \hat{L}_\tau - \hat{L}_u \right| \geq c a^{\rho/2} \middle| \tau \right] \leq C(\theta)\, \mathrm{e}^{-c/\theta} .$$

Remark 4.3. Set $\mathbb{E}^*_{(r)} = \int \bar{\mathrm{P}}^r_x(dw) \mathbb{E}^*_w$, where $\bar{\mathrm{P}}^r_x$ is the law of $\bar{\xi}$ under $\bar{\mathrm{P}}_x$ killed at time r. Let τ be a stopping time with respect to the filtration generated by ζ. By the strong Markov property of the Brownian snake at time τ, we see that under $\mathbb{N}_x[\tau < \infty, \cdot]$, conditionally on ζ_τ, $(W_{\tau+s}, s \geq 0)$ is distributed according to $\mathbb{E}^*_{(\zeta_\tau)}$.

Proof. Let $\alpha_p = c_0 (p+1) 2^{-p\rho/2}$ and c_0 such that $\sum_{p \geq 0} \alpha_p = 1$. Using the continuity of the path $(\hat{L}_s, s \geq 0)$, we have for $r > 0$,

$$\mathbb{E}^*_{(r)} \left[\sup_{s \leq a} \left| \hat{L}_s - \hat{L}_0 \right| \geq c a^{\rho/2} \right] \leq \sum_{p \geq 0} \sum_{l=1}^{2^p} \mathbb{E}^*_{(r)} \left[\left| \hat{L}_{(l-1)2^{-p}a} - \hat{L}_{l 2^{-p} a} \right| \geq \alpha_p c a^{\rho/2} \right].$$

Using the Brownian snake property, we see that conditionally on the lifetime process ζ, $\hat{L}_{(l-1)2^{-p}a} - \hat{L}_{l2^{-p}a}$ is distributed as $L^{(1)}_{t_1} - L^{(2)}_{t_2}$ where $L^{(1)}$ and $L^{(2)}$ are independent and distributed according to $\int \bar{\mathrm{P}}^{t_0}_x(dw) \bar{\mathrm{P}}_{w(t_0)}$ where $t_0 = \inf\{\zeta_u; u \in [(l-1)2^{-p}a, l2^{-p}a]\}$, $t_1 = \zeta_{(l-1)2^{-p}a} - t_0$ and $t_2 = \zeta_{l2^{-p}a} - t_0$. Thus $\left| L^{(1)}_{t_1} - L^{(2)}_{t_2} \right|$ is stochastically dominated by $L_{t_1 \vee t_2} (< L_{t_1 + t_2})$ under $\bar{\mathrm{P}}_0$. For $h > 0, \delta > 0$, we have

$$\bar{\mathrm{P}}_0[L_t \geq h] = \bar{\mathrm{P}}_0[S_h \leq t] \leq \bar{\mathrm{E}}_0 \left[\mathrm{e}^{-\delta S_h + \delta t} \right] = \mathrm{e}^{\delta t - c^*_\rho \delta^\rho h} . \tag{12}$$

With $t = t_1 + t_2$ and $h = \alpha_\rho c a^{\rho/2}$, we deduce that for $\delta > 0$,

$$\mathbb{E}^*_{(r)}\left[\left|\hat{L}_{(l-1)2^{-p}a} - \hat{L}_{l2^{-p}a}\right| \geq \alpha_p c a^{\rho/2}\right] \leq \mathrm{P}_r\left[\mathrm{e}^{\delta(t_1+t_2)}\right]\mathrm{e}^{-c^*_\rho \delta^\rho \alpha_p c a^{\rho/2}}$$
$$= \mathrm{P}_0\left[\mathrm{e}^{\delta\bar{\zeta}_{2^{-p}a}}\right]\mathrm{e}^{-c^*_\rho \delta^\rho \alpha_p c a^{\rho/2}},$$

where under P_u, ζ is a linear Brownian motion started at u and $\bar{\zeta}_v = \zeta_v - 2\inf\{\zeta_u; u \leq v\}$ is a 3-dimensional Bessel process started at 0 under P_0. Take $\delta = b(2^{-p}a)^{-1/2}$. By scaling, we have

$$\mathrm{P}_0\left[\mathrm{e}^{\delta\bar{\zeta}_{2^{-p}a}}\right]\mathrm{e}^{-c^*_\rho \alpha_p c a^{\rho/2}\delta^\rho} = c_1(b)\,\mathrm{e}^{-c^*_\rho \alpha_p c 2^{p\rho/2} b^\rho},$$

where $c_1(b)$ depends only on b. Thus we have

$$\mathbb{E}^*_{(r)}\left[\sup_{s\in[0,a]}\left|\hat{L}_s - \hat{L}_0\right| \geq c a^{\rho/2}\right] \leq \sum_{p=0}^{\infty}\sum_{l=1}^{2^p} c_1(b)\,\mathrm{e}^{-c^*_\rho \alpha_p c 2^{p\rho/2} b^\rho}$$
$$\leq c_1(b)\,\mathrm{e}^{-c^*_\rho c_0 c b^\rho}\sum_{p=0}^{\infty} 2^p\,\mathrm{e}^{-c^*_\rho c_0 c b^\rho p}$$
$$= c_2(\theta)\,\mathrm{e}^{-c/\theta}, \tag{13}$$

where we take $b = [c^*_\rho c_0 \theta]^{-1/\rho}$ for the last equality. Since the result is independent of $r > 0$, the lemma is then a consequence of the remark before the beginning of this proof. □

Let $n \geq 1$ be an integer. We define inductively a sequence of stopping time $(\tau_i, i \geq 0)$ by

$$\tau_0 = 0 \quad \text{and} \quad \tau_{i+1} = \inf\{v > \tau_i;\, \left|\zeta_v - \zeta_{\tau_i}\right| = 2^{-n/\rho}\}.$$

Let $N = \inf\{i > 0; \tau_i = 0\}$. Recall that, conditionally on $\{\tau_1 < \infty\}$, the sequence $(\zeta_{\tau_i}, i \geq 1)$ is a simple random walk on $2^{-n/\rho}\mathbb{Z}_+$ stopped when it reaches 0. Therefore, we have for $i_0 > 1$,

$$\sum_{i=1}^{\infty}\mathbb{N}_x\left[\zeta_{\tau_i} = i_0 2^{-n/\rho}\right] = \mathbb{N}_x\left[\tau_1 < \infty\right]\mathbb{N}_x\left[\sum_{i=1}^{\infty}\mathbf{1}_{\{\zeta_{\tau_i} = i_0 2^{-n/\rho}\}}\,\Big|\,\tau_1 < \infty\right]$$
$$= 2\mathbb{N}_x\left[\sup_{s\geq 0}\zeta_s > 2^{-n/\rho}\right] = 2^{n/\rho}.$$

Lemma 4.4. *Let $\lambda > 0$. There exist two constants $C_* > 0$, $c_* > 0$ such that for any integer $n \geq 1$, for every $M > m \geq 2^{-n/\rho}$, we have*

$$\mathbb{N}_x\left[\exists i \in \{1,\cdots,N-1\}, m \leq \zeta_{\tau_i} \leq M, \sup_{s\in[\tau_i,\tau_{i+1}]}\left|\hat{W}_s - \hat{W}_{\tau_i}\right| \geq c_* n^{1+\frac{\rho}{4}} 2^{-n/2}\right]$$
$$\leq C_* M 2^{2n/\rho}\,\mathrm{e}^{-\lambda n},$$

$$\mathbb{N}_x\left[\exists i\in\{1,\cdots,N-1\}, m\le\zeta_{\tau_i}\le M, \sup_{s\in[\tau_i,\tau_{i+1}]}\left|\hat{L}_s-\hat{L}_{\tau_i}\right|\ge c_* n^{1+\frac{\rho}{2}}2^{-n}\right]$$
$$\le C_* M 2^{2n/\rho}\,\mathrm{e}^{-\lambda n}.$$

Proof. Let c_1, c_* be two positive constants whose value will be fixed later. We set $a = c_1 2^{-2n/\rho}\log(2^{n/\rho})$. Let $k\ge 1$. We have

$$\begin{aligned}\mathbb{N}_x&\left[\zeta_{\tau_i}=k2^{-n/\rho}, \sup_{s\in[\tau_i,\tau_{i+1}]}\left|\hat{W}_s-\hat{W}_{\tau_i}\right|\ge c_* n^{1+\frac{\rho}{4}}2^{-n/2}\right]\\ &\le \mathbb{N}_x\left[\zeta_{\tau_i}=k2^{-n/\rho}, \tau_{i+1}-\tau_i>a\right]\\ &\quad+\mathbb{N}_x\left[\zeta_{\tau_i}=k2^{-n/\rho}, \sup_{s\in[\tau_i,\tau_i+a]}\left|\hat{W}_s-\hat{W}_{\tau_i}\right|\ge c_* n^{1+\frac{\rho}{4}}2^{-n/2}\right].\end{aligned}$$

The law of $\tau_{i+1}-\tau_i$ knowing $\{i<N\}$ is the law of the first exit time from $[-2^{-n/\rho}, 2^{-n/\rho}]$ for a standard linear Brownian motion started at 0. Thus there exist two positive constants a_1, a_2 (independent of n, c_1) such that:

$$\mathbb{N}_x\left[\zeta_{\tau_i}=k2^{-n/\rho}, \tau_{i+1}-\tau_i>a\right]\le\mathbb{N}_x\left[\zeta_{\tau_i}=k2^{-n/\rho}\right]a_1 2^{-a_2c_1n/\rho}.$$

Set $\alpha_p=c_0(p+1)2^{-p\rho/4}$ for $p\ge 0$ and c_0 is so that $\sum_{p=0}^{\infty}\alpha_p=1$. For $r>0$, we have

$$\begin{aligned}I_n&=\mathbb{E}^*_{(r)}\left[\sup_{s\in[0,a]}\left|\hat{W}_s-\hat{W}_a\right|\ge c_* n^{1+\frac{\rho}{4}}2^{-n/2}\right]\\ &\le\sum_{p=0}^{\infty}\sum_{l=1}^{2^p}\mathbb{E}^*_{(r)}\left[\left|\hat{W}_{(l-1)2^{-p}a}-\hat{W}_{l2^{-p}a}\right|\ge\alpha_p c_* n^{1+\frac{\rho}{4}}2^{-n/2}\right].\end{aligned}$$

Conditionally on $(L_t(W_s), t\in[0,\zeta_s), s\ge 0)$, $\hat{W}_{(l-1)2^{-p}a}-\hat{W}_{l2^{-p}a}$ is a centered Gaussian random variable with variance

$$V^2=\hat{L}_{(l-1)2^{-p}a}+\hat{L}_{l2^{-p}a}-2\inf_{s\in[(l-1)2^{-p}a,l2^{-p}a]}\hat{L}_s.$$

If Z is a d-dimensional centered Gaussian random variable with variance V^2, then

$$\mathrm{P}[|Z|>b]\le 2^{d/2}\,\mathrm{e}^{-b^2/4V^2}.$$

Let $V_0^2=(p+1)n2^{-p\rho/2}a^{\rho/2}$. We have

$$\begin{aligned}\mathbb{E}^*_{(r)}&\left[\left|\hat{W}_{(l-1)2^{-p}a}-\hat{W}_{l2^{-p}a}\right|\ge\alpha_p c_* n^{1+\frac{\rho}{4}}2^{-n/2}, V^2<V_0^2\right]\\ &\le 2^{d/2}\,\mathrm{e}^{-n(p+1)c_2c_*^2c_1^{-\rho/2}},\end{aligned}$$

where c_2 depends only on ρ. From the proof of lemma 4.2 (see (13)), we deduce that for $\theta \in (0,1)$,

$$\mathbb{E}^*_{(r)}\left[V^2 \geq V_0^2\right] \leq \mathbb{E}^*_{(r)}\left[\mathbb{E}^*_{(\zeta_{(l-1)}2^{-p}a)}\left[\sup_{s\leq 2^{-p}a}\left|\hat{L}_s - \hat{L}_0\right| \geq V_0^2/3\right]\right]$$
$$\leq c_3(\theta)\,\mathrm{e}^{-(p+1)n/3\theta},$$

where c_3 depends only on θ. Thus we have

$$I_n \leq 2^{d/2}\,\mathrm{e}^{-n(p+1)c_2c_*^2c_1^{-\rho/2}} + c_3(\theta)\,\mathrm{e}^{-(p+1)n/3\theta}.$$

Let $\lambda > 0$ be fixed. We can choose c_1, c_*, θ^{-1} large enough so that for every $n \geq 1$, $M > m \geq 2^{-n/\rho}$,

$$\mathbb{N}_x\left[\exists i \in \{1, \cdots, N-1\}, m \leq \zeta_{\tau_i} \leq M, \sup_{s\in[\tau_i,\tau_{i+1}]}\left|\hat{W}_s - \hat{W}_{\tau_i}\right| \geq c_* n^{1+\frac{\rho}{4}} 2^{-n/2}\right]$$
$$\leq \sum_{k=1}^{[M2^{n/\rho}]+1}\sum_{i=1}^{\infty}\mathbb{N}_x[\zeta_{\tau_i} = k2^{-n/\rho}](a_1 2^{-a_2c_1n/\rho} + I_n)$$
$$\leq C_* M 2^{2n/\rho}\,\mathrm{e}^{-\lambda n},$$

where C_* is a constant independent of n, M and m. This ends the proof of the first inequality.

The second inequality is proved in a similar way. □

We are now going to give three lemmas which describe the behavior of the paths W_s for $s \geq 0$, near their end-point.

For a path $w \in \mathcal{W}$, we set for $A_0 > 0$ and integers $n > n_0 \geq 1$,

$$F^{A_0}_{n_0,n}(w) = \mathbf{1}_{\{\hat{L}(w)\geq 2^{-n_0+1}\}}\frac{1}{n-n_0}\sum_{k=n_0}^{n-1}\mathbf{1}_{\{\sup_{t\in[0,2^{-k-1}]}|\gamma_{\hat{L}(w)-t}(w)-\hat{w}|>A_0 2^{-k/2}\}}.$$

We have the following lemma :

Lemma 4.5. *Let $\delta \in (0,1]$. For every $\lambda > 0$, we can choose $A_0 > 0$ such that there exists a constant K_1 and for every integers $n \geq 3$, $n_0 \in [1, n-\sqrt{n}]$, for every $M > m \geq 2^{-n/\rho}$, $x \in \mathbb{R}^d$,*

$$\mathbb{N}_x\left[\exists s \geq 0;\ m \leq \zeta_s \leq M,\ \hat{L}_s > 2^{-n_0+1},\ F^{A_0}_{n_0,n}(W_s) > \delta\right] \leq K_1 M 2^{2n/\rho} e^{-\lambda(n-n_0)}.$$

Proof. For $A > 0$, $n > n_0 \geq 1$, $w \in \mathcal{W}$, we set

$$\tilde{F}^{A}_{n_0,n}(w) = \mathbf{1}_{\{\hat{L}(w)\geq 2^{-n_0}\}}\frac{1}{n-n_0}\sum_{k=n_0}^{n-1}\mathbf{1}_{\{\sup_{t\in[0,2^{-k}]}|\gamma_{\hat{L}(w)-t}(w)-\hat{w}|>A 2^{-k/2}\}}.$$

From the remark following lemma 4.2, we have for $k > 0$,

$$I = \mathbb{N}_x\left[\hat{L}_{\tau_i} > 2^{-n_0}, \tilde{F}^A_{n_0,n}(W_{\tau_i}) > \delta \mid \zeta_{\tau_i} = k2^{-n/\rho}\right]$$
$$= \mathbb{E}^*_{(k2^{-n/\rho})}\left[\hat{L}_{\tau_i} > 2^{-n_0}, \tilde{F}^A_{n_0,n}(W_{\tau_i}) > \delta\right].$$

Conditionally on $\hat{L}_{\tau_i}$, $(\gamma_{\hat{L}_{\tau_i}-t}(W_{\tau_i}) - \gamma_{\hat{L}_{\tau_i}}(W_{\tau_i}), t \in [0, \hat{L}_{\tau_i}])$ is under $\mathbb{E}^*_{(k2^{-n/\rho})}$ a standard Brownian motion. Thanks to lemma 0 in [21] and a scaling argument, we easily get $I \le \mathrm{e}^{(dc_0-\delta A)(n-n_0)}$, where c_0 is a universal constant. Hence, summing over $k \in \{1, \cdots, [M2^{n/\rho}] + 1\}$ and $i \ge 1$, we have for $M > m \ge 2^{-n/\rho}$,

$$\mathbb{N}_x\left[\exists i \in \{1, \cdots, N-1\}; m \le \zeta_{\tau_i} \le M, \hat{L}_{\tau_i} > 2^{-n_0}, \tilde{F}^A_{n_0,n}(W_{\tau_i}) > \delta\right]$$
$$\le 2M2^{2n/\rho}\,\mathrm{e}^{(dc_0-\delta A)(n-n_0)}. \tag{14}$$

We will now interpolate between τ_i and τ_{i+1}. Let $A_0 > 1$, $\lambda > 0$. We consider the two constants c_*, C_* defined in lemma 4.4. We write

$$\mathcal{A}_1 = \bigcap_{i\in\{1,\dots,N-1\}}\left\{\sup_{r\in[\tau_i,\tau_{i+1}]}\left|\hat{W}_r - \hat{W}_{\tau_i}\right| \le c_* n^{1+\frac{\rho}{4}}2^{-n/2}\right\}$$
$$\mathcal{A}_2 = \bigcap_{i\in\{1,\dots,N-1\}}\left\{\sup_{r\in[\tau_i,\tau_{i+1}]}\left|\hat{L}_r - \hat{L}_{\tau_i}\right| \le c_* n^{1+\frac{\rho}{2}}2^{-n}\right\}.$$

Fix $n > n_0 \ge 1$. Assume there is $s_0 > 0$ such that $\hat{L}_{s_0} \ge 2^{-n_0+1}$ and $m \le \zeta_{s_0} \le M$. There is a unique $i \in \{1, \dots, N-1\}$ such that $s_0 \in [\tau_i, \tau_{i+1})$. We want to compare $\tilde{F}^A_{n_0,n}(W_{\tau_i})$ and $F^{A_0}_{n_0,n}(W_{s_0})$ on $\mathcal{A}_1 \cap \mathcal{A}_2$. Let $s_1 \in [\tau_i, \tau_{i+1}]$ such that $\zeta_s \ge \zeta_{s_1}$ for $s \in [\tau_i, \tau_{i+1}]$. All the paths W_s for $s \in [\tau_i, \tau_{i+1}]$ coincide up to time ζ_{s_1}. From the snake property, we have on $\mathcal{A}_1$,

$$\sup_{t\in[0,\hat{L}_{s_0}-\hat{L}_{s_1}]}\left|\gamma_{\hat{L}_{s_0}-t}(W_{s_0}) - \hat{W}_{\tau_i}\right| \le \sup_{s\in[\tau_i,\tau_{i+1}]}\left|\hat{W}_s - \hat{W}_{\tau_i}\right| \le c_* n^{1+\frac{\rho}{4}}2^{-n/2}.$$

Notice there exists c_1 (depending only on c_*) such that if $n_0 \le k \le n - c_1\log n$, then $2^{-k-1} \ge c_* n^{1+\frac{\rho}{2}}2^{-n}$ and $2^{-\frac{k}{2}-1} \ge c_* n^{1+\frac{\rho}{4}}2^{-n/2}$. For $n_0 \le k \le n - c_1\log n$, we have on $\mathcal{A}_2$, $\hat{L}_{s_0} - 2^{-k-1} \ge \hat{L}_{\tau_i} - 2^{-k} > 0$. Since the path $(\gamma_t(W_{s_0}), t \ge 0)$ and $(\gamma_t(W_{\tau_i}), t \ge 0)$ coincide up to time $\hat{L}_{s_1}$, we get on $\mathcal{A}_2$,

$$\left\{\gamma_t(W_{s_0});\, \hat{L}_{s_0} - 2^{-k-1} \le t \le \hat{L}_{s_1}\right\} \subset \left\{\gamma_t(W_{\tau_i});\, \hat{L}_{\tau_i} - 2^{-k} \le t \le \hat{L}_{\tau_i}\right\}.$$

We deduce that for $n_0 \le k \le n - c_1 \log n$, on $\mathcal{A}_1 \cap \mathcal{A}_2$,

$$\begin{aligned}
\sup_{t\in[0,2^{-k-1}]} \left|\gamma_{\hat{L}_{s_0}-t}(W_{s_0}) - \hat{W}_{s_0}\right| &\le \sup_{s\in[\tau_i,\tau_{i+1}]} \left|\hat{W}_s - \hat{W}_{\tau_i}\right| \\
&\quad + \sup_{t\in[0,\hat{L}_{s_0}-\hat{L}_{s_1}]} \left|\gamma_{\hat{L}_{s_0}-t}(W_{s_0}) - \hat{W}_{\tau_i}\right| \\
&\quad + \sup_{t\in[\hat{L}_{s_0}-\hat{L}_{s_1},2^{-k-1}]} \left|\gamma_{\hat{L}_{s_0}-t}(W_{s_0}) - \hat{W}_{\tau_i}\right| \\
&\le 2c_* n^{1+\frac{\rho}{4}} 2^{-n/2} + \sup_{t\in[0,2^{-k}]} \left|\gamma_{\hat{L}_{\tau_i}-t}(W_{\tau_i}) - \hat{W}_{\tau_i}\right| .
\end{aligned}$$

Therefore on $\mathcal{A}_1 \cap \mathcal{A}_2$, we have $F^{A_0}_{n_0,n}(W_{s_0}) \le \tilde{F}^{A_0/2}_{n_0,n}(W_{\tau_i}) + c_1 \frac{\log n}{n-n_0}$. Let $\delta > 0$ be fixed. For n large enough, and $n_0 \in [1, n - \sqrt{n}]$, we have $c_1 \frac{\log n}{n-n_0} \le c_1 \frac{\log n}{\sqrt{n}} \le \delta/2$. Decomposing on the sets $\mathcal{A}_1 \cap \mathcal{A}_2$, $\mathcal{A}_1^c$ and $\mathcal{A}_2^c$, we get

$$\begin{aligned}
&\mathbb{N}_x \left[\exists s \ge 0,\ m \le \zeta_s \le M,\ \hat{L}_s > 2^{-n_0+1},\ F^{A_0}_{n_0,n}(W_s) > \delta\right] \\
&\le \mathbb{N}_x \left[\exists i \in \{1, \dots, N-1\},\ m \le \zeta_{\tau_i} \le M,\ \hat{L}_{\tau_i} > 2^{-n_0},\ \tilde{F}^{A_0/2}_{n_0,n}(W_{\tau_i}) > \frac{\delta}{2}\right] \\
&+ \mathbb{N}_x \left[\exists i \in \{1, \dots, N-1\},\ m \le \zeta_{\tau_i} \le M,\ \sup_{r\in[\tau_i,\tau_{i+1}]} \left|\hat{W}_r - \hat{W}_{\tau_i}\right| \ge c_* n^{1+\frac{\rho}{4}} 2^{-n/2}\right] \\
&+ \mathbb{N}_x \left[\exists i \in \{1, \dots, N-1\},\ m \le \zeta_{\tau_i} \le M,\ \sup_{r\in[\tau_i,\tau_{i+1}]} \left|\hat{L}_r - \hat{L}_{\tau_i}\right| \ge c_* n^{1+\frac{\rho}{2}} 2^{-n}\right] \\
&\le 2M 2^{2n/\rho} e^{\left(dc_0 - \frac{\delta A_0}{4}\right)(n-n_0)} + 2C_* M 2^{2n/\rho} e^{-\lambda n}
\end{aligned}$$

by formula (14) and lemma 4.4.

It suffices now to take A_0 large enough so that $\delta \frac{A_0}{4} - dc_0 > \lambda$ to get the right member bounded from above by

$$2(C_* + 1) M 2^{2n/\rho} e^{-\lambda(n-n_0)}. \qquad \square$$

Let $\gamma_{[0,r]} = (\gamma_t, t \in [0, r])$ a path in $\mathbb{R}^d$. For $a_0 > 0$ and an integer $k \ge 1$, we set

$$\mathcal{A}^{a_0}_k(\gamma_{[0,r]}) = \left\{\exists t \in \left[r - \frac{15}{16} 2^{-k}, r - \frac{7}{8} 2^{-k}\right],\ d(\gamma_t, D^c) < a_0 2^{-k/2}\right\}$$

and

$$\phi^{a_0}_{n_0,n} = \mathbf{1}_{\{r \ge 2^{-n_0+1}\}} \frac{1}{n-n_0} \sum_{k=n_0}^{n-1} \mathbf{1}_{\mathcal{A}^{a_0}_k(\gamma_{[0,r]})}.$$

We then have the following lemma :

Lemma 4.6. *For every $\lambda > 0$, we can choose $a_0 > 0$ such that there exists a constant K_2 and for every integers $n \geq 3, n_0 \in [1, n-\sqrt{n}]$, for every $M > m \geq 2^{-n/\rho}$, $x \in D$,*

$$\mathbb{N}_x\left[\exists s \geq 0;\ m \leq \zeta_s \leq M,\ \hat{L}_s > 2^{-n_0+1},\ \phi^{a_0}_{n_0,n}\left(\gamma_{[0,\hat{L}_s]}(W_s)\right) > \frac{1}{6}, \tau_D(W_s) = \zeta_s\right] \\ \leq K_2 M 2^{2n/\rho} 2^{n-n_0} e^{-\lambda(n-n_0)}.$$

Proof. Let us set

$$\tilde{\mathcal{A}}^{a_0}_k(\gamma_{[0,r]}) = \left\{\exists t \in \left[r - 2^{-k}, r - \frac{3}{4}2^{-k}\right];\ d(\gamma(t), D^c) < a_0 2^{-k/2}\right\}$$

and for $n_1 > n_0 \geq 1$,

$$\tilde{\phi}^{a_0}_{n_0,n_1}(\gamma_{[0,r]}) = \mathbf{1}_{\{r > 2^{-n_0}\}} \frac{1}{n_1 - n_0} \sum_{k=n_0}^{n_1-1} \mathbf{1}_{\tilde{\mathcal{A}}^{a_0}_k(\gamma_{[0,r]})}.$$

From [2] p.265, it is easy to see that for $r > 2^{-n_0}$, $x \in D$,

$$\mathrm{P}_x\left[\{\gamma_t \in D;\, t \in [0, r - 2^{-n_1-1}]\} \cap \{\tilde{\phi}^{a_0}_{n_0,n_1}(\gamma_{[0,r]}) > 1/12\}\right] \leq 2^{n_1-n_0} g_1(a_0)^{n_1-n_0},$$

where g_1 is a nondecreasing function (independent of r) such that $\lim_{a\downarrow 0} g_1(a) = 0$. We take $a_0 > 0$ such that $g_1(a_0) \leq \mathrm{e}^{-2\lambda}$. Conditionally on ζ_{τ_i}, $\hat{L}_{\tau_i}$, the process $\gamma_{[0,\hat{L}_{\tau_i}]}(W_{\tau_i}) = \left(\gamma_t(W_{\tau_i}), t \in [0, \hat{L}_{\tau_i}]\right)$ is a standard Brownian motion started at x. Hence, we have for $k \geq 1$,

$$\mathbb{N}_x\Big[\zeta_{\tau_i} = k2^{-n/\rho}, \hat{L}_{\tau_i} > 2^{-n_0}, \tilde{\phi}^{a_0}_{n_0,n_1}(\gamma_{[0,\hat{L}_{\tau_i}]}(W_{\tau_i})) > 1/12, \kappa_D(W_{\tau_i}) \\ > \hat{L}_{\tau_i} - 2^{-n_1-1}\Big] \leq \mathbb{N}_x\left[\zeta_{\tau_i} = k2^{-n/\rho}\right] 2^{n_1-n_0}\, \mathrm{e}^{-2\lambda(n_1-n_0)}.$$

Summing over $i \geq 1$ and $k \in \{1, \cdots, [M2^{n/\rho}]+1\}$, we have for $M \geq m \geq 2^{-n/\rho}$,

$$\mathbb{N}_x\Big[\exists i \in \{1, \cdots, N-1\};\, m \leq \zeta_{\tau_i} \leq M, \hat{L}_{\tau_i} > 2^{-n_0}, \\ \tilde{\phi}^{a_0}_{n_0,n_1}(\gamma_{[0,\hat{L}_{\tau_i}]}(W_{\tau_i})) > 1/12, \kappa_D(W_{\tau_i}) > \hat{L}_{\tau_i} - 2^{-n_1-1}\Big] \\ \leq 2M 2^{2n/\rho} 2^{n_1-n_0}\, \mathrm{e}^{-2\lambda(n_1-n_0)}.$$

We will now interpolate between τ_i and τ_{i+1}. We consider the two constants c_*, C_* defined in lemma 4.4. We write

$$\mathcal{A}_2 = \bigcap_{i\in\{1,\ldots,N-1\}} \left\{\sup_{r\in[\tau_i,\tau_{i+1}]} \left|\hat{L}_r - \hat{L}_{\tau_i}\right| \leq c_* n^{1+\frac{\rho}{2}} 2^{-n}\right\}.$$

Fix $n > n_0 \geq 1$. Assume there is $s_0 > 0$ such that $\hat{L}_{s_0} \geq 2^{-n_0+1}$ and $m \leq \zeta_{s_0} \leq M$. There is a unique $i \in \{1, \dots, N-1\}$ such that $s_0 \in [\tau_i, \tau_{i+1})$. We want to compare $\tilde{\phi}^{A}_{n_0,n_1}(W_{\tau_i})$ and $\phi^{A_0}_{n_0,n}(W_{s_0})$ on $\mathcal{A}_2$. Let $s_1 \in [\tau_i, \tau_{i+1}]$ such that $\zeta_s \geq \zeta_{s_1}$ for $s \in [\tau_i, \tau_{i+1}]$. All the paths W_s for $s \in [\tau_i, \tau_{i+1}]$ coincide up to time ζ_{s_1}.

Notice there exists c_1 (depending only on c_*) such that if $n_0 \leq k \leq n - c_1 \log n$, then $\frac{1}{16}\, 2^{-k} \geq c_* n^{1+\frac{\rho}{2}} 2^{-n}$. For $n_0 \leq k \leq n - c_1 \log n$, we have on $\mathcal{A}_2$,

$$\hat{L}_{\tau_i} - 2^{-k} \leq \hat{L}_{s_0} - \frac{15}{16}2^{-k} \leq \hat{L}_{s_0} - \frac{7}{8}2^{-k} \leq \hat{L}_{\tau_i} - \frac{3}{4}2^{-k}.$$

And since $\hat{L}_{\tau_i} - \frac{3}{4}2^{-k} \leq \hat{L}_{s_1}$, we have

$$\left\{\gamma_t(W_{s_0});\, t \in \left[\hat{L}_{s_0} - \frac{15}{16}2^{-k}, \hat{L}_{s_0} - \frac{7}{8}2^{-k}\right]\right\} \subset \left\{\gamma_t(W_{\tau_i});\, t \in \left[\hat{L}_{\tau_i} - 2^{-k}, \hat{L}_{\tau_i} - \frac{3}{4}2^{-k}\right]\right\}.$$

Notice we also have $\hat{L}_{\tau_i} > 2^{-n_0}$ since $\hat{L}_{s_0} > 2^{-n_0+1}$. Let n_1 be the largest integer smaller than $n - c_1 \log n$. From the snake property, since $\kappa_D(W_{s_0}) = \hat{L}_{s_0}$, we have that $\kappa_D(W_s) \geq \hat{L}_{s_1}$ for $s \in [\tau_i, \tau_{i+1}]$. And thus we get on $\mathcal{A}_2$, $\kappa_D(W_{\tau_i}) \geq \hat{L}_{s_1} \geq \hat{L}_{\tau_i} - 2^{-n_1-1}$. For n large enough, $n_1 > n_0$. The previous remarks lead to

$$\begin{aligned}\phi^{a_0}_{n_0,n}\left(\gamma_{[0,\tilde{L}_{s_0}]}(W_{s_0})\right) &\leq \frac{n_1 - n_0}{n - n_0}\tilde{\phi}^{a_0}_{n_0,n_1}\left(\gamma_{[0,\hat{L}_{\tau_i}]}(W_{\tau_i})\right) + c_1\frac{\ln n}{n - n_0}\\ &\leq \tilde{\phi}^{a_0}_{n_0,n_1}\left(\gamma_{[0,\hat{L}_{\tau_i}]}(W_{\tau_i})\right) + \frac{1}{12}\end{aligned}$$

for n large enough. Decomposing on the sets $\mathcal{A}_2$ and $\mathcal{A}_2^c$, we get for n large enough,

$$\begin{aligned}&\mathbb{N}_x\left[\exists s \geq 0;\; m \leq \zeta_s \leq M,\; \hat{L}_s \geq 2^{-n_0+1},\; \phi^{a_0}_{n_0,n}\left(\gamma_{[0,\hat{L}_s]}(W_s)\right) > \frac{1}{6},\; \tau_D(W_s) = \zeta_s\right]\\ &\leq \mathbb{N}_x\Bigg[\exists i \in \{1, \dots, N-1\};\; m \leq \zeta_{\tau_i} \leq M,\; \hat{L}_{\tau_i} \geq 2^{-n_0},\\ &\qquad \tilde{\phi}^{a_0}_{n_0,n_1}\left(\gamma_{[0,\hat{L}_{\tau_i}]}(W_{\tau_i})\right) > \frac{1}{12}, \kappa_D(W_{\tau_i}) \geq \hat{L}_{\tau_i} - 2^{-n_1-1}\Bigg]\\ &\qquad + \mathbb{N}_x\left[\exists i \in \{1, \dots, N-1\};\; m \leq \zeta_{\tau_i} \leq M,\; \sup_{r \in [\tau_i,\tau_{i+1}]}\left|\hat{L}_r - \hat{L}_{\tau_i}\right| \geq c_* n^{(1+\frac{\rho}{2})} 2^{-n}\right]\\ &\leq 2M 2^{2n/\rho} 2^{n_1 - n_0}\, \mathrm{e}^{-\lambda(n_1-n_0)} + C_* M 2^{2n/\rho} e^{-\lambda n}\\ &\leq (2 + C_*) M 2^{2n/\rho} 2^{n-n_0} e^{-\lambda(n-n_0)},\end{aligned}$$

where we use that $\sqrt{n} \geq 2c_1 \log n$ implies $2(n_1 - n_0) \geq n - n_0$ for the last inequality. □

Let $S_{[0,r)} = (S_t, t \in [0,r))$ be a càdlàg path in $\mathbb{R}$. We define for $a_1 > 0$ and $n > n_0 \geq 1$,

$$\psi^{a_1}_{n_0,n}(S_{[0,r)}) = \mathbf{1}_{\{r>2^{-n_0+1}\}} \frac{1}{n-n_0} \sum_{k=n_0}^{n-1} \mathbf{1}_{\left\{S_{\left(r-\frac{7}{8}2^{-k}\right)-} - S_{\left(r-\frac{15}{16}2^{-k}\right)-} < a_1 2^{-k/\rho}\right\}}.$$

Lemma 4.7. *For every $\lambda > 0$, we can choose a_1 large enough such that there exists a constant K_3 and for every integers $n \geq 3$, $n_0 \in [1, \dots n - \sqrt{n}]$, for every $M > m \geq 2^{-n/\rho}$, $x \in \mathbb{R}^d$,*

$$\mathbb{N}_x\left[\exists s > 0;\ m \leq \zeta_s \leq M,\ \hat{L}_s > 2^{-n_0+1},\ \phi^{a_1}_{n_0,n}\left(S_{[0,\hat{L}_s)}(W_s)\right) > \frac{1}{6}\right]$$
$$\leq K_3 M 2^{2n/\rho} 2^{n-n_0} e^{-\lambda(n-n_0)}.$$

Proof: the same ideas of the proof of lemma 4.6 lead to define

$$\tilde{\psi}^{a_1}_{n_0,n}(S_{[0,r)}) = \mathbf{1}_{\{r>2^{-n_0}\}} \frac{1}{n-n_0} \sum_{k=n_0}^{n-1} \mathbf{1}_{\left\{S_{(r-\frac{3}{4}2^{-k})-} - S_{(r-2^{-k})-} < a_1 2^{-k/\rho}\right\}}.$$

Using the strong Markov property at time τ_i for the Brownian snake, we get

$$\mathbb{N}_x\left[\zeta_{\tau_i} = k2^{-n/\rho}, \hat{L}_{\tau_i} > 2^{-n_0}, \tilde{\psi}^{a_0}_{n_0,n}(S_{[0,\hat{L}_{\tau_i})}(W_{\tau_i})) > 1/12\right]$$
$$= \mathbb{N}_x\left[\zeta_{\tau_i} = k2^{-n/\rho}\right] \bar{\mathrm{P}}_x\left[L_{k2^{-n/\rho}} > 2^{-n_0}, \tilde{\psi}^{a_0}_{n_0,n}(S_{[0,L_{k2^{-n/\rho}})}) > 1/12\right].$$

From the lemma 9.1 in the appendix we know that for $r > 0$, $(S_t, t \in [0, L_r))$ and $(S_{L_r-} - S_{(L_r-t)-}, t \in [0, L_r))$ are identically distributed under $\bar{\mathrm{P}}_x$. Let q the integer part of $(n-n_0)/12$. The set

$$\{L_{k2^{-n/\rho}} > 2^{-n_0}, \frac{1}{n-n_0} \sum_{k=n_0}^{n-1} \mathbf{1}_{\{S_{2^{-k}} - S_{\frac{3}{4}2^{-k}} < a_1 2^{-k/\rho}\}} > 1/12\}$$

is a subset of

$$\bigcup_{n_0 \leq k_1 < \cdots < k_q < n} \quad \bigcap_{j=1}^{q} \left\{S_{2^{-k_j}} - S_{\frac{3}{4}2^{-k_j}} < a_1 2^{-k_j/\rho}\right\}.$$

Since the increments of the process S are independent, we have by scaling that the probability of the last event is $g_2(a_1)^{n-n_0}$, where g_2 is a function such that $\lim_{a\downarrow 0} g_2(a) = 0$. We take $a_1 > 0$ so that $g_2(a_1) \leq \mathrm{e}^{-\lambda}$. Notice there are less than 2^{n-n_0} possible choices for $k_1, \dots, k_q$. Thus we have

$$\mathbb{N}_x\left[\zeta_{\tau_i} = k2^{-n/\rho}, \hat{L}_{\tau_i} > 2^{-n_0}, \tilde{\psi}^{a_0}_{n_0,n}(S_{[0,\hat{L}_{\tau_i})}(W_{\tau_i})) > 1/12\right]$$
$$\leq \mathbb{N}_x\left[\zeta_{\tau_i} = k2^{-n/\rho}\right] 2^{n-n_0}\, \mathrm{e}^{-\lambda(n-n_0)}.$$

And summing over $i \geq 1$ and $k \in \{1, \cdots, [M2^{n/\rho}] + 1\}$, we have for $M \geq m \geq 2^{-n/\rho}$,

$$\mathbb{N}_x \Big[\exists i \in \{1, \cdots, N-1\};\ m \leq \zeta_{\tau_i} \leq M, \hat{L}_{\tau_i} > 2^{-n_0}, \tilde{\psi}^{a_0}_{n_0,n}(S_{[0,\hat{L}_{\tau_i})}(W_{\tau_i})) > 1/12\Big] \\ \leq 2M2^{2n/\rho}2^{n-n_0}\, \mathrm{e}^{-\lambda(n-n_0)}\,.$$

The end of the proof is similar to the one of lemma 4.6. □

5. Proof of the uniqueness theorem 1.1

The proof of this theorem relies heavily on the lemmas of the previous section and on the excursion decomposition of the snake.

Proof of Theorem 1.1. Let $B_\varepsilon = B_{\partial D}(y_0, \varepsilon)$, where $y_0 \in \partial D$. We denote by v_ε the maximal nonnegative solution of (4) and u_ε the minimal nonnegative solution. In the first section we recall a representation of those functions in terms of the super-process X. From the characterization of $\mathcal{R}_D$ (this a projection of the graph $\mathcal{G}_D$ on $\mathbb{R}^d$) in 2.2 C from [12], the Poisson representation of proposition 2.2 and lemma 5.2 in [8] (with the Brownian motion replaced by the Brownian motion stopped at its exit time of D), we get for $x \in D$, $v_\varepsilon(x) \leq \mathbb{N}_x[T < \infty]$, where

$$T = \inf\{s > 0,\ \zeta_s = \tau_D(W_s) \text{ and } \hat{W}_s \in \overline{B_\varepsilon}\}.$$

(In fact we will see the above inequality is an equality.) We deduce from (2) and the snake representation of the exit measure that $u_\varepsilon(x) = \mathbb{N}_x[Y_D(B_\varepsilon) > 0]$. The strong Markov property applied at the stopping time T gives

$$u_\varepsilon(x) = \mathbb{N}_x[T < \infty, Y_D(B_\varepsilon) > 0] = \mathbb{N}_x[T < \infty, \mathbb{E}^*_{W_T}(Y_D(B_\varepsilon) > 0)].$$

Thus, to prove the uniqueness, it is enough to prove that $\mathbb{E}^*_{W_T}(Y_D(B_\varepsilon) > 0)) = 1$ $\mathbb{N}_x$-a.e. on $\{T < \infty\}$. Using proposition 2.2 on $\{T < \infty\}$, we have

$$\mathbb{E}^*_{W_T}(Y_D(B_\varepsilon) > 0) = 1 - \exp\left\{-\int_0^{\zeta_T} \mathbb{N}_{W_T(t)}(Y_D(B_\varepsilon) > 0)dt\right\}.$$

Thanks to the snake property, it is clear that $\mathbb{N}_x$-a.e. for every $s \in (0, \sigma)$, $L(W_s) = (L_t(W_s), t \in [0, \zeta_s])$ is continuous nondecreasing and the path $(\Gamma_t(W_s), t \in [0, \zeta_s])$ is constant on intervals where $L(W_s)$ itself is constant. Therefore the time change $S_s(W_T) = t$ implies

$$\mathbb{E}^*_{W_T}(Y_D(B_\varepsilon) > 0) = 1 - \exp\left\{-\int_0^{\hat{L}_T} \mathbb{N}_{\gamma_s(W_T)}(Y_D(B_\varepsilon) > 0)dS_s(W_T)\right\}.$$

Notice that $\gamma_s(W_T) \in D$ for $s \in [0, \hat{L}_T)$ and $\hat{W}_T \in \overline{B_\varepsilon}$. Now, let $A, a, a' > 0$. We set $J = J(A, a, a')$ the set of integers k such that $2^{-k+1} \leq \hat{L}_T$ and

$$\left|\gamma_s(W_T) - \hat{W}_T\right| \leq A2^{-k/2} \qquad \text{for} \quad s \in \left[0, \hat{L}_T - 2^{-k}\right],$$

$$d\big(\gamma_s(W_T), D^c\big) > a2^{-k/2} \qquad \text{for} \quad s \in \left[\hat{L}_T - \frac{15}{16}2^{-k}, \hat{L}_T - \frac{7}{8}2^{-k}\right],$$

$$\text{and} \quad S_{\hat{L}_T - \frac{15}{16}2^{-k}}(W_T) - S_{\hat{L}_T - \frac{7}{8}2^{-k}}(W_T) \geq a'2^{-k/\rho}.$$

Lemmas 4.5, 4.6 and 4.7 show that we can choose A, a, a' such that J is infinite $\mathbb{N}_x$-a.e. Moreover, lemma 3.2 gives for $\varepsilon > 0$ small enough that there exists $c > 0$ such that if $k \in J$ and if $t \in \left[\hat{L}_T - \frac{15}{16}2^{-k}, \hat{L}_T - \frac{7}{8}2^{-k}\right]$, then we have

$$\mathbb{N}_{\gamma_t(W_T)}(Y_D(B_\varepsilon) > 0) \geq c2^{k/\rho}.$$

We deduce that

$$\begin{aligned}
&\int_0^{\hat{L}_T} \mathbb{N}_{\gamma_s(W_T)}(Y_D(B_\varepsilon) > 0) dS_s(W_T) \\
&\quad \geq \sum_{k\in J} \int_{\hat{L}_T - \frac{15}{16}2^{-k}}^{\hat{L}_T - \frac{7}{8}2^{-k}} \mathbb{N}_{\gamma_s(W_T)}(Y_D(B_\varepsilon) > 0) dS_s(W_T) \\
&\quad \geq \sum_{k\in J} c2^{k/\rho}(S_{\hat{L}_T - \frac{15}{16}2^{-k}}(W_T) - S_{\hat{L}_T - \frac{7}{8}2^{-k}}(W_T)) \\
&\quad \geq \sum_{k\in J} ca'2^{k/\rho}2^{-k/\rho} = +\infty.
\end{aligned}$$

This implies that $\mathbb{E}^*_{W_T}(Y_D(B_\varepsilon) > 0) = 1$ $\mathbb{N}_x$-a.e., which in turn implies $v_\varepsilon = u_\varepsilon$ in D. □

We end this section with a lemma which will be useful later. Let $K \subset D$ be a compact set.

Lemma 5.1. *Let $\lambda > 0$. There exist $\delta_0 > 0$, $C > 0$ such that for all $x \in K$, $\delta \in (0, \delta_0]$,*

$$\mathbb{N}_x\left[\exists s \in (0,\sigma); \kappa_D(W_s) < \delta\right] \leq C\delta^\lambda,$$
$$\mathbb{N}_x\left[\exists s \in (0,\sigma); \zeta_s < \delta^{2/\rho}, \hat{L}_s > \delta\right] \leq C\delta^\lambda.$$

The proof of the first inequality uses an uniqueness result in a parabolic setting similar to theorem 1.1. The proof of the second inequality is more involved.

Proof. Let $\mathcal{G} = \{(\hat{L}_s, \hat{W}_s), s \in (0,\sigma)\}$ be the graph of the Brownian snake. Using the Brownian snake property on $[s, \inf\{u > s; \zeta_u = \tau_D(W_s)\}]$, we see that the set $\mathcal{A}_1 = \{\exists s \in (0,\sigma); \kappa_D(W_s) < \delta\}$ is a subset of $\{\mathcal{G} \cap [0,\delta) \times D^c \neq \emptyset\}$. Let O be a smooth domain such that $\overline{D^c} \subset O$ and $K \subset (\bar{O})^c$. Then we have

$$\mathcal{A}_1 \subset \{\mathcal{G} \cap [0,\delta) \times O \neq \emptyset\} \subset \bigcap_{t\in[0,\delta)\cap\mathbb{Q}} \{\mathcal{G} \cap \{t\} \times O \neq \emptyset\}.$$

We consider the stopping time for the Brownian snake

$$T_t = \inf\left\{s > 0; \zeta_s = \tau_{\mathbb{R}^+\times[0,t)\times\mathbb{R}^d}(W_s) \quad \text{and} \quad \hat{W}_s \in O\right\},$$

where we use the notation of section 2.3. Let Y_t be the exit measure of the Brownian snake of $\mathbb{R}^+ \times [0,t) \times \mathbb{R}^d$. We have $\{Y_t(O) > 0\} \subset \{T_t < \infty\}$. Arguing as in the proof of theorem 1.1 (mainly lemma 9.1 has to be replaced by the duality lemma p.45 of [3]), we can prove that for $x \in \mathbb{R}^d$,

$$\mathbb{N}_x[T_t < \infty] = \mathbb{N}_x[Y_t(O) > 0].$$

Therefore we have using theorem 8 of [4] and the right continuity of X for $\delta > 0$,

$$\begin{aligned}\mathbb{N}_x[\mathcal{A}_1] &\leq \mathbb{N}_x[\mathcal{G} \cap \{t\} \times O \neq \emptyset \quad \text{for some} \quad t \in [0,\delta) \cap \mathbb{Q}] \\ &\leq \mathbb{N}_x[Y_t(O) > 0 \quad \text{for some} \quad t \in [0,\delta) \cap \mathbb{Q}] \\ &\leq -\log\left(1 - \mathbb{P}^X_{\delta_x}[X_t(O) \neq 0 \quad \text{for some} \quad t \in [0,\delta)]\right).\end{aligned}$$

The first inequality of the lemma is then a consequence of theorem 9.2.4. of [6].

The proof of the second inequality is more involved. We set $m = \delta^{2/\rho}$ and $\mathcal{A}_2 = \{\exists s \in (0,\sigma);\, \zeta_s < m, \hat{L}_s > \delta\}$. We have

$$\mathbb{N}_x[\mathcal{A}_2] \leq \sum_{k=0}^{\infty} \mathbb{N}_x\left[\exists s \in (0,\sigma);\, \zeta_s \in (m2^{-k-1}, m2^{-k}], \hat{L}_s > \delta\right].$$

For each $k \in \mathbb{Z}_+$, we define inductively a sequence of stopping time $(\tau_i^k, i \geq 0)$ by

$$\tau_0^k = 0, \quad \text{and} \quad \tau_{i+1}^k = \inf\left\{v > \tau_i^k;\, \left|\zeta_v - \zeta_{\tau_i^k}\right| = m2^{-k-1}\right\}.$$

Let $N_k = \inf\{i > 0;\, \tau_i^k = \infty\}$. Recall that $\mathbb{N}_x[\tau_1^k < \infty] = m^{-1}2^k$. Conditionally on $\{\tau_1^k < \infty\}$, the sequence $(\zeta_{\tau_i^k}, i \geq 1)$ is a simple random walk on $m2^{-k-1}\mathbb{Z}_+$ stopped when it reached 0. We have for $j_0 \geq 1$,

$$\begin{aligned}\mathbb{N}_x\left[\sum_{i=1}^{\infty} \mathbf{1}_{\{\zeta_{\tau_i^k} = j_0 m2^{-k-1}\}}\right] &= \mathbb{N}_x\left[\tau_1^k < \infty\right] \mathbb{N}_x\left[\sum_{i=1}^{\infty} \mathbf{1}_{\{\zeta_{\tau_i^k} = j_0 m2^{-k-1}\}} \middle| \tau_1^k < \infty\right] \\ &= m^{-1}2^{k+1}. \qquad (15)\end{aligned}$$

We have

$$\begin{aligned}&\mathbb{N}_x\left[\exists s \in (0,\sigma);\, \zeta_s \in (m2^{-k-1}, m2^{-k}] \quad \text{and} \quad \hat{L}_s > \delta\right] \\ &\leq \sum_{j=1}^{2} \mathbb{N}_x\left[\exists i \in \{1,\cdots,N_k - 1\};\, \zeta_{\tau_i^k} = jm2^{-k-1} \ \text{ and } \ \exists s \in [\tau_i^k, \tau_{i+1}^k], \ \text{ s.t. } \ \hat{L}_s > \delta\right] \\ &\leq \sum_{j=1}^{2} \sum_{i=1}^{\infty} \mathbb{N}_x\left[\zeta_{\tau_i^k} = jm2^{-k-1}, \exists s \in [\tau_i^k, \tau_{i+1}^k] \quad \text{s.t.} \quad \hat{L}_s > \delta\right].\end{aligned}$$

We consider only $j \in \{1, 2\}$. Let $c_1 > 0$ be a constant whose value will be chosen later. We set $a = c_1(m2^{-k-1})^2 \log(2^{k+1}/m)$ and $c_2 = c_1^{-2/\rho} 2^{(k+1)3\rho/4} m^{-\rho/4}$. For

δ small enough, notice that $c_2 a^{\rho/2} < \delta/2$ for every $k \in \mathbb{Z}_+$. We have

$$\begin{aligned}
&\mathbb{N}_x \left[\zeta_{\tau_i^k} = jm2^{-k-1}, \exists s \in [\tau_i^k, \tau_{i+1}^k] \quad \text{s.t.} \quad \hat{L}_s > \delta \right] \\
&\quad \le \mathbb{N}_x \left[\zeta_{\tau_i^k} = jm2^{-k-1}, \tau_{i+1}^k - \tau_i^k > a \right] \\
&\qquad + \mathbb{N}_x \left[\zeta_{\tau_i^k} = jm2^{-k-1}, \sup_{s \in [\tau_i^k, \tau_i^k + a]} \left| \hat{L}_s - \hat{L}_{\tau_i^k} \right| > c_2 a^{\rho/2} \right] \\
&\qquad + \mathbb{N}_x \left[\zeta_{\tau_i^k} = jm2^{-k-1}, \hat{L}_{\tau_i^k} > \delta - c_2 a^{\rho/2} \right].
\end{aligned}$$

We write $I_k^{(l)}$ for the l-th term of the right member. The distribution of $\tau_{i+1}^k - \tau_i^k$ knowing $\{i < N_k\}$ is the law of the first exit time from $[-m2^{-k-1}, m2^{-k-1}]$ for a standard linear Brownian motion started at 0. Thus there exist two positive constants a_1, a_2 such that

$$\begin{aligned}
I_k^{(1)} &= \mathbb{N}_x \left[\zeta_{\tau_i^k} = jm2^{-k-1}, \tau_{i+1}^k - \tau_i^k > a \right] \\
&\le \mathbb{N}_x \left[\zeta_{\tau_i^k} = jm2^{-k-1} \right] a_1 \, \mathrm{e}^{-a_2 c_1 \log(m^{-1} 2^{k+1})} .
\end{aligned}$$

For $\delta < 1$ and $k \ge 0$, we have $c_2 > c_1^{-2/\rho} = \theta$. We deduce from lemma 4.2 that

$$\begin{aligned}
I_k^{(2)} &= \mathbb{N}_x \left[\zeta_{\tau_i^k} = jm2^{-k-1}, \sup_{s \in [\tau_i^k, \tau_i^k + a]} \left| \hat{L}_s - \hat{L}_{\tau_i^k} \right| > c_2 a^{\rho/2} \right] \\
&\le \mathbb{N}_x \left[\zeta_{\tau_i^k} = jm2^{-k-1} \right] c_3 \, \mathrm{e}^{-m^{-\rho/4} 2^{(k+1)\rho/4}},
\end{aligned}$$

where c_3 depends only on c_1.

Conditionally on $\zeta_{\tau_i^k} = jm2^{-k-1}$, the path $W_{\tau_i^k}$ is distributed as $\bar{\xi}$ under $\bar{\mathrm{P}}_x^{jm2^{-k-1}}$. So, we get for $b > 0$,

$$\begin{aligned}
I_k^{(3)} &= \mathbb{N}_x \left[\zeta_{\tau_i^k} = jm2^{-k-1}, \hat{L}_{\tau_i^k} > \delta - c_2 a^{\rho/2} \right] \\
&\le \mathbb{N}_x \left[\zeta_{\tau_i^k} = jm2^{-k-1} \right] \bar{\mathrm{P}}_x [L_{jm2^{-k-1}} > \delta - c_2 a^{\rho/2}] \\
&\le \mathbb{N}_x \left[\zeta_{\tau_i^k} = jm2^{-k-1} \right] \mathrm{e}^{bjm2^{-k-1} - c_\rho^* b^\rho (\delta - c_2 a^{\rho/2})},
\end{aligned}$$

where we used (12). Now take $b = (c_\rho^*)^{-1/\rho} m^{-1} 2^{k+1}$ and use the fact that $c_2 a^{\rho/2} < \delta/2 = m^{\rho/2}/2$ to get

$$I_k^{(3)} \le \mathbb{N}_x \left[\zeta_{\tau_i^k} = jm2^{-k-1} \right] c_4 \, \mathrm{e}^{-m^{-\rho/2} 2^{(k+1)\rho}/2} .$$

We have

$$\mathbb{N}_x[\mathcal{A}_2] \le \sum_{j=1}^{2}\sum_{k=0}^{\infty}\sum_{i=1}^{\infty}[I_k^{(1)} + I_k^{(2)} + I_k^{(3)}].$$

We deduce from (15) and the upper bounds on $I_k^{(1)}$, $I_k^{(2)}$ and $I_k^{(3)}$, that for $\lambda > 0$ given, we can choose c_1 and C large enough so that $\mathbb{N}_x[\mathcal{A}_2] \le C\delta^\lambda$. □

6. An upper bound for the hitting probability of small balls for Y_D in the critical dimension d_c

The theorem 1.3 is a direct consequence of the next inequality and the snake representation of the exit measure (proposition 2.4). Once again the proof relies on the uniform behavior of the paths W_s near its end point.

Proposition 6.1. *Let $d = d_c$, $K \subset D$ be a compact set. There exist two positive constants C_K and ε_K such that for all $x \in K$, $y \in \partial D$, $\varepsilon \in (0, \varepsilon_K]$,*

$$\mathbb{N}_x\left[Y_D(B_{\partial D}(y,\varepsilon)) > 0\right] \le C_K\left(\log(1/\varepsilon)\right)^{-1}.$$

Proof of Proposition 6.1. Let $d = d_c$. Recall the notation at the beginning of section 3. By formula (6), we have

$$\begin{aligned}\mathbb{N}_x\left[Y_D(B_{\partial D}(y,\varepsilon))\right] &= \bar{\mathrm{E}}_x\left[\mathbf{1}_{B_{\partial D}(y,\varepsilon)}(\Gamma_{\tau_D})\right]\\ &= \mathrm{E}_x\left[\mathbf{1}_{B_{\partial D}(y,\varepsilon)}(\gamma_{\kappa_D})\right]\\ &= \int_{B_{\partial D}(y,\varepsilon)}\theta(dz)P_D(x,z),\end{aligned}$$

where θ is the surface measure on ∂D and P_D is the Poisson kernel. From (9), we see that if K is a compact subset of D, there exist positive constants C_K and ε_K such that for every $x \in K$, $y \in \partial D$, $\varepsilon \in (0, \varepsilon_K]$,

$$\mathbb{N}_x\left[Y_D(B_{\partial D}(y,\varepsilon))\right] \le C_K\varepsilon^{d_c-1}.$$

Then we consider the stopping time

$$T = \inf\{s > 0; \tau_D(W_s) = \zeta_s \quad \text{and} \quad \hat{W}_s \in B_{\partial D}(y,\varepsilon)\}.$$

We have from the construction of Y_D,

$$\{Y_D(B_{\partial D}(y,\varepsilon)) > 0\} \subset \{T < \infty\}.$$

Consequently, using the strong Markov property at time T, we get

$$\mathbb{N}_x\left[Y_D(B_{\partial D}(y,\varepsilon))\right] = \mathbb{N}_x\left[T < \infty; \mathbb{E}^*_{W_T}\left[Y_D(B_{\partial D}(y,\varepsilon))\right]\right].$$

Thus we see that a lower bound for $\mathbb{E}^*_{W_T}\,[Y_D(B_{\partial D}(y,\varepsilon))]$ with the previous upper bound of $\mathbb{N}_x\,[Y_D(B_{\partial D}(y,\varepsilon))]$ yield an upper bound for $\mathbb{N}_x\,[T<\infty]$, that is for $\mathbb{N}_x\,[Y_D(B_{\partial D}(y,\varepsilon))>0]$. By proposition 2.2 and relation (6), we have

$$\begin{aligned}\mathbb{E}^*_{W_T}\,[Y_D(B_{\partial D}(y,\varepsilon))] &= 2\int_0^{\zeta_T} dt\ \mathbb{N}_{W_T(t)}\,[Y_D(B_{\partial D}(y,\varepsilon))]\\ &= 2\int_0^{\zeta_T} dt\ \bar{\mathrm{P}}_{W_T(t)}\left[\Gamma_{\tau_D}\in B_{\partial D}(y,\varepsilon)\right]\\ &= 2\int_0^{\zeta_T} dt\ \mathrm{P}_{\Gamma_t(W_T)}\left[\gamma_{\kappa_D}\in B_{\partial D}(y,\varepsilon)\right]\\ &= 2\int_0^{\zeta_T} dt\int_{B_{\partial D}(y,\varepsilon)}\theta(dz)P_D(\Gamma_t(W_T),z).\end{aligned}$$

The time change $S_v(W_T)=t$ and (9) imply

$$\begin{aligned}\mathbb{E}^*_{W_T}\,[Y_D(B_{\partial D}(y,\varepsilon))] &= 2\int_0^{\kappa_D(W_T)} dS_v(W_T)\int_{B_{\partial D}(y,\varepsilon)}\theta(dz)P_D(\gamma_v(W_T),z)\\ &\geq 2c\int_0^{\kappa_D(W_T)} dS_v(W_T)\ d(\gamma_v(W_T),\partial D)\\ &\quad\times\int_{B_{\partial D}(y,\varepsilon)}\theta(dz)\,|\gamma_v(W_T)-z|^{-d}\,.\end{aligned}$$

Remark 6.2. Let ε be small, and consider the integer $n\geq 1$ such that $2^{-n}\leq\varepsilon^2<2^{-n+1}$. Let V_n be the set of integer $k\in\{0,\dots,n\}$ such that, for all $v\in[\hat{L}_T-2^{-k},\hat{L}_T-2^{-(k+1)}]$, we have

$$\left|\gamma_v(W_T)-\hat{W}_T\right|<A_0 2^{-k/2},\quad d\big(\gamma_v(W_T),\partial D\big)>a_0 2^{-k/2}. \tag{16}$$

As mentioned in remark 4.1, for a fixed time t, the paths $S(W_t)$ and $\gamma(W_t)$ are distributed respectively according to the law of a subordinator of index ρ and a Brownian motion. If this were also true for W_T, then we would get

$$\mathbb{E}^*_{W_T}\,[Y_D(B_{\partial D}(y,\varepsilon))]\geq c\varepsilon^{d_c-1}\sum_{k\in V_n}\int_{[\hat{L}_T-2^{-k},\hat{L}_T-2^{-(k+1)})} dS_v(W_T)\ 2^{k(d_c-1)/2}.$$

By the scaling property of subordinators and lemma 9.1, we would have

$$\mathbb{E}^*_{W_T}\,[Y_D(B_{\partial D}(y,\varepsilon))]\geq c\varepsilon^{d_c-1}\mathrm{Card}\ (V_n)^{1/\rho}S_1$$

where S_1 is a subordinator of index ρ. Moreover, the scaling property for the Brownian motion and Borel-Cantelli lemma give that Card $V_n\geq cn$ and finally we get the upper bound for the hitting probability $c[\log(1/\varepsilon)]^{-1/\rho}$. Unfortunately, there is no reason for the law of $S(W_T)$, where T is random, to be the law of a subordinator, nor for the law of $\gamma(W_T)$ to be the law of a Brownian motion. (For the usual Brownian snake, the law of W_T, where T is the first hitting time of a ball is the law of a diffusion but not of a Brownian motion. This example from Le Gall can be found in [9] proposition 1.4.). That is why we need the lemmas of section 4 which say that the estimates of (16) are true sufficiently often along the path $\gamma(W_T)$ and that we have similar estimates for the path $S(W_T)$. However, this remark suggests that the accurate upper bound for the hitting probability should be $c[\log(1/\varepsilon)]^{-1/\rho}$.

Let us go back to the proof. Let ε be small, and consider the integer $n \geq 1$ such that $2^{-n} \leq \varepsilon^2 < 2^{-n+1}$. Let n_0 be the integer part of $n/2$. Let $\lambda > 0$ be large enough. Let us assume that ε is small enough so that $c_* n^{1+\frac{\rho}{2}} 2^{-n} < 2^{-n_1}$ where c_* is defined in lemma 4.4 and $n_1 > n_0$ is the integer part of $11n/12$. Consider the set

$$\mathcal{B} = \{\zeta_T \geq 2 * 2^{-n/\rho}\} \cap \{\hat{L}_T > 2 * 2^{-n_0}\}$$

Let U_n be the set of integers $k \in \{n_0, \cdots, n_1\}$ such that for all $v \in [\hat{L}_T - \frac{15}{16} 2^{-k}, \hat{L}_T - \frac{7}{8} 2^{-k}]$, we have

$$\left|\gamma_v(W_T) - \hat{W}_T\right| < A_0 2^{-k/2}, \quad d(\gamma_v(W_T), \partial D) > a_0 2^{-k/2}, \tag{17}$$

and $S_{(\hat{L}_T - \frac{7}{8}2^{-k})-}(W_T) - S_{(\hat{L}_T - \frac{15}{16}2^{-k})-}(W_T) > a_1 2^{-k/\rho}$, where A, a_0, a_1 are defined in lemma 4.5, 4.6 and 4.7. On $\mathcal{B}$, we then have for $\varepsilon > 0$ small enough,

$$\begin{aligned} &\mathbb{E}^*_{W_T}\left[Y_D(B_{\partial D}(y,\varepsilon))\right] \\ &\geq \sum_{k \in U_n} \int_{[\hat{L}_T - \frac{15}{16}2^{-k}, \hat{L}_T - \frac{7}{8}2^{-k})} dS_v(W_T) a_0 2^{-k/2} \\ &\quad \times \int_{B_{\partial D}(y,\varepsilon)} \theta(dz)[A_0 2^{-k/2} + 4 * 2^{-n/2}]^{-d} \geq c' \varepsilon^{d_c - 1} \text{Card } U_n, \end{aligned}$$

where the constant $c' > 0$ is independent of W, n and $x \in K$. Notice that on

$$\begin{aligned} \mathcal{B}_1 = \mathcal{B} \cap \{\zeta_T \leq 2^{n/\rho}\} \cap \{F^{A_0}_{n_0,n_1}(W_T) < 1/6\} \cap \{\phi^{a_0}_{n_0,n_1}(W_T) < 1/6\} \\ \cap \{\psi^{a_1}_{n_0,n_1}(W_T) < 1/6\}, \end{aligned}$$

Card $U_n > n/3 \geq c'' \log(1/\varepsilon)$. Thus we deduce from the previous inequalities that there exist a constant C such that for any ε small enough and $x \in K$,

$$C\varepsilon^{d_c - 1} \geq \mathbb{N}_x[T < \infty; \mathcal{B}_1] \varepsilon^{d_c - 1} \log(1/\varepsilon).$$

The set $\mathcal{B}_1^c$ is a subset of $\cup_{i=1}^6 \mathcal{H}_i$, where

$$\begin{aligned} \mathcal{H}_1 &= \left\{\sup_{s \geq 0} \zeta_s \geq M\right\} \quad \text{with} \quad M = 2^{n/\rho}; \\ \mathcal{H}_2 &= \left\{\exists s \in (0,\sigma); \kappa_D(W_s) < 4.2^{-n_0}\right\} \supset \left\{\hat{L}_T \leq 2 * 2^{-n_0}\right\}; \\ \mathcal{H}_3 &= \left\{\exists s \in (0,\sigma); \zeta_s < 2 * 2^{-n/\rho}, \hat{L}_s > 2^{-n_0}\right\} \\ &\supset \left\{\zeta_T < 2 * 2^{-2n/\rho}\right\} \cap \left\{\hat{L}_T > 2 * 2^{n_0}\right\}; \\ \mathcal{H}_4 &= \left\{\exists s \in (0,\sigma), 2 * 2^{-n/\rho} \leq \zeta_s \leq M, F^{A_0}_{n_0,n_1}(W_s) > 1/6\right\}; \\ \mathcal{H}_5 &= \left\{\exists s \in (0,\sigma), 2 * 2^{-n/\rho} \leq \zeta_s \leq M, \phi^{a_0}_{n_0,n_1}(W_s) > 1/6\right\}; \\ \mathcal{H}_6 &= \left\{\exists s \in (0,\sigma), 2 * 2^{-n/\rho} \leq \zeta_s \leq M, \psi^{a_0}_{n_0,n_1}(W_s) > 1/6\right\}. \end{aligned}$$

Using the normalization of $\mathbb{N}_x$ for $\mathcal{H}_1$, lemma 5.1 for $\mathcal{H}_2$ and $\mathcal{H}_3$, lemmas 4.5, 4.6 and 4.7 respectively for $\mathcal{H}_4$, $\mathcal{H}_5$ and $\mathcal{H}_6$, we see we can choose A_0, a_0 and a_1 so that $\mathbb{N}_x[\mathcal{B}_1^c] \leq c'\varepsilon^\delta$ for some constants $c' > 0, \delta > 0$. So we deduce that for $x \in K$, $\varepsilon > 0$ small enough

$$\mathbb{N}_x\left[Y_D(B_{\partial D}(y,\varepsilon)) > 0\right] \leq \mathbb{N}_x[T < \infty] \leq C\left[\log 1/\varepsilon\right]^{-1} + c'\varepsilon^\delta,$$

which ends the proof. □

7. Lower bound of dim supp X_D

Using the snake representation of X_D, we see that a lower bound for the Hausdorff dimension of the support of Y_D will provide a lower bound for the Hausdorff dimension of the support of X_D.

Proposition 7.1. *Let $d \geq 2$. Let $x \in D$. $\mathbb{N}_x$-a.e. on $\{Y_D \neq 0\}$, we have*

$$\text{dim supp } Y_D \geq \frac{2}{\alpha - 1} \wedge (d-1).$$

Proof. We set $d_0 = \frac{2}{\alpha-1} \wedge (d-1)$. Following the idea of [8], we will first prove that for $\varepsilon \in (0, d_0/3)$,

$$\mathbb{N}_x\left[\int Y_D(dz)\ F_{d_0-3\varepsilon}(z, Y_D)\right] = 0,$$

where if $\theta > 0$, F_θ is the measurable function on $\mathbb{R}^d \times M_f$ defined by

$$F_\theta(y,\nu) = \mathbf{1}_{\left\{\limsup_{n\to\infty} \nu(B_{\partial D}(y, 2^{-n}))2^{n\theta} > 0\right\}}.$$

By proposition 2.3, we have

$$\mathbb{N}_x\left[\int Y_D(dy)F_\theta\,(y, Y_D)\right] = \int \bar{\mathrm{P}}_x^D(dw)\mathbb{E}\left[F_\theta\left(\hat{w}, \int \mathcal{N}_w(d\mathrm{W})\ Y_D(\mathrm{W})\right)\right]. \tag{18}$$

In order to use the Borel-Cantelli lemma, we first bound $\int \mathbb{P}(d\omega)\mathbf{1}_{A_n}(w,\omega)$, where

$$A_n := \left\{(w,\omega);\ 2^{n(d_0-3\varepsilon)}\int \mathcal{N}_w(\omega)(d\mathrm{W})\ Y_D(\mathrm{W})\left(B_{\partial D}(\hat{w}, 2^{-n})\right) \geq C_{d_0}2^{-n\varepsilon}\right\}$$

and $C_{d_0} = C_{d_0}(w)$ is a finite positive constant that does not depend on n and ω, and depends only on w. Its value will be fixed later. Recall that τ_D is the exit time of D for the process Γ and κ_D is the exit time of D for the process γ. Using the Markov inequality, we get for $\bar{\mathrm{P}}_x^D$-a.e. paths w,

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{A_n}\right] &\leq \mathbb{E}\left[C_{d_0}^{-1}2^{n(d_0-2\varepsilon)}\int \mathcal{N}_w(d\mathrm{W})Y_D(\mathrm{W})\left(B_{\partial D}(\hat{w},2^{-n})\right)\right] \\ &= 2^{n(d_0-2\varepsilon)}C_{d_0}^{-1}4\int_0^{\zeta_w} dv\ \mathbb{N}_{w(v)}\left[Y_D\left(B_{\partial D}(y,2^{-n})\right)\right]_{y=\hat{w}} \\ &= 4\ 2^{n(d_0-2\varepsilon)}C_{d_0}^{-1}\int_0^{\tau_D(w)} dv\ \bar{\mathrm{P}}_{w(v)}^D\left[\hat{w}\in B_{\partial D}(y,2^{-n})\right]_{y=\hat{w}} \\ &= 4\ 2^{n(d_0-2\varepsilon)}C_{d_0}^{-1}\int_{[0,\kappa_D(w))} dS_u(w)\ \mathrm{P}_{\gamma_u(w)}\left[\gamma_{\kappa_D}\in B_{\partial D}(y,2^{-n})\right]_{y=\gamma_{\kappa_D}(w)},\end{aligned} \tag{19}$$

where γ is under P_x a Brownian motion in $\mathbb{R}^d$ started at x. In the first equality we used the form of the intensity of the Poisson measure $\mathcal{N}_w$. In the second one, we applied (6). In the third one, we made the formal change of variable $v=S_u$, using the specific properties of the process ξ under $\bar{\mathrm{P}}_x^D$, and in particular the fact that $\Gamma=\gamma_L$ is constant over each interval (S_{u-},S_u).

Let $r\in(0,1]$, we have for $0\leq u<\kappa_D$

$$\mathrm{P}_{\gamma_u}\left[\gamma_{\kappa_D}\in B_{\partial D}(y,r)\right]_{y=\gamma_{\kappa_D}} = \int_{B_{\partial D}(\gamma_{\kappa_D},r)} P_D(\gamma_u,y')\theta(dy').$$

We deduce from (9) that for $(y,y')\in D\times\partial D$,

$$P_D(y,y')\leq c_1 d(y,\partial D)\left|y-y'\right|^{-d}\leq c_1 d(y,\partial D)^{-(d_0-\varepsilon)}\left|y-y'\right|^{(d_0-\varepsilon)+1-d}.$$

Notice also there exists a positive constant c_2 such that for all $(y,y'')\in D\times\partial D$, $r\in(0,1]$,

$$\int_{B_{\partial D}(y'',r)}\left|y-y'\right|^{(d_0-\varepsilon)+1-d}\theta(dy')\leq c_2 r^{d_0-\varepsilon}.$$

Thus we deduce that for every $r\in(0,1]$,

$$\mathrm{P}_{\gamma_u}\left[\gamma_{\kappa_D}\in B_{\partial D}(y,r)\right]_{y=\gamma_{\kappa_D}}\leq c_1c_2\ r^{d_0-\varepsilon}d(\gamma_u,\partial D)^{-(d_0-\varepsilon)}. \tag{20}$$

The proof of the next lemma is postponed to the end of this section.

Lemma 7.2. *Let $\theta>0$, then $\bar{\mathrm{P}}_x^D$-a.s. we have*

$$\sup_{u\in[0,\kappa_D)}\frac{(\kappa_D-u)^{\theta+1/2}}{d(\gamma_u,\partial D)}<\infty.$$

The proof of the following lemma relies on an integration by part and on the path properties of the subordinator S (see lemma 3.2.3 in [8]).

Lemma 7.3. *Let $d' \in [0, 2/\rho)$, then $\bar{\mathrm{P}}_x^D(dw)$-a.s. we have*

$$\int_{[0,\kappa_D)} (\kappa_D - u)^{-d'/2} dS_u < \infty.$$

As a consequence of those two lemmas, the variable

$$C_{d_0} = \int_{[0,\kappa_D)} dS_u \; d(\gamma_u, \partial D)^{-(d_0-\varepsilon)}$$

is finite $\bar{\mathrm{P}}_x^D$-a.s. Thus plugging (20) into (19), we get that for every $n \geq 1$,

$$\mathbb{E}\left[\mathbf{1}_{A_n}\right] \leq 4c_1c_2 \; 2^{-n\varepsilon}.$$

Applying the Borel-Cantelli lemma to the sequence $(A_n, n \geq 1)$, we get $\bar{\mathrm{P}}_x^D$-a.s., $\mathbb{P}$-a.s.

$$\limsup_{n\to\infty} 2^{n(d_0-3\varepsilon)} \int \mathcal{N}_w(d\mathrm{W}) Y_D(\mathrm{W}) \left(B_{\partial D}(\hat{w}, 2^{-n})\right) = 0.$$

Hence by the definition of F_θ and (18), we get

$$\mathbb{N}_x \left[\int Y_D(dy) F_{d_0-3\varepsilon}(y, Y_D)\right] = 0.$$

We deduce from theorem 4.9 of [16], that $\mathbb{N}_x$-a.e. on $\{Y_D \neq 0\}$,

$$\dim \; \operatorname{supp} Y_D \geq d_0 - 3\varepsilon.$$

Since ε is arbitrary, the lower bound of the proposition follows. □

Proof of Lemma 7.2. It is enough to prove the result under P_x. Let $\theta \in (0, 1/2)$ and $D_\varepsilon = \{y \in D; d(y, \partial D) > \varepsilon\}$. For simplicity we write $\kappa = \kappa_D$ and $\kappa_\varepsilon = \kappa_{D_\varepsilon}$. We will first derive an upper bound for

$$\mathrm{P}_x\left[\kappa - \kappa_\varepsilon \geq \varepsilon^{2-\theta}\right].$$

For $\varepsilon > 0$ small enough, we have using the Markov property at time κ_ε:

$$\begin{aligned} \mathrm{P}_x\left[\kappa - \kappa_\varepsilon \geq \varepsilon^{2-\theta}\right] &\leq \left(1 - \mathrm{e}^{-1}\right)^{-1} \left[1 - \mathrm{E}_x\left[\mathrm{e}^{-\varepsilon^{-2+\theta}(\kappa-\kappa_\varepsilon)}\right]\right] \\ &\leq \left(1 - \mathrm{e}^{-1}\right)^{-1} \sup_{y\in D,\; d(y,\partial D)=\varepsilon} \left[1 - \mathrm{E}_y\left[\mathrm{e}^{-\varepsilon^{-2+\theta}\kappa}\right]\right] \qquad (21) \end{aligned}$$

Since the domain D is bounded C^2, we have the uniform exterior sphere condition. There exists $h > 0$ such that for each point $y_0 \in \partial D$, we can find $y_1 \in D^c$ so that $y_0 \in \partial B(y_1, h)$ and $B(y_1, h) \subset D^c$, where $B(y, r)$ is the ball centered at y with radius r. For $y \in D$ there exists $y_0 \in \partial D$ such that $d(y, \partial D) = |y - y_0|$. Clearly, under P_y, $\kappa \leq \kappa_{B(y_1,h)}$, when y_1 is defined as above. Thus

$$\left[1 - \mathrm{E}_y\left[\mathrm{e}^{-\varepsilon^{-2+\theta}\kappa}\right]\right] \leq \left[1 - \mathrm{E}_y\left[\mathrm{e}^{-\varepsilon^{-2+\theta}\kappa_{B(y_1,h)}}\right]\right].$$

On the other hand, following [18] (p. 88) (see also [24]), it is easy to prove that for $y' \in \mathbb{R}^d$, $|y'| > h$, $\beta \geq 0$,

$$\mathrm{E}_{y'}\left[\mathrm{e}^{-\beta\kappa_{B(0,h)}}\right] = \frac{|y'|^{-\nu} K_\nu(\sqrt{2\beta}\,|y'|)}{|h|^{-\nu} K_\nu(\sqrt{2\beta}h)},$$

where $\nu = (d/2) - 1$ and K_ν is the second modified Bessel function. Since $K_\nu(r) = \sqrt{\pi/2r}\,\mathrm{e}^{-r}[1 + O(1/r)]$ (see [23] p. 202), it easy to deduce from (21) and the previous inequality (take $\beta = \varepsilon^{-2+\theta}$ and $y' = y - y_1$, where $d(y, \partial D) = \varepsilon$ and $|y'| = h + \varepsilon$) that for ε small enough,

$$\mathrm{P}_x\left[\kappa - \kappa_\varepsilon \geq \varepsilon^{2-\theta}\right] \leq c\varepsilon^{\theta/2},$$

where the constant c is independent of ε. Now thanks to the Borel-Cantelli lemma we get that P_x-a.s. the sequence $\left(2^{n(2-\theta)}(\kappa - \kappa_{2^{-n}}), n \geq 1\right)$ is bounded.

On the other hand notice that for $u \in [\kappa_{2^{-n+1}}, \kappa_{2^{-n}}]$ we have $d(\gamma_u, \partial D) \geq 2^{-n}$ and $\kappa - u \leq \kappa - \kappa_{2^{-n+1}}$. Thus we have

$$\frac{\kappa - u}{d(\gamma_u, \partial D)^{2-\theta}} \leq 4\; 2^{(n-1)(2-\theta)}(\kappa - \kappa_{2^{-n+1}}).$$

Since the right hand side is uniformly bounded in n, we get the lemma. □

8. Proof of Theorem 1.6 on the connected component of X_D

The proof of Theorem 1.6 mimic the proof of Theorem 2.4 in [8]. It relies on the next two lemmas. We only give the proof of Lemma 8.2 because it differs from its analogue in [8].

Lemma 8.1. *We consider the product measure* $\mathbb{N}_{x_1} \otimes \mathbb{N}_{x_2}$ *on the space* $C(\mathbb{R}^+, \mathcal{W})^2$. *The canonical process on this space is denoted by* (W^1, W^2). *Assume* $d > 2d_c - 1$. *Then for every* $(x_1, x_2) \in D^2$, *we have* $\mathbb{N}_{x_1} \otimes \mathbb{N}_{x_2}$*-a.e.*

$$\operatorname{supp} Y_D(W^1) \cap \operatorname{supp} Y_D(W^2) = \emptyset.$$

Lemma 8.2. *For* $\varepsilon > 0$, $\delta > 0$, *set*

$$g_\varepsilon(\delta) = \sup \mathbb{N}_y\left[\operatorname{supp} Y_D \cap \partial D \backslash B_{\partial D}(z, \varepsilon) \neq \emptyset\right],$$

where the supremum is taken over $(y, z) \in D \times \partial D$, *such that* $d(y, \partial D) = |y - z| < \delta$. *Then for every* $\varepsilon > 0$, $\lim_{\delta \downarrow 0} g_\varepsilon(\delta) = 0$.

Proof. Since the boundary of D is C^2, we have the uniform exterior sphere condition. There exists $\delta_0 \in (0, \varepsilon/3)$, for every $z \in \partial D$, we can find $z_0 \in D^c$ (unique) such that $B(z_0, \delta_0) \subset D^c$ and $\partial B(z_0, \delta_0) \cap \partial D = \{z\}$. We define $B_r = B(z_0, r\delta_0)$. We have for $y \in B_2 \backslash B_1$, $\mathbb{N}_y$-a.e.

$$\begin{aligned}
&\{\text{supp } Y_D \cap \partial D \backslash B_{\partial D}(z, \varepsilon) \neq \emptyset\} \\
&\quad \subset \left\{\exists s \in (0, \sigma); \zeta_s = \tau_D(W_s) \quad \text{and} \quad \hat{W}_s \in \partial D \backslash B_{\partial D}(z, \varepsilon)\right\} \\
&\quad \subset \left\{\exists s \in (0, \sigma); \tau_{\bar{B}_3^c}(W_s) < \infty, \tau_{\bar{B}_3^c}(W_s) < \tau_{B_1}(W_s)\right\}.
\end{aligned}$$

The first inclusion is a consequence of the definition of $L^{\mathbb{R}^+ \times \mathbb{R}^+ \times D}$ and the second is a consequence of the snake property. By the special Markov property (cf [4] proposition 7), if N is the number of excursions of the Brownian snake outside $\mathbb{R}^+ \times \mathbb{R}^+ \times B_2 \backslash B_1$ that reach $\mathbb{R}^+ \times \mathbb{R}^+ \times B_3^c$ before $\mathbb{R}^+ \times \mathbb{R}^+ \times \overline{B_1}$, then we have

$$\begin{aligned}
&\mathbb{N}_y \left[\exists s \in (0, \sigma); \tau_{\bar{B}_3^c}(W_s) < \infty, \tau_{\bar{B}_3^c}(W_s) < \tau_{B_1}(W_s)\right] \\
&\quad = \mathbb{N}_y[N > 0] \\
&\quad \leq \mathbb{N}_y[N] \\
&\quad = \mathbb{N}_y \left[\int Y_{B_2 \backslash B_1}(dy') \mathbb{N}_{y'}[\tau_{\bar{B}_3^c}(W_s) < \infty, \tau_{\bar{B}_3^c} < \tau_{B_1}]\right] \\
&\quad \leq \mathbb{N}_y \left[\int_{\partial B_2} Y_{B_2 \backslash B_1}(dy') \mathbb{N}_{y'}[\tau_{\bar{B}_3^c} < +\infty]\right].
\end{aligned}$$

We used the fact that if $y' \in \partial B_1$, then from the snake property, we have $\mathbb{N}_{y'}$-a.e. for all $s \in (0, \sigma)$, $\tau_{B_1(W_s)} = 0$. By symmetry, we get that $\mathbb{N}_{y'}[\tau_{\bar{B}_3^c} < +\infty] = c_0$ is independent of $y' \in \partial B_2$. It is also finite since $(\hat{W}_s, s \geq 0)$ is continuous under $\mathbb{E}_{(0,0,y')}$. We then deduce from (6) that

$$\mathbb{N}_y \left[\text{supp } Y_D \cap \partial D \backslash B_{\partial D}(z, \varepsilon) \neq \emptyset\right] \leq c_0 \mathrm{E}_y[\kappa_{B_2} < \kappa_{B_1}].$$

Thus we get that for $\delta \in (0, \delta_0)$,

$$g_\varepsilon(\delta) \leq c_0 \mathrm{E}_y[\kappa_{B(0,2\delta_0)} < \kappa_{B(0,\delta_0)}],$$

where $|y| = \delta_0 + \delta$. The lemma is then a consequence of classical results on Brownian motion. □

Proof of Theorem 1.6. Let $(D_k, k \geq 0)$ be an increasing sequence of open subsets of D such that $\bar{D}_k \subset D_{k+1}$ and $d(y, \partial D) \leq 1/k$ for all $y \in \partial D_k$. From the special Markov property (see [4] proposition 7) and proposition 2.4, we get that the law X_D under $\mathbb{P}_\nu^X$ is the same as the law of $\sum_{i \in I} Y_D(W^i)$, where conditionally on X_{D_k}, the random measure $\sum_{i \in I} \delta_{W^i}$ is a Poisson measure on $C(\mathbb{R}^+, \mathcal{W})$ with intensity $\int X_{D_k}(dy)\mathbb{N}_y[\cdot]$. With a slight abuse of notation, we may assume that the point measure $\sum_{i \in I} Y_D(W^i)$ is also defined under $\mathbb{P}_\nu^X$. It follows from lemma 8.1 and properties of Poisson measures that a.s. for every $i \neq j$,

$$\text{supp } Y_D(W^i) \cap \text{supp } Y_D(W^j) = \emptyset.$$

For $\varepsilon > 0$, let U_ε denote the event "supp X_D is contained in a finite union of disjoint compact sets of ∂D with diameter less than ε". It is easy to check that U_ε is measurable. Let k be large enough. Furthermore, by the previous observations, and denoting by $y_i \in D_k$ the common starting point of the paths W_s^i, and by z_i the only point in ∂D such that $|y_i - z_i| = d(y_i, \partial D)$, we have

$$\begin{aligned}\mathbb{P}_\nu^X[U_\varepsilon] &\geq \mathbb{P}_\nu^X\left[\forall i \in I, \text{diam }(\text{supp } Y_D(W^i)) \leq \varepsilon\right]\\ &\geq \mathbb{P}_\nu^X\left[\forall i \in I, \text{supp } Y_D(W^i) \subset B_{\partial D}(z_i, \varepsilon/2)\right]\\ &= \mathbb{E}_\nu^X\left[\exp - \int X_{D_k}(dy)\mathbb{N}_y[\text{supp } Y_D \cap \partial D \backslash B_{\partial D}(z, \varepsilon/2) \neq \emptyset]\right]\\ &\geq \mathbb{E}_\nu^X\left[\exp - g_{\varepsilon/2}(1/k)(X_{D_k}, \mathbf{1})\right],\end{aligned}$$

where for $B \in \mathcal{B}(\mathbb{R}^d)$, diam $(B) = \sup\{\left|x - x'\right|; (x, x') \in B \times B\}$. We can now let k go to $+\infty$, using lemma 8.2, to conclude that $\mathbb{P}_\nu^X[U_\varepsilon] = 1$. Since this holds for every $\varepsilon > 0$, we conclude that supp X_D is totally disconnected $\mathbb{P}_\nu^X$-a.s. □

9. Appendix

Lemma 9.1. *Let $(S_t, t \geq 0)$ be a stable subordinator. For $r > 0$, let $L_r = \inf\{u > 0, S_u > r\}$. Then $(S_t, t \in [0, L_r))$ and $(S_{L_r-} - S_{(L_r-t)-}, t \in [0, L_r))$ are identically distributed.*

We write P for the law of the subordinator $S = (S_t, t \geq 0)$ started at 0. We recall that the Laplace transform of S is given by $\eta(\lambda) = c_\rho^* \lambda^\rho$, where $c_\rho^* = 2^{-\rho}/\Gamma(1+\rho)$. Its Lévy measure is given by $\Pi(ds) = \mathbf{1}_{(0,\infty)}(s)[2^\rho\Gamma(\rho)\Gamma(1-\rho)]^{-1}s^{-1-\rho}ds$. Notice that L_r is the last exit time of $[0, r]$ for S. Let $Q = (Q_t, t \geq 0)$ be the transition kernel of S and $U = \int_0^\infty Q_t \; dt$ its potential. The transition kernels and the potential are absolutely continuous with respect to the Lebesgue measure l on $\mathbb{R}$. And we have $Q_t(x, dy) = q_t(y - x)dy$ and $U(x, dy) = u(y - x)dy$, where $u(y) = \rho 2^\rho y^{\rho-1}\mathbf{1}_{y\geq 0}$. Let $\hat{Q} = (\hat{Q}_t, t \geq 0)$ be the transition kernel of $(-S_t, t \geq 0)$. This is the dual kernel of Q with respect to l. We consider the process V defined by

$$V_t = \begin{cases} S_{(L_r-t)-} & \text{if} \quad 0 \leq t < L, \\ \Delta & \text{if} \quad t \geq L, \end{cases}$$

where Δ is a cemetery point added to $\mathbb{R}$. Notice the law of S_0 is δ_0, the Dirac mass at 0, and thus, the density of $\delta_0 U$ w.r.t. the reference measure l is just u. Thanks to XVIII 45 and 51 of [7], the process V is under P a Markov process with kernel $(\tilde{Q}_t, t \geq 0)$ defined as the u-transform of $\hat{Q}$, that is

$$\tilde{Q}_t(x, dy) = \frac{1}{u(x)}\, u(y) q_t(x - y) dy.$$

We define the process Y by

$$Y_t = \begin{cases} V_0 - V_t & \text{if} \quad 0 \le t < L, \\ \Delta & \text{if} \quad t \ge L. \end{cases}$$

Notice that $Y_0 = 0$ P-a.s. and the process Y is right continuous and nondecreasing up to its lifetime. We want to prove that Y and the process S killed at time L_r have the same law. It will be enough to check that for every integer $n \ge 1$, every sequence $t_n > \cdots > t_1 > 0$, and $f_1, \dots, f_n$, measurable nonnegative functions on $\mathbb{R}$,

$$\mathrm{E}\left[f_1(Y_{t_1}) \dots f_n(Y_{t_n})\right] = \mathrm{E}\left[f_1(S_{t_1}) \dots f_n(S_{t_n}) \mathbf{1}_{S_{t_n} < r}\right].$$

Using the transition kernel of V, we get

$$\begin{aligned} I &= \mathrm{E}\left[f_1(Y_{t_1}) \dots f_n(Y_{t_n})\right] \\ &= \mathrm{E}\left[f_1(V_0 - V_{t_1}) \dots f_n(V_0 - V_{t_n})\right] \\ &= \int_{\mathbb{R}} \nu(dv_0) \int_{\mathbb{R}} \tilde{Q}_{t_1}(v_0, dv_1) f_1(v_0 - v_1) \dots \int_{\mathbb{R}} \tilde{Q}_{t_n - t_{n_1}}(v_{n-1}, dv_n) f_n(v_0 - v_n), \end{aligned}$$

where ν is the law of $V_0 = S_{L_r -}$. Thanks to [3] proposition 2 p.76, we have that

$$\nu(dv_0) = u(v_0) \mathbf{1}_{v_0 < r} dv_0 \int_{r - v_0}^{\infty} \Pi(ds) = c'_\rho u(v_0)(r - v_0)^{-\rho} \mathbf{1}_{v_0 < r} dv_0.$$

Thus we have

$$\begin{aligned} I = & c'_\rho \int_{\mathbb{R}} dv_0 \; u(v_0)(r - v_0)^{-\rho} \mathbf{1}_{v_0 < r} \int_{\mathbb{R}^n} dv_1 \dots dv_n \; \frac{u(v_1)}{u(v_0)} q_{t_1}(v_0 - v_1) f_1(v_0 - v_1) \dots \\ & \times \frac{u(v_n)}{u(v_{n-1})} q_{t_n - t_{n-1}}(v_{n-1} - v_n) f_n(v_0 - v_n) \\ = & c'_\rho \int_{\mathbb{R}} dv_0 \; (r - v_0)^{-\rho} \mathbf{1}_{v_0 < r} \int_{\mathbb{R}^n} dv_1 \dots dv_n \; u(v_n) \; q_{t_1}(v_0 - v_1) f_1(v_0 - v_1) \dots \\ & \times q_{t_n - t_{n-1}}(v_{n-1} - v_n) f_n(v_0 - v_n). \end{aligned}$$

We use the change of variable $z = v_0$, $y_1 = v_0 - v_1, \cdots, y_n = v_0 - v_n$, and the definition of u to get

$$\begin{aligned} I &= c'_\rho \int_{\mathbb{R}^n} dy_1 \dots dy_n \; q_{t_1}(y_1) f_1(y_1) \dots q_{t_n - t_{n-1}}(y_n - y_{n-1}) f_n(y_n) \\ & \quad \times \int_{\mathbb{R}} dz \; (r - z)^{-\rho} \rho 2^\rho (z - y_n)^{\rho - 1} \mathbf{1}_{r > z > y_n} \\ &= \mathrm{E}\left[f_1(S_{t_1}) \dots f_n(S_{t_n}) \mathbf{1}_{S_{t_n} < r}\right], \end{aligned}$$

because $c'_\rho \int_{\mathbb{R}} dz \; (r - z)^{-\rho} \rho 2^\rho (z - y_n)^{\rho - 1} \mathbf{1}_{r > z > y_n} = \mathbf{1}_{r > y_n}$. □

Notations

$d_c = (\alpha+1)/(\alpha-1)$ critical dimension.
$\theta(dy)$ Lebesgue measure on ∂D.
$B_\varepsilon = B_{\partial D}(y_0, \varepsilon)$ ball on ∂D.
P_D Poisson kernel of D.
G_D Green function of D.
γ_t Brownian motion in $\mathbb{R}^d$.
E_x law of γ started at x.
$\rho = (\alpha - 1)$.
S_t ρ-stable subordinator.
ξ_t residual life time of S.
L_t time change, inverse of S.
$\Gamma_t = \gamma_{L_t}$ "freezed" Brownian motion.
$E = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^d$: state space of $\overline{\xi} = (\xi, L, \Gamma)$.
$\overline{\mathrm{P}}_z$ law of $\overline{\xi}_t$ started at $z \in E$.
$\overline{\mathrm{P}}_x$ law of $\overline{\xi}$ started at $(0, 0, x)$.
$\overline{\mathrm{P}}_x^D$ law of $\overline{\xi}$ killed out of $\mathbb{R}^+ \times \mathbb{R}^+ \times D$.
$\overline{\mathrm{P}}_x^r$ law of $\overline{\xi}$ killed at time r.
κ_B exit time of B for γ.
τ_B exit time of B for $\Gamma = \gamma_L$.
$w = (w, \zeta)$ E-valued path with life time ζ; for $t \in [0, \zeta)$, we write $w(t) = (\xi_t(w), L_t(w), \Gamma_t(w))$.
$\tau_B(w)$ exit time of B for $\Gamma(w)$.
$\kappa_B(w)$ exit time of B for $\Gamma_{L^{-1}}(w)$.
$\hat{w} = \Gamma_\zeta(w)$ spatial end point.

Notations for the snake

ζ_s life time of the snake at time s.
W_s snake at time s; for $t \in [0, \zeta_s)$, $W_s(t) = (\xi_t(W_s), L_t(W_s), \Gamma_t(W_s))$.
$S_t(W_s)$ inverse of the time change $L_t(W_s)$.
$\gamma_t(W_s) = \Gamma_{S_t(W_s)}(W_s)$ spatial motion of the snake path W_s.
$\hat{W}_s = \Gamma_{\zeta_s}(W_s)$ end point of the spatial motion of the snake path W_s.
$\hat{L}_s = L_{\zeta_s}(W_s)$ end point of the time change of the snake path W_s.
$\mathbb{E}_w$ law of W_s started at path w.
$\mathbb{E}_w^*$ law of W_s started at path w and killed when its life ζ_s time reaches 0.
$\mathbb{E}_{(r)}^* = \int \overline{\mathrm{P}}_x^r(dw)\mathbb{E}_w^*$ law of W_s killed when its life time reaches 0 and started with a typical (random) path of life time r.
$\mathbb{N}_z$ excursion measure of the snake started at the trivial path $z \in E$.
$\mathbb{N}_x$ excursion measure of the snake started at the trivial path $(0, 0, x)$.
σ duration of the snake excursion.
L^D exit local time of D.
Y_D exit measure of the snake.

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