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Fragmentation associated with Lévy processes using snake

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Abstract We consider the height process of a Lévy process with no negative jumps, and its associated continuous tree representation. Using Lévy snake tools developed by Le Gall-Le Jan and Duquesne-Le Gall, with an underlying Poisson process, we construct a fragmentation process, which in the stable case corresponds to the self-similar fragmentation described by Miermont. For the general fragmentation process we compute a family of dislocation measures as well as the law of the size of a tagged fragment. We also give a special Markov property for the snake which is of its own interest.

Keywords Fragmentation · Lévy snake · Dislocation measure · Stable processes · Special Markov property

Mathematics Subject Classification (2000) 60J25 · 60G57

1 Introduction

We present a fragmentation process associated with continuous random trees (CRT) with general critical or sub-critical branching mechanism ψ, which were introduced by Le Gall and Le Jan [15] and developed later by Duquesne and Le Gall [11]. This extends

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previous work from Miermont [18] on stable CRT (i.e. $\psi(\lambda) = \lambda^\alpha$ for $\alpha \in (1, 2)$). Although the underlying ideas are the same in both constructions, the arguments in the proofs are very different. Following Abraham and Serlet [3] who deal with the particular case of Brownian CRT, our arguments rely on Lévy Poisson snake processes. Those path processes are Lévy snakes (see [11]) with underlying spatial motion a Poisson process. This Lévy Poisson snake puts marks on the CRT where it is cut in order to construct the fragmentation process. In [3], the CRT is associated with Brownian motion (i.e. $\psi(\lambda) = \lambda^2$) and the marks are put on the skeleton of the tree. On the contrary, we focus here on the case where the branching mechanism has no Brownian part, which implies that the marks lie on the nodes of the CRT. The construction of the Lévy Poisson snake can surely be extended to the case of a branching mechanism that contains a Brownian part but some marks would then be on the skeleton whereas the others would lie on the nodes, which makes the study of the fragmentation more involved.

This construction provides non trivial examples of non self-similar fragmentations, and the tools developed here could give further results on the fragmentation associated with CRT. For instance, using this construction in [1], we gave the asymptotics for the small fragments which was an open question even for the fragmentation at nodes of the stable CRT.

The next three subsections give a brief presentation of the mathematical objects and state the mains results. The last one describes the organization of the paper.

1.1 Exploration process

The coding of a tree by its height process is now well-known. For instance, the height process of Aldous' CRT [4] is a normalized Brownian excursion. In [15], Le Gall and Le Jan associated with a Lévy process $X = (X_t, t \geq 0)$ with no negative jumps that does not drift to infinity, a continuous state branching process (CSBP) and a Lévy CRT which keeps track of the genealogy of the CSBP. Let ψ denote the Laplace exponent of X. We shall assume here that there is no Brownian part, that is

$$\psi(\lambda) = \alpha_0\lambda + \int_{(0,+\infty)} \pi(d\ell)\left[\mathrm{e}^{-\lambda\ell} - 1 + \lambda\ell\right],$$

with $\alpha_0 \geq 0$ and the Lévy measure π is a positive σ-finite measure on $(0, +\infty)$ such that $\int_{(0,+\infty)}(\ell \wedge \ell^2)\pi(d\ell) < \infty$. Following [11], we shall also assume that X is of infinite variation a.s. which implies that $\int_{(0,1)} \ell\pi(d\ell) = \infty$. Notice those assumptions are fulfilled in the stable case: $\psi(\lambda) = \lambda^\alpha$, $\alpha \in (1, 2)$.

Informally, for the height process $H = (H_t, t \geq 0)$ associated with X, H_t gives the distance (which can be understood as the number of generations) between the individual labeled t and the root 0 of the CRT. An individual labeled t is an ancestor of $s \geq t$ if $H_t = \inf\{H_r, r \in [t, s]\}$, and $\inf\{H_r, r \in [s, t]\}$ is the "generation" of the most recent common ancestor of s and t. The height process is a key tool in this construction but it is not a Markov process. The so-called exploration process $\rho = (\rho_t, t \geq 0)$ is a càd-làg Markov process taking values in $\mathcal{M}_f(\mathbb{R}_+)$, the set of measures with finite mass on $\mathbb{R}_+$ endowed with the topology of weak convergence. The height process can easily be recovered from the exploration process as $H_t = H(\rho_t)$, where $H(\mu)$ denotes the supremum of the closed support of the measure μ (with the convention that $H(0) = 0$).

To understand what the exploration process means, let us use the queuing system representation of [15]. We consider a LIFO (Last In, First Out) queue with one server. A jump of X at time s corresponds to the arrival of a new customer requiring a service equal to $\Delta_s := X_s - X_{s-}$. The server interrupts his current job and starts immediately the service of this new customer (LIFO procedure). When this new service is finished, the server will resume the previous job. As we assume that π is infinite, all services will suffer interruptions. The customer (arrived at time) s will still be in the system at time $t > s$ if and only if $X_{s-} < \inf_{s\le r\le t} X_r$ and, in this case, the quantity $\rho_t(H_s)$ represents the remaining service required by the customer s at time t. Observe that $\rho_t([0, H_t])$ corresponds to the load of the server at time t and is equal to $X_t - I_t$ where

$$I_t = \inf\{X_u, 0 \le u \le t\}.$$

Another process of interest will be the dual process $(\eta_t, t \ge 0)$ which is also a measure-valued process. In the queuing system description, for a customer s still present in the system at time t, the quantity $\eta_t(H_s)$ represents the amount of service of customer s already completed at time t, so that $\rho_t(H_s) + \eta_t(H_s) = \Delta_s$ holds for any customer s still present in the system at time t.

Definition and properties of the height process, the exploration process and the dual process are recalled in Sect. 2.

1.2 Fragmentation

A fragmentation process is a Markov process which describes how an object with given total mass evolves as it breaks into several fragments randomly as time passes. Notice there may be loss of mass but no creation. This kind of processes has been widely studied in the recent years, see Bertoin [9] and references therein. To be more precise, the state space of a fragmentation process is the set of the non-increasing sequences of masses with finite total mass

$$\mathcal{S}^{\downarrow} = \left\{ s = (s_1, s_2, \dots);\ s_1 \ge s_2 \ge \cdots \ge 0 \quad \text{and} \quad \Sigma(s) = \sum_{k=1}^{+\infty} s_k < +\infty \right\}.$$

If we denote by P_s the law of a $\mathcal{S}^{\downarrow}$-valued process $\Lambda = (\Lambda^\theta, \theta \ge 0)$ starting at $s = (s_1, s_2, \dots) \in \mathcal{S}^{\downarrow}$, we say that Λ is a fragmentation process if it is a Markov process such that $\theta \mapsto \Sigma(\Lambda^\theta)$ is non-increasing and if it fulfills the fragmentation property: the law of $(\Lambda^\theta, \theta \ge 0)$ under P_s is the non-increasing reordering of the fragments of independent processes of respective laws $P_{(s_1,0,\dots)}$,$P_{(s_2,0,\dots)}$, … In other words, each fragment behaves independently of the others, and its evolution depends only on its initial mass. As a consequence, to describe the law of the fragmentation process with any initial condition, it suffices to study the laws $P_r := P_{(r,0,\ldots)}$ for any $r \in (0, +\infty)$, that is the law of the fragmentation process starting with a single mass r.

A fragmentation process is said to be self-similar of index $\alpha \in \mathbb{R}$ if, for any $r > 0$, the law of the process $(\Lambda^\theta, \theta \geq 0)$ under P_r is the law of the process $(r\Lambda^{r^\alpha\theta}, \theta \geq 0)$ under P_1. Bertoin [8] proved that the law of a self-similar fragmentation is characterized by: the index of self-similarity α, an erosion coefficient which corresponds (when $\alpha = 0$) to a deterministic rate of loss of mass, and a dislocation measure ν on $\mathcal{S}^\downarrow$ which describes sudden dislocations of a fragment of mass 1.

Connections between fragmentation processes and random trees or Brownian excursion have been pointed out by several authors. Let us mention the work of Bertoin [7] who constructed a fragmentation process by looking at the lengths of the excursions above level t of a Brownian excursion. Aldous and Pitman [5] constructed another fragmentation process, which is related to the additive coalescent process, by cutting Aldous' Brownian CRT. Their proofs rely on projective limits on trees. These results have been generalized by Miermont [17,18] to CRT associated with stable Lévy processes, using path transformations of the Lévy process. Concerning the Aldous-Pitman's fragmentation process, Abraham and Serlet [3] gave an alternative construction using Poisson snakes. Our presentation follows their ideas. However, we give next a more intuitive presentation which is in fact equivalent.

We set I the infimum process of the Lévy process X and we consider an excursion of the reflected process $X-I$ away from 0, which corresponds also to an excursion of the exploration process (and the height process) away from 0. Let $\mathbb{N}$ be the corresponding excursion measure and σ denote the length of those excursions under $\mathbb{N}$. Intuitively, σ represents the "size" of the total progeny of the root 0. Let $\mathcal{J} = \{t \in [0, \sigma];\ X_t \neq X_{t-}\}$ be the set of jumping times of X or nodes of the CRT, and consider $(T_t; t \in \mathcal{J})$ a countable family of independent random variables such that T_t is distributed (conditionally on X) according to an exponential law with parameter $\Delta_t = X_t - X_{t-}$. At time T_t, the node corresponding to the jump Δ_t is cut from the CRT. Two individuals, say $u \leq v$, belong to the same fragment at time θ if no node has been cut before time θ between them and their most recent common ancestor which is defined as $u \curlywedge v = \inf\big\{t \in [0, u];\ \min\{H_r, r \in [u, v]\} = \min\{H_r, r \in [t, u]\}\big\}$. Let Λ^θ denote the family of decreasing positive Lebesgue measures of the fragments, completed by zeros if necessary so that $\Lambda^\theta \in \mathcal{S}^\downarrow$. See Sect. 4.4 for a precise construction.

Cutting nodes at time $\theta > 0$ may also be viewed as adding horizontal lines under the epigraph of H (see Fig. 1). We then consider the excursions obtained after cutting the initial excursion along the horizontal lines and gluing together the corresponding pieces of paths (for instance, the bold piece of the path of H in Fig. 1 corresponds to the bold excursion in Fig. 2). The lengths of these excursions, ranked in decreasing order, form the fragmentation process as θ increases. Of course, the figures are caricatures as the process H is very irregular and the number of fragments is infinite.

Remark that, for $\theta = 0$, no mark has appeared and Λ^0 has only one non-zero term: the length of the initial excursion. In order to study the fragmentation starting from a single fixed mass, we need to work under the law of an excursion conditioned by its length. We know, (cf [6], Sect. VII) that the right continuous inverse, $(\tau_r, r \geq 0)$, of $-I$ is a subordinator with Laplace exponent ψ^{-1}. This subordinator has no drift as

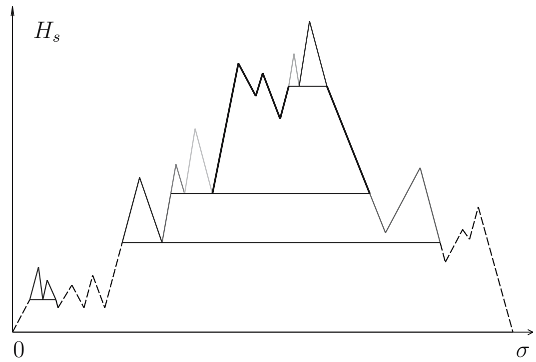

Fig. 1 Cutting at nodes

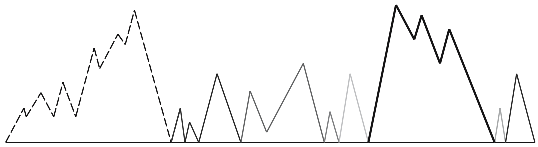

Fig. 2 Fragmentation of the excursion

$\lim_{\lambda\to\infty}\lambda^{-1}\psi^{-1}(\lambda)=0$ (see (3)). We denote by π_* its Lévy measure: for $\lambda\geq 0$

$$\psi^{-1}(\lambda)=\int\limits_{(0,\infty)}\pi_*(d\ell)(1-\mathrm{e}^{-\lambda\ell}).$$

And the length of the excursion, σ, under the excursion measure $\mathbb{N}$ is distributed according to the measure π_*. By decomposing the measure $\mathbb{N}$ w.r.t. the distribution of σ, we get that $\mathbb{N}[d\mathcal{E}]=\int_{(0,\infty)}\pi_*(dr)\mathbb{N}_r[d\mathcal{E}]$, where $(\mathbb{N}_r, r\in(0,\infty))$ is a measurable family of probability measures on the set of excursions such that $\mathbb{N}_r[\sigma=r]=1$ for π^*-a.e. $r>0$. One can use Theorem V.8.1 in [19] and the fact that the set of excursions can be seen as a Borel subset of Skorohod space of càd-làg functions with compact support, to ensure the existence of a regular version of such a decomposition.

The next theorem asserts that the process $(\Lambda^\theta,\theta\geq 0)$ is a fragmentation process: let us denote by P_r the law of the process $(\Lambda^\theta,\theta\geq 0)$ under $\mathbb{N}_r$.

Theorem 1.1 *For $\pi_*(dr)$-almost every r, under P_r, the process $\Lambda=(\Lambda^\theta,\theta\geq 0)$ is a $\mathcal{S}^\downarrow$-valued fragmentation process. More precisely, the law under P_r of the process $(\Lambda^{\theta+\theta'},\theta'\geq 0)$ conditionally on $\Lambda^\theta=(\Lambda_1,\Lambda_2,\ldots)$ is given by the decreasing reordering of independent processes of respective law $\mathrm{P}_{\Lambda_1},\mathrm{P}_{\Lambda_2},\ldots$.*

The proof of this Theorem relies on the study of a tagged fragment, in fact the one which contains 0, and the corresponding height process (that is the dashed lines of Figs. 1 and 2) and exploration process. We shall refer to this exploration process as the pruned exploration process. Another key ingredient is the special Markov property for the underlying exploration process, see Sect. 3.5 for precise statements. This result has the same flavor as the special Markov property of [11] but for the fact that the cutting is on the nodes instead of being on the branches.

There is no loss of mass thanks to the following proposition:

Proposition 1.2 *For $\pi_*(dr)$ almost every r, P_r-a.s., for every $\theta \geq 0$, $\sum_{i=1}^{+\infty} \Lambda_i^\theta = r$.*

Remark 1.3 A more regular version of the family of conditional probability laws $(\mathbb{N}_r, r > 0)$ would allow us to get results in Theorem 1.1 and Proposition 1.2 for all $r \geq 0$ instead of $\pi_*(dr)$ almost everywhere. This is for instance the case when the Lévy process is stable (for which it is possible to construct the measure $\mathbb{N}_r$ from $\mathbb{N}_1$ by a scaling property) or when we can construct this family via a Vervaat's transform of the Lévy bridge (see [16]).

We now describe the dislocation measures of the fragmentation at nodes. Let $\mathcal{T} = \{\theta \geq 0;\ \Lambda^\theta \neq \Lambda^{\theta-}\}$ denote the jumping times of the process Λ and consider the dislocation process of the CRT fragmentation at nodes: $\sum_{\theta \in \mathcal{T}} \delta_{(\theta, \Lambda^\theta)}$. As a direct consequence of Sect. 4.6, the dislocation process is a point process with intensity $\tilde{\nu}_{\Lambda^{\theta-}}(ds)d\theta$, where $(\tilde{\nu}_x, x \in \mathcal{S}^\downarrow)$ is a family of σ-finite measures on $\mathcal{S}^\downarrow$. We refer to [13] for the definition of intensity of a random point measure. Furthermore there exists a family $(\nu_r, r > 0)$ of σ-finite measures on $\mathcal{S}^\downarrow$, which we call dislocation measures of the fragmentation Λ, such that $\nu_r(ds)$-a.e. $\Sigma(s) = r$ (i.e. there is no loss of mass at the fragmentation) and for any $x = (x_1, x_2, \ldots) \in \mathcal{S}^\downarrow$ and any non-negative measurable function, F, defined on $\mathcal{S}^\downarrow$,

$$\int F(s)\tilde{\nu}_x(ds) = \sum_{i \geq 1; x_i > 0} \int F(x^{i,s})\nu_{x_i}(ds),$$

where $x^{i,s}$ is the decreasing reordering of the merging of the sequences $s \in \mathcal{S}^\downarrow$ and x, where x_i has been removed of the sequence x. This last property means that only one element of x fragments and the fragmentation depends only on the size of this very fragment. The same family of dislocation measures, up to a scaling factor, appears for the fragmentation at height of the CRT, see [10].

In the general case, the fragmentation is not self-similar. But in the stable case, $\psi(\lambda) = \lambda^\alpha$ with $\alpha \in (1, 2)$, using scaling properties, we get that the fragmentation is self-similar with index $1/\alpha$ and we recover the results of Miermont [18], see Corollary 4.6. In particular the dislocation measure ν of a fragment of size 1 is given by: for any measurable non-negative function F on $\mathcal{S}^\downarrow$,

$$\int F(x)\nu(dx) = \frac{\alpha(\alpha-1)\Gamma\left(1-\alpha^{-1}\right)}{\Gamma(2-\alpha)} \mathbb{E}\left[S_1 F(\Delta S_t/S_1,\ t \leq 1)\right],$$

where $(S_t, t \geq 0)$ is a stable subordinator with Laplace exponent $\psi^{-1}(\lambda) = \lambda^{1/\alpha}$, and $F(\Delta S_t/S_1,\ t \leq 1)$ has to be understood as F applied to the decreasing reordering of the sequence $(\Delta S_t/S_1, 0 \leq t \leq 1)$, where $(\Delta S_t, t \geq 0)$ are the jumps of the subordinator.

In order to give the corresponding dislocation measures for the CRT fragmentation at nodes in general, we need to consider $(\Delta S_t, t \geq 0)$ the jumps of a subordinator S with Laplace exponent ψ^{-1}. Let μ be the measure on $\mathbb{R}_+ \times \mathcal{S}^\downarrow$ such that for any non-negative measurable function, F, on $\mathbb{R}_+ \times \mathcal{S}^\downarrow$,

$$\int_{\mathbb{R}_+ \times \mathcal{S}^\downarrow} F(r, x)\mu(dr, dx) = \int \pi(dv)\mathbb{E}[F(S_v, (\Delta S_t, t \leq v))], \tag{1}$$

where $(\Delta S_t, t \leq v)$ has to be understood as the family of jumps of the subordinator up to time v ranked in decreasing size.

Intuitively, μ is the joint law of S_T and the jumps of S up to time T, where T and S are independent, and T is distributed according to the infinite measure π.

Theorem 1.4 *There exists a family of dislocation measures $(\nu_r, r > 0)$ on $\mathcal{S}^\downarrow$ s.t.*

$$r\mu(dr, dx) = \nu_r(dx)\pi_*(dr).$$

In particular, $\pi_*(dr)$-a.e. we have that $\nu_r(dx)$-a.e. $\Sigma(x) = r$. The dislocation process of the CRT fragmentation at nodes $(\Lambda^\theta = (\Lambda_i^\theta, i \geq 1), \theta \geq 0)$ is under $\mathbb{N}$ a point process with intensity $\sum_{i\geq 1} \mathbf{1}_{\{\Lambda_i^{\theta-}>0\}}\nu_{\Lambda_i^{\theta-}}(dx)\, d\theta$.

For self-similar fragmentations with no loss of mass, the dislocation measure (together with the index of self-similarity) characterizes the law of the fragmentation process. In the general case, although we can define the family of dislocation measures in a similar way, the fact that this family of measures characterizes the law of the fragmentation remains an open problem.

1.3 Law of the pruned exploration process

In order to use snake techniques, we define a measure-valued process $\mathcal{S} := ((\rho_t, M_t), t \geq 0)$ called the Lévy Poisson snake, where the process ρ is the usual exploration process whereas the process M keeps track of the cut nodes on the CRT which allows to construct the fragmentation (see Sect. 3 for a precise definition).

In order to prove the fragmentation property (Theorem 1.1), we need several intermediate results on the Lévy Poisson snake that are interesting on their own. As they are not the main purpose of this paper, their proofs are postponed at the end of the paper.

In particular, we study the size of a tagged fragment, for instance the one that contains the root of the CRT. So, we set A_t the Lebesgue measure of the set of the individuals prior to t who belongs to the tagged fragment at a given time $\theta > 0$ (see (19) for a precise definition), its right-continuous inverse $C_t = \inf\{r > 0; A_r \geq t\}$ and we define the pruned exploration process $\tilde{\rho}$ by

$$\tilde{\rho}_t = \rho_{C_t} \quad \text{for } t \geq 0.$$

The pruned exploration process $\tilde{\rho}$ corresponds to the exploration process associated with the dashed height process of Figs. 1 and 2. We introduce the following Laplace exponent of a Lévy process (with no negative jumps that does not drift to infinity), $\psi^{(\theta)}$ defined for $\lambda \geq 0$

$$\psi^{(\theta)}(\lambda) = \psi(\theta + \lambda) - \psi(\theta).$$

Notice that $\psi^{(\theta)} = \alpha_0^{(\theta)} + \int_{(0,\infty)} \pi^{(\theta)}(d\ell) \left[\mathrm{e}^{-\lambda\ell} - 1 + \lambda\ell \right]$, where

$$\alpha^{(\theta)} = \alpha_0 + \int_{(0,\infty)} (1 - \mathrm{e}^{-\theta\ell})\ell\pi(d\ell) \quad \text{and} \quad \pi^{(\theta)}(d\ell) = \mathrm{e}^{-\theta\ell}\,\pi(d\ell).$$

Theorem 1.5 *The pruned exploration process $\tilde{\rho}$ is distributed as the exploration process associated with a Lévy process with Laplace exponent $\psi^{(\theta)}$.*

The proof relies on the description of the height process given in [15], see (4). An alternative proof would be, as in [3], to use a martingale problem for $\tilde{\rho}$, see Remark (3.9).

Let $\tilde{\sigma}$ be the length of the excursion of $\tilde{\rho}$. In order to prove the fragmentation property, we need the law of $\tilde{\rho}$ conditionally on $\tilde{\sigma} = r$. The next result seems to be well known but, as we did not find any good reference for it, we will give a complete proof in Sect. 5.2.

Lemma 1.6 *The distribution of $\tilde{\rho}$ (resp. of a Lévy process with Laplace exponent $\psi^{(\theta)}$) under the excursion measure, $\mathbb{N}$, is absolutely continuous w.r.t. to distribution of ρ (resp. of X) with density given by $\mathrm{e}^{-\sigma\psi(\theta)}$, where σ denotes the length of the excursion under $\mathbb{N}$. Equivalently, for any non-negative measurable function G on the space of excursions, we have*

$$\mathbb{N}\left[\mathrm{e}^{\psi(\theta)\tilde{\sigma}}[1 - \mathrm{e}^{-G(\tilde{\rho})}]\right] = \mathbb{N}\left[1 - \mathrm{e}^{-G(\rho)}\right].$$

We deduce that $\pi_*^{(\theta)}(dr) = \mathrm{e}^{-r\psi(\theta)}\,\pi_*(dr)$, where $\pi_*^{(\theta)}$ is the Lévy measure corresponding to the Laplace exponent $(\psi^{(\theta)})^{-1}$. And we have $\pi_*(dr)$-a.e., conditionally on the length of the excursion being equal to r, the law of the excursion of the pruned exploration process is the law of the excursion of the exploration process.

Finally, we give the joint law of length of the exploration process and the length of the pruned exploration process. This result allows to compute the law of a tagged fragment for the fragmentation process (that is the law of $\tilde{\sigma}$ conditionally on $\sigma = r$) at a given time $\theta > 0$.

Proposition 1.7 *For all non-negative γ, κ, θ, we have*

$$\mathbb{N}\left[1 - \mathrm{e}^{-\psi(\gamma)\sigma - \kappa\tilde{\sigma}}\right] = \psi^{-1}(\kappa + \psi(\gamma + \theta)) - \theta.$$

1.4 Organization of the paper

In Sect. 2, we recall the construction of the Lévy CRT and give the properties we shall use in this paper. Section 3 is devoted to the definition and some properties of the Lévy Poisson snake and the special Markov property. From this Lévy Poisson snake, we define in Sect. 4 the fragmentation process associated with the Lévy CRT and prove the fragmentation property, Theorem 1.1, and check there is no loss of mass, Proposition 1.2. The proof relies on the special Markov property. We also compute in this section the dislocation measures of this fragmentation. Finally, we collect in Sect. 5 most of the technical proofs on the Lévy Poisson snake as well as the proof of the special Markov property (Sect. 5.3). In particular proofs of Theorem 1.5 (restated in Theorem 3.8), Lemma 1.6 are given in Sect. 5.2 and the proof of Proposition 1.7 is given in Sect. 5.4 as it relies on the special Markov property.

2 Lévy snake: notations and properties

We recall here the construction of the Lévy continuous random tree (CRT) introduced in [14,15] and developed later in [11]. We will emphasize on the height process and the exploration process which are the key tools to handle this tree. The results of this section are mainly extracted from [11].

2.1 The underlying Lévy process

We consider a $\mathbb{R}$-valued Lévy process $X = (X_t, t \geq 0)$ with no negative jumps, starting from 0. Its law is characterized by its Laplace transform: for $\lambda \geq 0$

$$\mathbb{E}\left[\mathrm{e}^{-\lambda X_t}\right] = \mathrm{e}^{t\psi(\lambda)},$$

where its Laplace exponent ψ is given by

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,+\infty)} \pi(d\ell)\left[\mathrm{e}^{-\lambda\ell} - 1 + \mathbf{1}_{\{\ell<1\}}\lambda\ell\right],$$

where $\beta \geq 0$ and the Lévy measure π is a positive σ-finite measure on $(0, +\infty)$ such that $\int_{(0,+\infty)}(1 \wedge \ell^2)\pi(d\ell) < \infty$. In this paper, we assume that X

- has first moments (i.e. $\int_{(0,+\infty)}(\ell \wedge \ell^2)\pi(d\ell) < \infty$),
- has no Brownian part (i.e. $\beta = 0$),
- is of infinite variation (i.e. $\int_{(0,1)} \ell\pi(d\ell) = +\infty$),
- does not drift to $+\infty$.

The Laplace exponent of X can then be written as

$$\psi(\lambda) = \alpha_0\lambda + \int_{(0,+\infty)} \pi(d\ell)\left[\mathrm{e}^{-\lambda\ell} - 1 + \lambda\ell\right],$$

with $\alpha_0 \geq 0$ (as X does not drift to $+\infty$) and the Lévy measure π is a positive σ-finite measure on $(0, +\infty)$ such that

$$\int_{(0,+\infty)} (\ell \wedge \ell^2)\pi(d\ell) < \infty \quad \text{and} \quad \int_{(0,1)} \ell\pi(d\ell) = \infty. \tag{2}$$

For $\lambda \geq 1/\varepsilon > 0$, we have $\mathrm{e}^{-\lambda\ell} - 1 + \lambda\ell \geq \frac{1}{2}\lambda\ell\mathbf{1}_{\{\ell\geq 2\varepsilon\}}$, which implies that $\lambda^{-1}\psi(\lambda) \geq \alpha_0 + \int_{(2\varepsilon,\infty)} \ell\,\pi(d\ell)$. We deduce that

$$\lim_{\lambda\to\infty} \frac{\lambda}{\psi(\lambda)} = 0. \tag{3}$$

We introduce some processes related to X. Let $\mathcal{J} = \{s \geq 0; X_s \neq X_{s-}\}$ be the set of jumping times of X. For $s \in \mathcal{J}$, we denote by

$$\Delta_s = X_s - X_{s-}$$

the jump of X at time s and $\Delta_s = 0$ otherwise. Let $I = (I_t, t \geq 0)$ be the infimum process of X, $I_t = \inf_{0\leq s\leq t} X_s$, and let $S = (S_t, t \geq 0)$ be the supremum process, $S_t = \sup_{0\leq s\leq t} X_s$. We will also consider for every $0 \leq s \leq t$ the infimum of X over $[s, t]$:

$$I_t^s = \inf_{s\leq r\leq t} X_r.$$

The point 0 is regular for the Markov process $X - I$, and $-I$ is the local time of $X - I$ at 0 (see [6], Chap. VII). Let $\mathbb{N}$ be the associated excursion measure of the process $X - I$ away from 0, and let $\sigma = \inf\{t > 0; X_t - I_t = 0\}$ be the length of the excursion of $X - I$ under $\mathbb{N}$. We will assume that under $\mathbb{N}$, $X_0 = I_0 = 0$.

Since X is of infinite variation, 0 is also regular for the Markov process $S - X$. The local time, $L = (L_t, t \geq 0)$, of $S - X$ at 0 will be normalized so that

$$\mathbb{E}[\mathrm{e}^{-\beta S_{L_t^{-1}}}] = \mathrm{e}^{-t\psi(\beta)/\beta},$$

where $L_t^{-1} = \inf\{s \geq 0; L_s \geq t\}$ (see also [6] Theorem VII.4 (ii)).

2.2 The height process and the Lévy CRT

For each $t \geq 0$, we consider the reversed process at time t, $\hat{X}^{(t)} = (\hat{X}_s^{(t)}, 0 \leq s \leq t)$ by:

$$\hat{X}_s^{(t)} = X_t - X_{(t-s)-} \quad \text{if} \quad 0 \leq s < t,$$

and $\hat{X}_t^{(t)} = X_t$. The two processes $(\hat{X}_s^{(t)}, 0 \le s \le t)$ and $(X_s, 0 \le s \le t)$ have the same law. Let $\hat{S}^{(t)}$ be the supremum process of $\hat{X}^{(t)}$ and $\hat{L}^{(t)}$ be the local time at 0 of $\hat{S}^{(t)} - \hat{X}^{(t)}$ with the same normalization as L.

Definition 2.1 ([11], Definition 1.2.1, Lemma 1.2.1 and Lemma 1.2.4) *There exists a* $[0, \infty]$*-valued lower semi-continuous process* $H = (H_t, t \ge 0)$*, called the height process, such that* $H_0 = 0$ *and for all* $t \ge 0$*, a.s.* $H_t = \hat{L}_t^{(t)}$*. And a.s. for all* $s < t$ *s.t.* $X_{s-} \le I_t^s$ *and for* $s = t$ *if* $\Delta_t > 0$ *then* $H_t < \infty$ *and for all* $t' > t \ge 0$*, the process* H *takes all the values between* H_t *and* $H_{t'}$ *on the time interval* $[t, t']$.

Remark 2.2 Those results can also be found in [15], see Proposition 4.3 and Lemma 4.6 as we assumed there is no Brownian part in X. We shall also use the following formula (see formula (4.5) in [15]): a.s. for a.e. $t \ge 0$,

$$H_t = \lim_{\varepsilon \downarrow 0} \frac{1}{\beta_\varepsilon} \operatorname{Card} \left\{ s \in [0, t],\ X_{s-} < I_t^s,\ \Delta X_s > \varepsilon \right\}, \tag{4}$$

where $\beta_\varepsilon = \int_{(\varepsilon, +\infty)} \ell \pi(d\ell)$.

The height process $(H_t, t \in [0, \sigma])$ under $\mathbb{N}$ codes a continuous genealogical structure, the Lévy CRT, via the following procedure.

(i) To each $t \in [0, \sigma]$ corresponds a vertex at generation H_t.
(ii) Vertex t is an ancestor of vertex t' if $H_t = H_{t,t'}$, where

$$H_{t,t'} = \inf\{H_u, u \in [t \wedge t', t \vee t']\}. \tag{5}$$

In general $H_{t,t'}$ is the generation of the last common ancestor of t and t'.
(iii) We put $d(t, t') = H_t + H_{t'} - 2H_{t,t'}$ and identify t and t' ($t \sim t'$) if $d(t, t') = 0$.

The Lévy CRT coded by H is then the quotient set $[0, \sigma]/\sim$, equipped with the distance d and the genealogical relation specified in (ii).

2.3 The exploration process

The height process is not Markov. But it is a very simple function of a measure-valued Markov process, the so-called exploration process.

If E is a polish space, let $\mathcal{B}(E)$ (resp. $\mathcal{B}_+(E)$) be the set of real-valued measurable (resp. and non-negative) functions defined on E endowed with its Borel σ-field, and let $\mathcal{M}(E)$ (resp. $\mathcal{M}_f(E)$) be the set of σ-finite (resp. finite) measures on E, endowed with the topology of vague (resp. weak) convergence. For any measure $\mu \in \mathcal{M}(E)$ and $f \in \mathcal{B}_+(E)$, we write

$$\langle \mu, f \rangle = \int f(x)\, \mu(dx).$$

The exploration process $\rho = (\rho_t, t \geq 0)$ is a $\mathcal{M}_f(\mathbb{R}_+)$-valued process defined as follows: for every $f \in \mathcal{B}_+(\mathbb{R}_+)$, $\langle \rho_t, f\rangle = \int_{[0,t]} d_s I_t^s f(H_s)$, or equivalently

$$\rho_t(dr) = \sum_{\substack{0<s\leq t\\ X_{s-}<I_t^s}} (I_t^s - X_{s-})\delta_{H_s}(dr). \tag{6}$$

In particular, the total mass of ρ_t is $\langle \rho_t, 1\rangle = X_t - I_t$.

For $\mu \in \mathcal{M}(\mathbb{R}_+)$, we set

$$H(\mu) = \sup \operatorname{Supp} \mu, \tag{7}$$

where Supp μ is the closed support of μ, with the convention $H(0) = 0$. We have

Proposition 2.3 ([11], Lemma 1.2.2 and formula (1.12)) *Almost surely, for every $t > 0$,*

- $H(\rho_t) = H_t$,
- $\rho_t = 0$ *if and only if* $H_t = 0$,
- *if* $\rho_t \neq 0$, *then* Supp $\rho_t = [0, H_t]$.
- $\rho_t = \rho_{t-} + \Delta_t \delta_{H_t}$, *where* $\Delta_t = 0$ *if* $t \notin \mathcal{J}$.

In the definition of the exploration process, as X starts from 0, we have $\rho_0 = 0$ a.s. To state the Markov property of ρ, we must first define the process ρ started at any initial measure $\mu \in \mathcal{M}_f(\mathbb{R}_+)$.

For $a \in [0, \langle \mu, 1\rangle]$, we define the erased measure $k_a\mu$ by

$$k_a\mu([0,r]) = \mu([0,r]) \wedge (\langle \mu, 1\rangle - a), \quad \text{for } r \geq 0.$$

If $a > \langle \mu, 1\rangle$, we set $k_a\mu = 0$. In other words, the measure $k_a\mu$ is the measure μ erased by a mass a backward from $H(\mu)$.

For $\nu, \mu \in \mathcal{M}_f(\mathbb{R}_+)$, and μ with compact support, we define the concatenation $[\mu, \nu] \in \mathcal{M}_f(\mathbb{R}_+)$ of the two measures by:

$$\big\langle [\mu, \nu], f\big\rangle = \big\langle \mu, f\big\rangle + \big\langle \nu, f(H(\mu) + \cdot)\big\rangle, \quad f \in \mathcal{B}_+(\mathbb{R}_+).$$

Finally, we set for every $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ and every $t > 0$, $\rho_t^\mu = \big[k_{-I_t}\mu, \rho_t\big]$. We say that $(\rho_t^\mu, t \geq 0)$ is the process ρ started at $\rho_0^\mu = \mu$, and write $\mathbb{P}_\mu$ for its law. Unless there is an ambiguity, we shall write ρ_t for ρ_t^μ.

Proposition 2.4 ([11], Proposition 1.2.3) *The process $(\rho_t, t \geq 0)$ is a càd-làg strong Markov process in $\mathcal{M}_f(\mathbb{R}_+)$.*

Remark 2.5 From the construction of ρ, we get that a.s. $\rho_t = 0$ if and only if $-I_t \geq \langle \rho_0, 1\rangle$ and $X_t - I_t = 0$. This implies that 0 is also a regular point for ρ. Notice that $\mathbb{N}$ is also the excursion measure of the process ρ away from 0, and that σ, the length of the excursion, is $\mathbb{N}$-a.e. equal to $\inf\{t > 0;\ \rho_t = 0\}$.

Remark 2.6 The process ρ is adapted to the filtration generated by the process X and ρ_0, completed the usual way. On the other hand, notice that a.s. the jumping times of ρ are also the jumping times of X, and for $s \in \mathcal{J}$, we have $\rho_s(\{H_s\}) = \Delta_s$. We deduce that $(\Delta_u, u \in (s, t])$ is measurable w.r.t. the σ-field $\sigma(\rho_u, u \in [s, t])$.

2.4 The dual process and representation formula

We shall need the $\mathcal{M}_f(\mathbb{R}_+)$-valued process $\eta = (\eta_t, t \geq 0)$ defined by

$$\eta_t(dr) = \sum_{\substack{0<s\leq t \\ X_{s-}<I_t^s}} (X_s - I_t^s)\delta_{H_s}(dr).$$

The process η is the dual process of ρ under $\mathbb{N}$ (see Corollary 3.1.6 in [11]). We write (recall $\Delta_s = X_s - X_{s-}$)

$$\kappa_t(dr) = \rho_t(dr) + \eta_t(dr) = \sum_{\substack{0<s\leq t \\ X_{s-}<I_s^t}} \Delta_s \delta_{H_s}(dr). \tag{8}$$

We recall the Poisson representation of (ρ, η) under $\mathbb{N}$. Let $\mathcal{N}(dx\, d\ell\, du)$ be a Poisson point measure on $[0, +\infty)^3$ with intensity

$$dx\, \ell\pi(d\ell)\mathbf{1}_{[0,1]}(u)du.$$

For every $a > 0$, let us denote by $\mathbb{M}_a$ the law of the pair (μ_a, ν_a) of measures on $\mathbb{R}_+$ with finite mass defined by: for any $f \in \mathcal{B}_+(\mathbb{R}_+)$

$$\langle \mu_a, f\rangle = \int \mathcal{N}(dx\, d\ell\, du)\mathbf{1}_{[0,a]}(x)u\ell f(x), \tag{9}$$

$$\langle \nu_a, f\rangle = \int \mathcal{N}(dx\, d\ell\, du)\mathbf{1}_{[0,a]}(x)\ell(1-u) f(x). \tag{10}$$

Remark 2.7 In particular $\mu_a(dr) + \nu_a(dr)$ is defined as $\mathbf{1}_{[0,a]}(r)d_r W_r$, where W is a subordinator with Laplace exponent $\psi' - \alpha_0$.

We finally set $\mathbb{M} = \int_0^{+\infty} da\ \mathrm{e}^{-\alpha_0 a}\, \mathbb{M}_a$.

Proposition 2.8 ([11], Proposition 3.1.3) *For every non-negative measurable function F on $\mathcal{M}_f(\mathbb{R}_+)^2$,*

$$\mathbb{N}\left[\int_0^\sigma F(\rho_t, \eta_t)\, dt\right] = \int \mathbb{M}(d\mu\, d\nu)F(\mu, \nu),$$

where $\sigma = \inf\{s > 0; \rho_s = 0\}$ denotes the length of the excursion.

We shall also give a Bismut formula for the height process. (Notice the proof of Lemma 3.4 in [12] does not require the continuity of the height process, whereas this assumption is done in [12] for other results.)

Proposition 2.9 ([12], Lemma 3.4)
For every non-negative measurable function F defined on $\mathcal{B}_+([0,\infty])^2$

$$\mathbb{N}\left[\int_0^\sigma ds\ F((H_{(s-t)_+}, t \geq 0), (H_{(s+t)\wedge\sigma}, t \geq 0))\right] = \int \mathbb{M}(d\mu\, d\nu)\mathbb{E}[F(H_1^{(\mu)}, H_2^{(\nu)})],$$

where $H_1^{(\mu)}$ and $H_2^{(\nu)}$ are independent and distributed as H under $\mathbb{P}_\mu^*$ and $\mathbb{P}_\nu^*$ respectively.

We shall also use later the next result.

Proposition 2.10 ([11], Lemma 3.2.2)
Let τ be an exponential variable of parameter $\lambda > 0$ independent of X defined under the measure $\mathbb{N}$. Then, for every $F \in \mathcal{B}_+(\mathcal{M}_f(\mathbb{R}_+))$, we have

$$\mathbb{N}\left(F(\rho_\tau)\mathbf{1}_{\tau\leq\sigma}\right) = \lambda \int \mathbb{M}(d\mu\, d\nu) F(\mu)\, \mathrm{e}^{-\psi^{-1}(\lambda)\langle\nu,1\rangle}.$$

Exponential formula for the Poisson point process of jumps of the inverse subordinator of $-I$ gives (see also the beginning of Sect. 3.2.2. [11]) that for $\lambda > 0$

$$\mathbb{N}\left[1 - \mathrm{e}^{-\lambda\sigma}\right] = \psi^{-1}(\lambda). \tag{11}$$

3 The Lévy Poisson snake

As in [3], we want to construct a Poisson snake in order to cut the Lévy CRT at its nodes. For this, we will construct a consistent family $(m^{(\theta)} = (m_t^{(\theta)}, t \geq 0), \theta \geq 0)$ of measure-valued processes. For fixed θ and t, $m_t^{(\theta)}$ will be a point-measure whose atoms mark the atoms of the measure ρ_t and such that the set of atoms of $m_t^{(\theta+\theta')}$ contains those of $m_t^{(\theta)}$. To achieve this, we attach to each jump of X a Poisson process indexed by θ, with intensity equal to this jump. In fact only the first jump of the Poisson processes will be necessary to build the fragmentation process but we consider Poisson processes in order to have the additive property of Proposition 3.2.

3.1 Definition and properties

Conditionally on the Lévy process X, we consider a family $(\sum_{u>0}\delta_{V_{s,u}}, s \in \mathcal{J})$ of independent Poisson point measures on $\mathbb{R}_+$ with respective intensity $\Delta_s\ \mathbf{1}_{\{u>0\}}du$. We define the $\mathcal{M}(\mathbb{R}_+^2)$-valued process $M = (M_t, t \geq 0)$ by

$$M_t(dr, dv) = \sum_{\substack{0<s\leq t\\ X_{s-}<I_t^s}} (I_t^s - X_{s-})\left(\sum_{u>0}\delta_{V_{s,u}}(dv)\right)\delta_{H_s}(dr). \tag{12}$$

Notice that a.s.

$$M_t(dr, dv) = \rho_t(dr)M_{t,r}(dv), \tag{13}$$

where $M_{t,r} = \sum_{u>0} \delta_{V_{s,u}}$ with $s > 0$ s.t. $X_{s-} < I_t^s$ and $H_s = r$.

Let $\theta > 0$. For $t \geq 0$, notice that

$$M_t(\mathbb{R}_+ \times [0, \theta]) \leq \sum_{0<s\leq t} \Delta_s \xi_s,$$

with $\xi_s = \text{Card}\ \{u > 0;\ V_{s,u} \leq \theta\}$. In particular, we have for $T > 0$,

$$\sup_{t\in[0,T]} M_t(\mathbb{R}_+ \times [0, \theta]) \leq \sum_{0<s\leq T} \Delta_s \xi_s. \tag{14}$$

Notice the variable ξ_s are, conditionally on X, independent and distributed as Poisson random variables with parameter $\theta \Delta_s$. We have $\mathbb{E}[\sum_{0<s\leq T} \Delta_s \xi_s | X] = \theta \sum_{0<s\leq T} \Delta_s^2$. As $\int_{(0,\infty)} (\ell^2 \wedge \ell)\pi(d\ell)$ is finite, this implies the quantity $\sum_{0<s\leq T} \Delta_s^2$ is finite a.s. In particular we have a.s.

$$\sup_{t\in[0,T]} M_t(\mathbb{R}_+ \times [0, \theta]) < \infty,$$

and M_t is a σ-finite measure on $\mathbb{R}_+^2$.

We call the process $\mathcal{S} = ((\rho_t, M_t), t \geq 0)$ the Lévy Poisson snake started at $\rho_0 = 0$, $M_0 = 0$. To get the Markov property of the Lévy Poisson snake, we must define the process $\mathcal{S}$ started at any initial value $(\mu, \Pi) \in \mathbb{S}$, where $\mathbb{S}$ is the set of pair (μ, Π) such that $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ and $\Pi(dr, dv) = \mu(dr)\Pi_r(dv)$, $(\Pi_r, r > 0)$ being a measurable family of σ-finite measures on $\mathbb{R}_+$, such that $\Pi(\mathbb{R}_+ \times [0, \theta]) < \infty$ for all $\theta \geq 0$. We set $H_t^\mu = H(k_{-I_t}\mu)$. Then, we define the process $M^{\mu,\Pi} = (M_t^{\mu,\Pi}, t \geq 0)$ by: for any $\varphi \in \mathcal{B}_+(\mathbb{R}_+^2)$,

$$\langle M_t^{\mu,\Pi}, \varphi\rangle = \int_{(0,\infty)^2} \varphi(r, v)k_{-I_t}\mu(dr)\Pi_r(dv) + \int_{(0,\infty)^2} \varphi(r + H_t^\mu, v)M_t(dr, dv).$$

We shall write M for $M^{\mu,\Pi}$. By construction and since ρ is an homogeneous Markov process, the Lévy Poisson snake $\mathcal{S} = (\rho, M)$ is an homogeneous Markov process.

We now denote by $\mathbb{P}_{\mu,\Pi}$ the law of the Lévy Poisson snake starting at time 0 from (μ, Π), and by $\mathbb{P}^*_{\mu,\Pi}$ the law of the Lévy Poisson snake killed when ρ reaches 0. We deduce from (14), that a.s.

$$\mathbb{E}_{\mu,\Pi}\left[\sup_{t\in[0,T]} M_t(\mathbb{R}_+ \times [0, \theta]) \,\middle|\, X\right] \leq \theta \sum_{0<s\leq T} \Delta_s^2 + \Pi(\mathbb{R}_+ \times [0, \theta]) < \infty. \tag{15}$$

Let $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$ be the filtration generated by $\mathcal{S}$ completed the usual way. Notice this filtration is also generated by the processes X and $(\sum_{s\in\mathcal{J},\, s\leq t}$

$\sum_{u\geq 0}\delta_{V_{s,u}}, t\geq 0)$. In particular the filtration $\mathcal{F}$ is right continuous. And by construction, we have that ρ is Markovian with respect to $\mathcal{F}$. The technical proof of the next result is postponed to the appendix.

Proposition 3.1 *The Lévy Poisson snake, $\mathcal{S}$, is a càd-làg strong Markov process in $\mathbb{S}\subset\mathcal{M}_f(\mathbb{R}_+)\times\mathcal{M}(\mathbb{R}_+^2)$.*

We shall use later the following property, which is a consequence of Poisson point measure properties.

Proposition 3.2 *Let $\theta>0$ and $M^\theta=(M_t^\theta, t\geq 0)$ be the measure-valued process defined by*

$$M_t^\theta(dr,[0,a])=M_t(dr,(\theta,\theta+a]),\quad \textit{for all } a\geq 0.$$

Then, given ρ, M^θ is independent of $M\mathbf{1}_{\mathbb{R}_+\times[0,\theta]}$ and is distributed as M.

3.2 Poisson representation of the snake

Notice that a.s. $(\rho_t, M_t)=(0,0)$ if and only if $\rho_t=0$. In particular, $(0,0)$ is a regular point for the Lévy Poisson snake. We still write $\mathbb{N}$ for the excursion measure of the Lévy Poisson snake away from $(0,0)$, with the same normalization as in Sect. 2.4.

We decompose the path of $\mathcal{S}$ under $\mathbb{P}^*_{\mu,\Pi}$ according to excursions of the total mass of ρ above its minimum, see Sect. 4.2.3 in [11]. More precisely let $(\alpha_i,\beta_i), i\in I$ be the excursion intervals of the process $\langle\rho,1\rangle$ above its minimum under $\mathbb{P}^*_{\mu,\Pi}$. For every $i\in I$, we define $h_i=H_{\alpha_i}$ and $\mathcal{S}^i=(\rho^i,M^i)$ by the formulas

$$\langle\rho_t^i,f\rangle=\int_{(h_i,+\infty)} f(x-h_i)\rho_{(\alpha_i+t)\wedge\beta_i}(dx)$$

$$\langle M_t^i,\varphi\rangle=\int_{(h_i,+\infty)\times[0,+\infty)}\varphi(x-h_i,v)M_{(\alpha_i+t)\wedge\beta_i}(dx,dv).$$

It is easy to adapt Lemma 4.2.4. of [11] to get the following Lemma.

Lemma 3.3 *Let $(\mu,\Pi)\in\mathcal{M}_f(\mathbb{R}_+)\times\mathcal{M}(\mathbb{R}_+^2)$. The point measure $\sum_{i\in I}\delta_{(h_i,\mathcal{S}^i)}$ is under $\mathbb{P}^*_{\mu,\Pi}$ a Poisson point measure with intensity $\mu(dr)\mathbb{N}[d\mathcal{S}]$.*

3.3 The process $m^{(\theta)}$

For $\theta\geq 0$, we define the $\mathcal{M}(\mathbb{R}_+)$-valued process $m^{(\theta)}=(m_t^{(\theta)}, t\geq 0)$ by

$$m_t^{(\theta)}(dr)=M_t(dr,(0,\theta]). \tag{16}$$

We make two remarks. We have for $s > 0$,

$$\mathbb{P}_{0,0}(m_s^{(\theta)} = 0|X) = \mathrm{e}^{-\theta \sum_{0<r\leq s,\ X_{r-}<I_s^r} \Delta_r} = \mathrm{e}^{-\theta\langle \kappa_s, 1\rangle}. \tag{17}$$

Notice that for $s \in \mathcal{J}$, i.e. $\Delta_s > 0$, we have $M_s(\{H_s\}, dv) = \Delta_s \sum_{u\geq 0} \delta_{V_{s,u}}(dv)$, where conditionally on X, $\sum_{u\geq 0} \delta_{V_{s,u}}(dv)$ is a Poisson point measure with intensity $\Delta_s \mathbf{1}_{\{u>0\}} du$. In particular, we have

$$\mathbb{P}_{\mu,\Pi}(m_s^{(\theta)}(\{H_s\}) > 0|X) = \mathbb{P}(M_s(\{H_s\} \times (0,\theta]) > 0|X) = 1 - \mathrm{e}^{-\theta\Delta_s}.$$

Recall that $\sum_{s\geq 0} \delta_{(s,\Delta_s)}$ is a Poisson point process with intensity π. From Poisson point measure properties, we get the following Lemma.

Lemma 3.4 *The random measure $\sum_{s\geq 0} \mathbf{1}_{\{m_s^{(\theta)}(\{H_s\})>0\}} \delta_{(s,\Delta_s)}$ is a Poisson point process with intensity*

$$n^\theta(d\ell) = (1 - \mathrm{e}^{-\theta\ell})\pi(d\ell). \tag{18}$$

Finally, the next Lemma on time reversibility can easily be deduced from Corollary 3.1.6 of [11] and the construction of M.

Lemma 3.5 *For every $\theta > 0$, under $\mathbb{N}$, the processes $((\rho_s, \eta_s, \mathbf{1}_{\{m_s^{(\theta)}=0\}}), s \in [0,\sigma])$ and $((\eta_{(\sigma-s)-}, \rho_{(\sigma-s)-}, \mathbf{1}_{\{m^{(\theta)}_{(\sigma-s)-}=0\}}), s \in [0,\sigma])$ have the same distribution.*

3.4 The pruned exploration process

In this section, we fix $\theta > 0$ and write m for $m^{(\theta)}$. We define the following continuous additive functional of the process $((\rho_t, m_t), t \geq 0)$: for $t \geq 0$

$$A_t = \int_0^t \mathbf{1}_{\{m_s=0\}}\, ds, \tag{19}$$

Lemma 3.6 *We have the following properties.*

(i) *For $\lambda > 0$, $\mathbb{N}[1 - \mathrm{e}^{-\lambda A_\sigma}] = \psi^{(\theta)^{-1}}(\lambda)$.*
(ii) *$\mathbb{N}$-a.e. 0 and σ are points of increase for A. More precisely, $\mathbb{N}$-a.e. for all $\varepsilon > 0$, we have $A_\varepsilon > 0$ and $A_\sigma - A_{(\sigma-\varepsilon)\vee 0} > 0$.*
(iii) *$\mathbb{N}$-a.e. the set $\{s; m_s \neq 0\}$ is dense in $[0,\sigma]$.*

The proof of this Lemma is postponed to Sect. 5.2.

We set $C_t = \inf\{r > 0; A_r > t\}$ the right continuous inverse of A, with the convention that $\inf \emptyset = \infty$. From excursion decomposition, see Lemma 3.3, (ii) of Lemma 3.6 implies the following Corollary.

Corollary 3.7 *For any initial measures μ, Π, $\mathbb{P}_{\mu,\Pi}$-a.s. the process $(C_t, t \geq 0)$ is finite. If $m_0 = 0$, then $\mathbb{P}_{\mu,\Pi}$-a.s. $C_0 = 0$.*

We define the pruned exploration process $\tilde{\rho} = (\tilde{\rho}_t = \rho_{C_t}, t \geq 0)$ and the pruned Lévy Poisson snake $\tilde{\mathcal{S}} = (\tilde{\rho}, \tilde{M})$, where $\tilde{M} = (M_{C_t}, t \geq 0)$. Notice C_t is a $\mathcal{F}$-stopping time for any $t \geq 0$ and is finite a.s. from Corollary 3.7. Notice the process $\tilde{\rho}$, and thus the process $\tilde{\mathcal{S}}$, is càd-làg. We also set $\tilde{H}_t = H_{C_t}$ and $\tilde{\sigma} = \inf\{t > 0; \tilde{\rho}_t = 0\}$.

Let $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_t, t \geq 0)$ be the filtration generated by the pruned Lévy Poisson snake $\tilde{\mathcal{S}}$ completed the usual way. In particular $\tilde{\mathcal{F}}_t \subset \mathcal{F}_{C_t}$, where if τ is an $\mathcal{F}$-stopping time, then $\mathcal{F}_\tau$ is the σ-field associated with τ.

We are now able to restate precisely Theorem 1.5.

Theorem 3.8 *For every measure μ with finite mass, the law of the pruned exploration process $\tilde{\rho}$ under $\mathbb{P}_{\mu,0}$ is the law of the exploration process associated with a Lévy process with Laplace exponent $\psi^{(\theta)}$ under $\mathbb{P}_\mu$.*

The proof relies on the approximation formula (4) and is postponed to Sect. 5.2.

Remark 3.9 An alternative proof would be, as in [3], to use a martingale problem for $\tilde{\rho}$. Indeed, there is a simple relation between the infinitesimal generator of ρ and those of $\tilde{\rho}$: Let $F, K \in \mathcal{B}(\mathcal{M}_f(\mathbb{R}_+))$ bounded such that, for any $\mu \in \mathcal{M}_f(R_+)$, $\mathbb{E}_\mu\left[\int_0^\sigma |K(\rho_s)|\, ds\right] < \infty$ and $M_t = F(\rho_{t\wedge\sigma}) - \int_0^{t\wedge\sigma} K(\rho_s)$, for $t \geq 0$, define an $\mathcal{F}$-martingale. In particular, notice that $\mathbb{E}_\mu\left[\sup_{t\geq 0}|M_t|\right] < \infty$. Thus, we can define for $t \geq 0$,

$$N_t = \mathbb{E}^*_\mu[M_{C_t}|\tilde{\mathcal{F}}_t].$$

Proposition 3.10 *The process $N = (N_t, t \geq 0)$ is an $\tilde{\mathcal{F}}$-martingale. We have for all $\mu \in \mathcal{M}_f(\mathbb{R}_+)$, $\mathbb{P}_\mu$-a.s.*

$$\int_0^{\tilde{\sigma}} du \int_{(0,\infty)} \left(1 - \mathrm{e}^{-\theta\ell}\right) \pi(d\ell)\, |F([\tilde{\rho}_u, \ell\delta_0]) - F(\tilde{\rho}_u)| < \infty,$$

and the representation formula for N_t:

$$N_t = F(\tilde{\rho}_{t\wedge\tilde{\sigma}}) - \int_0^{t\wedge\tilde{\sigma}} du \left(K(\tilde{\rho}_u) + \int_{(0,\infty)} \left(1 - \mathrm{e}^{-\theta\ell}\right) \pi(d\ell) \Big(F([\tilde{\rho}_u, \ell\delta_0]) - F(\tilde{\rho}_u) \Big) \right). \tag{20}$$

Let us also mention that we have computed the infinitesimal generator of ρ for exponential functionals in [2].

3.5 Special Markov property

We still work with fixed $\theta > 0$ and write m for $m^{(\theta)}$.

In order to define the excursion of the Lévy Poisson snake away from $\{s \geq 0;\ m_s = 0\}$, we define O as the interior of $\{s \geq 0,\ m_s \neq 0\}$. We shall see that the complementary of O has positive Lebesgue measure. Its Lebesgue measure corresponds to the length of the fragment at time θ which contains 0.

Lemma 3.11 $\mathbb{N}$-*a.e. the open set* O *is dense in* $[0, \sigma]$.

Proof Thanks to Lemma 3.6, (iii), $\{s \geq 0,\ m_s \neq 0\}$ is dense. For any element s of this set, there exists $u \leq H_s$ such that $m_s([0, u]) \neq 0$ and $\rho_s(\{u\}) > 0$. Then we consider $\tau_s = \inf\{t > s, \rho_t(\{u\}) = 0\}$. By the right continuity of ρ, $\tau_s > s$ and clearly $(s, \tau_s) \subset O$ $\mathbb{N}$-a.e. Therefore O in dense in $[0, \sigma]$. $\square$

We write $O = \bigcup_{i\in I}(\alpha_i, \beta_i)$ and say that $(\alpha_i, \beta_i)_{i\in I}$ are the excursions intervals of the Lévy Poisson snake $\mathcal{S} = (\rho, M)$ away from $\{s \geq 0,\ m_s = 0\}$. Using the right continuity of ρ and the definition of M, we get that for $i \in I$, $\alpha_i > 0$, $\alpha_i \in \mathcal{J}$ that is $\rho_{\alpha_i}(\{H_{\alpha_i}\}) = \Delta_{\alpha_i}$, $M_{\alpha_i}(\{H_{\alpha_i}\}, [0, \theta]) \geq 1$ and $M_{\alpha_i}([0, H_{\alpha_i}), [0, \theta]) = 0$. For every $i \in I$, let us define the measure-valued process $\mathcal{S}^i = (\rho^i, M^i)$ by: for every $f \in \mathcal{B}_+(\mathbb{R}_+)$, $\varphi \in \mathcal{B}_+(\mathbb{R}_+^2)$, $t \geq 0$,

$$\begin{aligned}
\langle \rho_t^i, f\rangle &= \int\limits_{[H_{\alpha_i},+\infty)} f(x - H_{\alpha_i})\rho_{(\alpha_i+t)\wedge\beta_i}(dx)\\
\langle M_t^i, \varphi\rangle &= \int\limits_{(H_{\alpha_i},+\infty)\times[0,+\infty)} \varphi(x - H_{\alpha_i}, v)M_{(\alpha_i+t)\wedge\beta_i}(dx, dv).
\end{aligned} \tag{21}$$

Notice that the mass located at H_{α_i} is kept in the definition of ρ^i whereas it is removed in the definition of M^i. In particular, $\rho_0^i = \Delta_i\delta_0$, with $\Delta_{\alpha_i} > 0$ and, for every $t < \beta_i - \alpha_i$, the measure ρ_t^i charges 0. On the contrary, as $M_0^i = 0$ we have for every $t < \beta_i - \alpha_i$, $M_t^i(\{0\} \times \mathbb{R}_+) = 0$. We call Δ_{α_i} the starting mass of $\mathcal{S}^i$.

Let $\tilde{\mathcal{F}}_\infty$ be the σ-field generated by $\tilde{\mathcal{S}} = ((\rho_{C_t}, M_{C_t}), t \geq 0)$ and $\mathbb{P}^*_{\mu,\Pi}(d\mathcal{S})$ denote the law of the snake $\mathcal{S}$ started at (μ, Π) and stopped when ρ reaches 0. For $\ell \in [0, +\infty)$, we will write $\mathbb{P}^*_\ell$ for $\mathbb{P}^*_{\delta_\ell,0}$. Recall (18) and define the measure N by

$$\mathrm{N}(d\mathcal{S}) = \int\limits_{(0,+\infty)} \pi(d\ell)\left(1 - \mathrm{e}^{-\theta\ell}\right)\mathbb{P}^*_\ell(d\mathcal{S}) = \int\limits_{(0,\infty)} n^{(\theta)}(d\ell)\mathbb{P}^*_\ell(d\mathcal{S}). \tag{22}$$

If Q is a measure on $\mathbb{S}$ and ϕ is a non-negative measurable function defined on a measurable space $\mathbb{R}_+ \times \Omega \times \mathbb{S}$, we denote by

$$Q[\phi(u, \omega, \cdot)] = \int\limits_{\mathbb{S}} \phi(u, \omega, \mathcal{S})Q(d\mathcal{S}).$$

In other words, the integration concerns only the third component of the function ϕ.

We can now state the Special Markov Property.

Theorem 3.12 (Special Markov property) *Let ϕ be a non-negative measurable function defined on $\mathbb{R}_+ \times \Omega \times \mathbb{S}$ such that $t \mapsto \phi(t, \omega, \mathcal{S})$ is progressively $\tilde{\mathcal{F}}_\infty$-measurable for any $\mathcal{S} \in \mathbb{S}$. Then, we have $\mathbb{P}_{0,0}$-a.e.*

$$\mathbb{E}_{0,0}\left[\exp\left(-\sum_{i\in I}\phi(A_{\alpha_i}, \omega, \mathcal{S}^i)\right)\Bigg|\, \tilde{\mathcal{F}}_\infty\right]$$
$$= \exp\left(-\int_0^\infty du \int \mathrm{N}(d\mathcal{S})\left[1 - \mathrm{e}^{-\phi(u,\omega,\mathcal{S})}\right]\right). \tag{23}$$

Furthermore, the law of the excursion process $\sum_{i\in I}\delta_{(A_{\alpha_i}, \rho_{\alpha_i-}, \mathcal{S}^i)}$, given $\tilde{\mathcal{F}}_\infty$, is the law of a Poisson point measure of intensity $\mathbf{1}_{\{u\geq 0\}} du\ \delta_{\tilde{\rho}_u}(d\mu)\ \mathrm{N}(d\mathcal{S})$.

Informally speaking, this theorem gives the law of the Lévy Poisson snake 'above' the tagged fragment that contains the root. It allows to prove that the fragments evolve independently and have the same law.

The proof of this theorem is postponed to Sect. 5.3.

4 Link between Lévy snake and fragmentation processes at nodes

4.1 Construction of the fragmentation process

Let $\mathcal{S} = (\rho, M)$ be a Lévy Poisson snake. For fixed $\theta > 0$, let us consider the following equivalence relation $\mathcal{R}_\theta$ on $[0, \sigma]$, defined under $\mathbb{N}$ or $\mathbb{N}_\sigma$ (see definition of the law $\mathbb{N}_\sigma$ of the excursion conditioned to have length σ in the introduction) by:

$$s\mathcal{R}_\theta t \iff m_s^{(\theta)}\big([H_{s,t}, H_s]\big) = m_t^{(\theta)}\big([H_{s,t}, H_t]\big) = 0, \tag{24}$$

where $H_{s,t} = \inf_{u\in[s,t]} H_u$ (recall Definition (5)). Intuitively, two points s and t belongs to the same class of equivalence (i.e. the same fragment) at time θ if there is no cut on their lineage down to their most recent common ancestor (that is $m_s^{(\theta)}$ puts no mass on $[H_{s,t}, H_s]$ nor $m_t^{(\theta)}$ on $[H_{s,t}, H_t]$). Notice cutting occurs on branching points, that is at node of the CRT. Each node of the CRT corresponds to a jump of the underlying Lévy process X. The cutting times are, conditionally on the CRT, independent exponential random times, with parameter equal to the jump of the corresponding node.

Let us index the different equivalent classes in the following way: For any $s \leq \sigma$, let us define $H_s^0 = 0$ and recursively for $k \in \mathbb{N}$,

$$H_s^{k+1} = \inf\left\{u \geq 0 \;\big|\; m_s^\theta\big((H_s^k, u]\big) > 0\right\},$$

with the usual convention $\inf \emptyset = +\infty$. We set

$$K_s = \sup\{j \in \mathbb{N},\ H_s^j < +\infty\}.$$

Remark 4.1 Notice that we have $K_s = \infty$ if $M_s(\cdot, [0, \theta])$ has infinitely many atoms. By construction of M using Poisson point measures, this happens $\mathbb{N}[d\mathcal{S}]$ ds-a.e., if and only if the intensity measure $\rho_s + \eta_s$ is infinite. Since $\mathbb{N}[d\mathcal{S}]$-a.e., ρ and η are processes taking values in the set of measures with finite mass, we get that $\mathbb{N}[d\mathcal{S}]$-a.e., $K_s < \infty$.

Let us remark that $s\mathcal{R}_\theta t$ implies $K_s = K_t$. We denote, for any $j \in \mathbb{N}$, $(R^{j,k}, k \in J_j)$ the family of equivalent classes with positive Lebesgue measure such that $K_s = j$. For $j \in \mathbb{N}$, $k \in J_j$ we set

$$A_t^{j,k} = \int_0^t \mathbf{1}_{\{s \in R^{j,k}\}} ds \quad \text{and} \quad C_t^{j,k} = \inf\{u \geq 0,\ A_u^{j,k} > t\},$$

with the convention $\inf \emptyset = \sigma$. And we define the corresponding Lévy snake, $\tilde{\mathcal{S}}^{j,k} = (\tilde{\rho}^{j,k}, \tilde{M}^{j,k})$ by: for every $f \in \mathcal{B}_+(\mathbb{R}_+)$, $\varphi \in \mathcal{B}_+(\mathbb{R}_+ \times \mathbb{R}_+)$, $t \geq 0$,

$$\langle \tilde{\rho}_t^{j,k}, f \rangle = \int_{(H_{C_0^{j,k}}, +\infty)} f(x - H_{C_0^{j,k}}) \rho_{C_t^{j,k}}(dx)$$

$$\langle \tilde{M}_t^{j,k}, \varphi \rangle = \int_{(H_{C_0^{j,k}}, +\infty) \times (\theta, +\infty)} \varphi(x - H_{C_0^{j,k}}, v - \theta) M_{C_t^{j,k}}(dx, dv).$$

Let $\tilde{\sigma}^{j,k} = A_\infty^{j,k}$ be the length of the excursion $\tilde{\mathcal{S}}^{j,k}$.

Remark 4.2 In view of the computation of the dislocation measures, we introduce the set $\mathcal{L}^{(\theta)} = (\tilde{\rho}^{(j,k)}, j \in \mathbb{N}, k \in J_j)$ of fragments of Lévy snake as well as the the set $\mathcal{L}^{(\theta-)}$ defined similarly but for the equivalence relation where $\mathcal{R}_\theta$ in (24) is replaced by $\mathcal{R}_{\theta-}$ defined as

$$s\mathcal{R}_{\theta-}t \iff M_s\big([H_{s,t}, H_s] \times (0, \theta)\big) = M_s\big([H_{s,t}, H_t] \times (0, \theta)\big) = 0. \tag{25}$$

Notice that $m_s^{(\theta)}(\cdot) = M_s\big(\cdot, (0, \theta]\big)$. So the two equivalence relations are equal $\mathbb{N}$-a.e. for fixed θ, but may differ if M has an atom in $\{\theta\} \times \mathbb{R}_+$.

Let us now define the process of interest. Let us denote by $\Lambda^\theta = (\Lambda_1^\theta, \Lambda_2^\theta, \ldots)$ the sequence of positive Lebesgue measures of the equivalent classes of $\mathcal{R}_\theta$, $(\tilde{\sigma}^{j,k}, j \in \mathbb{N}, k \in J_j)$, ranked in decreasing order. Notice this sequence is at most countable. If it is finite, we complete the sequence with zeros, so that $\mathbb{N}$-a.s. and $\mathbb{N}_\sigma$-a.s.

$$\Lambda^\theta \in \mathcal{S}^\downarrow = \Big\{(x_1, x_2, \ldots),\ x_1 \geq x_2 \geq \cdots \geq 0,\ \sum x_i < \infty\Big\}.$$

For $\pi^*(d\sigma)$-a.e. $\sigma > 0$, let P_σ denote the law of $(\Lambda^\theta, \theta \geq 0)$ under $\mathbb{N}_\sigma$. By convention P_0 is the Dirac mass at $(0, 0, \ldots) \in \mathcal{S}^\downarrow$.

4.2 The fragmentation property: proof of Theorem 1.1

We keep the notations of the previous Section. The fact that $(\Lambda^\theta, \theta \geq 0)$ is a fragmentation process is a direct consequence of the Lemma 4.3 below and the fact that $\mathbb{N}(\cdot) = \int_{(0,+\infty)} \pi_*(dr)\mathbb{N}_r(\cdot)$ which implies that the result of Lemma 4.3 holds $\mathbb{N}_r$-a.s. for $\pi_*(dr)$ almost every r.

Lemma 4.3 *Under $\mathbb{N}$, the law of the family $(\tilde{\mathcal{S}}^{j,k}, j \in \mathbb{N}, k \in J_j)$, conditionally on $(\tilde{\sigma}^{j,k}, j \in \mathbb{N}, k \in J_j)$, is the law of independent Lévy Poisson snakes, and the conditional law of $\tilde{\mathcal{S}}^{j,k}$ is $\mathbb{N}_{\tilde{\sigma}^{j,k}}$.*

Proof For $j = 0$, notice that J_0 has only one element, say 0. And $\tilde{\mathcal{S}}^{0,0}$ is just the Lévy snake $\tilde{\mathcal{S}} := (\rho_{C_t}, M_{C_t})$ associated with the pruned exploration process and defined in Sect. 1. Of course, we have $\tilde{\sigma}^{0,0} = \tilde{\sigma}$. From the special Markov property (Theorem 3.12) and Proposition 3.2, we deduce that conditionally on $\tilde{\sigma}^{0,0}$, $\tilde{\mathcal{S}}^{0,0}$ and the family $(\mathcal{S}^i, i \in I)$ of excursions of $\mathcal{S}$ away from $\{s \geq 0; m_s^\theta = 0\}$ (as defined in Sect. 5.3) are independent.

From Lemma 1.6 (and the comments below this lemma) for the exploration process and Proposition 3.2 for the underlying Poisson process, we deduce that, conditionally on $\tilde{\sigma}^{0,0}$, $\tilde{\mathcal{S}}^{0,0}$ is distributed according to $\mathbb{N}_{\tilde{\sigma}^{0,0}}$.

Furthermore, from the special Markov property (Theorem 3.12), the conditional law of $\mathcal{S}^i$ is given by N, defined in (22). Now we give a Poisson decomposition of the measure N.

For $\mathcal{S}' = (\rho', M')$ distributed according to N, we consider $(\alpha'_l, \beta'_l)_{l\in I'}$ the excursion intervals of the Lévy Poisson snake, $\mathcal{S}'$, away from $\{H'_s = 0\}$. For $l \in I'$, we set $\mathcal{S}'^l = (\rho'^l, M'^l)$ where for $s \geq 0$,

$$\rho'^l_s(dr) = \rho'_{(s+\alpha'_l)\wedge\beta'_l}(dr)\mathbf{1}_{(0,+\infty)}(r),$$
$$M'^l_s(dr, dv) = M'_{(s+\alpha'_l)\wedge\beta'_l}(dr, dv)\mathbf{1}_{(0,+\infty)}(r).$$

Let us remark that in the above definition ρ'^l and M'^l do not have mass at $\{0\}$ and $\{0\} \times \mathbb{R}_+$.

As a direct consequence of the Poisson decomposition of $\mathbb{P}^*_\ell$ (see Lemma 3.3), we get the following Lemma.

Lemma 4.4 *Under* N*, the point measure $\sum_{i'\in I'} \delta_{\mathcal{S}'^{i'}}$ is a Poisson point measure with intensity $C_\theta \mathbb{N}(d\mathcal{S})$ where $C_\theta = \int_{(0,\infty)}(1 - \mathrm{e}^{-\theta\ell})\ell\pi(d\ell) = \psi'(\theta) - \psi'(0)$.*

By this Poisson representation, each process $\mathcal{S}^i$ is composed of i.i.d. excursions of law $\mathbb{N}$. Thus we get, conditionally on $\tilde{\sigma}^{0,0}$, a family $(\mathcal{S}^{1,k}, k \in J_1)$ of i.i.d. excursions distributed as the atoms of a Poisson point measure with intensity $\tilde{\sigma}^{0,0} C_\theta \mathbb{N}$. Now, we can repeat the above arguments for each excursion $\mathcal{S}^{1,k}$, $k \in J_1$: so that conditionally on $\tilde{\sigma}^{0,0}$, we can

- check that $\tilde{\mathcal{S}}^{1,k}$ is built from $\mathcal{S}^{1,k}$ as $\tilde{\mathcal{S}}$ from $\mathcal{S}$ in Sect. 5.3,
- get a family $(\mathcal{S}^{2,k',k}, k' \in J_2^k)$, which are, conditionally on $\tilde{\sigma}^{1,k}$, distributed as the atoms of a Poisson point measure with intensity $\tilde{\sigma}^{1,k} C_\theta \mathbb{N}$. and are independent of $\tilde{\mathcal{S}}^{1,k}$.

If we set $J_2 = \cup_{k \in J_1} J_2^k \times \{k\}$, we get that conditionally on $\tilde{\sigma}^{0,0}$, and $(\tilde{\sigma}^{1,k}, k \in J_1)$,

- the excursions $\tilde{\mathcal{S}}^{0,0}$ and $(\tilde{\mathcal{S}}^{1,k}, k \in J_1)$, are independent,
- $\tilde{\mathcal{S}}^{i,k}$ is distributed as $\mathbb{N}_{\tilde{\sigma}^{j,k}}$, for $j \in \{0,1\}, k \in J_j$,
- $(\mathcal{S}^{2,k'}, k' \in J_2)$, are distributed as the atoms of a Poisson point measure with intensity $\sum_{k \in J_1} \tilde{\sigma}^{1,k} C_\theta \mathbb{N}$, and are independent of $\tilde{\mathcal{S}}^{0,0}$ and $(\tilde{\mathcal{S}}^{1,k}, k \in J_1)$.

Finally, the result follows by induction. □

4.3 There is no loss of mass: proof of Proposition 1.2

Let $\theta > 0$. We use the notations of the proof of Lemma 4.3. For $n \in \mathbb{N}$, we have $\mathbb{N}$-a.e.

$$\sigma = \sum_{k=0}^{n} \sum_{j \in J_k} \tilde{\sigma}^{j,k} + \int_0^\sigma \mathbf{1}_{\{K_s \geq n+1\}} \, ds.$$

By monotone convergence, we deduce from Remark 4.1 that we get as $n \to +\infty$, $\mathbb{N}$-a.e. ,

$$\sigma = \sum_{k=0}^{\infty} \sum_{j \in J_k} \tilde{\sigma}^{j,k}.$$

As the decreasing reordering of $(\tilde{\sigma}^{j,k}, j \in \mathbb{N}, k \in J_j)$ is Λ^θ, we get that $\mathbb{N}$-a.e. $\sum_{i=1}^{+\infty} \Lambda_i^\theta = \sigma$. As the sequence $(\sum_{i=1}^{\infty} \Lambda_i^\theta, \theta \geq 0)$ is non increasing, we deduce that the previous equality holds for any $\theta > 0$, $\mathbb{N}$-a.e.

Here again the result for P_r is deduced from the one under $\mathbb{N}$.

4.4 Another representation of the fragmentation

Following the ideas in [3,5], we give an other representation of the fragmentation process described in Sect. 4, using a Poisson point measure under the epigraph of the height process.

We consider a fragmentation process, as time θ increases, of the CRT, by cutting at nodes (set of points (s, a) such that $\kappa_s(\{a\}) > 0$, where κ is defined in (8)). More precisely, we consider, conditionally on the CRT or equivalently on the exploration process ρ, a Poisson point process, $\mathcal{Q}(d\theta, ds, da)$ under the epigraph of H, with intensity $d\theta \; q_\rho(ds, da)$, where

$$q_\rho(ds, da) = \frac{ds \; \kappa_s(da)}{d_{s,a} - g_{s,a}}, \tag{26}$$

with $d_{s,a} = \sup\{u \geq s, \min\{H_v, v \in [s,u]\} \geq a\}$ and $g_{s,a} = \inf\{u \leq s, \min\{H_v, v \in [u,s]\} \geq a\}$. (The set $[g_{s,a}, d_{s,a}] \subset [0,\sigma]$ represent the individuals who have a common ancestor with the individual s after or at generation a.)

Notice that from this representation, the cutting times of the nodes are, conditionally on the CRT, independent exponential random times, and their parameter is equal to the mass of the node (defined as the mass of κ or equivalently as the value of the jump of X corresponding to the given node).

We say two points $s, s' \in [0,\sigma]$ belongs to the same fragment at time θ, if there is no cut on their lineage down to their most recent common ancestor $H_{s,s'}$: that is for $v = s$ and $v = s'$,

$$\int \mathbf{1}_{[H_{s,s'},H_v]}(a)\mathbf{1}_{[g_{v,a},d_{v,a}]}(u)\mathcal{Q}([0,\theta], du, da) = 0.$$

This define an equivalence relation, and we call fragment an equivalent class. Let Λ^θ be the sequences of Lebesgue measures of the corresponding equivalent classes ranked in decreasing order.

It is clear that conditionally on the CRT, the process $(\Lambda^\theta, \theta \geq 0)$ has the same distribution as the fragmentation process defined in Sect. 4. Roughly speaking, in Sect. 3 (which leads to the fragmentation of Sect. 4) we mark the node as they appear: that is, for a given level a, the node $\{s; \kappa_s(\{a\}) > 0\}$ is marked at $g_{s,a}$. Whereas in this Section the same node is marked uniformly on $[g_{s,a}, d_{s,a}]$. In both case, the cutting times of the nodes are, conditionally on the CRT, independent exponential random times, and their parameter is equal to the mass of the node (defined as the common value of $\kappa_u(\{a\})$ for $u \in \{s; \kappa_s(\{a\}) > 0\}$, or equivalently as the value of the jump of X corresponding to the given node).

Now, we define the fragments of the Lévy snake corresponding to the cutting of ρ according to the measure q_ρ. For (s,a) chosen according to the measure $q_\rho(ds, da)$, we can define the following Lévy snake fragments $(\rho^i, i \in \tilde{I})$ of ρ by considering

- the open intervals of excursion after s of H above level a: $((\alpha_i, \beta_i), i \in \tilde{I}_+)$, which are such that $\alpha_i > s$, $H_{\alpha_i} = H_{\beta_i} = a$, and for $s' \in (\alpha_i, \beta_i)$ we have $H_{s'} > a$ and $H_{s',s} = a$ (recall Definition (5));
- the open intervals of excursion before s of H above level a: $((\alpha_i, \beta_i), i \in \tilde{I}_-)$, which are such that $\beta_i < s$, $H_{\alpha_i} = H_{\beta_i} = a$, and for $s' \in (\alpha_i, \beta_i)$ we have $H_{s'} > a$ and $H_{s',s} = a$;
- the excursion, i_s, of H above level a that straddle s: $(\alpha_{i_s}, \beta_{i_s})$, which is such that $\alpha_{i_s} < s < \beta_{i_s}$, $H_{\alpha_{i_s}} = H_{\beta_{i_s}} = a$, and for $s' \in (\alpha_{i_s}, \beta_{i_s})$ we have $H_{s'} > a$ and $H_{s',s} = a$;
- the excursion, i_0, of H under level a: $\{s \in [0,\sigma]; H_{s',s} < a\} = [0, \alpha_{i_0}) \cup (\beta_{i_0}, \sigma]$.

For $i \in \tilde{I}_+ \cup \tilde{I}_- \cup \{i_s\}$, we set $\rho^i = (\rho^i_s, s \geq 0)$ where

$$\int f(r)\rho^i_s(dr) = \int f(r-a)\mathbf{1}_{\{r>a\}}\rho_{(\alpha_i+s)\wedge\beta_i}(dr)$$

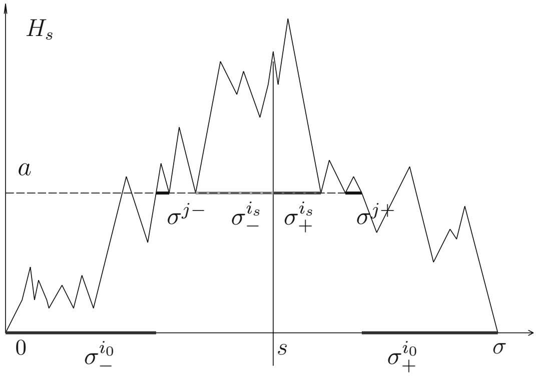

Fig. 3 Different fragments, with $j+ \in \tilde{I}^+$ and $j- \in \tilde{I}^-$

for $f \in \mathcal{B}_+(\mathbb{R})$. For i_0, we set $\rho^{i_0} = (\rho_s^{i_0}, s \geq 0)$ where $\rho_s^{i_0} = \rho_s$ if $s < \alpha_{i_0}$ and $\rho_s^{i_0} = \rho_{s-\beta_{i_0}+\alpha_{i_0}}$ if $s > \beta_{i_0}$. Finally, we set $\tilde{I} = \tilde{I}_+ \cup \tilde{I}_- \cup \{i_s, i_0\}$. And $(\rho^i, i \in \tilde{I})$ correspond to the fragments of the Lévy snake corresponding to the cutting of ρ according to one point chosen with the measure q_ρ. We shall denote $\tilde{\nu}_\rho$ the distribution of $(\rho^i, i \in \tilde{I})$ under $\mathbb{N}$.

In Sect. 4.6, we shall use σ^i, the length of fragment ρ^i. For $i \in \tilde{I}_- \cup \tilde{I}_+$, we have $\sigma^i = \beta_i - \alpha_i$. We also have $\sigma^{i_s} = \sigma_-^{i_s} + \sigma_+^{i_s}$ (resp. $\sigma^{i_0} = \sigma_-^{i_0} + \sigma_+^{i_0}$), where $\sigma_-^{i_s} = s - \alpha_{i_s}$ (resp. $\sigma_-^{i_0} = \alpha_{i_0}$) is the length of the fragment before s and $\sigma_+^{i_s} = \beta_{i_s} - s$ (resp. $\sigma_+^{i_0} = \sigma - \beta_{i_0}$) is the length of the fragment after s. Notice that $\mathbb{N}$-a.e. $\sigma = \sum_{i \in \tilde{I}} \sigma^i$. The Fig. 3 should help to visualize the different lengths.

4.5 The dislocation process is a point process

Let $\mathcal{T}$ be the set of jumping times of the Poisson process $\mathcal{Q}$. For $\theta \in \mathcal{T}$, consider $\mathcal{L}^{(\theta-)} = (\rho_i, i \in I^{(\theta-)})$ and $\mathcal{L}^{(\theta)} = (\rho_i, i \in I^{(\theta)})$ the families of Lévy snakes defined in Remark 4.2. The lengths, ranked in decreasing order, of those families of Lévy snakes correspond respectively to the fragmentation process just before time θ and at time θ. Notice that for $\theta \in \mathcal{T}$ the families $\mathcal{L}^{(\theta-)}$ and $\mathcal{L}^{(\theta)}$ agree but for only one snake $\rho^{i_\theta} \in \mathcal{L}^{(\theta-)}$ which fragments in a family $\{\rho^i, i \in \tilde{I}^{(\theta)}\} \subset \mathcal{L}^{(\theta)}$. Thus we have

$$\mathcal{L}^{(\theta)} = \left(\mathcal{L}^{(\theta-)} \backslash \{\rho^{i_\theta}\}\right) \bigcup \left\{\rho^i, i \in \tilde{I}^{(\theta)}\right\}.$$

From the representation of the previous Section, this fragmentation is given by cutting the Lévy snake according to the measure q_ρ: that is the measure $\tilde{\nu}_\rho$ defined at the end of Sect. 4.4. We refer to [13] for the definition of intensity of a random point measure. From Lemma 4.3 and the construction of the Lévy Poisson Snake, we deduce that

$$\sum_{\theta \in \mathcal{T}} \delta_{\left(\theta, \mathcal{L}^{(\theta-)}, (\rho^i, i \in \tilde{I}^{(\theta)})\right)}$$

is a point process with intensity $d\theta\ \delta_{\mathcal{L}^{(\theta-)}} \sum_{\rho\in\mathcal{L}^{(\theta-)}} \tilde{\nu}_\rho$. Using a projection argument (by taking the expectation over the snakes conditionally on their length), we get that the process

$$\sum_{\theta\in\mathcal{T}} \delta_{\left(\theta,(\sigma(\rho),\rho\in\mathcal{L}^{(\theta-)}),(\sigma(\rho^i),i\in\tilde{I}^{(\theta)})\right)}$$

is a point process with intensity $d\theta\ \delta_{(\sigma(\rho),\rho\in\mathcal{L}^{(\theta-)})} \sum_{\rho\in\mathcal{L}^{(\theta-)}} \nu_{\sigma(\rho)}$, where $\nu_{\sigma(\rho)}$ is the distribution of the decreasing lengths of Lévy snakes under $\tilde{\nu}_\rho$, integrated w.r.t. to the law of ρ conditionally on $\sigma(\rho)$. More precisely we have $\pi_*(dr)$-a.e.

$$\int_{\mathcal{S}^\downarrow} F(x)\nu_r(dx) = \mathbb{N}_r\left[\int F((\sigma^i, i\in\tilde{I}))\tilde{\nu}_\rho(d(\rho^i, i\in\tilde{I}))\right],$$

for any non-negative measurable function F defined on $\mathcal{S}^\downarrow$, where $(\sigma^i, i\in\tilde{I})$ as to be understood as the family of length, of the fragments $(\rho^i, i\in\tilde{I})$, ranked in decreasing size.

This prove that the dislocation process is a point process. And we will now explicit the family of dislocation measures $(\nu_r, r>0)$. As computations are more tractable under $\mathbb{N}$ than under $\mathbb{N}_r$, we shall compute for $\lambda\geq 0$, and any non-negative measurable function, F, defined on $\mathcal{S}^\downarrow$

$$\int_{\mathbb{R}_+\times\mathcal{S}^\downarrow} \mathrm{e}^{-\lambda r}\ F(x)\pi_*(dr)\nu_r(dx).$$

From the definition of $\tilde{\nu}_\rho$, and using the notation at the end of Sect. 4.4, we get that this last quantity is equal to

$$A = \mathbb{N}\left[\mathrm{e}^{-\lambda\sigma}\int q_\rho(ds,da)F\big((\sigma^i, i\in\tilde{I})\big)\right], \tag{27}$$

where $(\sigma^i, i\in\tilde{I})$ as to be understood as the family of lengths ranked in decreasing size. As this family is completely characterized by the measure

$$\sum_{i\in\tilde{I}} \delta_{\sigma^i},$$

we also write with a slight abuse of notation

$$A = \mathbb{N}\left[\mathrm{e}^{-\lambda\sigma}\int q_\rho(ds,da)F\Big(\sum_{i\in\tilde{I}}\delta_{\sigma^i}\Big)\right].$$

4.6 Computation of dislocation measures

From Proposition 2.9 on Bismut formula, and Poisson representation formula for the snake (see Lemma 3.3 or Sect. 4.2.3 in [11]), we get, thanks to Remark 2.7,

$$\begin{aligned} B &:= \mathbb{N}\left[\int_0^\sigma ds \int \kappa_s(da) F_1(\sigma^{i_0}) F_2(\sigma^{i_s}) F_3\left(\sum_{i\in\tilde{I}_-\cup\tilde{I}_+} \delta_{\sigma^i}\right)\right] \\ &= \int_0^{+\infty} db\, \mathrm{e}^{-\alpha_0 b}\, \mathbb{E}\left[\sum_{0\le a\le b;\, \Delta W_a>0} \Delta W_a F_1(S_{W_a-}) F_2(S_{W_b} - S_{W_a}) \right. \\ &\qquad \left. \times F_3\left(\sum_{W_{a-}<u\le W_a;\, \Delta S_u>0} \delta_{\Delta S_u}\right)\right], \end{aligned}$$

where W is a subordinator with Laplace exponent $\psi' - \alpha_0$ and S is a subordinator with Laplace exponent ψ^{-1} independent of W. Notice W has no drift and Lévy measure $\ell\pi(d\ell)$. Palm formula conditionally on S for the jumps of W and the independence of the increments of S imply that

$$\begin{aligned} B = &\int_0^{+\infty} db\, \mathrm{e}^{-\alpha_0 b} \int_0^b da \int \ell^2\pi(d\ell) \mathbb{E}\left[F_1(S_{W_a})\right] \mathbb{E}\left[F_2(S_{W_{b-a}})\right] \\ &\times \mathbb{E}\left[F_3\left(\sum_{0\le u\le \ell;\, \Delta S_u>0} \delta_{\Delta S_u}\right)\right]. \end{aligned}$$

Observe that $\int_0^\infty \mathrm{e}^{-\alpha_0 u}\, \mathbb{E}\left[\mathrm{e}^{-\lambda S_{W_u}}\right] = \int_{(0,\infty)} r\pi_*(dr)\, \mathrm{e}^{-\lambda r}$. Thus, we have

$$\begin{aligned} B = &\int \ell^2\pi(d\ell) \int_{(0,\infty)} r\pi_*(dr) F_1(r) \int_{(0,\infty)} r'\pi_*(dr') F_2(r') \\ &\times \mathbb{E}\left[F_3\left(\sum_{0\le u\le \ell;\, \Delta S_u>0} \delta_{\Delta S_u}\right)\right]. \end{aligned}$$

So, we can use these results to compute A defined in (27). Notice that $q_\rho(ds, da) = \dfrac{\kappa_s(da)ds}{d_{s,a} - g_{s,a}}$ and $d_{s,a} - g_{s,a} = \sigma^{i_s} + \sum_{i \in \tilde{I}_- \cup \tilde{I}_+} \sigma^i$, to get

$$A = \int \ell^2 \pi(d\ell) \int_{(0,\infty)} r\pi_*(dr) \int_{(0,\infty)} r'\pi_*(dr') \\ \times \mathbb{E}\left[\frac{\mathrm{e}^{-\lambda(r+r'+S_\ell)}}{r' + S_\ell} F\left(\delta_r + \delta_{r'} + \sum_{0 \le u \le \ell; \Delta S_u > 0} \delta_{\Delta S_u} \right) \right].$$

We now use the following fact. Fix $t > 0$ and $(\Delta S_u, 0 \le u \le t)$; pick randomly a jump L among the $(\Delta S_u, 0 \le u \le t)$ in such a way that the probability $L = \Delta S_u$ is $\Delta S_u / S_t$. The Palm formula implies that

$$\mathbb{E}\left[F(L) G\left(\sum_{0 \le u \le t; \Delta S_u > 0} \delta_{\Delta S_u} \right) \right] \\ = t \int_{(0,\infty)} \pi_*(dr) F(r) \mathbb{E}\left[\frac{r}{r + S_t} G\left(\delta_r + \sum_{0 \le u \le t; \Delta S_u > 0} \delta_{\Delta S_u} \right) \right].$$

Apply this result twice to finally get

$$A = \int \pi(d\ell) \mathbb{E}\left[S_\ell \, \mathrm{e}^{-\lambda S_\ell} \, F\left(\sum_{0 \le u \le \ell; \Delta S_u > 0} \delta_{\Delta S_u} \right) \right].$$

From Sect. 4.5, we deduce that

$$\int_{\mathbb{R}_+ \times \mathcal{S}^\downarrow} \mathrm{e}^{-\lambda r} \, F(x) \pi_*(dr) \nu_r(dx) = \int \pi(dv) \mathbb{E}\Big[S_v \, \mathrm{e}^{-\lambda S_v} \, F\big((\Delta S_u, u \le v) \big) \Big].$$

From definition (1) of μ, we deduce that

$$\int_{\mathbb{R}_+ \times \mathcal{S}^\downarrow} \mathrm{e}^{-\lambda r} \, F(x) \pi_*(dr) \nu_r(dx) = \int \mathrm{e}^{-\lambda r} \, F(x) \, r \mu(dr, dx).$$

This ends the proof of Theorem 1.4.

Remark 4.5 It is easy to check that the dislocation measures of the fragmentation at nodes associated to $\psi^{(\theta)}$, $(\nu_r^{(\theta)}, r > 0)$, is equal to $(\nu_r, r > 0)$, $\pi_*(dr)$-a.e.

4.7 The stable case

For the stable CRT (with $\psi(\lambda) = \lambda^\alpha$ and $\alpha \in (1, 2)$), thanks to scaling properties, the corresponding fragmentation is self similar with index $1/\alpha$, and we can recover the result of [18].

Corollary 4.6 *Let $\psi(\lambda) = \lambda^\alpha$ and $\alpha \in (1, 2)$. The fragmentation at nodes is self-similar, with index $1/\alpha$, that is $\int_{\mathcal{S}_r^\downarrow} F(x)\nu_r(dx) = r^\gamma \int_{\mathcal{S}_1^\downarrow} F(rx)\nu_1(dx)$ holds for any non-negative measurable function on $\mathcal{S}^\downarrow$. And the dislocation measure ν_1 on $\mathcal{S}_1^\downarrow$ is s.t.*

$$\int F(x)\nu_1(dx) = \frac{\alpha(\alpha-1)\Gamma(1-\alpha^{-1})}{\Gamma(2-\alpha)}\mathbb{E}[S_1 \; F((\Delta S_t/S_1, t \le 1))],$$

holds for any non-negative measurable function F on $\mathcal{S}_1^\downarrow$, where $(\Delta S_t, t \ge 0)$ are the jumps of a stable subordinator $S = (S_t, t \ge 0)$ of Laplace exponent $\psi^{-1}(\lambda) = \lambda^{1/\alpha}$, ranked by decreasing size.

Proof For $\psi(\lambda) = \lambda^\alpha$, we get $\pi(dr) = \alpha(\alpha-1)\Gamma(2-\alpha)^{-1}r^{-1-\alpha}dr$ as well as $\pi_*(dr) = \left[\alpha\Gamma(1-\alpha^{-1})\right]^{-1} r^{-(1+\alpha)/\alpha}dr$. In particular, we have for a non-negative measurable function, F, defined on $\mathbb{R}_+ \times \mathcal{S}_1^\downarrow$,

$$\begin{aligned}\int F(r,x)\, r\mu(dr,dx) &= \mathbb{E}\left[\int \pi(dv)\; S_v F(S_v, (\Delta S_t, t \le v))\right]\\ &= \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\mathbb{E}\left[\int \frac{dv}{v^{1+\alpha}}\; S_v F(S_v, (\Delta S_t, t \le v))\right]\\ &= \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\mathbb{E}\left[\int \frac{dv}{v}\; S_1 F(v^\alpha S_1, v^\alpha S_1(\Delta S_t/S_1, t \le 1))\right]\\ &= \frac{\alpha-1}{\Gamma(2-\alpha)}\int \mathbb{E}[S_1 \; F(y, y(\Delta S_t/S_1, t \le 1))]\frac{dy}{y},\end{aligned}$$

where we used the scaling property of S, that is $(\Delta S_t, t \le r)$ is distributed as $(r^\alpha \Delta S_t, t \le 1)$, for the third equality, and the change of variable $y = v^\alpha S_1$ for the fourth equality. From Theorem 1.4, we have that

$$\begin{aligned}&\int \frac{1}{\alpha\Gamma(1-\alpha^{-1})}\frac{dr}{r^{(1+\alpha)/\alpha}}\nu_r(dx)\; F(r,x)\\ &= \int \frac{\alpha-1}{\Gamma(2-\alpha)}\mathbb{E}[S_1 \; F(y, (y\Delta S_t, t \le 1))]\frac{dy}{y}.\end{aligned}$$

This implies that for a.e. $r > 0$,

$$\int \nu_r(dx)\; F(x) = \frac{\alpha(\alpha-1)\Gamma(1-\alpha^{-1})}{\Gamma(2-\alpha)} r^{1/\alpha}\mathbb{E}[S_1 F(r(\Delta S_t/S_1, t \le 1))],$$

and thus $\int \nu_r(dx)\ F(x) = r^{1/\alpha} \int \nu_1(dx)\ F(rx)$, with

$$\int \nu_1(dx)\ F(x) = \frac{\alpha(\alpha-1)\Gamma(1-\alpha^{-1})}{\Gamma(2-\alpha)} \mathbb{E}[S_1 F((\Delta S_t/S_1, t \le 1))].$$

$\square$

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5 Appendix

5.1 Proof of Proposition 3.1

We first check the process M is right continuous. Recall (13). We have by construction a.s. for all $t' > t$,

$$M_{t'}(dr, dv) = k_{X_t - I^t_{t'}} \rho_t(dr) M_{t,r}(dv) + \rho_{t'}(dr) \mathbf{1}_{\{r > H_{t,t'}\}} M_{t',r}(dv),$$

where $H_{t,t'}$ is defined by (5). Thanks to (14), we have, for $\theta > 0$,

$$\int_{\mathbb{R}_+} \rho_{t'}(dr) \mathbf{1}_{\{r > H_{t,t'}\}} M_{t',r}([0,\theta]) \le \sum_{t < s \le t'} \Delta_s \xi_s.$$

In particular this quantity decreases to 0 as $t' \downarrow t$ a.s. By the properties of the exploration process, we recall that a.s. $k_{X_t - I^t_{t'}} \rho_t = \rho_{t''}$, where $t'' = \inf\{s \in [t, t']; I^t_s = I^t_{t'}\}$. From the right continuity of ρ, we deduce that a.s. for the vague convergence

$$\lim_{t' \downarrow t} M_{t'} = M_t.$$

This implies the right continuity of the process M for the vague topology on $\mathcal{M}(\mathbb{R}_+^2)$.

Now, we check the process M has left limits. Let $t < t'$. For $r \in [0, H_{t,t'}]$, we have $k_{X_t - I^t_{t'}} \rho_t(dr) M_{t,r} = \mathbf{1}_{\{r \le H_{t,t'}\}} \rho_{t'}(dr) M_{t',r}$, as well as

$$M_t(dr, dv) = \mathbf{1}_{\{r \le H_{t,t'}\}} \rho_{t'}(dr) M_{t',r}(dv) + [\rho_t(dr) - k_{X_t - I^t_{t'}} \rho_t(dr)] M_{t,r}(dv).$$

If ρ is continuous at t', then either $\rho_{t'}(\{H_{t'}\}) = 0$ or $H_{t,t'} = H_{t'}$ for t close enough to t'. In particular, since $\lim_{t \to t'} H_{t,t'} = H_{t'}$, we have $\lim_{t \uparrow t'} \mathbf{1}_{\{r \le H_{t,t'}\}} \rho_{t'}(dr) = \rho_{t'}(dr)$. If ρ is not continuous at t', this implies that $\rho_{t'}(dr) = \rho_{t'-}(dr) + \Delta_{t'} \delta_{H_{t'}}(dr)$ and for t close enough to t', $H_{t,t'} < H_{t'}$. Then, we get $\lim_{t \uparrow t'} \mathbf{1}_{\{r \le H_{t,t'}\}} \rho_{t'}(dr) = \rho_{t'-}(dr)$. In any case, we have a.s. for the vague convergence

$$\lim_{t \uparrow t'} \mathbf{1}_{\{r \le H_{t,t'}\}} \rho_{t'}(dr) M_{t',r}(dv) = \rho_{t'-}(dr) M_{t',r}(dv).$$

Now, we check that for the vague topology

$$\lim_{t\uparrow t'}[\rho_t(dr) - k_{X_t - I^t_{t'}}\rho_t(dr)]M_{t,r}(dv) = 0.$$

For this purpose, we remark that

$$\begin{aligned}
&\mathbb{E}_{\mu,\Pi}\left[\int_{\mathbb{R}_+}[\rho_t(dr) - k_{X_t - I^t_{t'}}\rho_t(dr)]M_{t,r}([0,\theta])|X\right] \\
&= \theta\int_{\mathbb{R}_+}[\rho_t(dr) - k_{X_t - I^t_{t'}}\rho_t(dr)](\rho_t(\{r\}) + \eta_t(\{r\})) \\
&\leq \theta(\langle\rho_t + \eta_t, 1\rangle)\int_{\mathbb{R}_+}[\rho_t(dr) - k_{X_t - I^t_{t'}}\rho_t(dr)] \\
&= \theta\,(\langle\rho_t + \eta_t, 1\rangle)\,(X_t - I^t_{t'}).
\end{aligned}$$

As ρ and η are respectively càd-làg and càg-làd process, they are bounded over any finite interval a.s. Since $\lim_{t\uparrow t'} X_t - I^t_{t'} = 0$, we deduce that

$$\lim_{t\uparrow t'}\mathbb{E}_{\mu,\Pi}\left[\int_{\mathbb{R}_+}[\rho_t(dr) - k_{X_t - I^t_{t'}}\rho_t(dr)]M_{t,r}([0,\theta])|X\right] = 0.$$

Thanks to (15) and Fatou's Lemma, we deduce that

$$\lim_{t\uparrow t'}\int_{\mathbb{R}_+}[\rho_t(dr) - k_{X_t - I^t_{t'}}\rho_t(dr)]M_{t,r}([0,\theta]) = 0.$$

Therefore, we conclude that for vague topology,

$$\lim_{t\uparrow t'} M_t = M_{t'-}.$$

We deduce that for the vague topology on $\mathcal{M}(\mathbb{R}^2_+)$, the process M is a.s. càd-làg. This implies the process $\mathcal{S}$ is a.s. càd-làg.

We check the strong Markov property of $\mathcal{S}$. Mimicking the proof of Proposition 1.2.3 in [11], and using properties of Poisson point measure, one gets that, for any $\mathcal{F}$-stopping time T, we have a.s. for every $t > 0$,

$$\begin{aligned}
\rho_{T+t} &= \left[k_{-I^{(T)}_t}\rho_T, \rho^{(T)}_t\right] \\
M_{T+t}(dr, dv) &= k_{-I^{(T)}_t}\rho^{(T)}_t(dr)M_{T,r}(dv) + M^{(T)}_t(dr + H(k_{-I^{(T)}_t}\rho_T), dv)
\end{aligned}$$

where $I^{(T)}$, $\rho^{(T)}$ and $M^{(T)}$ are the analogues of I, ρ and M with X replaced by the shifted process $X^{(T)} = (X_{T+t} - X_T, t \geq 0)$. This implies the strong Markov property.

5.2 Law of the pruned exploration process

5.2.1 Proof of Lemma 3.6

We first prove (i). Let $\lambda > 0$. Before computing $v = \mathbb{N}[1 - \exp -\lambda A_\sigma]$, notice that $A_\sigma \leq \sigma$ implies, thanks to (11), that $v \leq \mathbb{N}[1 - \exp -\lambda\sigma] = \psi^{-1}(\lambda) < +\infty$. We have

$$v = \lambda \mathbb{N}\left[\int_0^\sigma dA_t \; \mathrm{e}^{-\lambda \int_t^\sigma dA_u}\right] = \lambda \mathbb{N}\left[\int_0^\sigma dA_t \; \mathbb{E}^*_{\rho_t,0}[\mathrm{e}^{-\lambda A_\sigma}]\right],$$

where we replaced $\mathrm{e}^{-\lambda \int_t^\sigma dA_u}$ in the last equality by $\mathbb{E}^*_{\rho_t,M_t}[\mathrm{e}^{-\lambda A_\sigma}]$, its optional projection, and used that $M_t(\mathbb{R}_+, [0,\theta]) = 0 dA_t$-a.e. to replace $\mathbb{E}^*_{\rho_t,M_t}$ by $\mathbb{E}^*_{\rho_t,0}$, as m under $\mathbb{E}_{\mu,\Pi}$ is distributed as m under $\mathbb{E}_{\mu,0}$ if $\Pi(\mathbb{R}_+, [0,\theta]) = 0$. In order to compute this last expression, we use the decomposition of $\mathcal{S}$ under $\mathbb{P}^*_{\mu,0}$ according to excursions of the total mass of ρ above its minimum, see Lemma 3.3. Using the same notations as in this Lemma, notice that under $\mathbb{P}^*_{\mu,0}$, we have $A_\sigma = A_\infty = \sum_{i \in I} A^i_\infty$, where for every $T \geq 0$,

$$A^i_T = \int_0^T \mathbf{1}_{\{M^i_t(\mathbb{R}_+ \times [0,\theta]) = 0\}} dt. \tag{28}$$

By Lemma 3.3, we get

$$\mathbb{E}^*_{\mu,0}[\mathrm{e}^{-\lambda A_\sigma}] = \mathrm{e}^{-\langle \mu, 1\rangle \mathbb{N}[1 - \exp -\lambda A_\sigma]} = \mathrm{e}^{-v\langle \mu, 1\rangle}.$$

Now, for fixed t, recall (17). By conditioning with respect to X or to ρ thanks to Remark 2.6, we have

$$\begin{aligned} v = \lambda \mathbb{N}\Big[\int_0^\sigma dA_t \; \mathrm{e}^{-v\langle \rho_t, 1\rangle}\Big] &= \lambda \mathbb{N}\Big[\int_0^\sigma dt \, \mathbf{1}_{\{m_t = 0\}} \, \mathrm{e}^{-v\langle \rho_t, 1\rangle}\Big] \\ &= \lambda \mathbb{N}\Big[\int_0^\sigma dt \, \mathrm{e}^{-(v+\theta)\langle \rho_t, 1\rangle - \theta\langle \eta_t, 1\rangle}\Big]. \end{aligned}$$

Now we use Proposition 2.8 to get

$$
\begin{aligned}
v &= \lambda \int_0^{+\infty} da\ \mathrm{e}^{-\alpha_0 a}\ \mathbb{M}_a[\mathrm{e}^{-(v+\theta)\langle\mu,1\rangle-\theta\langle\nu,1\rangle}] \\
&= \lambda \int_0^{+\infty} da\ \mathrm{e}^{-\alpha_0 a} \exp\Big\{-\int_0^a dx \int_0^1 du \int_{(0,\infty)} \ell\pi(d\ell)\Big[1-\mathrm{e}^{-(v+\theta)u\ell-\theta(1-u)\ell}\Big]\Big\} \\
&= \lambda \int_0^{+\infty} da\ \exp\Big\{-a\int_0^1 du\ \psi'(\theta+vu)\Big\} \qquad (29)\\
&= \lambda \frac{v}{\psi(\theta+v)-\psi(\theta)}. \qquad (30)
\end{aligned}
$$

where, for the third equality, we used

$$
\psi'(\lambda) = \alpha_0 + \int_{(0,\infty)} \pi(d\ell)\ \ell(1-\mathrm{e}^{-\lambda\ell}). \tag{31}
$$

Notice that if $v=0$, then (29) implies $v=\lambda/\psi'(\theta)$, which is absurd. Therefore we have $v\in(0,\infty)$, and we can divide (30) by v to get $\psi^{(\theta)}(v)=\lambda$. This proves (i).

Now, we prove (ii). If we let $\lambda\to\infty$ in (i) and use that $\lim_{r\to\infty}\psi^{(\theta)}(r)=+\infty$, then we get that $\mathbb{N}[A_\sigma>0]=+\infty$. Notice that for $(\mu,\Pi)\in\mathbb{S}$, we have under $\mathbb{P}^*_{\mu,\Pi}$, $A_\infty\geq\sum_{i\in I}A^i_\infty$, with A_i defined by (28). Thus Lemma 3.3 imply that if $\mu\neq 0$, then $\mathbb{P}^*_{\mu,\Pi}$-a.s. I is infinite and $A_\infty>0$. Using the Markov property at time t of the snake under $\mathbb{N}$, we get that for any $t>0$, $\mathbb{N}$-a.e. on $\{\sigma>t\}$, we have $A_\sigma-A_t>0$. This implies that σ is a point of increase of A $\mathbb{N}$-a.e. By time reversibility, see Lemma 3.5, we also get that 0 is a point of increase of A $\mathbb{N}$-a.e.

To prove (iii), recall that $\int_{(0,1)}\ell\pi(d\ell)=+\infty$ implies that $\mathcal{J}=\{s\geq 0;\ \Delta_s>0\}$ is dense in $\mathbb{R}_+$ a.s. Moreover, for every $t>r\geq 0$,

$$
\sum_{r\leq s\leq t}\Delta_s = +\infty\ a.s.
$$

Now, by the properties of Poisson point measures, we have

$$
\mathbb{P}(\forall s\in[r,t],\ m_s=0) = \mathbb{E}\Big[\mathrm{e}^{-\theta\sum_{r\leq s\leq t}\Delta_s}\Big] = 0
$$

which proves (iii).

5.2.2 Proof of Theorem 3.8 (and Theorem 1.5)

Let $\varepsilon > 0$. Let us define by induction the following stopping times:

$$\begin{aligned} & T_0^\varepsilon = 0 \\ \forall k \geq 0, \qquad & S_{k+1}^\varepsilon = \inf\left\{s > T_k^\varepsilon,\ m_s(\{H_s\}) > 0,\ \rho_s(\{H_s\}) > \varepsilon\right\} \\ & T_{k+1}^\varepsilon = \inf\left\{s > S_{k+1}^\varepsilon,\ X_s = X_{S_{k+1}^\varepsilon -}\right\}. \end{aligned}$$

We set

$$E_\varepsilon = \bigcup_{k \in \mathbb{N}} [T_k^\varepsilon, S_{k+1}^\varepsilon)$$

the set of times for which no mass of size greater than ε is marked, and, for every $t \geq 0$, we set

$$R_t^\varepsilon = \inf\left\{s \geq 0,\ \int_0^s \mathbf{1}_{E_\varepsilon}(u)du > t\right\}.$$

Finally, let us define the process $X^\varepsilon = (X_t^\varepsilon, t \geq 0)$ by $X_t^\varepsilon = X_{R_t^\varepsilon}$. The strong Markov property implies that X^ε is a Lévy process. Informally X^ε is distributed as X but for the jumps of size larger than ε, say Δ, which are removed with probability $1 - \mathrm{e}^{-\theta\Delta}$. A standard calculation shows that the Laplace exponent of X^ε is given by

$$\psi^{\theta,\varepsilon}(\lambda) = \psi(\lambda) + \int_{(\varepsilon,+\infty)} \pi(d\ell)\left(1 - \mathrm{e}^{-\theta\ell}\right)\left(1 - \mathrm{e}^{-\lambda\ell}\right).$$

Notice that the set E_ε decreases to $\{s; m_s = 0\}$ as ε goes down to 0. This implies the process X^ε converges a.s. point wise to the process $\widetilde{X} := (X_{C_t}, t \geq 0)$ as ε goes down to 0. Moreover, $\psi^{\theta,\varepsilon}$ converges to $\psi^{(\theta)}$. This implies $\widetilde{X}$ is a Lévy process with Laplace exponent $\psi^{(\theta)}$.

It remains to prove that $\tilde{\rho}$ is the exploration process associated with $\tilde{X}$. Formulas (4) and (6) provide a measurable functional Υ such that $\rho = \Upsilon(X)$. Recall that $\beta_{\varepsilon'} = \int_{(\varepsilon',+\infty)} \ell\pi(d\ell)$. Formula (4) implies that a.s. for all $t \geq 0$,

$$H_{C_t} = \lim_{\varepsilon' \to 0} \frac{1}{\beta_{\varepsilon'}} \mathrm{Card}\left\{s \in [0, C_t],\ X_{s-} < I_{s,C_t},\ \Delta X_s > \varepsilon'\right\}. \tag{32}$$

By definition of T_k^ε, for any integer $k \geq 1$, all the jumps of X in the time interval $[S_k^\varepsilon, T_k^\varepsilon]$ are erased at time T_k^ε, that is a.s. $X_{s-} \geq I_{s,t}$ for all $s \in E_\varepsilon^c$, $t \in E_\varepsilon$ and $s < t$. As $C_t \in E_\varepsilon$, we get that

$$\begin{aligned} & \mathrm{Card}\left\{s \in [0, C_t], X_{s-} < I_{s,C_t}, \Delta X_s > \varepsilon'\right\} \\ & \quad = \mathrm{Card}\left\{s \in [0, C_t] \cap E_\varepsilon, X_{s-} < I_{s,C_t}, \Delta X_s > \varepsilon'\right\}. \end{aligned}$$

Letting ε goes down to 0 and using an obvious time change, we get

$$\begin{aligned}&\mathrm{Card}\left\{s\in[0,C_t],\ X_{s-}<I_{s,C_t},\ \Delta X_s>\varepsilon'\right\}\\&\quad=\mathrm{Card}\left\{s\in[0,t],\ \widetilde{X}_{s-}<\widetilde{I}_{s,t},\ \Delta\widetilde{X}_s>\varepsilon'\right\},\end{aligned}$$

where $\widetilde{I}_{s,t}=\inf_{s\le r\le t}\widetilde{X}_r$. The Lévy measure of $\widetilde{X}$, $\pi^{(\theta)}$ is given by $\pi^{(\theta)}(d\ell)=\mathrm{e}^{-\theta\ell}\pi(d\ell)$. as $\int_{(0,1)}\ell\pi(d\ell)=\infty$ and $\int_{[1,\infty)}\ell\pi(d\ell)<\infty$, we deduce that $\lim_{\varepsilon'\to 0}\beta^{(\theta)}_{\varepsilon'}/\beta_{\varepsilon'}=1$, where

$$\beta^{(\theta)}_{\varepsilon'}=\int\limits_{(\varepsilon',\infty)}\ell\pi^{(\theta)}(d\ell).$$

We deduce from (32) that a.s. for all $t\ge 0$,

$$H(\tilde{\rho}_t)=\lim_{\varepsilon'\to 0}\frac{1}{\beta^{(\theta)}_{\varepsilon'}}\mathrm{Card}\left\{s\in[0,t],\ \widetilde{X}_{s-}<\widetilde{I}_{s,t},\ \Delta\widetilde{X}_s>\varepsilon'\right\}.$$

This, combined with (6) implies that a.s., $\tilde{\rho}=\Upsilon(\widetilde{X})$, which proves Theorem 3.8.

5.2.3 Proof of Lemma 1.6

Let $\theta>0$. We set $X^{(\theta)}=(X^{(\theta)}_t,t\ge 0)$ the Lévy process with Laplace exponent $\psi^{(\theta)}$. Notice that $(\mathrm{e}^{-\theta X_t-t\psi(\theta)},t\ge 0)$ is a martingale w.r.t. the natural filtration generated by X, $(\mathcal{H}_t,t\ge 0)$. We define a new probability by

$$d\mathbb{P}^{(\theta)}_{|\mathcal{H}_t}=\mathrm{e}^{-\theta X_t-t\psi(\theta)}\,d\mathbb{P}_{|\mathcal{H}_t}.$$

The law of $(X_u,u\in[0,t])$ under $\mathbb{P}^{(\theta)}$ is the law of $(X^{(\theta)}_u,u\in[0,t])$. Therefore, we have for any non-negative measurable function on the path space

$$\mathbb{E}\left[F(X^{(\theta)}_{\le t})\,\mathrm{e}^{\theta X^{(\theta)}_t+t\psi(\theta)}\right]=\mathbb{E}[F(X_{\le t})]. \tag{33}$$

We define $-I^{(\theta)}_t=-\inf_{u\in[0,t]}X^{(\theta)}_u$, and $\tau^{(\theta)}$ its right-continuous inverse. In particular, it is a subordinator of Laplace exponent $\psi^{(\theta)^{-1}}$. Since $\psi^{(\theta)^{-1}}(\lambda)=\psi^{-1}(\lambda+\psi(\theta))-\theta$, we have

$$\mathbb{E}\left[\mathrm{e}^{-\lambda\tau^{(\theta)}_r}\right]=\mathrm{e}^{-r[\psi^{-1}(\lambda+\psi(\theta))-\theta]}.$$

Furthermore, this equality holds for $\lambda\ge-\psi(\theta)$. With $\lambda=-\psi(\theta)$, we get $\mathbb{E}\left[\mathrm{e}^{\psi(\theta)\tau^{(\theta)}_r}\right]=\mathrm{e}^{\theta r}$.

From (33), we get that the process $(Q_t, t \geq 0)$, where $Q_t = \mathrm{e}^{\theta X_t^{(\theta)} + t\psi(\theta)}$ is a martingale. Since $M_{\tau_r^{(\theta)}} = \mathrm{e}^{-\theta r + \psi(\theta)\tau_r^{(\theta)}}$ is integrable and $\mathbb{E}[M_{\tau_r^{(\theta)}}] = 1$, we deduce from (33) that

$$\mathbb{E}\left[F(X^{(\theta)}_{\leq \tau_r^{(\theta)}})\,\mathrm{e}^{-\theta r + \psi(\theta)\tau_r^{(\theta)}}\right] = \mathbb{E}[F(X_{\leq \tau_r})]. \tag{34}$$

Let $\mathcal{E}_i = (X_{t+\alpha_i} - I_{\alpha_i}, t \in [\alpha_i, \alpha_i + \sigma_i])$, $i \in I$, be the excursions of X above its minimum, up to time τ_r. With F such that $F(X_{\leq \tau_r}) = \mathrm{e}^{-\sum_{i\in I} G(\mathcal{E}_i)}$, we get

$$\mathbb{E}[F(X_{\leq \tau_r})\,\mathrm{e}^{-\lambda \tau_r}] = \mathrm{e}^{-r\mathbb{N}[1-\mathrm{e}^{-G(\mathcal{E})-\lambda\sigma}]}.$$

We deduce from (34) that

$$\mathrm{e}^{-\theta r}\,\mathrm{e}^{-r\mathbb{N}[1-\mathrm{e}^{-G(\mathcal{E}^{(\theta)})+\psi(\theta)\sigma^{(\theta)}}]} = \mathrm{e}^{-r\mathbb{N}[1-\mathrm{e}^{-G(\mathcal{E})}]},$$

where $\mathcal{E}^{(\theta)}$ is an excursion of $X^{(\theta)}$ above its minimum, that is

$$\mathbb{N}[1 - \mathrm{e}^{-G(\mathcal{E}^{(\theta)})+\psi(\theta)\sigma^{(\theta)}}] = \mathbb{N}[1 - \mathrm{e}^{-G(\mathcal{E})}] - \theta.$$

Subtracting $\mathbb{N}[1 - \mathrm{e}^{\psi(\theta)\sigma^{(\theta)}}] = -\theta$, in the above equality, we get

$$\mathbb{N}\left[\mathrm{e}^{\psi(\theta)\sigma^{(\theta)}}[1 - \mathrm{e}^{-G(\mathcal{E}^{(\theta)})}]\right] = \mathbb{N}\left[1 - \mathrm{e}^{-G(\mathcal{E})}\right].$$

5.3 Proof of the Special Markov Property

In order to simplify the notations, we will write $\mathbb{P}$ instead of $\mathbb{P}_{0,0}$ and $\mathbb{E}$ instead of $\mathbb{E}_{0,0}$. Recall that $\theta > 0$ is fixed.

Fix $t > 0$. Let us remark that to prove Theorem 3.12, we may only consider functions ϕ satisfying the assumptions of Theorem 3.12 and these three conditions:

- (h_1) $\phi(s, \omega, \mathcal{S}) = 0$ if the starting mass of $\mathcal{S}$ is less than η, that is $\langle \rho_0, 1\rangle \leq \eta$, for a fixed positive real number $\eta > 0$.
- (h_2) $s \mapsto \phi(s, \omega, \mathcal{S})$ is uniformly continuous.
- (h_3) $\phi(u, \omega, \mathcal{S}) = 0$ for any $u > t$.

Indeed if (23) holds for such functions then by Monotone Class Theorem and monotonicity it holds also for every function satisfying the assumptions of Theorem 3.12.

The proof now goes along 3 steps.

Step 1. Approximation of the functional.

Recall the definition of the stopping times S_k^ε and T_k^ε of Sect. 5.2.2. For every $k \geq 1$, we define the measure-valued process $\mathcal{S}^{k,\varepsilon} = (\rho^{k,\varepsilon}, M^{k,\varepsilon})$ in a similar way as the processes (ρ^i, M^i) in (21): for every non-negative continuous functions f and φ, and $s \geq 0$,

$$\langle \rho_s^{k,\varepsilon}, f\rangle = \int_{[H_{S_k^\varepsilon},+\infty)} f(x - H_{S_k^\varepsilon})\rho_{(S_k^\varepsilon+s)\wedge T_k^\varepsilon}(dx)$$

$$\langle M_s^{k,\varepsilon}, \varphi\rangle = \int_{(H_{S_k^\varepsilon},+\infty)\times[0,+\infty)} \varphi(x - H_{S_k^\varepsilon}, v)M_{(S_k^\varepsilon+s)\wedge T_k^\varepsilon}(dx, dv).$$

We call $\Delta_{S_k^\varepsilon}$ the starting mass of $\mathcal{S}^{k,\varepsilon}$. Notice that $\rho_0^{k,\varepsilon} = \delta_{\Delta_{S_k^\varepsilon}}$ and $\Delta_{S_k^\varepsilon} \geq \varepsilon$.

Lemma 5.1 *$\mathbb{P}$-a.s., we have for $\varepsilon > 0$ small enough*

$$\sum_{i\in I}\phi(A_{\alpha_i}, \omega, \mathcal{S}^i) = \sum_{k\geq 1}\phi(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon}). \tag{35}$$

Proof Let I_η be the set of indexes $i \in I$, such that the starting mass of $\mathcal{S}^i$ is larger than η and $A_{\alpha_i} \leq t$. Because of (h_1) and (h_3), we have

$$\sum_{i\in I}\phi(A_{\alpha_i}, \omega, \mathcal{S}^i) = \sum_{i\in I_\eta}\phi(A_{\alpha_i}, \omega, \mathcal{S}^i).$$

Let $\varepsilon < \eta$. Then, for any $i \in I_\eta$, there exists $k \in \mathbb{N}^*$, such that $\mathcal{S}^{k,\varepsilon} = \mathcal{S}^i$. Furthermore, all the others excursions $\mathcal{S}^{k,\varepsilon}$ which do not belong to $\{\mathcal{S}^i, i \in I_\eta\}$ either have a starting mass less than η or $A_{S_k^\varepsilon} \geq t$ (and thus $\phi(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon}) = 0$), or have a starting mass greater that η but $m_{S_k^\varepsilon}([0, H_{S_k^\varepsilon})) > 0$. But, as the set $\{0 < s \leq t, \Delta_s > \eta\}$ is finite, there exists only a finite number of excursions $\mathcal{S}^i$ which straddle a time $s \leq t$ such that $\Delta_s > \eta$. Therefore, the minimum over those excursions of their starting mass, say η', is positive a.s. and, if we choose $\varepsilon < \eta'$, there are no excursions $\mathcal{S}^{k,\varepsilon}$ with initial mass greater than η and $A_{S_k^\varepsilon} < t$ which do not correspond to a $\mathcal{S}^i$ for $i \in I_\eta$.

Consequently, if we choose $\varepsilon < \eta \wedge \eta'$, we have

$$\sum_{i\in I}\phi(A_{\alpha_i}, \omega, \mathcal{S}^i) = \sum_{k\geq 1}\phi(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon}).$$

□

For $k \geq 1$, we consider the σ-field $\mathcal{F}^{(\varepsilon),k}$ generated by the family of processes

$$\left(\mathcal{S}_{(T_l^\varepsilon+s)\wedge S_{l+1}^\varepsilon-},\ s > 0\right)_{l\in\{0,\ldots,k-1\}}.$$

Notice that for $k \geq 1$, $\mathcal{F}^{(\varepsilon),k} \subset \mathcal{F}_{S_k^\varepsilon}$. It is easy to check the following measurable result.

Lemma 5.2 *For any $\varepsilon > 0$, $k \in \mathbb{N}^*$, the function $\phi(A_{S_k^\varepsilon}, \omega, \cdot)$ is $\mathcal{F}^{(\varepsilon),k}$-measurable.*

Step 2. Computation of the conditional expectation of the approximation.

Lemma 5.3 *For every $\tilde{\mathcal{F}}_\infty$-measurable non-negative random variable Z, we have*

$$\mathbb{E}\left[Z \exp\left(-\sum_{k\geq 1}\phi(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon})\right)\right] = \mathbb{E}\left[Z \prod_{k\geq 1} \mathrm{N}\left[\mathrm{e}^{-\phi(A_{S_k^\varepsilon},\omega,\cdot)} \;\Big|\; \rho_0 > \varepsilon\right]\right].$$

Remark 5.4 Let us note that the right-hand side of the previous equality does not give the conditional expectation of the functional given $\tilde{\mathcal{F}}_\infty$ as the obtained random variable is only $\mathcal{F}^{(\varepsilon),k}$-measurable according to Lemma 5.2. However, we will obtain the desired result by letting ε goes down to 0 in the next Step.

Proof For every integer $p \geq 1$, we consider a non-negative random variable Z of the form $Z = Z_0 Z_1$, where $Z_0 \in \mathcal{F}^{(\varepsilon),p}$ and $Z_1 \in \sigma(\mathcal{S}_{(T_k^\varepsilon+s)\wedge S_{k+1}^\varepsilon-},\ s \geq 0,\ k \geq p)$ are bounded non-negative.

To compute $D = \mathbb{E}\left[Z \exp\left(-\sum_{k=1}^{p}\phi(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon})\right)\right]$, we first apply the strong Markov property at time T_p^ε. We obtain

$$D = \mathbb{E}\left[Z_0 \exp\left(-\sum_{k=1}^{p}\phi(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon})\right)\mathbb{E}^*_{\rho_{T_p^\varepsilon},0}[Z_1]\right].$$

Notice that $\rho_{T_p^\varepsilon} = \rho_{S_p^\varepsilon-}$, and consequently $\rho_{T_p^\varepsilon}$ is measurable with respect to $\mathcal{F}_{S_p^\varepsilon}$. So, when we use the strong Markov property at time S_p^ε, we get thanks to Lemma 5.2 and since $\mathcal{F}^{(\varepsilon),k} \subset \mathcal{F}_{S_k^\varepsilon}$,

$$D = \mathbb{E}\left[Z_0 \exp\left(-\sum_{k=1}^{p-1}\phi(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon})\right)\mathbb{E}^*_{\rho_0^{p,\varepsilon},0}\left[\mathrm{e}^{-\phi(A_{S_p^\varepsilon},\omega,\cdot)}\right]\mathbb{E}^*_{\rho_{T_p^\varepsilon},0}[Z_1]\right].$$

Conditionally on $\mathcal{F}_{T_{p-1}^\varepsilon}$, the measure $\rho_0^{p,\varepsilon}$ is a Dirac mass and, by the Poisson representation of Lemma 3.4, the mass of $\rho_0^{p,\varepsilon}$ is distributed according to the law $n^\theta(d\ell \mid \ell > \varepsilon)$. From Poisson point measure properties, notice that $\rho_0^{p,\varepsilon}$ is also independent of $\sigma(\mathcal{S}_t, t < S_p^\varepsilon)$ and thus of $\mathcal{F}^{(\varepsilon),p}$.

Therefore, $\rho_0^{p,\varepsilon}$ is independent of Z_0, $\rho_{T_p^\varepsilon} = \rho_{S_p^\varepsilon-}$ and, thanks to Lemma 5.2 of $\phi(A_{S_p^\varepsilon}, \omega, \cdot)$. So, by conditioning with respect to $\mathcal{F}^{(\varepsilon),p}$, we get

$$D = \mathbb{E}\left[Z_0 \exp\left(-\sum_{k=1}^{p-1}\phi(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon})\right)\mathrm{N}\left[\mathrm{e}^{-\phi(A_{S_p^\varepsilon},\omega,\cdot)} \;\Big|\; \rho_0 > \varepsilon\right]\mathbb{E}^*_{\rho_{T_p^\varepsilon},0}[Z_1]\right]. \tag{36}$$

Now, using one more time the strong Markov property at time T_p^ε, we get from (36)

$$D = \mathbb{E}\left[Z\mathrm{N}\left[\mathrm{e}^{-\phi(A_{S_p^\varepsilon},\omega,\cdot)} \;\middle|\; \rho_0 > \varepsilon\right] \exp\left(-\sum_{k=1}^{p-1} \phi(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon})\right)\right].$$

From monotone class Theorem, this equality holds also for any $Z \in \mathcal{F}^{(\varepsilon),\infty}$ non-negative. Thanks to Lemma 5.2, the non-negative random variable $Z' = Z\mathrm{N}[\mathrm{e}^{-\phi(A_{S_p^\varepsilon},\omega,\cdot)} | \rho_0 > \varepsilon]$ is measurable w.r.t. $\mathcal{F}^{(\varepsilon),\infty}$. So, we may iterate the previous argument and eventually get that for any non-negative random variable $Z \in \mathcal{F}^{(\varepsilon),\infty}$, we have

$$\mathbb{E}\left[Z \exp\left(-\sum_{k=1}^{p} \phi(A_{S_k^\varepsilon}, \omega, \mathcal{S}^{k,\varepsilon})\right)\right] = \mathbb{E}\left[Z \prod_{k=1}^{p} \mathrm{N}\left[\mathrm{e}^{-\phi(A_{S_k^\varepsilon},\omega,\cdot)} \;\middle|\; \rho_0 > \varepsilon\right]\right].$$

Let p goes to infinity and notice that $\tilde{\mathcal{F}}_\infty \subset \mathcal{F}^{(\varepsilon),\infty}$ to end the proof. □

Step 3. Computation of the limit.
We define

$$n_\varepsilon = n^\theta(\ell > \varepsilon) = \int_{(\varepsilon,+\infty)} \pi(d\ell)\left(1 - \mathrm{e}^{-\theta\ell}\right). \tag{37}$$

Let $g_\varepsilon(u,\omega) = -n_\varepsilon \log(1 - n_\varepsilon^{-1}\mathrm{N}[1 - \exp -\phi(u,\omega,\cdot)])$ for $\varepsilon > 0$ $u \geq 0$.

Lemma 5.5 *There exists a positive sequence* $(\varepsilon_j, j \in \mathbb{N}^*)$ *decreasing to* 0, *such that* $\mathbb{P}$-*a.s.*

$$\lim_{j\to\infty} \frac{1}{n_{\varepsilon_j}} \sum_{k\geq 1} g_{\varepsilon_j}(A_{S_k^{\varepsilon_j}}, \omega) = \int_0^{+\infty} g(u,\omega)\, du,$$

where $g(u,\omega) = \mathrm{N}\left[1 - \mathrm{e}^{-\phi(u,\omega,\cdot)}\right]$.

Proof The assumptions made on ϕ imply that g_ε uniformly converges to g.
Recall the definition of the set E_ε of Sect. 5.2.2 and let us set

$$A_t^\varepsilon = \int_0^t \mathbf{1}_{E_\varepsilon}(u)du.$$

Then, we have $A_{S_k^\varepsilon}^\varepsilon = \sum_{l=0}^{k-1} e_l^\varepsilon$ with $e_l^\varepsilon = S_{l+1}^\varepsilon - T_l^\varepsilon$. From point Poisson measure property, notice that, conditionally on $\tilde{\mathcal{F}}_\infty$, the random variables $(e_k^\varepsilon, k \geq 1)$ are independent and distributed as exponential variables with mean n_ε^{-1}. Moreover, a.s. for every $s \geq 0$, $\lim_{\varepsilon\to 0} A_s^\varepsilon = A_s$ and, by Dini theorem, this convergence is uniform on $[0, t]$.

Let $\mathcal{N} = \sum_{j\in J} \delta_{(u_j,r_j)}$ be a Poisson point process in $[0, +\infty)^2$ with intensity the Lebesgue measure. We assume that $\mathcal{N}$ is independent of $\tilde{\mathcal{F}}_\infty$. The previous remarks show that, conditionally on $\tilde{\mathcal{F}}_\infty$, the random variable

$$B_\varepsilon = n_\varepsilon^{-1} \sum_{k\geq 1} g_\varepsilon(A^\varepsilon_{S_k^\varepsilon}, \omega)$$

has the same distribution as $n_\varepsilon^{-1} \sum_{j\in J} g_\varepsilon(u_j, \omega)\mathbf{1}_{[0,n_\varepsilon]}(r_j)$. The exponential formula for Poisson point process implies that B_ε converges in distribution to $\int_0^\infty g(u, \omega)du$, which is a $\tilde{\mathcal{F}}_\infty$-measurable random variable. Therefore, we can find a sub-sequence $(\varepsilon_j,\ j \geq 1)$ (of $\tilde{\mathcal{F}}_\infty$-measurable random variables) such that, a.s. conditionally on $\tilde{\mathcal{F}}_\infty$,

$$\lim_{j\to+\infty} B_{\varepsilon_j} = \int_0^\infty g(u)du.$$

Use that A^ε converges uniformly to A over any compact set (with $A^\varepsilon_t \geq A_t$), the uniform continuity of g_ε (see condition (h_2)) and (h_3) to end the proof. □

We can now finish the proof of the theorem. Let $Z \in \tilde{\mathcal{F}}_\infty$ bounded and non-negative. We have

$$\begin{aligned}
\mathbb{E}\left[Z \exp\left(-\sum_{i\in I} \phi(A_{\alpha_i}, \omega, \mathcal{S}^i)\right)\right] &= \lim_{j\to\infty} \mathbb{E}\left[Z \exp\left(-\sum_{k\geq 1} \phi(A_{S_k^{\varepsilon_j}}, \omega, \mathcal{S}^{k,\varepsilon_j})\right)\right] \\
&= \lim_{j\to\infty} \mathbb{E}\left[Z \prod_{k\geq 1} \mathrm{N}\left[\mathrm{e}^{-\phi(A_{S_k^{\varepsilon_j}},\omega,\cdot)} \,\Big|\, \rho_0 > \varepsilon_j\right]\right],
\end{aligned}$$

where we used Lemma 5.1 and dominated convergence for the first equality, Lemma 5.3 for the second equality. We have

$$\begin{aligned}
\prod_{k\geq 1} \mathrm{N}\left[\mathrm{e}^{-\phi(A_{S_k^{\varepsilon_j}},\omega,\cdot)} \,\Big|\, \rho_0 > \varepsilon\right] &= \prod_{k\geq 1} \left(1 - n_\varepsilon^{-1}\mathrm{N}\left[1 - \mathrm{e}^{-\phi(A_{S_k^{\varepsilon_j}},\omega,\cdot)}\right]\right) \\
&= \mathrm{e}^{-\frac{1}{n_\varepsilon}\sum_{k\geq 1} g_\varepsilon(A_{S_k^\varepsilon},\omega)}.
\end{aligned}$$

Using the sequence $(\varepsilon_j, j \geq 1)$ of Lemma 5.5, we have that $\mathbb{P}$-a.s.

$$\lim_{j\to\infty} \prod_{k\geq 1} \mathrm{N}\left[\mathrm{e}^{-\phi(A_{S_k^{\varepsilon_j}},\omega,\cdot)} \,\Big|\, \rho_0 > \varepsilon_j\right] = \exp\left(-\int_0^{+\infty} du\, \mathrm{N}\left[1 - \mathrm{e}^{-\phi(u,\omega,\cdot)}\right]\right).$$

Use dominated convergence to get

$$\mathbb{E}\left[Z\exp\left(-\sum_{i\in I}\phi(A_{\alpha_i},\omega,\mathcal{S}^i)\right)\right]=\mathbb{E}\left[Z\exp\left(-\int_0^\infty \mathrm{N}\left[1-\mathrm{e}^{-\phi(u,\omega,\cdot)}\right]du\right)\right].$$

To end the proof, it suffices to remark that $\exp\left(-\int_0^\infty \mathrm{N}\left[1-\mathrm{e}^{-\phi(u,\omega,\cdot)}\right]du\right)$ is $\tilde{\mathcal{F}}_\infty$-measurable and so this is $\mathbb{P}$-a.s. equal to the conditional expectation of $\exp\left(-\sum_{i\in I}\phi(A_{\alpha_i},\omega,\mathcal{S}^i)\right)$ w.r.t. $\tilde{\mathcal{F}}_\infty$. That is (23) holds.

5.4 Proof of Proposition 1.7

Using the special Markov property, Theorem 3.12, with $\phi(\mathcal{S})=\psi(\gamma)\sigma$, we have

$$\begin{aligned}
v&=\mathbb{N}\left[1-\mathrm{e}^{-\kappa\tilde{\sigma}-\psi(\gamma)\sigma}\right]\\
&=\mathbb{N}\left[1-\mathrm{e}^{-(\kappa+\psi(\gamma))\tilde{\sigma}-\psi(\gamma)\int_0^\sigma \mathbf{1}_{\{m_s\neq 0\}}\,ds}\right]\\
&=\mathbb{N}\left[1-\mathrm{e}^{-(\kappa+\psi(\gamma))\tilde{\sigma}-\tilde{\sigma}\int_{(0,+\infty)}\pi(d\ell)(1-\mathrm{e}^{-\theta\ell})\mathbb{E}_\ell^*[1-\exp(-\psi(\gamma)\sigma)]}\right].
\end{aligned}$$

Notice that σ under $\mathbb{P}_\ell^*$ is distributed as τ_ℓ, the first time for which the infimum of X, started at 0, reaches $-\ell$. Since τ_ℓ is distributed as a subordinator with Laplace exponent ψ^{-1} at time ℓ, we have

$$\mathbb{E}_\ell^*[1-\mathrm{e}^{-\psi(\gamma)\sigma}]=\mathbb{E}\left[1-\mathrm{e}^{-\psi(\gamma)\tau_\ell}\right]=1-\mathrm{e}^{-\ell\gamma}.$$

and

$$\begin{aligned}
\int_{(0,+\infty)}\pi(d\ell)(1-\mathrm{e}^{-\theta\ell})\mathbb{E}_\ell^*[1-\mathrm{e}^{-\psi(\gamma)\sigma}]&=\int_{(0,+\infty)}\pi(d\ell)(1-\mathrm{e}^{-\theta\ell})(1-\mathrm{e}^{-\gamma\ell})\\
&=\psi^{(\theta)}(\gamma)-\psi(\gamma).
\end{aligned}$$

We get

$$v=\mathbb{N}\left[1-\mathrm{e}^{-\tilde{\sigma}(\kappa+\psi^{(\theta)}(\gamma))}\right]=\psi^{(\theta)^{-1}}(\kappa+\psi^{(\theta)}(\gamma)).$$

Using the definition of $\psi^{(\theta)}$, we have $\psi(v+\theta)=\kappa+\psi(\gamma+\theta)$, which gives the result.

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