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Jean-François Delmas · Klaus Fleischmann

On the hot spots of a catalytic super-Brownian motion

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Abstract. Consider the catalytic super-Brownian motion X^ϱ (reactant) in $\mathbb{R}^d$, $d \leq 3$, which branching rates vary randomly in time and space and in fact are given by an ordinary super-Brownian motion ϱ (catalyst). Our main object of study is the collision local time $L = L_{[\varrho, X^\varrho]}\big(\mathrm{d}(s,x)\big)$ of catalyst and reactant. It determines the covariance measure in the martingale problem for X^ϱ and reflects the occurrence of "hot spots" of reactant which can be seen in simulations of X^ϱ. In dimension 2, the collision local time is absolutely continuous in time, $L\big(\mathrm{d}(s,x)\big) = \mathrm{d}s\, K_s(\mathrm{d}x)$. At fixed time s, the collision measures $K_s(\mathrm{d}x)$ of ϱ_s and X_s^ϱ have carrying Hausdorff dimension 2. Spatial marginal densities of L exist, and, via self-similarity, enter in the long-term random ergodic limit of L (diffusiveness of the 2-dimensional model). We also compare some of our results with the case of super-Brownian motions with deterministic time-independent catalysts.

1. Introduction

The ordinary *super-Brownian motion* $\varrho = (\varrho_t\,,\ t \geq 0)$ in Euclidean space $\mathbb{R}^d$ can be obtained as a limit of branching particles systems. In such branching particles system, the particles evolve according to independent Brownian motions in $\mathbb{R}^d$, and additionally, with constant rate $\gamma > 0$, each particle splits independently into 2 or 0 particles with equal probability (this is a critical binary branching mechanism).

We now interpret ϱ as a *catalyst process*: $\varrho_t(\mathrm{d}x)$ is the amount of catalytic "particles" at time t in the volume element $\mathrm{d}x$ of $\mathbb{R}^d$. We then let a super-Brownian motion $X^\varrho = (X_t^\varrho,\ t \geq 0)$ evolve in this *catalytic random medium* ϱ. Intuitively X^ϱ describes *reactant* "particles" which are evolving according to independent Brownian motions and which are performing critical binary branching, but at

J.-F. Delmas: MSRI, 1000 Centennial Drive, Berkeley, CA 94720, U.S.A. and ENPC-CERMICS, 6 av. Blaise Pascal, Champs-sur-Marne, 77455 Marne La Vallée, France. e-mail: `delmas@cermics.enpc.fr`

K. Fleischmann: Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, D-10117 Berlin, Germany. e-mail: `fleischmann@wias-berlin.de`

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random time-space varying rates given by $\varrho_t(\mathrm{d}x)$. In fact, the rate of branching of an intrinsic reactant particle with Brownian path W is controlled by the *collision local time* $L_{[\varrho,W]}$ of ϱ and W, defined as the measure

$$L_{[\varrho,W]}(\mathrm{d}s) \ := \ \lim_{\varepsilon\downarrow 0} \mathrm{d}s \int \varrho_s(\mathrm{d}y)\, p(\varepsilon, y - W_s),$$

where p is the standard heat kernel

$$p(t,x) \ := \ [2\pi t]^{-d/2} \exp\Big[-|x|^2/2t\Big], \qquad (t,x) \in (0,\infty) \times \mathbb{R}^d. \qquad (1)$$

According to [EP94], this collision local time $L_{[\varrho,W]}$ makes sense non-trivially in dimensions $d \le 3$, and vanishes for $d \ge 4$. In other words, for $d \ge 4$, the Brownian reactant particle does not hit the catalyst ϱ and X^ϱ degenerates to the heat flow. Thus we restrict our attention to $d \le 3$.

Catalytic superprocesses had been studied in various settings, see, for instance, [Fle90], or, for a recent survey, [DF00]. The *catalytic super-Brownian motion* X^ϱ was constructed in [DF97a]. For simplicity, we let ϱ and X^ϱ start at time 0 with Lebesgue measures ℓ_c and ℓ_r, respectively. In [FK99] it was shown that in dimensions $d = 2, 3$, given the catalyst ϱ, the reactant X^ϱ has almost surely a *density field* ξ^ϱ :

$$X_t^\varrho(\mathrm{d}x) \ = \ \xi_t^\varrho(x)\,\mathrm{d}x, \qquad t > 0.$$

Moreover, off the closed time-space support of the catalyst ϱ (which is a Lebesgue zero set), ξ^ϱ can be chosen as a C^∞-function that solves the heat equation. This was intuitively expected from the results on catalytic super-Brownian motion with a point catalyst [FL95] and with higher-dimensional deterministic time-independent catalysts [Del96].

Simulations of (ϱ, X^ϱ) in dimension $d = 2$ (see the figure in [FK99]) confirm the *heuristic picture* one has. Namely, at late times T,

- the reactant X_T^ϱ is rather uniform outside of the catalyst ϱ_T ,
- it is absent inside of the clumps of ϱ_T (since a huge rate of branching causes mainly killing),
- but occasionally also some *hot spots* of the reactant occur in the *interface* of ϱ_T and X_T^ϱ , that is in the boundary region of the catalytic clumps.

But so far the investigations on the catalytic super-Brownian motion X^ϱ do not reflect anything on the hot spots seen in the pictures. Our approach to gain some information about them is to study the *collision local time* $L := L_{[\varrho,X^\varrho]}$ of ϱ and X^ϱ defined as the limit of

$$L^\varepsilon\big(\mathrm{d}(s,x)\big) \ := \ \mathrm{d}s\ \varrho_s(\mathrm{d}x) \int X_s^\varrho(\mathrm{d}y)\, p(\varepsilon, x-y), \qquad (2)$$

as $\varepsilon \downarrow 0$.

Actually there is a further motivation to study this collision local time $L_{[\varrho,X^\varrho]}$. It occurs indeed in the description of the *martingale problem* for the process X^ϱ (see

Corollary 4 below). For martingale problems of catalytic super-Brownian motions, see also [DF94, Del96, Led97]. Moreover, the study of collision local times is a rapidly developing area (see, e.g., [EP98]).

Let us briefly present the results. We prove that in all dimensions of non-trivial existence of X^ϱ the collision local time $L = L_{[\varrho, X^\varrho]}$ of catalyst ϱ and reactant X^ϱ makes non-trivially sense (see Theorem 3 below). This non-trivial existence of L reflects the high fluctuations of X^ϱ in the interface of catalyst and reactant, seen as hot spots in simulations. Of course, in dimension one, $L\big(\mathrm{d}(s,x)\big)$ simplifies to $\mathrm{d}s\ \theta_s(x) X_s^\varrho(\mathrm{d}x)$ where $\big\{\theta_s(x) : s > 0,\ x \in \mathbb{R}\big\}$ is the jointly continuous density field of ϱ (cf. [KS88]).

Our *main result* however is that for $d=2$ and for fixed times $s>0$, the *collision measures*

$$K_s(\mathrm{d}x) := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{[(s-\varepsilon)_+, s]} L\big(\mathrm{d}(s,x)\big)$$

of ϱ_s and X_s^ϱ *exist* and have *carrying Hausdorff dimension* 2. Note that with the approximation of L by L^ε from (2), $K_s(\mathrm{d}x)$ is also the formal limit of the approximated collision measures $\varrho_s(\mathrm{d}x) \int X_s^\varrho(\mathrm{d}y)\ p(\varepsilon, x-y)$ as $\varepsilon \downarrow 0$. Moreover, there is a measurable version of the family $\{K_s : s > 0\}$ of these collision measures such that the *representation* $L_{[\varrho, X^\varrho]}\big(\mathrm{d}(s,x)\big) = \mathrm{d}s\ K_s(\mathrm{d}x)$ holds (Theorem 8). Note that this is in contrast, for instance, to the (one-dimensional) single-point catalytic model of [DF94], say X^{δ_0}, where the collision local time $L_{[\delta_0, X^{\delta_0}]}\big(\mathrm{d}(s,x)\big)$ has the form $\vartheta(\mathrm{d}s)\,\delta_0(\mathrm{d}x)$ with ϑ a singular measure on $\mathbb{R}_+$ with full carrying Hausdorff dimension ([DFLM95, FL95]).

Again in dimension 2, the marginal measures $L_{[\varrho, X^\varrho]}\big([0,T] \times (\cdot)\big)$ are *absolutely continuous.* Via *self-similarity* of $L_{[\varrho, X^\varrho]}$, which follows from the self-similarity of (ϱ, X^ϱ), this implies that $T^{-1} L\big([0,T] \times (\cdot)\big)$ has a *random ergodic limit* as $T \uparrow \infty$ (Theorem 5). This reflects the *diffusive* features in the long-term behavior in $d = 2$ (for the long-term limit of X_T^ϱ, see [FK99]).

It remains *open* whether also in dimension 3 collision measures exist or some absolute continuity results hold, since our L^2–approach fails in this case (see, for instance, Remark 6 below).

We will compare our results on the absolute continuity of the spatial marginal measures $L_{[\varrho, X^\varrho]}\big([0,T] \times (\cdot)\big)$ also with the case of a catalytic super-Brownian motion X^σ in $\mathbb{R}^d$ where the catalyst σ is a deterministic time-independent measure. Clearly, if σ is singular (as we mentioned already the case $\sigma = \delta_0$ in $d = 1$), the spatial marginals of $L = L_{[\sigma, X^\sigma]}$ are almost surely singular, too. But if σ is absolutely continuous, then the spatial marginals of $L_{[\sigma, X^\sigma]}\big([0,T] \times (\cdot)\big)$ are absolutely continuous if and only if $d \leq 3$ (Theorem 9). This is in contrast to our aforementioned result ($d = 2$) where the random catalyst ϱ is singular, but the spatial marginals $L_{[\varrho, X^\varrho]}\big([0,T] \times (\cdot)\big)$ are absolutely continuous.

Concerning the collision local time $L_{[\varrho, \sigma]}$ between a super-Brownian motion ϱ and a measure σ, or the collision local time $L_{[\varrho, \varrho']}$ between ϱ an independent copy ϱ' of ϱ, we refer to the discussion in Remarks 12 and 13 below, respectively.

The *outline* of the paper is as follows. In Section 2 we introduce formal definitions of the processes ϱ and X^ϱ and state the results on existence and properties of the collision local time between ρ and X^ϱ. Subsection 2.6 contains a digression to related models. The following four sections are then devoted to the proofs of our four theorems. In an appendix we collect some results on ordinary and catalytic super-Brownian motions used in the proofs.

2. Statement of results

2.1. Notation

The lower index $+$ on a set will always refer to the collection of all its nonnegative members. Similarly, f_+ is the nonnegative part of f. The supremum norm is denoted by $\|\cdot\|_\infty$. Let c always refer to a (finite) constant which value may vary from place to place. c with an index instead denotes a specific constant.

We denote by $\mathcal{B}(E)$ the space of all real Borel measurable functions defined on a Polish space E. But we also denote by $\mathcal{B}(E)$ the Borel σ–field of E.

For a fixed constant $q > d$, introduce the reference function $\phi_q \in \mathcal{B}_+(\mathbb{R}^d)$:

$$\phi_q(x) := \Big[1 + |x|^2\Big]^{-q/2}, \qquad x \in \mathbb{R}^d.$$

Set $\mathcal{B}^q := \big\{f \in \mathcal{B}(\mathbb{R}^d);\ \|f/\phi_q\|_\infty < \infty\big\}$ and write b$\mathcal{B}$ for the set of bounded f in $\mathcal{B}(\mathbb{R}^d)$.

If ν is a Radon measure on $\mathbb{R}^d$, we write (ν, f) for $\int \nu(\mathrm{d}x)\, f(x)$ (if the integral makes sense). Let $\mathcal{M}_q$ denote the set of all Radon measures ν on $\mathbb{R}^d$ such that $(\nu, \phi_q) < \infty$. This space of tempered measures is endowed with the coarsest topology such that the maps $\nu \mapsto (\nu, f)$ are continuous for all continuous f in b$\mathcal{B}$ with compact support and for $f = \phi_q$, getting a Polish space. Since $q > d$, Lebesgue measure belongs to $\mathcal{M}_q$.

We consider the Polish space $\mathcal{C} := C(\mathbb{R}_+, \mathcal{M}_q)$ of all continuous paths from $\mathbb{R}_+$ to $\mathcal{M}_q$ equipped with the topology of uniform convergence on compacta.

Let $(P_t,\ t \geq 0)$ denote the semigroup of standard heat flow on $\mathbb{R}^d$ [recall (1)]:

$$P_t[f](x) := \int \mathrm{d}y\ p(t, x-y) f(y), \qquad t > 0, \quad f \in \mathcal{B}_+(\mathbb{R}^d).$$

2.2. Catalyst and reactant process

We start by introducing the catalyst process.

Definition 1 (Catalyst process). *Let $\gamma > 0$ and $\nu \in \mathcal{M}_q$. There exists a unique probability measure $\mathbb{P}_\nu$ on $\big(\mathcal{C}, \mathcal{B}(\mathcal{C})\big)$, such that the coordinate process $\varrho = (\varrho_t,\ t \geq 0)$ on $\mathcal{C}$ is a* super-Brownian motion with constant branching rate γ *and starting measure ν. That is, ϱ is a continuous time-homogeneous strong Markov process with the following properties:*

– $\mathbb{P}_\nu$*-almost surely, $\varrho_0 = \nu$,*

– *for every* $f \in \mathcal{B}_+^q$, $t \geq r \geq 0$, *we have*

$$\mathbb{E}_\nu\Big[\mathrm{e}^{-(\varrho_t\,,f)}\;\Big|\;\sigma\big(\varrho_s\,,\ s\in[0,r]\big)\Big] \;=\; \mathrm{e}^{-\left(\varrho_r\,,\ w(t-r)\right)},$$

where w *is the unique nonnegative solution on* $\mathbb{R}_+ \times \mathbb{R}^d$ *of the log-Laplace equation*

$$w(t,x) + \gamma \int_0^t \mathrm{d}s\ P_s[w^2(t-s)](x) \;=\; P_t\,[f]\,(x).$$

We write $\mathbb{P}$ *for* $\mathbb{P}_\nu$ *in the case* $\nu = i_{\mathrm{c}}\ell$, *where* $i_{\mathrm{c}} > 0$ *and* ℓ *is the (normalized) Lebesgue measure on* $\mathbb{R}^d$. □

From now on we assume that $d \leq 3$, and that ϱ is distributed according to $\mathbb{P}$ (see [FK99] for a more general class of starting measures for the catalyst process). Next we recall the definition of the catalytic super-Brownian motion X^ϱ in the random medium ϱ (see [DF97a] for details).

Definition 2 (Catalytic super-Brownian motion). *Fix* $(r,\mu) \in \mathbb{R}_+ \times \mathcal{M}_q$ *and a constant* $\kappa > 0$. *For convenience, set* $\mathcal{C}' := C\big([r,\infty),\mathcal{M}_q\big)$. *There exists a (measurable) probability kernel* $\varrho \mapsto \mathrm{P}_{r,\mu}^\varrho$ *from* $\big(\mathcal{C},\mathcal{B}(\mathcal{C})\big)$ *to* $\big(\mathcal{C}',\mathcal{B}(\mathcal{C}')\big)$ *such that the coordinate process* $X^\varrho = (X_t^\varrho,\ t \geq r)$ *on* $\mathcal{C}'$ *is, under* $\mathrm{P}_{r,\mu}^\varrho$, *a* super-Brownian motion in the catalytic medium ϱ. *That is,* $\mathbb{P}$*–a.s. under* $\mathrm{P}_{r,\mu}^\varrho$, *the process* X^ϱ *is continuous time-inhomogeneous Markov with the following properties:*

– $\mathrm{P}_{r,\mu}^\varrho$*-almost surely,* $X_r^\varrho = \mu$,
– *for every* $f \in \mathcal{B}_+^q$, $t \geq s \geq r$, *we have*

$$\mathrm{E}_{r,\mu}^\varrho\Big[\mathrm{e}^{-(X_t^\varrho,f)}\;\Big|\;\sigma\big(X_u^\varrho\,,\,u\in[r,s]\big)\Big] \;=\; \mathrm{e}^{-\left(X_s^\varrho\,,\ v_t(s)\right)},$$

where v_t *is the unique nonnegative solution on* $[r,\infty) \times \mathbb{R}^d$ *of the catalytic log-Laplace equation*

$$v(s,x) + \kappa \int_s^\infty \mathrm{d}u \int \varrho_u(\mathrm{d}y)\ p(u-s,x-y)\,v^2(u,y) \;=\; J(s,x), \tag{3}$$

with $J(s) := 1_{\{t\geq s\}}\,P_{t-s}[f]$.

Often, we also pass from the quenched distributions $\mathrm{P}_{r,\mu}^\varrho$ *to the annealed laws* $\mathbb{E}\big[\mathrm{P}_{r,\mu}^\varrho\big]$. □

2.3. *Existence of collision local time of catalyst and reactant*

For our constant $q > d$, we introduce the function space $H^q := \bigcup_{T\geq 0} H_T^q$, where

$$H_T^q \;:=\; \Big\{g \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d);\ g(t,\cdot) = 0\ \forall t > T,\ \|g/\phi_q\|_\infty < \infty\Big\},$$

with $\|g/\phi_q\|_\infty = \sup_{(s,x)\in\mathbb{R}_+\times\mathbb{R}^d} |g(s,x)|/\phi_q(x)$.

Recall the approximated collision local time L^ε of ϱ and X^ϱ introduced in (2). We are now ready to state our first result: the existence of the collision local time $L = L_{[\varrho,X^\varrho]}$ of ϱ and X^ϱ in dimension $d \leq 3$. Recall that $d \leq 3$ and $(r,\mu) \in \mathbb{R}_+ \times \mathcal{M}_q$.

Theorem 3 (Collision local time). *There exists a random variable denoted by $L = L_{[\varrho, X^\varrho]}$ defined on $\big(\mathcal{C}\times\mathcal{C}', \mathcal{B}(\mathcal{C}\times\mathcal{C}')\big)$, taking values in the set of Radon measures on $[r,\infty)\times\mathbb{R}^d$ with the following properties:*

(a) (Tempered measure) *For every $T \geq r$, we have $\mathbb{E}\big[\mathrm{P}^{\varrho}_{r,\mu}(L, 1_{[r,T]}\,\phi_q)\big] < \infty$.*

(b) (Existence via convergence) *For every $\varphi \in H^{2q}$,*

$$\lim_{\varepsilon\downarrow 0}(L^\varepsilon,\varphi) = (L,\varphi), \quad \mathbb{E}[\mathrm{P}^{\varrho}_{r,\mu}]\text{–a.s.}$$

(c) (Regularity) *For every $\varphi \in H^{2q}$, and $\mathbb{E}\left[\mathrm{P}^{\varrho}_{r,\mu}\right]$–a.s., the process $((L, 1_{[r,t]}\varphi)$, $t \geq r$ is continuous and adapted to the filtration*

$$\Big(\mathcal{F}_t := \sigma(\varrho)\vee\sigma\left(X^{\varrho}_s,\ s\in[r,t]\right),\ t\geq r\Big).$$

(d) (Moments) *For every $m \geq 1$, $\varphi \in H^{2q}$, $\mathbb{P}$–a.s.,*

$$\begin{aligned}&\mathrm{E}^{\varrho}_{r,\mu}\left[\left[\int_{[r,\infty)\times\mathbb{R}^d} L\big(\mathrm{d}(s,x)\big)\,\varphi(s,x)\right]^m\right]\\ &\quad = m!\sum_{k=1}^{m}\frac{1}{k!}\sum_{\substack{n_1,\dots,n_k\geq 1,\\ n_1+\dots+n_k=m}}\prod_{i=1}^{k}\big(\mu,\chi_{n_i}(r)\big),\end{aligned} \tag{4}$$

where the functions χ_n, $n\geq 1$, belong to H^q and are recursively defined by

$$\begin{aligned}\chi_n(s,x) := \kappa\int_s^\infty \mathrm{d}u\int\varrho_u(\mathrm{d}y)\,p(u-s,x-y)&\\ \left[\sum_{i=1}^{n-1}\chi_i(u,y)\,\chi_{n-i}(u,y)\right]&, \qquad n\geq 2,\end{aligned} \tag{5}$$

with initial condition

$$\chi_1(s,x) := \int_s^\infty \mathrm{d}u\int\varrho_u(\mathrm{d}y)\ p(u-s,x-y)\,\varphi(u,y), \quad (s,x)\in\mathbb{R}_+\times\mathbb{R}^d.$$

The proof of this theorem is postponed to Section 3.

As an *application*, we can now describe the covariance measure of the martingale measure associated with X^ϱ. Let $C_b^{1,2}$ denote the set of bounded functions $\varphi\in\mathcal{B}(\mathbb{R}_+\times\mathbb{R}^d)$ such that the partial derivatives $\frac{\partial\varphi}{\partial s}$ and $\frac{\partial^2\varphi}{\partial x_i\partial x_j}$ exist, are continuous and bounded. It is easy to check that under $\mathbb{E}\left[\mathrm{P}^{\varrho}_{r,\mu}\right]$ the process $(M\varphi_{r,t},\ t\geq r)$ defined by

$$M\varphi_{r,t} := \big(X^\varrho_t,\varphi(t)\big) - \big(X^\varrho_r,\varphi(r)\big) - \int_r^t \mathrm{d}s\,\left(X^\varrho_s,\ \frac{\partial\varphi}{\partial s}(s) + \frac{1}{2}\,\Delta\varphi(s)\right),$$

is an $(\mathcal{F}_t\,,\ t \geq r)$-*martingale* [note that $\mathcal{F}_r = \sigma(\varrho) \vee \sigma(X_r)$]. Thanks to the Markov property of X^ϱ (given ϱ), and the moment formula (A.5) for X^ϱ stated in the appendix, we get that for φ, ψ in $C_b^{1,2}$, $\mathbb{P}$–a.s. for all $s \geq r$ and $t \geq r$,

$$\mathrm{E}^{\varrho}_{r,\mu}\left[M\varphi_{r,s} M\psi_{r,t}\right] \;=\; 2\kappa \int \mu(\mathrm{d}x) \int_r^{s\wedge t} \mathrm{d}u \int \varrho_u(\mathrm{d}y) \times\; p(u-r, x-y)\,\varphi(u,y)\,\psi(u,y). \tag{6}$$

The functional $M : \varphi \mapsto M\varphi$ defined on $C_b^{1,2}$ can be extended to an *orthogonal martingale measure* on H^q. Let $\langle M \rangle$ denote its *covariance measure*. Now we show how $\langle M \rangle$ can be expressed in terms of the collision local time $L = L_{[\varrho, X^\varrho]}$. Recall that $d \leq 3$ and that $(r,\mu) \in \mathbb{R}_+ \times \mathcal{M}_q$.

Corollary 4 (Covariance measure). *For every $\varphi \in H^q$, $\mathbb{E}\left[\mathrm{P}^{\varrho}_{r,\mu}\right]$–a.s. for every $t \geq r$, we have*

$$\langle M\varphi \rangle_{r,t} \;=\; 2\kappa \int_{[r,t]\times\mathbb{R}^d} L\big(\mathrm{d}(s,y)\big)\,\varphi^2(s,y). \tag{7}$$

Proof. Using the Markov property of X^ϱ (given ϱ) and an obvious extension of the second moment formula (6), we obtain for $\varphi \in H^q$, $\mathbb{P}$–a.s. for all $t \geq s \geq r$,

$$\mathbb{E}\mathrm{E}^{\varrho}_{r,\mu}\left[(M\varphi_{r,t})^2 \;\middle|\; \mathcal{F}_s\right] = (M\varphi_{r,s})^2 + 2\kappa \int X^{\varrho}_s(\mathrm{d}x) \int_s^t \mathrm{d}u \times \int \varrho_u(\mathrm{d}y)\; p(u-s, x-y)\,\varphi^2(u,y).$$

Notice that

$$\left(\int_{[r,t]\times\mathbb{R}^d} L\big(\mathrm{d}(s,y)\big)\,\varphi^2(s,y),\ t \geq r\right)$$

is in t non-decreasing and continuous, is adapted to $(\mathcal{F}_t\,,\ t \geq r)$, and zero for $t = r$. Then we deduce from the moment formula (4) with $m = 1$, that

$$\left(\langle M\varphi \rangle_{r,t} - 2\kappa \int_{[r,t]\times\mathbb{R}^d} L\big(\mathrm{d}(s,y)\big)\,\varphi^2(s,y),\ t \geq r\right)$$

is a continuous martingale under $\mathbb{E}\left[\mathrm{P}^{\varrho}_{r,\mu}\right]$ with bounded variation starting at time $t = r$ from 0. This martingale is then constant and, in fact, equal to 0, giving the claim (7). □

2.4. Collision local time in dimension two

We now state results for the collision local time $L = L_{[\varrho, X^\varrho]}$ in the "critical" dimension $d = 2$. For simplicity, we focus on the situation $r = 0$ and $\mu = i_{\mathrm{r}}\ell$ where $i_{\mathrm{r}} > 0$. For convenience, we introduce the following abbreviation for the annealed law:

$$\mathbf{P} \;:=\; \mathbb{E}\left[\mathrm{P}_{0,i_{\mathrm{r}}\ell}\right] \;=\; \mathbb{E}_{i_{\mathrm{c}}\ell}\left[\mathrm{P}_{0,i_{\mathrm{r}}\ell}\right] \qquad (\text{where } i_{\mathrm{c}}\,, i_{\mathrm{r}} > 0).$$

Theorem 5 (Two-dimensional collision local time). *Let* $d = 2$.

(a) (Spatial L^2–marginal densities) *For every* $t \geq s \geq 0$ *and* $z \in \mathbb{R}^2$,

$$\left(\int_{[s,t]\times\mathbb{R}^d} L\big(\mathrm{d}(r, y)\big)\, p(\varepsilon, z - y),\ \varepsilon > 0 \right)$$

converges in $L^2(\mathbf{P})$ *as* $\varepsilon \downarrow 0$ *to a random variable denoted by* $\lambda_{[s,t]}(z)$. *It has expectation*

$$\mathbf{E}\big[\lambda_{[s,t]}(z)\big] \;=\; i_{\mathrm{c}} i_{\mathrm{r}}\,(t - s),$$

and its finite variance is non-zero provided that $s < t$.

(b) (Spatial absolute continuity) *For* $t \geq s \geq 0$, *we have the representation*

$$L\big([s, t] \times \mathrm{d}x\big) \;=\; \lambda_{[s,t]}(x)\,\mathrm{d}x, \quad \mathbf{P}\text{–a.s.},$$

where we take a measurable version, with respect to the σ*–field* $\mathcal{B}(\mathbb{R}^2) \times \mathcal{F}_t$, *of the family* $\big\{\lambda_{[s,t]}(z) :\ z \in \mathbb{R}^2\big\}$ *defined in* (a).

(c) (Self-similarity) *Under* $\mathbf{P}$, *the laws of the scaled collision local times*

$$k^{-2} L\big(k(\cdot) \times k^{1/2}(\cdot)\big)$$

are independent of the scaling factor $k > 0$.

(d) (Random ergodic limit) *The following convergence in* $\mathcal{M}_q$ *holds in law with respect to* $\mathbf{P}$:

$$\lim_{T\uparrow\infty} T^{-1} L\big([0, T] \times (\cdot)\big) \;=\; \lambda_{[0,1]}(0)\,\ell$$

(*with* ℓ *the Lebesgue measure and* $0 < \mathbf{V}\mathrm{ar}\big[\lambda_{[0,1]}(0)\big] < \infty$).

Consequently, in dimension 2, the spatial marginal measures $L\big([s, t] \times (\cdot)\big)$ of the collision local time $L_{[\varrho, X^\varrho]}$ of catalyst and reactant have non-degenerated random densities $\lambda_{[s,t]}(z)$ at each fixed site z (provided that $s < t$). Moreover, $\lambda_{[0,1]}(0)$ enters as random factor of Lebesgue measure in the long-term ergodic limit. Recall that this reflects diffusive features of the hot spots.

Remark 6 (**Dimension three).** The $L^2(\mathbf{P})$–convergence in part (a) does *not* hold for $d = 3$. In fact, in the three-dimensional case an infinite term would be involved in our calculations, see the remark following (19) in the proof of Lemma 15 below. Recall on the other hand that in dimension one, $L_{[\varrho,X^\varrho]}(\mathrm{d}(s, x)) = \mathrm{d}s\ \theta_s(x) X^\varrho_s(\mathrm{d}x)$, where $(s, x) \mapsto \theta_s(x)$ is the jointly continuous density field of ϱ (see [DFR91] for the absolute continuity of the measures X^ϱ_s for fixed $s > 0$).

Remark 7 (**Regularity).** It is an open problem whether the spatial collision density functions $z \mapsto \lambda_{[s,t]}(z)$ have some regularities properties in the space variable z. Note also that the exceptional set in the $\mathbf{P}$–a.s. statement in (b) depends on $[s, t]$. One would also like to know whether this situation can be improved.

The statement (c) follows from the self-similarity of (ϱ, X^ϱ) by standard arguments (compare with [DF97b, Subsections 4.1 and 4.2]). Otherwise the proof of Theorem 5 will be provided in Section 4.

2.5. *Existence of collision measures in dimension two*

The assumptions imposed in the beginning of Subsection 2.4 are still in force. Using an L^2–approach, we prove the existence of collision measures in dimension $d = 2$. For this purpose, fix a function $f \in L^1_+(\mathbb{R})$ such that $f = 0$ outside a compact subset of $\mathbb{R}$ and $\int \mathrm{d}u \; f(u) = 1$. For $t, \varepsilon > 0$, set

$$f_{\varepsilon,t}(s) \;:=\; \frac{1}{\varepsilon}\, f\Big(\varepsilon^{-1}(s-t)\Big), \qquad s \in \mathbb{R}. \tag{8}$$

Note that the finite measures $\mathbf{1}_{\mathbb{R}_+} f_{\varepsilon,t}(s)\mathrm{d}s$ on $\mathbb{R}$ converge weakly to the Dirac measure at t as ε decreases to 0. We also define measures K^ε_t in $\mathcal{M}_q$ by

$$(K^\varepsilon_t, \varphi) \;:=\; \int_{\mathbb{R}_+\times\mathbb{R}^2} L\big(\mathrm{d}(s,y)\big)\, \varphi(y)\, f_{\varepsilon,t}(s), \qquad \varphi \in \mathcal{B}^q_+ . \tag{9}$$

Theorem 8 (Two-dimensional collision measures). *Let* $d = 2$.

(a) (Existence of collision measures): *For each* $t > 0$ *there is a random measure* K_t *in* $\mathcal{M}_q$ *such that for any* $\varphi \in \mathcal{B}^q_+$*, the following* $L^2(\mathbf{P})$*–convergence holds:*

$$(K^\varepsilon_t, \varphi) \underset{\varepsilon\downarrow 0}{\longrightarrow} (K_t, \varphi).$$

(b) (Carrying Hausdorff dimension): *For each* $t > 0$ *fixed,* K_t *has carrying Hausdorff dimension two,* $\mathbf{P}$*–a.s.*

(c) (Representation of collision local time): *To the family* $K = \{K_t : t > 0\}$ *of random measures of* (a)*, there is an* $(\mathcal{F}_t, t > 0)$*–adapted version denoted by the same symbol* K*, such that*

$$L\big(\mathrm{d}(s,y)\big) \;=\; \mathrm{d}s\, K_s(\mathrm{d}y), \quad \mathbf{P}\text{–a.s.}$$

Note that the closed support of ϱ_s is a supporting set of K_s. Therefore K_s is supported by a Lebesgue null set, although its carrying Hausdorff dimension is 2.

The proof of Theorem 8 is given in Section 5 below. As in Remark 6, the $L^2(\mathbf{P})$–approach to prove part (a) fails for $d = 3$ (see Remark 21).

2.6. *Digression*

So far we restricted our attention to the model of a super-Brownian reactant X^ϱ with a super-Brownian catalyst ϱ. What about collision local time questions for *related catalytic models?*

At the first place we think of a catalyst described by a time-independent deterministic measure $\sigma(\mathrm{d}x)$ on $\mathbb{R}^d$. Intuitively, the corresponding catalytic SBM $X^\sigma = (X^\sigma_t, t \geq 0)$ in $\mathbb{R}^d$ describes a cloud of particles which are evolving according to independent Brownian motions and which are performing a critical binary branching whose rate is $\sigma(\mathrm{d}x)$ at site x. We refer to [Del96] for the construction and properties of such process, and keep the same framework. In particular, we assume there exists a $\beta \in (0, 1)$, such that

$$\sup_{x\in\mathbb{R}^d} \int_{B(x,1)} \frac{\sigma(\mathrm{d}y)}{|x-y|^{d-2+2\beta}} < \infty,$$

where $B(x, 1)$ denotes the ball in $\mathbb{R}^d$ centered at x, with radius 1. This condition on σ is rather general. In particular if $d = 1$, all finite measures σ satisfy this condition (with $\beta = 1/2$ for example), as well as some locally infinite measures $\sigma(\mathrm{d}x)$ as $|x|^{-\alpha}\,\mathrm{d}x$, with $0 < \alpha < 2$. Furthermore in all dimensions the Lebesgue measure satisfies this condition with $\beta \in (0, 1)$.

Let $\mathcal{M}_\mathrm{f}$ denote the space of finite Radon measures on $\mathbb{R}^d$ endowed with the topology of weak convergence. We write P^σ_η for the law of the catalytic super-Brownian motion X^σ started from $X^\sigma_0 = \eta \in \mathcal{M}_\mathrm{f}$. Recall that X^σ is a continuous $\mathcal{M}_\mathrm{f}$–valued process. As we stick to the presentation of [Del96], we will keep the $\mathcal{M}_\mathrm{f}$–version of X^σ instead of working with an $\mathcal{M}_q$–version. The existence of collision local time $L = L_{[\sigma,X^\sigma]}$ of the catalyst σ and the reactant X^σ was proved in the sense of an L^u–limit ($u > 0$), as ε decreases to 0, of

$$L^\varepsilon\big(\mathrm{d}(r,y)\big) = \mathrm{d}r\,\sigma(\mathrm{d}y) \int X^\sigma_r(\mathrm{d}x)\ p(\varepsilon, x-y).$$

Recall that the collision local time also describes the covariance measure of the martingale measure associated to X^σ (see Section 9 in [Del96]).

The moment formula for L can be deduced from the moment formula for L^ε (see Lemma 5.2 and equation (32) in [Del96], with $\varrho = \sigma$ and $V_\varrho(\varepsilon) = L^\varepsilon$) as ε decreases to 0. In particular, for $\varphi \in \mathrm{b}\mathcal{B}_+$ we have

$$\begin{aligned}\mathrm{E}^\sigma_\eta\big[(L, \varphi 1_{[s,t]})\big] &= \mathrm{E}^\sigma_\eta\Big[\int L\big([s,t]\times \mathrm{d}y\big)\,\varphi(y)\Big]\\ &= \int \eta(\mathrm{d}x) \int_s^t \mathrm{d}r \int \sigma(\mathrm{d}y)\, p(r, x-y)\varphi(y).\end{aligned}$$

If $L\big([s,t]\times \mathrm{d}y\big)$ is a.s. absolutely continuous, then the latter first moment formula implies that $\sigma(\mathrm{d}y)\int \eta(\mathrm{d}x)\int_s^t \mathrm{d}r\ \ p(r, x-y)$ is also absolutely continuous. There is no choice but to consider diffuse catalysts σ. This differs from the previous section where the random and time-dependent catalyst ϱ is singular for $d = 2$, nevertheless the spatial absolute continuity property holds for the collision local time. If $\sigma(\mathrm{d}y) = g(y)\,\mathrm{d}y$, it is easy to check that the collision local time is in fact

$$L\big(\mathrm{d}(r,y)\big) = \mathrm{d}r\ g(y) X^\sigma_r(\mathrm{d}y). \tag{10}$$

Therefore the absolute continuity of the spatial marginal measures is directly implied by the absolute continuity of the weighted occupation time measures $\int_s^t \mathrm{d}r\ X^\sigma_r$. The main result of this subsection is:

Theorem 9 (Weighted occupation time measures). *Assume $d \le 3$ and that the catalyst σ is absolutely continuous. Let $t > s > 0$ and $\eta \in \mathcal{M}_\mathrm{f}$.*

(a) (L^2–occupation densities): *For almost all* $z \in \mathbb{R}^d$, *as* ε *decreases to* 0,

$$\left(\int_s^t \mathrm{d}r \int X_r^\sigma(\mathrm{d}y)\, p(\varepsilon, z-y),\ \varepsilon > 0\right)$$

converges in $L^2(\mathrm{P}_\eta^\sigma)$ *to a random variable* $\lambda_{[s,t]}(z)$ *with expectation*

$$\mathrm{E}_\eta^\sigma\, \lambda_{[s,t]}(z) \;=\; \int \eta(\mathrm{d}x) \int_s^t \mathrm{d}r\ p(r, x-z),$$

and non-zero finite variance provided that $\eta \neq 0$ *and* $\sigma \neq 0$.

(b) (Absolute continuity): *There exists a measurable version of* $z \mapsto \lambda_{[s,t]}(z)$ *such that* P_η^σ*–a.s. we have*

$$\int_s^t \mathrm{d}r\ X_r^\sigma(\mathrm{d}z) \;=\; \lambda_{[s,t]}(z)\,\mathrm{d}z.$$

The proof of this theorem is given in Section 6. As a direct consequence of (10) and this theorem, we have

Corollary 10 (Spatial marginals of the collision local time). *Suppose* $d \leq 3$ *and that* σ *is absolutely continuous. Let* $t > s > 0$ *and* $\eta \in \mathcal{M}_\mathrm{f}$. *The random measure* $L\big([s,t] \times (\cdot)\big)$ *on* $\mathbb{R}^d$ *is absolutely continuous* P_η^σ*–a.s.*

*Remark 11 (***Singularity of spatial marginals in high dimensions).** Note that if $\sigma = \gamma\ell$ with the constant $\gamma > 0$ and ℓ the Lebesgue measure on $\mathbb{R}^d$, then X^σ is the super-Brownian motion ϱ of Definition 1, and by (10) we have $L\big(\mathrm{d}(r,y)\big) = \gamma\, \mathrm{d}r\, \varrho_r(\mathrm{d}y)$. It is well-known that the weighted occupation time measures $\int_s^t \mathrm{d}r\ \varrho_r$ are singular if $d \geq 4$. This suggests that for $d \geq 4$ and general catalyst σ, the measures $L_{[\sigma, X^\sigma]}([s,t] \times (\cdot))$ on $\mathbb{R}^d$ are singular, too.

We end our discussion by some remarks related to non-catalytic models. ◇

*Remark 12 (***Collision local time between** ϱ **and a measure** $\boldsymbol{\mu}$**).** The absolute continuity of the spatial marginal measures of the collision local time $L_{[\varrho,\mu]}$ of a super-Brownian motion ϱ in $\mathbb{R}^d$ and a deterministic measure μ on $\mathbb{R}^d$ (which does not act as a catalyst) holds if and only if $d \leq 3$. In fact, $L_{[\varrho,\mu]}$ is the measure Γ_μ in [Del96, Section 5] in the case of the catalytic measure ℓ. Then computing the first moment of $L_{[\varrho,\mu]}([s,t] \times (\cdot))$, one checks as in the proof of Theorem 9 that the spatial absolute continuity of $L_{[\varrho,\mu]}$ holds a.s. if and only if μ is absolutely continuous. If $\mu(\mathrm{d}y) = h(y)\,\mathrm{d}y$, then we have $L_{[\varrho,\mu]}\big(\mathrm{d}(r,y)\big) = \mathrm{d}r\, \varrho_r(\mathrm{d}y)\, h(y)$ a.s. Therefore the spatial absolute continuity property is true for $L_{[\varrho,\mu]}$ if it holds for $\int_s^t \mathrm{d}r\ \varrho_r$, that is if $d \leq 3$. ◇

*Remark 13 (***Collision between independent super-Brownian motions).** The collis-ion local time $L_{[\varrho,\varrho']}$ between two *independent* super-Brownian motions ϱ and ϱ' in $\mathbb{R}^d$ exists for $d \leq 5$ (see [BEP91]). The classical L^2–method can be used to prove that these collision local times enjoy the spatial absolute continuity property if $d \leq 2$, but it fails for $d \geq 3$. We refer to [Myt98] for existence of collision measures between independent super-Brownian motions and more general independent superprocesses. ◇

3. Existence of collision local time [proof of Theorem 3]

Recall that $d \le 3$. First of all we state the following lemma.

Lemma 14 (Approximated moment increments). *For every $m \ge 1$, $r \ge 0$, $\mu \in \mathcal{M}_q$, $T \ge 0$, $\xi \in (0, 1/4)$, $\mathbb{P}$–a.s. there exists a finite constant M_m (depending on ϱ) such that for every $\varphi \in H_T^{2q}$, $t' \ge t \ge 0$, $1 \ge \varepsilon' \ge \varepsilon > 0$,*

$$\mathrm{E}^{\varrho}_{r,\mu}\left[(L^{\varepsilon}, \varphi\, 1_{[t,t']})^{2m}\right] \le M_m \left\|\varphi/\phi_{2q}\right\|_{\infty}^{2m} \left[\left|t-t'\right|^{\xi}\left(1+\log_+\left(1/|t-t'|\right)\right)\right]^{2m}, \tag{11}$$

$$\mathrm{E}^{\varrho}_{r,\mu}\left[\left[(L^{\varepsilon}, \varphi) - (L^{\varepsilon'}, \varphi)\right]^{2m}\right] \le M_m \left\|\varphi/\phi_{2q}\right\|_{\infty}^{2m} \left[\left|\varepsilon-\varepsilon'\right|^{\xi}\left(1+\log_+\left(1/|\varepsilon-\varepsilon'|\right)\right)\right]^{2m}. \tag{12}$$

Based on this lemma, the proofs of Theorem 3 (b) and (c) are similar to the proof of Proposition 5.1 based on Lemma 5.2 in [Del96] with the obvious changes and are left to the reader. Claim (d) is not stated in Proposition 5.1, but it is a by-product of its proof [take the limit in (32) there]. Eventually, part (a) of Theorem 3 is proved by using the monotone convergence theorem with the moment formulas (4) and (A.2) (in the appendix) with $m = 1$ and the inequality (A.1).

Proof of Lemma 14. Fix $\mu \in \mathcal{M}_q$, $\xi \in (0, 1/4)$, and $T \ge r \ge 0$ (otherwise the moments disappear). We will verify (11); the proof of (12) is similar and is left to the reader.

Note first that for fixed $\varepsilon > 0$,

$$\sup_{x\in\mathbb{R}^d,\ y\in\mathbb{R}^d} \frac{\phi_q(y)\, p(\varepsilon, x-y)}{\phi_q(x)} < \infty.$$

Let $\varphi \in H_T^{2q}$. Since ϱ is $\mathbb{P}$–a.s. a continuous $\mathcal{M}_q$–valued path, it is then clear that the functions $(s, x) \mapsto \int \varrho_s(\mathrm{d}y)\, p(\varepsilon, x-y)\, \varphi(s, y)$ belong to H_T^{q}. Thanks to the remarks at the beginning of Subsection A.1 (in the appendix), we see that, for fixed t, t', ε, the function

$$(s,x) \mapsto J_{\varepsilon}(s,x) := \int_s^{\infty} \mathrm{d}u \int \mathrm{d}z\ p(u-s, x-z) \int \varrho_u(\mathrm{d}y)\, p(\varepsilon, z-y)\, \varphi(u,y)\, 1_{[t,t']}(u)$$

is well-defined and belongs to H_T^{q}.

We will now prove that $\mathbb{P}$–a.s. there exists a finite constant c such that for every $\varphi \in H_T^{2q}$, $t' \ge t \ge 0$, $1 \ge \varepsilon > 0$,

$$\left|J_{\varepsilon}(s,x)\right| \le c\, 1_{[0,T]}(s)\, \phi_q(x) \left\|\varphi/\phi_{2q}\right\|_{\infty} \left[\left|t-t'\right|^{\xi}\left(1+\log_+\left(1/|t-t'|\right)\right)\right]. \tag{13}$$

Clearly $\left|J_\varepsilon(s,x)\right| / \left\|\varphi/\phi_{2q}\right\|_\infty$ is bounded from above by

$$\mathsf{K}_1 \;:=\; 1_{[0,T]}(s)\int_s^T \mathrm{d}u \int \varrho_u(\mathrm{d}y)\, p(u-s+\varepsilon, x-y)\,\phi_{2q}(y)\,1_{[t,t']}(u).$$

We assume that $T \ge t$ (otherwise $\mathsf{K}_1 = 0$). Introduce the quantity

$$\mathsf{K}_2 \;:=\; 1_{[0,T\wedge t']}(s)\int_{s\vee t}^{T\wedge t'} \mathrm{d}u \int \varrho_u(\mathrm{d}y)\, p(u-s\vee t, x-y)\,\phi_{2q}(y).$$

Thanks to (A.4), we have $\mathsf{K}_2 \le 1_{[0,T]}(s)\, c_2 \left|t-t'\right|^\xi \phi_q(x)$. Now

$$\begin{aligned}\left|\mathsf{K}_1 - \mathsf{K}_2\right| \;\le\; & 1_{[0,T\wedge t']}(s)\int_{s\vee t}^{T\wedge t'} \mathrm{d}u \int \varrho_u(\mathrm{d}y)\\ & \left|p(u-s+\varepsilon, x-y) - p(u-s\vee t, x-y)\right| \phi_{2q}(y).\end{aligned}$$

Using the inequality

$$\left|p(v_1,z) - p(v_2,z)\right| \le c\int_{v_1}^{v_2} \mathrm{d}v\; v^{-1}\, p(2v,z),$$

where the constant c is independent of $z \in \mathbb{R}^d$ and $v_2 \ge v_1 > 0$, we get that

$$\begin{aligned}\left|\mathsf{K}_1 - \mathsf{K}_2\right| &\le c 1_{[0,T\wedge t']}(s)\int_{s\vee t}^{T\wedge t'} \mathrm{d}u \int \varrho_u(\mathrm{d}y)\phi_{2q}(y)\int_{u-s\vee t}^{u-s+\varepsilon} \mathrm{d}v\; v^{-1}\, p(2v, x-y)\\ &= c 1_{[0,T\wedge t']}(s)\int_0^{T\wedge t'-s+\varepsilon} \mathrm{d}v v^{-1}\int_{s\vee t\vee(v+s-\varepsilon)}^{T\wedge t'\wedge(v+s\vee t)} \mathrm{d}u \int \varrho_u(\mathrm{d}y)\phi_q^2(y) p(2v, x-y).\end{aligned}$$

In view of (A.3) and (A.1), we may continue with

$$\begin{aligned}\le\; & c\, 1_{[0,T\wedge t']}(s)\,\phi_q(x)\int_0^{T\wedge t'-s+\varepsilon} \mathrm{d}v\; v^{-1}\\ & \left|T\wedge t'\wedge(v+s\vee t) - s\vee t\vee(v+s-\varepsilon)\right|^\xi,\end{aligned}$$

where c is independent of t', t, ε, x. It is easy to check that

$$\begin{aligned}&\int_0^{T\wedge t'-s+\varepsilon} \mathrm{d}v\; v^{-1}\left|T\wedge t'\wedge(v+s\vee t) - s\vee t\vee(v+s-\varepsilon)\right|^\xi\\ &\quad\le q\; c\left|t'-t\right|^\xi\left(1+\log_+\left(1/|t'-t|\right)\right),\end{aligned}$$

where c is independent of t', t and ε. As a conclusion we obtain (13).

Using the estimate (A.4), a straight forward induction shows that all the functions χ_n, $n \ge 1$, of the recurrence relation (5) with initial condition $\chi_1 = J_\varepsilon$ belong to H_T^q and satisfy

$$\left|\chi_n(s,x)\right| \;\le\; c\, 1_{[0,T]}(s)\,\phi_q(x)\left\|\varphi/\phi_{2q}\right\|_\infty^n\left[\left|t-t'\right|^\xi\left(1+\log_+\left(1/|t-t'|\right)\right)\right]^n.$$

(Note that c is independent of φ, t, t' and ε.) Then the claim (11) is a consequence of (A.5) with $f = 0$ and

$$g(s,z) \;:=\; \int \varrho_s(\mathrm{d}y)\; p(\varepsilon, z-y)\; \varphi(s,y)\; 1_{[t,t']}(s),$$

finishing the proof of the lemma. □

4. Two-dimensional collision local time [proof of Theorem 5]

We now assume that $d = 2$.

4.1. Spatial marginal densities [proof of Theorem 5 (a)]

For the claimed L^2–convergence, it is enough to check that, for fixed s, t, z,

$$J^{\varepsilon,\varepsilon'} \;:=\; \mathbf{E}\Big[\int_{[s,t]\times\mathbb{R}^2} L\big(\mathrm{d}(r,y)\big)\; p(\varepsilon, z-y) \int_{[s,t]\times\mathbb{R}^2} L\big(\mathrm{d}(r',y')\big)\; p(\varepsilon', z-y')\Big] \tag{14}$$

converges in $\mathbb{R}_+$ as ε and ε' decrease to 0.

For $f \in L^1_+(\mathbb{R}^2)$ with $\int \mathrm{d}x\; f(x) = 1$, and $\varepsilon > 0$, $z \in \mathbb{R}^2$, we set

$$f_{\varepsilon,z}(x) := \varepsilon^{-1} f\Big(\varepsilon^{-1/2}(x-z)\Big).$$

Note that $f_{\varepsilon,z}(x)\,\mathrm{d}x$ converges weakly to $\delta_z(\mathrm{d}x)$, the Dirac mass at z, as ε decreases to 0. We will prove the following stronger result.

Lemma 15 (Convergence of $J^{\varepsilon,\varepsilon'}$). *For fixed $t \ge s > 0$, and $z, z' \in \mathbb{R}^2$, and f, f' in $L^1_+(\mathbb{R}^2)$ such that $\int \mathrm{d}x\; f(x) = 1 = \int \mathrm{d}x\; f'(x)$, the finite quantities*

$$J^{\varepsilon,\varepsilon'}(z,z') \;:=\; \mathbf{E}\Big[\int_{[s,t]\times\mathbb{R}^2} L\big(\mathrm{d}(r,y)\big)\; f_{\varepsilon,z}(y) \int_{[s,t]\times\mathbb{R}^2} L\big(\mathrm{d}(r',y')\big)\; f'_{\varepsilon',z'}(y')\Big], \quad \varepsilon, \varepsilon' > 0,$$

converge to a finite limit independent of f, f', as ε and ε' decrease to 0.

Note that we need the convergence for $z = z'$ to prove (14) and then (a). Note also that although f and f' are not in $\mathcal{B}^q$ a priori, we show that the $J^{\varepsilon,\varepsilon'}$ are finite.

Proof of Lemma 15. By a standard monotone class argument, we deduce from the quenched moment formula (4) for the collision local time with $m = 2$, that for $g \in \mathcal{B}_+\big((\mathbb{R}_+)^2 \times (\mathbb{R}^2)^2\big)$,

$$
\begin{aligned}
&\mathbf{E}\Big[\int_{\mathbb{R}_+\times\mathbb{R}^2} L\big(\mathrm{d}(r,y)\big)\int_{\mathbb{R}_+\times\mathbb{R}^2} L\big(\mathrm{d}(r',y')\big)\,g(r,r',y,y')\Big]\\
&= \mathbb{E}\Big[2\,i_{\mathrm{r}}\kappa\int\mathrm{d}x\int_0^\infty\mathrm{d}s_1\int\varrho_{s_1}(\mathrm{d}y_1)\,p(s_1,y_1-x)\int_{s_1}^\infty\mathrm{d}s_2\\
&\qquad\qquad\int\varrho_{s_2}(\mathrm{d}y_2)\,p(s_2-s_1,y_2-y_1)\\
&\qquad\qquad\int_{s_1}^\infty\mathrm{d}s_3\int\varrho_{s_3}(\mathrm{d}y_3)\,p(s_3-s_1,y_3-y_1)\,g(s_2,s_3,y_2,y_3)\\
&\qquad+\ i_{\mathrm{r}}^2\int\mathrm{d}x_1\int_0^\infty\mathrm{d}s_1\int\varrho_{s_1}(\mathrm{d}y_1)\,p(s_1,y_1-x_1)\\
&\qquad\qquad\int\mathrm{d}x_2\int_0^\infty\mathrm{d}s_2\int\varrho_{s_2}(\mathrm{d}y_2)\,p(s_2,y_2-x_2)\,g(s_1,s_2,y_1,y_2)\Big].
\end{aligned}
$$

Thus we can write

$$
J^{\varepsilon,\varepsilon'} = 2\,i_{\mathrm{r}}\kappa\,J_1^{\varepsilon,\varepsilon'} + i_{\mathrm{r}}^2\,J_2^{\varepsilon,\varepsilon'}, \tag{15}
$$

where

$$
\begin{aligned}
J_1^{\varepsilon,\varepsilon'}(z,z') &:= \int_0^t\mathrm{d}s_1\ \mathbb{E}\Big[\int_{s_1\vee s}^t\mathrm{d}s_2\int_{s_1\vee s}^t\mathrm{d}s_3\int\varrho_{s_1}(\mathrm{d}y_1)\int\varrho_{s_2}(\mathrm{d}y_2)\int\varrho_{s_3}(\mathrm{d}y_3)\\
&\qquad p(s_2-s_1,y_2-y_1)\,p(s_3-s_1,y_3-y_1)\,f_{\varepsilon,z}(y_2)\,f'_{\varepsilon',z'}(y_3)\Big]
\end{aligned}
$$

and

$$
J_2^{\varepsilon,\varepsilon'}(z,z') := \mathbb{E}\Big[\int_s^t\mathrm{d}s_1\int_s^t\mathrm{d}s_2\int\varrho_{s_1}(\mathrm{d}y_1)\int\varrho_{s_2}(\mathrm{d}y_2)\ f_{\varepsilon,z}(y_1)\,f'_{\varepsilon',z'}(y_2)\Big]
$$

are respectively third and second moment expressions of the catalyst process ϱ only. We easily compute $J_2^{\varepsilon,\varepsilon'}$ thanks to the moment formula (A.2) for ordinary super-Brownian motion (with $f=0$ and g properly chosen):

$$
\begin{aligned}
J_2^{\varepsilon,\varepsilon'}(z,z') &= 2\gamma\,i_{\mathrm{c}}\int\mathrm{d}x\int_0^t\mathrm{d}s_3\int_{s_3\vee s}^t\mathrm{d}s_1\int_{s_3\vee s}^t\mathrm{d}s_2\int\mathrm{d}y_1\int\mathrm{d}y_2\int\mathrm{d}y_3\\
&\qquad p(s_3,y_3-x)p(s_1-s_3,y_1-y_3)p(s_2-s_3,y_2-y_3)f_{\varepsilon,z}(y_1)f'_{\varepsilon',z'}(y_2)\\
&\qquad+\ i_{\mathrm{c}}^2\int\mathrm{d}x_1\int\mathrm{d}x_2\int_s^t\mathrm{d}s_1\int_s^t\mathrm{d}s_2\int\mathrm{d}y_1\int\mathrm{d}y_2\\
&\qquad p(s_1,y_1-x_1)\,p(s_2,y_2-x_2)\,f_{\varepsilon,z}(y_1)\,f'_{\varepsilon',z'}(y_2)\\
&= 2\gamma\,i_{\mathrm{c}}\int\mathrm{d}y_1\ f_{\varepsilon,z}(y_1)\int\mathrm{d}y_2\ f'_{\varepsilon',z'}(y_2)\\
&\qquad\int_0^t\mathrm{d}s_3\int_{s_3\vee s}^t\mathrm{d}s_1\int_{s_3\vee s}^t\mathrm{d}s_2\ p(s_1+s_2-2s_3,y_1-y_2)+\ i_{\mathrm{c}}^2\,(t-s)^2\\
&\le 2\gamma\,i_{\mathrm{c}}\int_0^t\mathrm{d}s_3\int_{s_3\vee s}^t\mathrm{d}s_1\int_{s_3\vee s}^t\mathrm{d}s_2\ p(s_1+s_2-2s_3,0)+\ i_{\mathrm{c}}^2\,(t-s)^2\\
&=: \mathsf{K}_2<\infty.
\end{aligned}
$$

As $(\varepsilon, \varepsilon') \downarrow 0$, the quantity $J_2^{\varepsilon,\varepsilon'}(z, z')$ converges to

$$\begin{aligned} J_2^0(z, z') \;:=\;& 2\gamma\, i_{\rm c} \int_0^t \mathrm{d}s_3 \int_{s_3 \vee s}^t \mathrm{d}s_1 \int_{s_3 \vee s}^t \mathrm{d}s_2 \; p(s_1 + s_2 - 2s_3, z - z') \\ &+ i_{\rm c}^2\, (t - s)^2 \;\le\; \mathsf{K}_2\,. \end{aligned} \tag{16}$$

We can also compute $J_1^{\varepsilon,\varepsilon'}$ using the Markov property of ϱ at time s_1 and twice the moment formula (A.2):

$$\begin{aligned} J_1^{\varepsilon,\varepsilon'}(z, z') \;=\; 2\gamma \int_0^t \mathrm{d}s_1\; \mathbb{E}\Big[& \int_{s_1}^t \mathrm{d}s_4 \int_{s_4 \vee s}^t \mathrm{d}s_2 \int_{s_4 \vee s}^t \mathrm{d}s_3 \int \varrho_{s_1}(\mathrm{d}y_1) \int \varrho_{s_1}(\mathrm{d}y_5) \\ & \int \mathrm{d}y_4 \int \mathrm{d}y_2 \int \mathrm{d}y_3\, p(s_4 - s_1, y_4 - y_5)\; p(s_2 - s_4, y_2 - y_4) \\ & p(s_3 - s_4, y_3 - y_4)\, p(s_2 - s_1, y_2 - y_1)\; p(s_3 - s_1, y_3 - y_1)\; f_{\varepsilon,z}(y_2) \\ & f'_{\varepsilon',z'}(y_3)\Big] \\ + \int_0^t \mathrm{d}s_1\; \mathbb{E}\Big[& \int_{s_1 \vee s}^t \mathrm{d}s_2 \int_{s_1 \vee s}^t \mathrm{d}s_3 \int \varrho_{s_1}(\mathrm{d}y_1) \int \varrho_{s_1}(\mathrm{d}y_4) \int \varrho_{s_1}(\mathrm{d}y_5) \\ & \int \mathrm{d}y_2 \int \mathrm{d}y_3\, p(s_2 - s_1, y_2 - y_4)\; p(s_3 - s_1, y_3 - y_5) \\ & p(s_2 - s_1, y_2 - y_1)\, p(s_3 - s_1, y_3 - y_1)\; f_{\varepsilon,z}(y_2)\; f'_{\varepsilon',z'}(y_3)\Big]. \end{aligned}$$

With obvious notation we write

$$J_1^{\varepsilon,\varepsilon'} \;=\; 2\gamma\, J_3^{\varepsilon,\varepsilon'} + J_4^{\varepsilon,\varepsilon'}. \tag{17}$$

Using again the moment formula, we get

$$J_3^{\varepsilon,\varepsilon'} \;=\; 2\gamma\, i_{\rm c}\, J_5^{\varepsilon,\varepsilon'} + i_{\rm c}^2\, J_6^{\varepsilon,\varepsilon'}, \tag{18}$$

where

$$\begin{aligned} J_5^{\varepsilon,\varepsilon'}(z, z') \;:=\; & \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_5 \int_{s_1}^t \mathrm{d}s_4 \int_{s_4 \vee s}^t \mathrm{d}s_2 \int_{s_4 \vee s}^t \mathrm{d}s_3 \int \mathrm{d}y_1 \int \mathrm{d}y_2 \int \mathrm{d}y_3 \int \mathrm{d}y_4 \\ & \int \mathrm{d}y_5 \int \mathrm{d}y_6\, p(s_1 - s_5, y_1 - y_6)\; p(s_1 - s_5, y_5 - y_6) \\ & p(s_4 - s_1, y_4 - y_5)\; p(s_2 - s_4, y_2 - y_4)\, p(s_3 - s_4, y_3 - y_4) \\ & p(s_2 - s_1, y_2 - y_1)\; p(s_3 - s_1, y_3 - y_1)\; f_{\varepsilon,z}(y_2)\; f_{\varepsilon',z'}(y_3) \end{aligned}$$

and

$$\begin{aligned} J_6^{\varepsilon,\varepsilon'}(z, z') \;:=\; & \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_4 \int_{s_4 \vee s}^t \mathrm{d}s_2 \int_{s_4 \vee s}^t \mathrm{d}s_3 \int \mathrm{d}y_1 \int \mathrm{d}y_2 \int \mathrm{d}y_3 \int \mathrm{d}y_4 \int \mathrm{d}y_5 \\ & p(s_4 - s_1, y_4 - y_5)\; p(s_2 - s_4, y_2 - y_4)\; p(s_3 - s_4, y_3 - y_4) \\ & p(s_2 - s_1, y_2 - y_1)\; p(s_3 - s_1, y_3 - y_1)\; f_{\varepsilon,z}(y_2)\; f_{\varepsilon',z'}(y_3). \end{aligned}$$

We now compute $J_6^{\varepsilon,\varepsilon'}$. Integrating with $\mathrm{d}y_1$, $\mathrm{d}y_5$, and $\mathrm{d}y_4$ gives

$$J_6^{\varepsilon,\varepsilon'}(z,z') = \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_4 \int_{s_4\vee s}^t \mathrm{d}s_2 \int_{s_4\vee s}^t \mathrm{d}s_3 \int \mathrm{d}y_2 \int \mathrm{d}y_3\, p(s_2+s_3-2s_4,\, y_2-y_3)$$
$$p(s_2+s_3-2s_1,\, y_2-y_3)\, f_{\varepsilon,z}(y_2)\, f_{\varepsilon',z'}(y_3).$$

The function

$$H_6(y_2,y_3) := \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_4 \int_{s_4\vee s}^t \mathrm{d}s_2 \int_{s_4\vee s}^t \mathrm{d}s_3\; p(s_2+s_3-2s_4,\, y_2-y_3)$$
$$p(s_2+s_3-2s_1,\, y_2-y_3)$$

is continuous in (y_2,y_3) and bounded from above by $H_6(y,y) =: \mathsf{K}_6$ which is finite since $d=2$. Thus $J_6^{\varepsilon,\varepsilon'}(z,z')$ is uniformly bounded by K_6. Using that $f_{\varepsilon,z}(y_2)\, f_{\varepsilon',z'}(y_3)\, \mathrm{d}y_2 \mathrm{d}y_3$ converges weakly to $\delta_z(\mathrm{d}y_2)\delta_{z'}(\mathrm{d}y_3)$, we deduce that $J_6^{\varepsilon,\varepsilon'}$ converges to

$$J_6^0(z,z') := H_6(z,z') \le \mathsf{K}_6\,. \tag{19}$$

Note that $H_6(z,z)=\infty$ if $d=3$, which implies that $J^{\varepsilon,\varepsilon'}(z,z)$ doesn't converge for $d=3$, however it is well-defined at least for $f(x)=f'(x)=p(1,x)$.

Similar arguments show that $J_5^{\varepsilon,\varepsilon'}(z,z')$ is uniformly bounded in $\varepsilon,\varepsilon' \in (0,1]$ and $z,z' \in \mathbb{R}^d$. As ε and ε' decrease to 0, it converges to

$$J_5^0(z,z') := \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_5 \int_{s_1}^t \mathrm{d}s_4 \int_{s_4\vee s}^t \mathrm{d}s_2 \int_{s_4\vee s}^t \mathrm{d}s_3\; h_5(s_1,s_2,s_3,s_4,s_5,z,z'), \tag{20}$$

where

$$h_5(s_1,s_2,s_3,s_4,s_5,z,z') := \int \mathrm{d}y_1 \int \mathrm{d}y_4\; p(s_1+s_4-2s_5,\, y_1-y_4)\, p(s_2-s_4,\, z-y_4)$$
$$p(s_3-s_4,\, z'-y_4)\, p(s_2-s_1,\, z-y_1)\, p(s_3-s_1,\, z'-y_1).$$

Finally, we study $J_4^{\varepsilon,\varepsilon'}$. Let $g \in \mathcal{B}_+\big((\mathbb{R}^2)^3\big)$ and $\overline{g}(x_1,x_2,x_3) := \sum_\pi g(x_{\pi(1)},x_{\pi(2)},x_{\pi(3)})$, where the sum is over all the permutations π of $\{1,2,3\}$. By a standard monotone class argument we deduce from the moment formula (A.2) for ϱ that

$$\mathbb{E}\left[\int \varrho_v(\mathrm{d}y_1) \int \varrho_v(\mathrm{d}y_4) \int \varrho_v(\mathrm{d}y_5)\; g(y_1,y_4,y_5)\right]$$
$$= 2\, i_{\mathrm{c}}\gamma^2 \int_0^v \mathrm{d}s_4 \int_{s_4}^v \mathrm{d}s_5 \int \mathrm{d}y_1 \int \mathrm{d}y_4 \int \mathrm{d}y_5 \int \mathrm{d}y_6\; p(v+s_5-2s_4,\, y_1-y_6)$$
$$p(v-s_5,\, y_4-y_6)\, p(v-s_5,\, y_5-y_6)\, \overline{g}(y_1,y_4,y_5)$$
$$+\; i_{\mathrm{c}}^2\, \gamma \int_0^v \mathrm{d}s_4 \int \mathrm{d}y_1 \int \mathrm{d}y_4 \int \mathrm{d}y_5\; p(2v-2s_4,\, y_1-y_4)\, \overline{g}(y_1,y_4,y_5)$$
$$+\; \frac{1}{3!}\,,\, i_{\mathrm{c}}^3 \int \mathrm{d}y_1 \int \mathrm{d}y_4 \int \mathrm{d}y_5\; \overline{g}(y_1,y_4,y_5).$$

This implies

$$J_4^{\varepsilon,\varepsilon'} = 2\, i_{\mathrm{c}} \gamma^2 J_7^{\varepsilon,\varepsilon'} + i_{\mathrm{c}}^2\, \gamma\, J_8^{\varepsilon,\varepsilon'} + \frac{1}{3!}\, i_{\mathrm{c}}^3\, J_9^{\varepsilon,\varepsilon'}, \tag{21}$$

where

$$\begin{aligned}
J_7^{\varepsilon,\varepsilon'}(z,z') := {}& 2\int_0^t \mathrm{d}s_1 \int_{s_1\vee s}^t \mathrm{d}s_2 \int_{s_1\vee s}^t \mathrm{d}s_3 \int_0^{s_1} \mathrm{d}s_4 \int_{s_4}^{s_1} \mathrm{d}s_5 \int \mathrm{d}y_1 \int \mathrm{d}y_4 \int \mathrm{d}y_5 \int \mathrm{d}y_2 \\
& \int \mathrm{d}y_3 \int \mathrm{d}y_6\, p(s_2-s_1, y_2-y_4)\, p(s_3-s_1, y_3-y_5)\, p(s_2-s_1, y_2-y_1) \\
& \quad p(s_3-s_1, y_3-y_1)\, f_{\varepsilon,z}(y_2)\, f'_{\varepsilon',z'}(y_3) \\
& \Big[p(s_1+s_5-2s_4, y_1-y_6)\, p(s_1-s_5, y_4-y_6)\, p(s_1-s_5, y_5-y_6) \\
& + \ p(s_1+s_5-2s_4, y_4-y_6)\, p(s_1-s_5, y_1-y_6)\, p(s_1-s_5, y_5-y_6) \\
& + \ p(s_1+s_5-2s_4, y_5-y_6)\, p(s_1-s_5, y_1-y_6)\, p(s_1-s_5, y_4-y_6) \Big]
\end{aligned}$$

and

$$\begin{aligned}
J_8^{\varepsilon,\varepsilon'}(z,z') := {}& 2\int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_4 \int_{s_1\vee s}^t \mathrm{d}s_2 \int_{s_1\vee s}^t \mathrm{d}s_3 \int \mathrm{d}y_1 \int \mathrm{d}y_4 \int \mathrm{d}y_5 \int \mathrm{d}y_2 \int \mathrm{d}y_3 \\
& p(s_2-s_1, y_2-y_4)\, p(s_3-s_1, y_3-y_5)\, p(s_2-s_1, y_2-y_1) \\
& p(s_3-s_1, y_3-y_1)\, f_{\varepsilon,z}(y_2)\, f'_{\varepsilon',z'}(y_3) \Big[p(2s_1-2s_4, y_1-y_4) \\
& + p(2s_1-2s_4, y_1-y_5) + p(2s_1-2s_4, y_4-y_5) \Big]
\end{aligned}$$

as well as

$$\begin{aligned}
J_9^{\varepsilon,\varepsilon'}(z,z') := {}& 3! \int_0^t \mathrm{d}s_1 \int_{s_1\vee s}^t \mathrm{d}s_2 \int_{s_1\vee s}^t \mathrm{d}s_3 \int \mathrm{d}y_1 \int \mathrm{d}y_4 \int \mathrm{d}y_5 \int \mathrm{d}y_2 \int \mathrm{d}y_3 \\
& p(s_2-s_1, y_2-y_4)\, p(s_3-s_1, y_3-y_5)\, p(s_2-s_1, y_2-y_1) \\
& p(s_3-s_1, y_3-y_1) f_{\varepsilon,z}(y_2)\, f'_{\varepsilon',z'}(y_3).
\end{aligned}$$

Arguments similar to those used for the convergence of $J_6^{\varepsilon,\varepsilon'}$ show that $J_7^{\varepsilon,\varepsilon'}$, $J_8^{\varepsilon,\varepsilon'}$ and $J_9^{\varepsilon,\varepsilon'}$ are uniformly bounded and converge as ε and ε' decrease to 0, respectively to

$$\begin{aligned}
J_7^0(z,z') := {}& 2\int_0^t \mathrm{d}s_1 \int_{s_1\vee s}^t \mathrm{d}s_2 \int_{s_1\vee s}^t \mathrm{d}s_3 \int_0^{s_1} \mathrm{d}s_4 \int_{s_4}^{s_1} \mathrm{d}s_5 \int \mathrm{d}y_1 \int \mathrm{d}y_6 \qquad (22) \\
& \Big[p(s_2-s_5, z-y_6)\, p(s_3-s_5, z'-y_6)\, p(s_2-s_1, z-y_1) \\
& \quad p(s_3-s_1, z'-y_1)\, p(s_1+s_5-2s_4, y_1-y_6)
\end{aligned}$$

$$
\begin{aligned}
&+ \;\; p(s_2+s_5-2s_4, z-y_6)\, p(s_3-s_5, z'-y_6)\, p(s_2-s_1, z-y_1) \\
&\qquad\qquad\qquad\qquad p(s_3-s_1, z'-y_1)\, p(s_1-s_5, y_1-y_6) \\
&+ \;\; p(s_2-s_5, z-y_6)\, p(s_3+s_5-2s_4, z'-y_6)\, p(s_2-s_1, z-y_1) \\
&\qquad\qquad\qquad\qquad p(s_3-s_1, z'-y_1)\, p(s_1-s_5, y_1-y_6)\Big],
\end{aligned}
$$

$$
\begin{aligned}
J_8^0(z,z') := 2\int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_4 \int_{s_1\vee s}^t \mathrm{d}s_2 \int_{s_1\vee s}^t \mathrm{d}s_3 \qquad\qquad\qquad\qquad (23)\\
\Big[\int \mathrm{d}y_1\; p(s_2+s_1-2s_4, y_1-z)\, p(s_2-s_1, z-y_1)\, p(s_3-s_1, z'-y_1) \\
+ \int \mathrm{d}y_1\; p(s_3+s_1-2s_4, z'-y_1)\, p(s_2-s_1, z-y_1)\, p(s_3-s_1, z'-y_1) \\
+ \;\; p(s_2+s_3-2s_1, z-z')\, p(s_2+s_3-2s_4, z'-z)\Big],
\end{aligned}
$$

and

$$
J_9^0(z,z') \;:=\; 6\int_0^t \mathrm{d}s_1 \int_{s_1\vee s}^t \mathrm{d}s_2 \int_{s_1\vee s}^t \mathrm{d}s_3\; p(s_2+s_3-2s_1, z-z'). \qquad (24)
$$

Altogether, for each $i \in \{1,\ldots,9\}$, the $J_i^{\varepsilon,\varepsilon'}$ are finite, uniformly bounded, and have a finite limit as $(\varepsilon,\varepsilon') \downarrow 0$. Thus, the $J^{\varepsilon,\varepsilon'}(z,z')$ are finite and converge in $\mathbb{R}_+$ as ε and ε' decrease to 0. This finishes the proof of Lemma 15. □

Completion of the proof of Theorem 5 (a). The claimed expectation expression for $\lambda_{[s,t]}(z)$ easily follows from the moment formula (4) for L in the case $m = 1$.

The second moment of $\lambda_{[s,t]}(z)$ is given by the limit J^0, independent of z, of $J^{\varepsilon,\varepsilon}(z,z)$ from Lemma 15 as $\varepsilon \downarrow 0$. By the formulas (15), (17), (18), and (21),

$$
\begin{aligned}
J^0 = 2\, i_{\mathrm{r}}\kappa \left[2\gamma\left(2\gamma\, i_{\mathrm{c}}\, J_5^0 + i_{\mathrm{c}}^2\, J_6^0\right) + \left\{2\, i_{\mathrm{c}}\gamma^2 J_7^0 + i_{\mathrm{c}}^2\,\gamma\, J_8^0 + \frac{1}{3!}\, i_{\mathrm{c}}^3\, J_9^0\right\}\right] \\
+ i_{\mathrm{r}}^2 J_2^0 \;<\; \infty \qquad\qquad (25)
\end{aligned}
$$

which, in the case $s < t$, is strictly larger than $\left(\mathbf{E}[\lambda_{[s,t]}(z)]\right)^2$, occurring from the J_2^0-term [see definition (16)]. This completes the proof of Theorem 5 (a). □

Remark 16 ***(Variance formula)****.* For $t \geq s \geq 0$ and $z \in \mathbb{R}^d$, from the representation formula (25) combined with (20), (19), (22), (23), (24), and (16), as well as the expectation formula in (a), we obtain the following formula for the *variance* of $\lambda_{[s,t]}(z)$:

$$
\begin{aligned}
&2 i_{\mathrm{c}}\, i_{\mathrm{r}} \left(i_{\mathrm{c}}^2\,\kappa + i_{\mathrm{r}}\gamma\right) \int_0^t \mathrm{d}s_1 \int_{s_1\vee s}^t \mathrm{d}s_2 \int_{s_1\vee s}^t \mathrm{d}s_3\; p(s_2+s_3-2s_1, 0) \\
&+ 8\, i_{\mathrm{c}}^2 i_{\mathrm{r}}\gamma\kappa \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \int_{s_2\vee s}^t \mathrm{d}s_3 \int_{s_2\vee s}^t \mathrm{d}s_4\; p(s_3+s_4-2s_2, 0)\, p(s_3+s_4-2s_1, 0)
\end{aligned}
$$

$$+ 8i_{\mathrm{c}}^2 i_{\mathrm{r}} \gamma\kappa \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \int_{s_2\vee s}^t \mathrm{d}s_3 \int_{s_2\vee s}^t \mathrm{d}s_4 \int \mathrm{d}y\, p(s_2+s_3-2s_1,\, y)\; p(s_3-s_2,\, y)\; p(s_4-s_2,\, y)$$

$$+\; 16\, i_{\mathrm{c}}\, i_{\mathrm{r}} \gamma^2 \kappa \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \int_{s_2}^t \mathrm{d}s_3 \int_{s_3\vee s}^t \mathrm{d}s_4 \int_{s_3\vee s}^t \mathrm{d}s_5 \int \mathrm{d}y_1 \int \mathrm{d}y_2\; p(s_2+s_3-2s_1,\, y_1-y_2)\; p(s_4-s_2,\, y_1)\; p(s_4-s_3,\, y_2)\; p(s_5-s_2,\, y_1)\; p(s_5-s_3,\, y_2)$$

$$+\; 16\, i_{\mathrm{c}}\, i_{\mathrm{r}} \gamma^2 \kappa \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \int_{s_2}^t \mathrm{d}s_3 \int_{s_3\vee s}^t \mathrm{d}s_4 \int_{s_3\vee s}^t \mathrm{d}s_5 \int \mathrm{d}y_1 \int \mathrm{d}y_2\; p(s_2+s_4-2s_1,\, y_2-y_1)\; p(s_4-s_3,\, y_2)\; p(s_3-s_2,\, y_1)\; p(s_5-s_2,\, y_2-y_1)\; p(s_5-s_3,\, y_2). \qquad \diamond$$

4.2. Spatial absolute continuity [proof of Theorem 5 (b)]

We first prove that

$$x \;\mapsto\; \phi_q(x) \int_{[s,t]\times\mathbb{R}^2} L\big(\mathrm{d}(r,y)\big)\; p(\varepsilon, x-y)$$

converges in $L^1(\ell\otimes\mathbf{P})$ as ε decreases to 0, to a limit, say $x \mapsto \phi_q(x)\xi(x)$. Moreover, for almost every x, $\mathbf{P}$–a.s., $\xi(x) = \lambda_{[s,t]}(x)$. Thanks to the statement (a) in the theorem, it is enough to check that the function

$$(x,\varepsilon) \mapsto \mathbf{E}\bigg[\int_{[s,t]\times\mathbb{R}^2} L\big(\mathrm{d}(r,y)\big)\; p(\varepsilon, x-y)\bigg],$$

is uniformly bounded on $\mathbb{R}^2\times(0,1]$. But this is clear since

$$\begin{aligned} &\mathbf{E}\bigg[\int_{[s,t]\times\mathbb{R}^2} L\big(\mathrm{d}(r,y)\big)\; p(\varepsilon, x-y)\bigg] \\ &\quad= \mathbf{E}\bigg[\int_s^t \mathrm{d}r\; i_{\mathrm{r}} \int \mathrm{d}z \int \varrho_r(\mathrm{d}y)\; p(r, z-y)\; p(\varepsilon, x-y)\bigg] \\ &\quad= i_{\mathrm{r}} i_{\mathrm{c}}\,(t-s). \end{aligned}$$

Statement (b) is then a straight forward consequence of the following criterion with $\nu(\mathrm{d}y)$ replaced by $L([s,t]\times\mathrm{d}y)$ [recall Theorem 3 (a)].

Proposition 17 (Sufficient criterion for absolute continuity). *Let $\nu \in \mathcal{M}_q$ be a random measure defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. We assume that $\mathbb{E}\big[(\nu,\phi_q)\big] < \infty$ and that*

$$\bigg((x,\omega) \;\mapsto\; \phi_q(x) \int \nu(\omega,\mathrm{d}y)\;\, p(\varepsilon, x-y),\;\;\; \varepsilon > 0\bigg)$$

converges in $L^1(\ell\otimes\mathbb{P})$ to some $\phi_q\,\xi$ as $\varepsilon\downarrow 0$. Then $\mathbb{P}$–a.s., the measure ν is absolutely continuous (with respect to Lebesgue measure) and has the density function ξ :

$$\nu(\mathrm{d}y) \;=\; \xi(y)\,\mathrm{d}y.$$

Proof. Let β be any bounded random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, and $f \in \mathcal{B}^q$ be continuous. Because of the assumed convergence in $L^1(\ell \otimes \mathbb{P})$, we get

$$J_\varepsilon := \int \mathrm{d}x\, f(x)\, \mathbb{E}\Big[\beta \int \nu(\mathrm{d}y)\ p(\varepsilon, x-y)\Big] \underset{\varepsilon\downarrow 0}{\longrightarrow} \int \mathrm{d}x\, f(x)\, \mathbb{E}\big[\beta\, \xi(x)\big].$$

On the other hand, the function

$$(y, \varepsilon) \mapsto \int \mathrm{d}x\ f(x)\ p(\varepsilon, x-y)$$

is bounded by $\phi_q(y)$ [thanks to (A.1)], continuous, and converges to f as $\varepsilon \downarrow 0$. By dominated convergence, we get that J_ε converges to $\mathbb{E}\big[\beta\,\langle \nu, f\rangle\big]$. Since β and f are arbitrary, the equality

$$\int \mathrm{d}x\ f(x)\, \mathbb{E}\big[\beta\, \xi(x)\big] = \mathbb{E}\big[\beta\,\langle \nu, f\rangle\big]$$

implies that ν is $\mathbb{P}$–a.s. absolutely continuous with respect to the Lebesgue measure, and that $\nu(\mathrm{d}y) = \xi(y)\,\mathrm{d}y$, $\mathbb{P}$–a.s., completing the proof. □

4.3. *Random ergodic limit [proof of Theorem 5 (d)]*

Let $f \in L^1_+(\mathbb{R}^2)$. Thanks to Lemma 15, we know that $T^{-1}\int_{[0,T]\times\mathbb{R}^2} L\big(\mathrm{d}(r,y)\big)\ f(y)$ is finite and even belongs to $L^2(\mathbf{P})$. By self-similarity it has the same law as

$$I_T := T \int_{[0,1]\times\mathbb{R}^2} L\big(\mathrm{d}(r,y)\big)\ f\big(y\sqrt{T}\,\big).$$

Thanks to Lemma 15 and Theorem 5 (a), we see that I_T converges in $L^2(\mathbf{P})$ as $T \uparrow \infty$ to $\lambda_{[0,1]}(0) \int \mathrm{d}x\ f(x)$. Thus we deduce that for any $f \in L^1_+(\mathbb{R}^2)$, the following convergence in law holds with respect to $\mathbf{P}$:

$$\lim_{T\uparrow\infty} \frac{1}{T} \int_{[0,T]\times\mathbb{R}^2} L\big(\mathrm{d}(r,y)\big)\ f(y) = \lambda_{[0,1]}(0) \int \mathrm{d}x\ f(x).$$

This ends the proof of (d). □

5. Two-dimensional collision measures [proof of Theorem 8]

5.1. *Representation of the collision local time [proof of Theorem 8 (a) and (c)]*

The proof relies on an L^2–approach and on the following lemma which is a time counterpart of Lemma 15. Recall the approximating collision measures K^ε_t introduced in (9).

Lemma 18 (Convergence of covariances). *Let $d = 2$. For fixed $t > 0$ and $\varphi, \varphi' \in \mathcal{B}^q_+$, the quantity*

$$J^{\varepsilon,\varepsilon'}_t := \mathbf{E}\Big[\langle K^\varepsilon_t, \varphi\rangle\langle K^{\varepsilon'}_t, \varphi'\rangle\Big]$$

converges to a finite limit, independent of f, as ε and ε' decrease to 0. Furthermore the convergence is uniform in t on any compact subset of $(0, +\infty)$.

The proof of this lemma will be postponed to Subsection 5.3. We first prove Theorem 8 (c) and then Theorem 8 (a) in Remark 19.

Proof of Theorem 8 (c). Let $A > a > 0$ and set $\mathrm{d}\mathbf{Q} := 1_{[a,A]}\,\mathrm{d}t\,\mathrm{d}\mathbf{P}$. We deduce from the uniform convergence in Lemma 18 that for any $\varphi \in \mathrm{b}\mathcal{B}_+$ the maps

$$\left(\int_{\mathbb{R}_+\times\mathbb{R}^2} L\big(\mathrm{d}(s,y)\big)\,\phi_q(y)\varphi(y)\,f_{\varepsilon,t}(s) : \ a \le t \le A \right)$$

converge in $L^2(\mathrm{d}\mathbf{Q})$ as $\varepsilon \downarrow 0$ to some limit $\Lambda(\varphi) = \big(\Lambda_t(\varphi) : \ a \le t \le A\big)$, say. It is clear that Λ is $\mathbf{Q}$–a.e. linear in φ and non-negative. In particular, $\Lambda(1)$ is finite $\mathbf{Q}$–a.e.

If (φ_m) is a non-decreasing sequence of functions of $\mathrm{b}\mathcal{B}_+$ which converge pointwise to $\varphi \in \mathrm{b}\mathcal{B}_+$, then the non-decreasing sequence $\Lambda(\varphi_m)$ bounded by $\Lambda(\varphi)$ converges $\mathbf{Q}$–a.e. to a limit Z, say. From the L^2–convergence, we deduce that

$$\begin{aligned}
\int_a^A \mathrm{d}t\ \mathbf{E}\left|\Lambda_t(\varphi_m) - \Lambda_t(\varphi)\right| &= \lim_{\varepsilon\downarrow 0} \int_a^A \mathrm{d}t\ \mathbf{E}\left|\left(K_t^\varepsilon, \phi_q\,(\varphi_m - \varphi)\right)\right| \qquad (26)\\
&\le \varlimsup_{\varepsilon\downarrow 0} \int_a^A \mathrm{d}t\ \mathbf{E}\left(K_t^\varepsilon, \phi_q\,|\varphi_m - \varphi|\right)\\
&= \varlimsup_{\varepsilon\downarrow 0} i_{\mathrm{r}} i_{\mathrm{c}} \int_a^A \mathrm{d}t \int_0^\infty \mathrm{d}u\ f_{\varepsilon,t}(u) \int \mathrm{d}y\,\phi_q(y)\\
&\quad \left|\varphi_m(y) - \varphi(y)\right|\\
&= i_{\mathrm{r}} i_{\mathrm{c}}\,(A-a)\,\left\|\phi_q\,(\varphi_m - \varphi)\right\|_{L^1} \underset{m\uparrow\infty}{\longrightarrow} 0.
\end{aligned}$$

By dominated convergence, we conclude that $Z = \Lambda(\varphi)$, $\mathbf{Q}$–a.e. From [Get74, Proposition 4.1], we deduce there exists a kernel $\tilde{\Lambda}$ from $\big(\Omega\times[a,A], \mathcal{F}\times\mathcal{B}([a,A])\big)$ to $\big(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)\big)$ such that for all $\varphi \in \mathrm{b}\mathcal{B}_+$, $\mathbf{Q}$–a.e. $\Lambda(\varphi) = \tilde{\Lambda}(\varphi)$. We then define the kernel $\big(K_t\,,\ t \in [a,A]\big)$ by $K_t(\varphi) = \tilde{\Lambda}_t(\varphi/\phi_q)$, for $\varphi \in \mathcal{B}^q$. By taking f such that $f = 0$ on $\mathbb{R}_+$, one can choose an $\mathcal{F}_t$–adapted version of $\big(K_t\,,\ t \in [a,A]\big)$. Since a, A are arbitrary, we can choose an adapted version of $(K_t\,,\ t > 0)$ which we still denote by $(K_t\,,\ t > 0)$.

Remark 19 (**Existence of collision measures).** Let $s > 0$ be fixed. By replacing $\mathrm{d}\mathbf{Q} := 1_{[a,A]}\,\mathrm{d}t\,\mathrm{d}\mathbf{P}$ by $\delta_s(\mathrm{d}t)\,\mathrm{d}\mathbf{P}$, where δ_s is the Dirac measure at s, one verifies the existence of a kernel K_s from $(\Omega, \mathcal{F})$ to $\big(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)\big)$ such that for any $\varphi \in \mathcal{B}^q$, $K_s^\varepsilon(\varphi)$ converges in L^2 to $K_s(\varphi)$ as $\varepsilon \downarrow 0$. This proves Theorem 8 (a). ◇

Now we check that $L\big(\mathrm{d}(s,y)\big) = \mathrm{d}s\ K_s(\mathrm{d}y)$, $\mathbf{P}$–almost surely. Since $L\big(\{0\}\times \mathbb{R}^2\big) = 0$ a.s., it is enough to verify that

$$I := \mathbf{E}\left[\left|\int_{\mathbb{R}_+\times\mathbb{R}^2} L\big(\mathrm{d}(s,y)\big)\,g(s)\varphi(y) - \int_0^\infty \mathrm{d}s\ g(s)\,K_s(\varphi)\right|\right]$$

equals 0, for every $\varphi \in \mathcal{B}^q$ and every continuous function g with compact support in $(0, +\infty)$. Let C denote a compact subset of $(0, +\infty)$ containing a small neighborhood of the support of g. Since g has compact support, we deduce from the L^2–convergence that

$$\begin{aligned} I &= \lim_{\varepsilon \downarrow 0} \mathbf{E}\Big[\Big|\int_{\mathbb{R}_+\times\mathbb{R}^2} L\big(\mathrm{d}(s,y)\big)\, g(s)\varphi(y) - \int_0^\infty \mathrm{d}t\; g(t)\, K_t^\varepsilon(\varphi)\Big|\Big] \\ &\le \overline{\lim_{\varepsilon \downarrow 0}}\, \mathbf{E}\Big[\int_{\mathbb{R}_+\times\mathbb{R}^2} L\big(\mathrm{d}(s,y)\big)\, \big|\varphi(y)\big| \Big| g(s) - \int_0^\infty \mathrm{d}t\; g(t) f_{\varepsilon,t}(s)\Big|\Big] \\ &\le \mathbf{E}\Big[\int_{C\times\mathbb{R}^2} L\big(\mathrm{d}(s,y)\big)\, \big|\varphi(y)\big|\Big]\, \overline{\lim_{\varepsilon \downarrow 0}}\, \sup_{u>0} \Big| g(u) - \int_0^\infty \mathrm{d}t\; g(t) f_{\varepsilon,t}(u)\Big|. \end{aligned}$$

Note that for $u > 0$, the finite measures $\mathrm{d}t\; f_{\varepsilon,t}(u)$ on $\mathbb{R}_+$ converge weakly to δ_u. Moreover, since g is continuous with compact support in $(0, \infty)$ and f vanishes outside a compact set in $\mathbb{R}$, we deduce that the latter limit superior expression disappears in the limit as $\varepsilon \downarrow 0$. Hence, $I = 0$. So we have $L\big(\mathrm{d}(s,y)\big) = \mathrm{d}s\; K_s(\mathrm{d}y)$, $\mathbf{P}$–a.s., finishing the proof of Theorem 8 (c). $\square$

5.2. Carrying Hausdorff dimension two [proof of Theorem 8 (b)]

In order to prove that at each fixed time $s > 0$ the carrying Hausdorff dimension of the collision measure K_s equals two, it is enough ([Fal90]) to verify the following lemma.

Lemma 20 (Sufficient condition). *Fix $s > 0$ and $R > 0$. The quantity*

$$I_\delta := \mathbf{E}\Big[\int_{\{|x|,|y|\le R\}} K_s(\mathrm{d}x) K_s(\mathrm{d}y)\; |x-y|^{-\delta}\Big] \tag{27}$$

is finite for all $\delta \in (0, 2)$.

Proof. From Lemma 18 and its proof in Subsection 5.3 below, from the L^2–convergence of K_s^ε as ε decrease to 0, we can deduce a formula for $\mathbf{E}\big[(K_s, \varphi)(K_s, \varphi')\big]$. By the monotone class theorem, this then implies a formula for I_δ from (27). We can write this formula as

$$I_\delta = 2 i_\mathrm{r} \kappa \left(2\gamma \left[2\gamma i_\mathrm{c} J_5 + i_\mathrm{c}^2 J_6\right] + \left[2 i_\mathrm{c} \gamma^2 J_7 + i_\mathrm{c}^2 \gamma J_8 + \frac{1}{6}\, i_\mathrm{c}^3 J_9\right]\right) + i_\mathrm{r}^2 J_2\,,$$

where the quantities J_i are given in the proof of Lemma 18 (in Subsection 5.3 below), but with $\varphi(y_2)\varphi'(y_3)$ replaced by $\phi(y_2, y_3) := 1_{\{|y_2|\le R,\, |y_3|\le R\}}\, |y_2 - y_3|^{-\delta}$. To derive finite upper bounds for the J_i is rather tedious, always the same technics are used. Therefore, we proceed with J_5 only.

From (30), (29), and (5.3) below with $\varphi(y_2)\varphi'(y_3)$ replaced by $\phi(y_2, y_3)$, we have

$$J_5 = \iiint \mathrm{d}s_1\mathrm{d}s_4\mathrm{d}s_5 \ 1_{\{0<s_5<s_1<s_4<t\}} \iiiint \mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3\mathrm{d}y_4 \ p(s_1+s_4-2s_5, y_1-y_4)$$
$$p(t-s_4, y_2-y_4)\, p(t-s_4, y_3-y_4)\, p(t-s_1, y_2-y_1)\, p(t-s_1, y_3-y_1)$$
$$1_{\{|y_2|\le R,\ |y_3|\le R\}} \ |y_2-y_3|^{-\delta}\,.$$

It is easy to check that J_5 is bounded from above by a constant times

$$\iiint \mathrm{d}s_1\mathrm{d}s_4\mathrm{d}s_5 \ 1_{\{0<s_5<s_1<s_4<t\}}[s_1+s_4-2s_5]^{-1}(t-s_4)^{-\delta/2}(t-s_1)^{-\delta/2}$$
$$[2t-s_1-s_4]^{-1} < \infty.$$

This gives $J_5 < \infty$, finishing the proof. □

5.3. *Convergence of covariances [proof of Lemma 18]*

Fix t, φ, φ', f as in the lemma. Decomposing as in (15) in the proof of Lemma 15 [with the obvious replacements as $\varphi(y) f_{\varepsilon,t}(s)$ instead of $1_{[s,t]} f_{\varepsilon,z}(y)$], we write

$$J_t^{\varepsilon,\varepsilon'} = 2i_{\mathrm{r}}\kappa\, J_1^{\varepsilon,\varepsilon'} + i_{\mathrm{r}}^2 J_2^{\varepsilon,\varepsilon'},$$

where

$$J_1^{\varepsilon,\varepsilon'} := \int_0^\infty \mathrm{d}s_1 \int_{s_1}^\infty \mathrm{d}s_2 \ f_{\varepsilon,t}(s_2) \int_{s_1}^\infty \mathrm{d}s_3 \ f_{\varepsilon',t}(s_3)$$
$$\Big[\int \varrho_{s_1}(\mathrm{d}y_1) \int \varrho_{s_2}(\mathrm{d}y_2) \int \varrho_{s_3}(\mathrm{d}y_3)$$
$$p(s_2-s_1, y_2-y_1)\, p(s_3-s_1, y_3-y_1)\, \varphi(y_2)\varphi'(y_3) \Big]$$

and for $t > 0$ and $\varepsilon > 0$,

$$J_2^{\varepsilon,\varepsilon'} := \int_0^\infty \mathrm{d}s_1 \ f_{\varepsilon,t}(s_1) \int_0^\infty \mathrm{d}s_2 \ f_{\varepsilon',t}(s_2) \Big[2\gamma i_{\mathrm{c}} \int \mathrm{d}y_1 \ \varphi(y_1) \int \mathrm{d}y_2 \ \varphi'(y_2)$$
$$\int_0^{s_1\wedge s_2} \mathrm{d}s_3 \ p(s_1+s_2-2s_3, y_1-y_2) \Big]$$
$$+ \ i_{\mathrm{c}}^2 \int_0^\infty ds_1 \ f_{\varepsilon,t}(s_1) \ \int_0^\infty ds_2 \ f_{\varepsilon,t}(s_2) \ \int \mathrm{d}y_1 \ \varphi(y_1) \int \mathrm{d}y_2 \ \varphi'(y_2).$$

Using the same decomposition as in (17), (18) and (21), we have

$$J_1^{\varepsilon,\varepsilon'} = 2\gamma \Big[2\gamma i_{\mathrm{c}} J_5^{\varepsilon,\varepsilon'} + i_{\mathrm{c}}^2 J_6^{\varepsilon,\varepsilon'} \Big] + \Big[2i_{\mathrm{c}}\gamma^2 J_7^{\varepsilon,\varepsilon'} + i_{\mathrm{c}}^2\gamma\, J_8^{\varepsilon,\varepsilon'} + \frac{1}{6}\, i_{\mathrm{c}}^3 J_9^{\varepsilon,\varepsilon'} \Big],$$

where we set

$$J_5^{\varepsilon,\varepsilon'} := \int_0^\infty \mathrm{d}s_2\; f_{\varepsilon,t}(s_2) \int_0^\infty \mathrm{d}s_3\; f_{\varepsilon',t}(s_3) \int_0^{s_2\wedge s_3} \mathrm{d}s_1 \int_{s_1}^{s_2\wedge s_3} \mathrm{d}s_4 \int_0^{s_1} \mathrm{d}s_5$$
$$\iiiint \mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3\mathrm{d}y_4\, p(s_1+s_4-2s_5,\, y_1-y_4)\, p(s_2-s_4,\, y_2-y_4)$$
$$p(s_3-s_4,\, y_3-y_4)\, p(s_2-s_1,\, y_2-y_1)\, p(s_3-s_1,\, y_3-y_1)\, \varphi(y_2)\varphi'(y_3)$$

$$J_6^{\varepsilon,\varepsilon'} := \int_0^\infty \mathrm{d}s_2\; f_{\varepsilon,t}(s_2) \int_0^\infty \mathrm{d}s_3\; f_{\varepsilon',t}(s_3) \int_0^{s_2\wedge s_3} \mathrm{d}s_1 \int_{s_1}^{s_2\wedge s_3} \mathrm{d}s_4 \iint \mathrm{d}y_2\mathrm{d}y_3$$
$$p(s_2+s_3-2s_4,\, y_2-y_3)\, p(s_2+s_3-2s_1,\, y_2-y_3)\, \varphi(y_2)\varphi'(y_3),$$

$$J_7^{\varepsilon,\varepsilon'} := \int_0^\infty \mathrm{d}s_2\; f_{\varepsilon,t}(s_2) \int_0^\infty \mathrm{d}s_3\; f_{\varepsilon',t}(s_3) \iiint \mathrm{d}s_1\mathrm{d}s_4\mathrm{d}s_5\; 1_{\{0<s_4<s_5<s_1<s_2\wedge s_3\}}$$
$$\iiiint \mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3\mathrm{d}y_6\, \varphi(y_2)\varphi'(y_3)\Big[p(s_2-s_5,\, y_2-y_6)\, p(s_3-s_5,\, y_3-y_6)$$
$$p(s_2-s_1,\, y_2-y_1)\, p(s_3-s_1,\, y_3-y_1)\, p(s_1+s_5-2s_4,\, y_1-y_6)$$
$$+\, p(s_2+s_5-2s_4,\, y_2-y_6)\, p(s_3-s_5,\, y_3-y_6)\, p(s_2-s_1,\, y_2-y_1)$$
$$p(s_3-s_1,\, y_3-y_1)\, p(s_1-s_5,\, y_1-y_6)$$
$$+\, p(s_2-s_5,\, y_2-y_6)\, p(s_3+s_5-2s_4,\, y_3-y_6)\, p(s_2-s_1,\, y_2-y_1)$$
$$p(s_3-s_1,\, y_3-y_1)\, p(s_1-s_5,\, y_1-y_6)\Big],$$

and

$$J_8^{\varepsilon,\varepsilon'} := 2\int_0^\infty \mathrm{d}s_2\; f_{\varepsilon,t}(s_2) \int_0^\infty \mathrm{d}s_3\; f_{\varepsilon',t}(s_3) \iint \mathrm{d}s_1\mathrm{d}s_4\; 1_{\{0<s_4<s_1<s_2\wedge s_3\}}$$
$$\iint \mathrm{d}y_2\mathrm{d}y_3\; \varphi(y_2)\varphi'(y_3)\Big[\int \mathrm{d}y_1\; p(s_2+s_1-2s_4,\, y_1-y_2)$$
$$p(s_2-s_1,\, y_2-y_1)\, p(s_3-s_1,\, y_3-y_1)$$
$$+\int \mathrm{d}y_1 p(s_2+s_1-2s_4,\, y_1-y_3) p(s_2-s_1,\, y_2-y_1)\, p(s_3-s_1,\, y_3-y_1)$$
$$+\, p(s_2+s_3-2s_1,\, y_2-y_3)\, p(s_2+s_3-2s_4,\, y_2-y_3)\Big],$$

as well as

$$J_9^{\varepsilon,\varepsilon'} := 6\int_0^\infty \mathrm{d}s_2\; f_{\varepsilon,t}(s_2) \int_0^\infty \mathrm{d}s_3\; f_{\varepsilon',t}(s_3)$$
$$\int_0^{s_2\wedge s_3} \mathrm{d}s_1 \iint \mathrm{d}y_2\mathrm{d}y_3\; \varphi(y_2)\varphi'(y_3)\, p(s_2+s_3-2s_1,\, y_2-y_3).$$

We will only prove the uniform convergence of $J_5^{\varepsilon,\varepsilon'}$, since it contains the main idea needed also for the proof of convergence of the other five terms.

1° (*Convergence of* $J_5^{\varepsilon,\varepsilon'}$). Since $f = 0$ outside a compact subset of $\mathbb{R}$, in the integrand of $J_5^{\varepsilon,\varepsilon'}$, the variables s_2, s_3 are bounded from above by a constant, say M, for $\varepsilon, \varepsilon' \in (0, 1]$. On $\{s_5 < s_1 < s_4 < s_2 \wedge s_3\}$, we set

$$\begin{aligned} h_5(s_1, s_2, s_3, s_4, s_5) := \iiiint \mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3\mathrm{d}y_4\ & p(s_1 + s_4 - 2s_5, y_1 - y_4) \\ & p(s_2 - s_4, y_2 - y_4) \\ p(s_3 - s_4, y_3 - y_4)\, p(s_2 - s_1, y_2 - y_1)\, & p(s_3 - s_1, y_3 - y_1)\, \varphi(y_2)\varphi'(y_3). \end{aligned} \tag{28}$$

Because $\|\varphi/\phi_q\|_\infty$ is finite and $\|\phi_q\|_\infty < 1$, we get by integrating over y_2,

$$\begin{aligned} h_5(s_1, s_2, s_3, s_4, s_5) \le \|\varphi/\phi_q\|_\infty \iiint \mathrm{d}y_1\mathrm{d}y_3\mathrm{d}y_4\ & p(s_1 + s_4 - 2s_5, y_1 - y_4) \\ & p(2s_2 - s_4 - s_1, y_1 - y_4)\, p(s_3 - s_4, y_3 - y_4) \\ & p(s_3 - s_1, y_3 - y_1)\, \left|\varphi'(y_3)\right|. \end{aligned}$$

Since

$$\begin{aligned} & p(s_1 + s_4 - 2s_5, y_1 - y_4)\, p(2s_2 - s_4 - s_1, y_1 - y_4) \\ & \quad \le c\, \left[(s_1 + s_4 - 2s_5)(2s_2 - s_4 - s_1)\right]^{-1}, \end{aligned}$$

and $\int \mathrm{d}y_3 \left|\varphi'(y_3)\right| < \infty$, we have

$$h_5(s_1, s_2, s_3, s_4, s_5) \le c\, \left[(s_1 + s_4 - 2s_5)(2s_2 - s_4 - s_1)\right]^{-1}.$$

Hence, for $0 < s_2, s_3 < M$,

$$H(s_2, s_3) := \iiint \mathrm{d}s_1\mathrm{d}s_4\mathrm{d}s_5\ \mathbf{1}_{\{0<s_5<s_1<s_4<s_2\wedge s_3\}}\, h_5(s_1, s_2, s_3, s_4, s_5) \tag{29}$$

is bounded from above by

$$\begin{aligned} \tilde{H}(s_2, s_3) := c \iiint \mathrm{d}s_1\mathrm{d}s_4\mathrm{d}s_5\ & \mathbf{1}_{\{0<s_5<s_1<s_4<s_2\wedge s_3\}} \\ & \left[(s_1 + s_4 - 2s_5)(2s_2 - s_4 - s_1)\right]^{-1}. \end{aligned}$$

Set $v_i := (s_2 \wedge s_3) - s_i$ for $i \in \{1, 4, 5\}$ to get

$$\begin{aligned} \tilde{H}(s_2, s_3) &= c \iiint \mathrm{d}v_1\mathrm{d}v_4\mathrm{d}v_5\ \mathbf{1}_{\{0<v_4<v_1<v_5<s_2\wedge s_3\}} \\ &\qquad \left[(2v_5 - v_1 - v_4)\big(v_4 + v_1 + 2s_2 - 2(s_2\wedge s_3)\big)\right]^{-1} \\ &\le c \iiint \mathrm{d}v_1\mathrm{d}v_4\mathrm{d}v_5\ \mathbf{1}_{\{0<v_4<v_1<v_5<s_2\wedge s_3\}} \left[(2v_5 - v_1 - v_4)(v_4 + v_1)\right]^{-1}. \end{aligned}$$

With the change of variables $u_1 := v_4,\ u_2 := v_1 - v_4,\ u_3 := v_5 - v_4$, it is easy to check that $\tilde{H}(s_2, s_3)$ is bounded for $0 \le s_2, s_3 < M$. It is clear that $(s_2, s_3) \mapsto H(s_2, s_3)$ is continuous. By the dominated convergence theorem, we conclude that

$$J_5^{\varepsilon,\varepsilon'} = \int_0^\infty \mathrm{d}s_2\ f_{\varepsilon,t}(s_2) \int_0^\infty \mathrm{d}s_3\ f_{\varepsilon',t}(s_3)\, H(s_2, s_3)$$

converges to

$$J_5 := H(t,t) \tag{30}$$

as $\varepsilon, \varepsilon'$ decrease to 0. The convergence is again uniform on any compact subset of $(0, \infty)$.

Remark 21 (**Dimension three).** The convergence of $J_5^{\varepsilon,\varepsilon'}$ fails for $d = 3$ where $H(t,t) = +\infty$ for $\varphi(y) = \varphi'(y) = 1_{\{|y| \le R\}}$. ◇

2° (*Limits of* $J_2^{\varepsilon,\varepsilon'}, J_6^{\varepsilon,\varepsilon'}, \ldots, J_9^{\varepsilon,\varepsilon'}$). Finally, we give the uniform limits J_i of $J_i^{\varepsilon,\varepsilon'}$ for i in $\{2, 6, 7, 8, 9\}$:

$$\begin{aligned} J_2 :=\ & \gamma i_c \int_0^{2t} \mathrm{d}s \iint \mathrm{d}y_1 \mathrm{d}y_2\ \varphi(y_1)\varphi'(y_2)\, p(s, y_1 - y_2) \\ & + i_c^2 \int \mathrm{d}y_1\ \varphi(y_1) \int \mathrm{d}y_2\ \varphi'(y_2), \end{aligned}$$

$$J_6 := \frac{1}{4} \int_0^{2t} \mathrm{d}v_1 \int_0^{v_1} \mathrm{d}v_4 \iint \mathrm{d}y_2 \mathrm{d}y_3\ p(v_1, y_2 - y_3)\, p(v_4, y_2 - y_3)\, \varphi(y_2)\varphi'(y_3),$$

$$\begin{aligned} J_7 := & \iiint \mathrm{d}s_1 \mathrm{d}s_4 \mathrm{d}s_5\ 1_{\{0<s_4<s_5<s_1<t\}} \iiiint \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3 \mathrm{d}y_6\ \varphi(y_2)\varphi'(y_3) \\ & \Big[p(t - s_5, y_2 - y_6)\, p(t - s_5, y_3 - y_6)\, p(t - s_1, y_2 - y_1) \\ & \quad p(t - s_1, y_3 - y_1)\, p(s_1 + s_5 - 2s_4, y_1 - y_6) \\ & + p(t + s_5 - 2s_4, y_2 - y_6)\, p(t - s_5, y_3 - y_6)\, p(t - s_1, y_2 - y_1) \\ & \quad p(t - s_1, y_3 - y_1)\, p(s_1 - s_5, y_1 - y_6) \\ & + p(t - s_5, y_2 - y_6)\, p(t + s_5 - 2s_4, y_3 - y_6)\, p(t - s_1, y_2 - y_1) \\ & \quad p(t - s_1, y_3 - y_1)\, p(s_1 - s_5, y_1 - y_6) \Big], \end{aligned}$$

$$\begin{aligned} J_8 := 2 & \iint \mathrm{d}v_1 \mathrm{d}v_4\ 1_{\{0<v_1<v_4<t\}} \iint \mathrm{d}y_2 \mathrm{d}y_3\ \varphi(y_2)\varphi'(y_3) \\ & \Big[\int \mathrm{d}y_1\ p(2v_4 - v_1, y_1 - y_2)\, p(v_1, y_2 - y_1)\, p(v_1, y_3 - y_1) \\ & + \int \mathrm{d}y_1\ p(2v_4 - v_1, y_1 - y_3)\, p(v_1, y_2 - y_1)\, p(v_1, y_3 - y_1) \\ & + p(2v_1, y_2 - y_3)\, p(2v_4, y_2 - y_3) \Big], \end{aligned}$$

$$J_9 \ := \ 3\int_0^{2t} \mathrm{d}v \iint \mathrm{d}y_2 \mathrm{d}y_3 \ \varphi(y_2)\varphi'(y_3)\ p(v, y_2 - y_3).$$

This finishes the proof of Lemma 18. □

6. Deterministic time-independent catalyst [proof of Theorem 9]

The proof is similar to the proof of Theorem 5 (a) – (b).

Proof of (a). It is enough to check that if σ is absolutely continuous and d ≤ 3, then for almost all $z \in \mathbb{R}^d$,

$$J^{\varepsilon,\varepsilon'} \ := \ \mathrm{E}^{\sigma}_{\eta}\left[\int_s^t \mathrm{d}r \int X^{\sigma}_r(\mathrm{d}y)\ p(\varepsilon, z-y) \int_s^t \mathrm{d}r' \int X^{\sigma}_{r'}(\mathrm{d}y')\ p(\varepsilon', z-y')\right]$$

converges in $\mathbb{R}_+$ to a non-zero limit as ε and ε' decrease to 0.

Remark 22 (**Smoothing procedure).** One could prove a stronger result similar to Lemma 15 but only for $d < 2(\beta+1)$. ◇

It is easy to deduce from Lemma 4.5 in [Del96] that

$$\mathrm{E}^{\sigma}_{\eta}\left[\int X^{\sigma}_r(\mathrm{d}y)\ p(\varepsilon, z-y) \int X^{\sigma}_{r'}(\mathrm{d}y') p(\varepsilon', z-y')\right] \ = \ I_1^{\varepsilon,\varepsilon'}(r,r') + I_2^{\varepsilon,\varepsilon'}(r,r'),$$

where

$$I_1^{\varepsilon,\varepsilon'}(r,r') \ := \ \int \eta(\mathrm{d}x)\ \ p(r+\varepsilon, x-z) \int \eta(\mathrm{d}x')\ \ p(r'+\varepsilon', x'-z),$$

and

$$\begin{aligned} I_2^{\varepsilon,\varepsilon'}(r,r') \ := \ & \int \eta(\mathrm{d}x) \int_0^{r\wedge r'} \mathrm{d}u \int \sigma(\mathrm{d}y)\ \ p(r-u+\varepsilon, y-z) \\ & p(r'-u+\varepsilon', y-z) p(u, x-y). \end{aligned}$$

We will show separately that $J_i^{\varepsilon,\varepsilon'} = \displaystyle\int_s^t \mathrm{d}r \int_s^t \mathrm{d}r'\ I_i^{\varepsilon,\varepsilon'}(r,r')$ converges, for $i \in \{1,2\}$.

1° (*Convergence of* $J_1^{\varepsilon,\varepsilon'}$). Since $\eta \in M_f$ and $r \wedge r' \geq s > 0$, we deduce that $I_1^{\varepsilon,\varepsilon'}(r,r')$ is bounded on $[s,t]^2$. It also converges pointwise to

$$I_1(r,r') \ := \ \int \eta(\mathrm{d}x)\ \ p(r, x-z) \int \eta(\mathrm{d}x')\ \ p(r', x'-z)$$

as ε and ε' decrease to 0. By dominated convergence, we deduce that $J_1^{\varepsilon,\varepsilon'}$ converges to $J_1 := \int_s^t \mathrm{d}r \int_s^t \mathrm{d}r'\ I_1(r,r')$ as $\varepsilon, \varepsilon'$ decrease to 0.

2° (*Convergence of* $J_2^{\varepsilon,\varepsilon'}$). We split the integral $I_2^{\varepsilon,\varepsilon'}$ according to $u \le s/2$ (integral $I_3^{\varepsilon,\varepsilon'}$) and $u > s/2$ (integral $I_4^{\varepsilon,\varepsilon'}$). We have for $r, r' \in [s,t]$

$$I_3^{\varepsilon,\varepsilon'}(r,r') = \int \eta(\mathrm{d}x) \int_0^{s/2} \mathrm{d}u \int \sigma(\mathrm{d}y) \\ p(r-u+\varepsilon, y-z)\, p(r'-u+\varepsilon', y-z)\, p(u, x-y).$$

This quantity is bounded from above by a constant time $\int \eta(\mathrm{d}x) \int_0^{s/2} \mathrm{d}u \int \sigma(\mathrm{d}y)$ $p(u, x-y)$ since $r, r' \in [s,t]$. This last quantity is finite because there exists a constant c such that for all $x \in \mathbb{R}^d$, $u > 0$,

$$\int \sigma(\mathrm{d}y)\; p(u, x-y) \;\le\; \frac{c}{(u \wedge 1)^{1-\beta}} \quad \text{(cf. (2) in [Del96]).} \tag{31}$$

Since $I_3^{\varepsilon,\varepsilon'}$ converges as $\varepsilon, \varepsilon'$ decrease to 0, we deduce from the dominated convergence theorem that $J_3^{\varepsilon,\varepsilon'} := \int_s^t \mathrm{d}r \int_s^t \mathrm{d}r'\; I_3^{\varepsilon,\varepsilon'}(r,r')$ converges to

$$J_3 := \int_s^t \mathrm{d}r \int_s^t \mathrm{d}r' \int \eta(\mathrm{d}x) \int_0^{s/2} \mathrm{d}u \int \sigma(\mathrm{d}y) \\ p(r-u, y-z)\, p(r'-u, y-z)\, p(u, x-y).$$

We write

$$\begin{aligned} J_4^{\varepsilon,\varepsilon'} &:= \int_s^t \mathrm{d}r \int_s^t \mathrm{d}r'\; I_4^{\varepsilon,\varepsilon'}(r,r') \\ &= \int_s^t \mathrm{d}r \int_s^t \mathrm{d}r' \int \eta(\mathrm{d}x) \int_{s/2}^{r\wedge r'} \mathrm{d}u \int \sigma(\mathrm{d}y) \\ &\qquad\qquad p(r-u+\varepsilon, y-z)\, p(r'-u+\varepsilon', y-z)\, p(u, x-y) \\ &= \int_{s/2}^t \mathrm{d}u \int_{(s-u)_+ + \varepsilon}^{t-u+\varepsilon} \mathrm{d}v \int_{(s-u)_+ + \varepsilon'}^{t-u+\varepsilon'} \mathrm{d}v' \int \eta(\mathrm{d}x) \int \sigma(\mathrm{d}y) \\ &\qquad\qquad p(v, y-z)\, p(v', y-z)\, p(u, x-y) \end{aligned}$$

where we set $v := r - u + \varepsilon$ and $v' := r' - u + \varepsilon'$, and continue the chain of equations with

$$= \int_{s/2}^t \mathrm{d}u \int_{(s-u)_+ + \varepsilon}^{t-u+\varepsilon} \mathrm{d}v \int_{(s-u)_+ + \varepsilon'}^{t-u+\varepsilon'} \mathrm{d}v'\; h(u, v, v'),$$

with $h(u,v,v') := \int \eta(\mathrm{d}x) \int \sigma(\mathrm{d}y)\; [2\pi(v+v')]^{-d/2}\; p\big(vv'/(v+v'), y-z\big)$ $p(u, x-y)$. To prove the convergence of $J_4^{\varepsilon,\varepsilon'}$, it is enough to check that the function h is integrable on $A := [s/2, t] \times [0, 2t]^2$.

Using (31), we get that $h(u,v,v') \le c\,(v+v')^{1-\beta-d/2} v^{\beta-1} v'^{\beta-1}$. This function is integrable over A for $d < 2(1+\beta)$. This is always satisfied if $d = 1$ or $d = 2$.

This is in fact the reason behind Remark 22. We now check that h is integrable if $d = 3$ for almost all $z \in \mathbb{R}^3$. Since $u \in [s/2, t]$, we have

$$H(u, v, v') = \int \mathrm{d}z\, h(u, v, v') \;=\; \frac{1}{[2\pi(v+v')]^{3/2}} \int \eta(\mathrm{d}x) \int \sigma(\mathrm{d}y)\; p(u, x-y)$$
$$\le\; \frac{c}{(v+v')^{3/2}} \int \eta(\mathrm{d}x) \int \sigma(\mathrm{d}y)\; p(a, y-x),$$

where a and c depend only on s, t. Using again (31), we get that $H(u, v, v') \le c(v+v')^{-3/2}$. Hence H is integrable over A. This implies that for almost all $z \in \mathbb{R}^3$, h is also integrable on A. This proves the convergence of $J_4^{\varepsilon,\varepsilon'}$ as $\varepsilon, \varepsilon'$ decrease to 0, for almost all $z \in \mathbb{R}^d$, $d \le 3$. The limit is

$$J_4 := \int_s^t \mathrm{d}r \int_s^t \mathrm{d}r' \int \eta(\mathrm{d}x) \int_{s/2}^{r\wedge r'} \mathrm{d}u \int \sigma(\mathrm{d}y)\, p(r-u, y-z)$$
$$p(r'-u, y-z)\, p(u, x-y).$$

Since $J_2^{\varepsilon,\varepsilon'} = J_3^{\varepsilon,\varepsilon'} + J_4^{\varepsilon,\varepsilon'}$ and $J^{\varepsilon,\varepsilon'} = J_1^{\varepsilon,\varepsilon'} + J_2^{\varepsilon,\varepsilon'}$, we deduced that $J^{\varepsilon,\varepsilon'}$ converges as $\varepsilon, \varepsilon'$ decrease to 0 for almost all $z \in \mathbb{R}^d$. Notice this limit is strictly bigger that J_1 unless $\eta = 0$ or $\sigma = 0$.

Proof of (b). Arguing as in the proof of Theorem 5 (b), one sees it is enough to check that $\mathrm{E}_\eta^\sigma\Big[\Big(\int_s^t \mathrm{d}r\; X_r^\sigma, \phi_q\Big)\Big] < \infty$, and that the function $(z, \varepsilon) \mapsto \mathrm{E}_\eta^\sigma\big[\int_s^t \mathrm{d}r \int X_r^\sigma(\mathrm{d}y)$ $p(\varepsilon, z-y)\big]$ is uniformly bounded on $\mathbb{R}^d \times (0, 1]$. We have

$$\mathrm{E}_\eta^\sigma\Big[\Big(\int_s^t \mathrm{d}r\; X_r^\sigma, \phi_q\Big)\Big] \le \mathrm{E}_\eta^\sigma\Big[\Big(\int_s^t \mathrm{d}r\; X_r^\sigma, 1\Big)\Big]$$
$$= \int_s^t \mathrm{d}r \int \eta(\mathrm{d}x) \int \mathrm{d}y\; p(r, x-y) < \infty,$$

and

$$\mathrm{E}_\eta^\sigma\Big[\int_s^t \mathrm{d}r \int X_r^\sigma(\mathrm{d}y)\; p(\varepsilon, z-y)\Big] \;=\; \int_s^t \mathrm{d}r \int \eta(\mathrm{d}x) \int \mathrm{d}y\; p(r, x-y)\, p(\varepsilon, z-y)$$
$$\le \int_s^t \mathrm{d}r\; p(r, 0)(\eta, 1).$$

This last constant is independent of $(z, \varepsilon) \in \mathbb{R}^d \times (0, 1]$. □

This finishes the proof of Theorem 9 altogether.

A. Appendix: Some basic properties of catalyst and reactant

A.1. Moment formulas for the catalyst

Let $d \ge 1$ and fix $\nu \in \mathcal{M}_q$. It is easy to check that for every $T > 0$, there exists a constant $c > 0$ such that for every $x \in \mathbb{R}^d$ and $\varepsilon \in (0, T]$,

$$\int \mathrm{d}y\; p(\varepsilon, x-y)\, \phi_q(y) \;\le\; c\, \phi_q(x). \tag{A.1}$$

Therefore we get that if $g \in H^q_T$, then the function $(r,x) \mapsto \int_r^\infty \mathrm{d}s\ P_{s-r}[g(s)](x)$ is well-defined and belongs to H^q_T. If $f \in \mathcal{B}^q$, then the function $(r,x) \mapsto 1_{\{t \geq r\}}\ P_{t-r}[f](x)$ is also well-defined and belongs to H^q_t.

It is well-known that for every $t \geq 0$, $g \in H^q$, $f \in \mathcal{B}^q$, and $m \geq 1$,

$$\mathbb{E}_\nu\Big[\Big[(\varrho_t\,, f) + \int_0^\infty \mathrm{d}s\ \big(\varrho_s\,, g(s)\big)\Big]^m\Big] \ = \ m! \sum_{k=1}^m \frac{1}{k!} \sum_{\substack{n_1,\cdots,n_k \geq 1,\\ n_1+\cdots+n_k=m}} \ \prod_{i=1}^k \big(\nu,\, \chi_{n_i}(0)\big), \tag{A.2}$$

where the sequence $(\chi_n\,,\ n \geq 1)$ is defined by the recurrence formula

$$\chi_n(r,x) \ := \ \gamma \int_r^\infty \mathrm{d}s \int \mathrm{d}y\ p(s-r, x-y) \left[\sum_{i=1}^{n-1} \chi_i(s,y)\,\chi_{n-i}(s,y)\right],$$

$(r,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $n \geq 2$, with initial condition

$$\chi_1(r,x) \ := \ 1_{\{t \geq r\}}\ P_{t-r}[f](x) + \int_r^\infty \mathrm{d}s\ P_{s-r}[g(s)](x), \qquad (r,x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

Thanks to the remark at the beginning of this subsection, we see that the functions χ_n, $n \geq 1$, are well-defined and belong to H^q.

A.2. Regularity properties of the catalyst

We now assume that $d \leq 3$. Recall that we write $\mathbb{P}$ for $\mathbb{P}_{i_c\ell}$. It is clear from the Hölder continuity Theorem 3 of [DF97a, p.254] that for every $\xi \in (0, 1/4)$, $T \geq 0$, $\mathbb{P}$–a.s. there exists a constant $c_1 := c_1(T, \varrho, \xi)$ such that for every $T \geq t \geq r \geq 0$, $f \in \mathcal{B}_+(\mathbb{R}^d)$,

$$\int_r^t \mathrm{d}s \int \varrho_s(\mathrm{d}z)\,\phi_q(z) f(z) \ \leq \ c_1\,|t-r|^\xi \int \mathrm{d}z\ f(z). \tag{A.3}$$

We have also (cf. Definition 2 b) and Theorem 4 of [DF97a, pp.224 and 259, respectively]) that for every $T \geq 0$, $\xi \in (0, 1/4)$, $\mathbb{P}$–a.s. there exists $c_2 := c_2(T, \varrho, \xi)$ such that for every $x \in \mathbb{R}^d$, $T \geq t \geq r \geq 0$,

$$\int_r^t \mathrm{d}s \int \varrho_s(\mathrm{d}z)\ p(s-r, x-z)\,\phi_q^2(z) \ \leq \ c_2\,|t-r|^\xi\ \phi_q(x). \tag{A.4}$$

A.3. Moment formulas for the reactant

Recall that $d \leq 3$. Using the Markov property of X^ϱ (given ϱ), it is easy to get that $\mathbb{P}$–a.s. for every $n \geq 1$, $t_n \geq \cdots \geq t_1 \geq 0$, and $f_n, \cdots, f_1 \in \mathcal{B}^q_+$,

$$\mathrm{E}^\varrho_{r,\mu}\Big[\exp\Big[-\sum_{t_i \geq r} (X^\varrho_{t_i}, f_i)\Big]\Big] \ = \ \mathrm{e}^{-(\mu,\, v(r))},$$

where v is the unique nonnegative solution of the catalytic log-Laplace equation (3) with $J(s) := \sum_{t_i \geq s} P_{t_i - s}[f_i]$. Using the continuity of X^ϱ, it can be shown that $\mathbb{P}$–a.s. for every nonnegative $g \in H^q$,

$$\mathrm{E}^{\varrho}_{r,\mu}\left[\exp\left[-\int_r^\infty \mathrm{d}s\ \left(X_s^\varrho, g(s)\right)\right]\right] = \mathrm{e}^{-\left(\mu, v(r)\right)},$$

where v is the unique nonnegative solution of (3) with $J(s) := \int_s^\infty \mathrm{d}u\ P_{u-s}[g(u)]$.

We deduce the next result on the *moments of the reactant process* X^ϱ from Theorem 4, Lemma 4 and Remark 2 of [DF97a, pp. 259 and 232, respectively]. We have $\mathbb{P}$–a.s. for every $t \geq 0$, $g \in H^q$, $f \in \mathcal{B}^q$, and $m \geq 1$,

$$\mathrm{E}^{\varrho}_{r,\mu}\left[\left[(X_t^\varrho, f) + \int_r^\infty \mathrm{d}s\ \left(X_s^\varrho,\ g(s)\right)\right]^m\right] = m! \sum_{k=1}^m \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k \geq 1,\\ n_1+\dots+n_k = m}} \prod_{i=1}^k \left(\mu, \chi_{n_i}(r)\right), \tag{A.5}$$

where $(\chi_n,\ n \geq 1)$ is defined by the recurrence formula (5) with initial condition

$$\chi_1(s,x) := 1_{\{t \geq s\}} P_{t-s}[f](x) + \int_s^\infty \mathrm{d}u\ P_{u-s}\left[g(u)\right](x) \qquad (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

Since $\chi_1 \in H^q$, inequality (A.4) implies that all the functions χ_n belong to H^q.

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