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Super Brownian motion with interactions

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Abstract

Using an approximating scheme with the Brownian snake, we prove the existence of solution to a martingale problem for super Brownian motion with interactions.

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1. Introduction

Let $\mathscr{B}(\mathbb{R}^d)$ be the set of real valued measurable functions defined on $\mathbb{R}^d$. Let $\mathscr{M}_{\mathrm{f}}$ be the set of finite measures on $\mathbb{R}^d$, endowed with the topology of the weak convergence. For $\mu \in \mathscr{M}_{\mathrm{f}}$ and $\varphi \in \mathscr{B}(\mathbb{R}^d)$, bounded, we denote $(\mu, \varphi) = \int \varphi(x)\mu(\mathrm{d}x)$.

Consider a super Brownian motion $X = (X_t,\ t \geqslant 0)$ started at $X_0 = \mu_0 \in \mathscr{M}_{\mathrm{f}}$. It is the unique solution of the martingale problem on $\mathscr{M}_{\mathrm{f}}$: for any bounded C^2 function, φ, with bounded derivatives,

$$X_0 = \mu_0,$$

$$(X_t, \varphi) = (X_0, \varphi) + \int_0^t \left(X_s, \frac{\Delta}{2}\varphi \right) \mathrm{d}s + M(\varphi)_t,$$

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where $M(\varphi)$ is a continuous martingale (with respect to the filtration generated by X) with quadratic variation

$$\langle M(\varphi)\rangle_t = 4\int_0^t (X_s, \varphi^2)\,\mathrm{d}s.$$

Let ϕ be a C^1 function defined on $\mathbb{R}^+$, s.t. $\phi(0)=0$ and $\phi'(t)>0$ for all $t\geqslant 0$. If we consider the changed time process $Y_t = X_{\phi(t)}$, for $t\geqslant 0$, then it is easy to check that $Y=(Y_t,\ t\geqslant 0)$ is a solution to the martingale problem on $\mathscr{M}_\mathrm{f}$: for any bounded C^2 function, φ, with bounded derivatives,

$$Y_0 = \mu_0,$$

$$(Y_t,\varphi) = (Y_0,\varphi) + \int_0^t \left(Y_s, \phi'(s)\,\frac{\Delta}{2}\varphi\right)\,\mathrm{d}s + M(\varphi)_t,$$

where $M(\varphi)$ is a continuous martingale with quadratic variation

$$\langle M(\varphi)\rangle_t = 4\int_0^t (Y_s, \phi'(s)\varphi^2)\,\mathrm{d}s.$$

By using the inverse of the time change ϕ, in order to recover X from Y, it is clear that the solution of this martingale problem is unique.

The aim of this paper is to use a random time change procedure to transform the martingale problem (see Ethier and Kurtz, 1986, Chapter 6). We prove in Theorem 1 the existence of solutions to the following martingale problem (MP) on $\mathscr{M}_\mathrm{f}$: for any bounded C^2 function, φ, with bounded derivatives,

$$Y_0 = \mu_0,$$

$$(Y_t,\varphi) = (Y_0,\varphi) + \int_0^t (Y_s, \theta(Y_s)A(Y_s)\varphi)\,\mathrm{d}s + M(\varphi)_t,$$

where $A(\mu)$ is the infinitesimal generator of a diffusion, with diffusion coefficient $\sigma(\mu,x)$ and drift $b(\mu,x)$, and $M(\varphi)$ is a continuous martingale with quadratic variation

$$\langle M(\varphi)\rangle_t = 4\int_0^t (Y_s, \theta(Y_s)\varphi^2)\,\mathrm{d}s.$$

The functions θ, b and σ are bounded continuous functions defined on $\mathscr{M}_\mathrm{f}\times\mathbb{R}^d$ taking values respectively in $\mathbb{R}, \mathbb{R}^d$ and $\mathbb{R}^{d\times d}$. And the functions θ and σ are positive.

The existence of solutions will be proved by using approximating schemes of the martingale problem. In fact we prove the tightness of the approximating scheme and that the limit points are a solution to the above martingale problem. Except in very particular cases, we were unable to prove the uniqueness of the limit, as well as the uniqueness of the solutions to the martingale problem (see Perkins, 1995, p. 7 on this last question).

Our approach relies on the Brownian snake representation of the super Brownian motion.

Roughly speaking, the super Brownian motion, X_t, can be described as the integral with respect to the local time of the snake lifetime process at level t of terminal points for the underlying paths. Now, using a random time change for each path, we integrate the terminal points of diffusions with respect to the local time of the snake lifetime process along a random curve instead of a deterministic line. This procedure modifies the underlying branching tree of the life time process. It was used in Bertoin et al. (1997) and Klenke (2003) to construct, from the Brownian snake, super Brownian motion with non-quadratic or catalytic branching mechanisms but without interactions.

In Perkins (1992, 1995), the interactions were introduced through a stochastic integral along the paths as well as a different weighting for each path. In particular, the structure of the underlying branching tree (that is the lifetime process for the Brownian snake representation) was the same for the superprocess and the interacting superprocess.

On the other side, Dhersin and Serlet (2000) see also Watanabe (1999), introduced a change in the underlying branching tree of the Brownian snake through a killing rate which depends on the path of the particle. This approach was a first step to introduce interaction in the underlying branching tree.

Let us mention also that similar martingale problems as (MP) appear as the limit of a discrete branching particles system with interactions, an approach developed by Métivier (1987) and Méléard and Roelly (1993).

2. The approximating scheme

2.1. The Brownian snake

We first recall the Brownian snake representation of the super Brownian motion.

Let $C=C(\mathbb{R}^+,\mathbb{R}^d)$ be the set of continuous functions defined on $\mathbb{R}^+$ with values in $\mathbb{R}^d$. We shall denote by $\mathbb{N}_{x,A}[\mathrm{d}W]$ the excursion measure on C of the Brownian snake $W=(W_s,\ s\geqslant 0)$ started at $x\in\mathbb{R}^d$ with underlying process a diffusion with infinitesimal generator A. We refer to Le Gall (1994) for the definition and properties of the Brownian snake. We recall that under $\mathbb{N}_{x,A}$, the law of the lifetime process $\zeta=(\zeta_s,\ s\geqslant 0)$ is the law of a positive excursion of linear Brownian motion. We take the normalization $\mathbb{N}_{x,A}[\sup_{s\geqslant 0}\zeta_s>\varepsilon]=1/2\varepsilon$. Under $\mathbb{N}_{x,A}$, conditionally on the lifetime process, W is a continuous C-valued Markov process started at the constant path equal to $x\in\mathbb{R}^d$. Conditionally, on the lifetime process and on $(W_u,\ u\in[0,s])$, the law of $W_{s'}$, with $s'\geqslant s$ is as follows: the two paths W_s and $W_{s'}$ coincide up to time $m=\inf_{u\in[s,s']}\zeta_u$, and $(W_{s'}(t+m),\ t\geqslant 0)$ is a diffusion with infinitesimal generator A, constant after $\zeta_{s'}-m$, which depends on $(W_u,\ u\in[0,s])$ only through its starting point $W_{s'}(m)=W_s(m)$.

Let $(\varepsilon_n,\ n\geqslant 1)$ be a sequence of positive numbers decreasing to 0. We define for $s\geqslant 0,\ t>0$, the local time L_s^t of the lifetime process, ζ, at level t up to time s:

$$L_s^t=\liminf_{n\to\infty}\frac{1}{\varepsilon_n}\int_0^s \mathbf{1}_{\{t-\varepsilon_n<\zeta_u<t\}}\,\mathrm{d}u,$$

and the measure valued process $(X_t(W),\ t\geqslant 0)$ defined by

$$X_t(W)=\int_{s\geqslant 0}\delta_{W_s(\zeta_s)}\,\mathrm{d}L_s^t, \tag{1}$$

where δ_y is the Dirac mass at point y. Notice the process $(X_t(W),\ t \geqslant 0)$ is always defined.

The family of local time $(L_s^t,\ s \geqslant 0,\ t > 0)$ is continuous $\mathbb{N}_{x,A}[\mathrm{d}W]$-a.e. and constant for s larger than σ, the duration of the excursion ζ under $\mathbb{N}_{x,A}$. In particular, the process $(X_t(W),\ t \geqslant 0)$ is finite and continuous under $\mathbb{N}_{x,A}[\mathrm{d}W]$.

Let $x \in \mathbb{R}^d$ and $\mu_0 \in \mathscr{M}_\mathrm{f}$. And consider the Poisson point measure on C, $\sum_{i\in I} \delta_{W^i}$, with intensity measure $\int \mu_0(\mathrm{d}x)\mathbb{N}_{x,A}[\mathrm{d}W]$. It is well known that the process $X = (X_t,\ t \geqslant 0)$ defined by $X_0 = \mu_0$, and

$$X_t = \sum_{i\in I} X_t(W^i)$$

is the usual superdiffusion started at μ_0 with underlying process a diffusion with infinitesimal generator A and branching mechanism $\psi(z) = 2z^2$.

We intend to replace the local time of the lifetime process at level t, by the local time along a random curve $\phi = (\phi_s^i,\ s \geqslant 0,\ i \in I)$, where $\phi_s^i \in [0,\infty]$. This curve ϕ needs to have particular properties (see also Rogers and Walsh (1991) for the definition of the local time along a random curve). This was already done for $\phi_s^i = \phi(W_s^i)$ defined as the first exit time of a domain D. The random measure associated to this curve is the so-called exit measure of D (see Le Gall, 1994). From this example we expect the following "tree property" to be in force:

If $\zeta_s^i > \phi_s^i$, where ζ_s^i is the lifetime of W_s^i, then for all s' such that $\inf_{u\in[s,s']} \zeta_u^i > \phi_s^i$, we have $\phi_{s'}^i = \phi_s^i$.

It is a natural condition when one deals with the excursion filtration (see also the definition of identifiable curve in Rogers and Walsh, 1991). Furthermore, in order to get the so-called special Markov property, we need that conditionally on what is "below" the curve ϕ, the excursions of the snake above the curve ϕ are distributed according to $\int X^\phi(\mathrm{d}x)\mathbb{N}_{x,A}[\mathrm{d}W]$, where X^ϕ is the exit measure of the superdiffusion above level ϕ. This will be stated precisely in property (B).

Eventually we will define for each t a random curve $\phi(t)$ and the corresponding exit measure X_t^ϕ in such a way that X_t^ϕ solves the martingale problem (MP).

The random time change will formally be given by the following equations:

- Stochastic differential equation and time change for the path W_s^i of the Brownian snake W^i:

$$\mathrm{d}_t V_s^i(t) = \sigma(Y_t, V_s^i(t))\,\mathrm{d}_t W_s^i(\phi_s^i(t)) + b(Y_t, V_s^i(t))\,\mathrm{d}_t\phi_s^i(t). \tag{2}$$

- Differential equation for the time change at time t:

$$\mathrm{d}_t\phi_s^i(t) = \theta(Y_t, V_s^i(t))\,\mathrm{d}t \quad \text{for } \phi_s^i(t) \leqslant \zeta_s^i. \tag{3}$$

- Definition of the random measure Y_t:

$$Y_t = \sum_{i\in I} \int_0^\infty \delta_{V_s^i(t)}\,\mathrm{d}_s L_s^{\phi_s^i(t),i}, \tag{4}$$

where $L_\cdot^{\phi_\cdot,i}$ is the "local time" of the lifetime of process of W^i on the random curve ϕ.

Notice that in general, the function $s \mapsto \phi_s^i(t)$ is not adapted to the filtration $(\mathscr{F}_s^i = \sigma(W_u^i,\ u \leqslant s), s \geqslant 0)$ generated by the snake, because the measure Y_t takes into account the paths $W_{s'}^i$ for $s' \geqslant s$.

We will present a discrete version of those equations and prove that X^ε, the discrete versions of Y, are tight and that any limit is solution to the martingale problem (MP).

We are now ready to present our approximating scheme.

2.2. The approximating scheme

Let θ, b and σ be bounded continuous functions defined on $\mathscr{M}_{\mathrm{f}} \times \mathbb{R}^d$ taking values respectively in $\mathbb{R}, \mathbb{R}^d$ and $\mathbb{R}^{d\times d}$. We also assume the functions θ and σ are positive. Let $\mu' \in \mathscr{M}_{\mathrm{f}}$ and $x' \in \mathbb{R}^d$. We will denote by $A(\mu', x')$ the infinitesimal generator of the d-dimensional Brownian motion with constant drift $b(\mu', x')$ and constant diffusion coefficient $\sigma(\mu', x')$.

Let $\mu_0 \in \mathscr{M}_{\mathrm{f}}$. We consider the Poisson point measure on $\mathbb{R}^d \times C$, $\sum_{i\in I} \delta_{(x^i, W^i)}$, with intensity measure $\mu_0(\mathrm{d}x)\mathbb{N}_{0,\Delta/2}[\mathrm{d}W]$. For $i \in I$, let σ_i be the duration of the lifetime process of the snake W^i. Recall W_s^i is the path at time s of the snake W^i.

Let $\varepsilon > 0$. We define by induction at time $k\varepsilon$ with $k \in \mathbb{N}$, the random time change $\phi^\varepsilon = ((\phi_s^{i,\varepsilon}(k\varepsilon),\ s \geqslant 0,\ i \in I), k \in \mathbb{N})$, the starting point $V = ((V_s^i(k\varepsilon),\ s \geqslant 0,\ i \in I), k \in \mathbb{N})$, the random measure $X^\varepsilon = (X_{k\varepsilon}^\varepsilon,\ k \in \mathbb{N})$ and the filtration $\mathscr{G}^\varepsilon = (\mathscr{G}_{k\varepsilon}^\varepsilon,\ k \in \mathbb{N})$ such that the following hypotheses (A) and (B) are in force.

(A) *$\phi^{i,\varepsilon}(k\varepsilon)$ enjoys the "tree property". And for all $i \in I$, the sets $\{s \in [0, \sigma^i];\ \phi_s^{i,\varepsilon}(k\varepsilon) < \zeta_s^i\}$ are open.*

The sets $\{s \in [0, \sigma^i]; \phi_s^{i,\varepsilon}(k\varepsilon) < \zeta_s^i\}$ can be described as the union of the open non-overlapping intervals (a^{i,j_k}, b^{i,j_k}) for $j_k \in J_k^i$, where the set J_k^i is possibly empty. We assume the family of indices J_l^i are non-overlapping for $i \in I,\ 0 \leqslant l \leqslant k$. Notice that from property (A), $\phi^{i,\varepsilon}(k\varepsilon)$ is constant over each interval (a^{i,j_k}, b^{i,j_k}). For $i \in I,\ j_k \in J_k,\ s \geqslant 0$, we consider the increments of the paths of the Brownian snake after time $\phi^{i,\varepsilon}(k\varepsilon)$:

$$\begin{aligned}\bar{W}_s^{i,j_k}(u) = {} & W^i_{(a^{i,j_k}+s)\wedge b^{i,j_k}}(u + \phi^{i,\varepsilon}_{(a^{i,j_k}+s)\wedge b^{i,j_k}}(k\varepsilon)) \\ & - W^i_{(a^{i,j_k}+s)\wedge b^{i,j_k}}(\phi^{i,\varepsilon}_{(a^{i,j_k}+s)\wedge b^{i,j_k}}(k\varepsilon)). \end{aligned} \tag{5}$$

And for $i \in I,\ j_k \in J_k^i$, we define the snake excursions $\bar{W}^{i,j_k} = (\bar{W}_s^{i,j_k},\ s \geqslant 0)$.

Let $\kappa_k^i(s) = \inf\{r \geqslant 0; \int_0^r \mathrm{d}u\, \mathbf{1}_{\{\phi_u^{i,\varepsilon}(k\varepsilon) \geqslant \zeta_u^i\}} > s\}$, the inverse of the time spent under $\phi^{i,\varepsilon}(k\varepsilon)$ by the life time of the snake W^i. We define the snake $W^{i;\kappa_k^i} = (W^i_{\kappa_k^i(s)},\ s \geqslant 0)$ and the σ-field $\mathscr{G}_{k\varepsilon}^\varepsilon$ generated by $(W^{i;\kappa_k^i},\ i \in I)$. Roughly speaking, $\mathscr{G}_{k\varepsilon}^\varepsilon$ represents all the information available on the Brownian snake up to level $\phi^{i,\varepsilon}(k\varepsilon)$.

(B) *The random measure $X_{k\varepsilon}^\varepsilon$ is $\mathscr{G}_{k\varepsilon}^\varepsilon$-measurable. The function $V_\cdot^i(k\varepsilon)$ is constant over each excursion interval (a^{i,j_k}, b^{i,j_k}), and let V^{i,j_k} be its value. Conditionally on $\mathscr{G}_{k\varepsilon}^\varepsilon$, the measure $\sum_{i\in I,\ j_k \in J_k^i} \delta_{(V^{i,j_k},\ \bar{W}^{i,j_k})}$ is a Poisson point measure with intensity $X_{k\varepsilon}^\varepsilon(\mathrm{d}x)\mathbb{N}_{0,\Delta/2}[\mathrm{d}W]$.*

We prove that (A) and (B) are in force by induction.

Initialization: For $k=0$, we set for $i\in I,\ s\in[0,\sigma_i]$:

- the time change: $\phi_s^{i,\varepsilon}(0)=0$,
- the measure: $X_0^\varepsilon=\mu_0$,
- the σ-field: $\mathscr{G}_0^\varepsilon=\sigma(X_0^\varepsilon)$,
- the excursion intervals: $(a^{i,j_0},b^{i,j_0})=(0,\sigma^i)$, where $j_0\in J_0^i=\{i\}$,
- the starting points: $V_s^{i,j_0}(0)=x^i$,
- the transformed snake above level $\phi_s^{i,\varepsilon}(0)$: for $i\in I,\ j_0\in J_0^i$, we set $U^{i,j_0}=(U_s^{i,j_0},\ s\geqslant 0)$, where for $s\geqslant 0,\ u\geqslant 0$,

$$U_s^{i,j_0}(u)=V_s^{i,j_0}+\sigma(X_0^\varepsilon,V_s^i(0))W_s^i(u)+b(X_0^\varepsilon,V_s^i(0))u.$$

Notice that conditionally on $\mathscr{G}_0^\varepsilon$, $\sum_{i\in I,\ j_0\in J_0^i}\delta_{(V^{i,j_0},U^{i,j_0})}$ is a Poisson point measure with intensity $\mu_0(\mathrm{d}x)\mathbb{N}_{x,A(\mu_0,x)}[\mathrm{d}W]$, with $\mu_0=X_0^\varepsilon$. Notice also that properties (A) and (B) are in force for $k=0$.

Let $k\geqslant 0$. Assume $\phi^{i,\varepsilon}(k\varepsilon)$, $X_{k\varepsilon}^\varepsilon$, $V^i(k\varepsilon)$ are built in such a way that properties (A) and (B) are satisfied. Let us now built $\phi^{i,\varepsilon}((k+1)\varepsilon)$, $X_{(k+1)\varepsilon}^\varepsilon$, $V^i((k+1)\varepsilon)$ and check that properties (A) and (B) are in force if k is replaced by $k+1$.

Step 1: Let us describe $\phi^{i,\varepsilon}((k+1)\varepsilon)$. We set for $s\in[a^{i,j_k},b^{i,j_k}],\ i\in I,\ j_k\in J_k^i$,

$$\phi_s^{i,\varepsilon}((k+1)\varepsilon)=\phi_s^{i,\varepsilon}(k\varepsilon)+\theta(X_{k\varepsilon}^\varepsilon,V_s^i(k\varepsilon))\,\varepsilon,$$

and $\phi_s^{i,\varepsilon}((k+1)\varepsilon)=+\infty$ if $s\notin[a^{i,j_k},b^{i,j_k}]$ for any $i\in I,\ j_k\in J_k^i$. This equation is the discrete version of (3).

Notice that on each interval $[a^{i,j_k},b^{i,j_k}]$, $\phi^{i,\varepsilon}((k+1)\varepsilon)$ is constant, and that outside those intervals $\phi_s^{i,\varepsilon}((k+1)\varepsilon)=+\infty>\zeta_s^i$. Therefore, property (A) is true for k replaced by $k+1$ by the continuity of ζ^i.

Step 2: We then describe the transformed snake U^{i,j_k} and built the random measure $X_{(k+1)\varepsilon}^\varepsilon$. We set for $i\in I,\ j_k\in J_k,\ s\geqslant 0,\ u\geqslant 0$,

$$U_s^{i,j_k}(u)=V^{i,j_k}+\sigma(X_{k\varepsilon}^\varepsilon,V^{i,j_k})\bar{W}_s^{i,j_k}(u)+b(X_{k\varepsilon}^\varepsilon,V^{i,j_k})u. \tag{6}$$

And for $i\in I,\ j\in J_k^i$, we define the snake excursions $U^{i,j_k}=(U_s^{i,j_k},\ s\geqslant 0)$. From property (B), we deduce that, conditional on $\mathscr{G}_{k\varepsilon}^\varepsilon$, the measure $\sum_{i\in I,\ j_k\in J_k^i}\delta_{(V^{i,j_k},U^{i,j_k})}$ is a Poisson point measure with intensity $\mu(\mathrm{d}x)\mathbb{N}_{x,A(\mu,x)}[\mathrm{d}W]$, where $\mu=X_{k\varepsilon}^\varepsilon$. Using (1), we can define the family of random measure

$$\tilde{X}_u^{i,j_k}=X_u(U^{i,j_k}). \tag{7}$$

Since we have at most a countable family of excursions $(U^{i,j_k},\ i\in I,\ j\in J_k^i)$, the family of measure valued process $(\tilde{X}^{i,j_k}=(\tilde{X}_u^{i,j_k},\ u>0)i\in I,\ j\in J_k^i)$ is well defined. Since, conditionally on $\mathscr{G}_{k\varepsilon}^\varepsilon$, $\sum_{i\in I,\ j_k\in J_k^i}\delta_{(V^{i,j_k},U^{i,j_k})}$ is distributed according to a Poisson point measure with intensity $\mu(\mathrm{d}x)\mathbb{N}_{x,A(\mu,x)}[\mathrm{d}W]$, we deduce that, conditionally on $\mathscr{G}_{k\varepsilon}^\varepsilon$,

$$\sum_{i\in I,\ j_k\in J_k^i}\delta_{(V^{i,j_k},\tilde{X}^{i,j_k})}$$

is distributed according to a Poisson point measure with intensity $\mu(\mathrm{d}x)\mathbb{N}_{x,A(\mu,x)}[\mathrm{d}X]$, where $\mathbb{N}_{x,A(\mu,x)}[\mathrm{d}X]$ is the measure of $(X_u(W),\ u>0)$ under $\mathbb{N}_{x,A(\mu,x)}[\mathrm{d}W]$.

Recall that on each interval $[a^{i,j_k}, b^{i,j_k}]$, $\phi^{i,\varepsilon}_{\cdot}((k+1)\varepsilon)$ and $\phi^{i,\varepsilon}_{\cdot}(k\varepsilon)$ are constant and $\mathscr{G}^{\varepsilon}_{k\varepsilon}$-measurable. In particular, the random measures

$$X^{i,j_k}_{(k+1)\varepsilon} = \tilde{X}^{i,j_k}_{u^{i,j_k}_k} = X_{u^{i,j_k}_k}(U^{i,j_k}), \tag{8}$$

where

$$u^{i,j_k}_k = \phi^{i,\varepsilon}_{\cdot}((k+1)\varepsilon) - \phi^{i,\varepsilon}_{\cdot}(k\varepsilon) = \varepsilon\theta(X^{\varepsilon}_{k\varepsilon}, V^{i,j_k}), \tag{9}$$

are well defined.

Let $\mathscr{E}^{i,j_k}$ be the σ-field generated by the snake $\bar{W}^{i,j_k}$ below level u^{i,j_k}_k. More precisely, $\mathscr{E}^{i,j_k}$ is the σ-field generated by $(\bar{W}^{i,j_k}_{\kappa^i(s)},\ s \geqslant 0)$, where we define $\kappa^i(s) = \inf\{r \geqslant 0; \int_{a^{i,j_k}}^{(a^{i,j_k}+r)\wedge b^{i,j_k}} \mathrm{d}u\, \mathbf{1}_{\{\phi^{i,\varepsilon}_u((k+1)\varepsilon) \geqslant \zeta^i_u\}} > s\}$ (the inverse of the time spent under $\phi^{i,\varepsilon}((k+1)\varepsilon)$ by the snake $\bar{W}^{i,j_k}$).

From the Markov property of the Brownian snake (see the first part of Proposition 13 in the appendix), we get that $X^{i,j_k}_{(k+1)\varepsilon}$ is measurable with respect to the σ-field generated by $\mathscr{G}^{\varepsilon}_{k\varepsilon}$ and $\mathscr{E}^{i,j_k}$. Notice the σ-field $\mathscr{G}^{\varepsilon}_{(k+1)\varepsilon}$, defined as $\mathscr{G}^{\varepsilon}_{k\varepsilon}$ with k replaced by $(k+1)$, is the σ-field generated by $\mathscr{G}^{\varepsilon}_{k\varepsilon}$ and the family $(\mathscr{E}^{i,j_k},\ i \in I,\ j_k \in J^i_k)$. In particular, the random measure

$$X^{\varepsilon}_{(k+1)\varepsilon} = \sum_{i \in I,\ j_k \in J_k} X^{i,j_k}_{(k+1)\varepsilon} \tag{10}$$

is measurable with respect to $\mathscr{G}^{\varepsilon}_{(k+1)\varepsilon}$. This gives that the first sentence of property (B) is true for k replaced by $k+1$. Notice the above definition is the discrete version of (4).

Step 3: To prove the second part, we have to define the functions $V^i((k+1)\varepsilon)$. Let us consider the excursions of the Brownian snake above level $\phi^{\cdot,\varepsilon}_{\cdot}((k+1)\varepsilon)$. We focus on the snake $\bar{W}^{i,j_k}$. The set $\{s \in (a^{i,j_k}, b^{i,j_k});\ \phi^{i,\varepsilon}_s((k+1)\varepsilon) < \zeta^i_s\}$ is open. It is the union of the open non-overlapping intervals $(a^{i,j_{k+1}}, b^{i,j_{k+1}})$ for $j_{k+1} \in J^{i,j_k}_{k+1}$, where the set of indices J^{i,j_k}_{k+1} is possibly empty. Recall $\phi^{i,\varepsilon}_s((k+1)\varepsilon)$ and $\phi^{i,\varepsilon}_s(k\varepsilon)$ are constant functions over $(a^{i,j_{k+1}}, b^{i,j_{k+1}})$. Using u^{i,j_k}_k defined in (9), we define for $s \in [a^{i,j_{k+1}}, b^{i,j_{k+1}}]$, $u \geqslant 0$,

$$\begin{aligned}\bar{W}^{i,j_{k+1}}_s(u) &= \bar{W}^{i,j_k}_{s'}(u + u^{i,j_k}_k) - \bar{W}^{i,j_k}_{s'}(u^{i,j_k}_k)\\ &= W^i_{s''}(u + \phi^{i,\varepsilon}_{s''}((k+1)\varepsilon)) - W^i_{s''}(\phi^{i,\varepsilon}_{s''}((k+1)\varepsilon)),\end{aligned}$$

with $s' = (a^{i,j_{k+1}} + s - a^{i,j_k}) \wedge (b^{i,j_{k+1}} - a^{i,j_k})$ and $s'' = (a^{i,j_{k+1}} + s) \wedge b^{i,j_{k+1}}$.

Define for $j_{k+1} \in J^{i,j_k}_{k+1}$ the snakes $\bar{W}^{i,j_{k+1}} = (\bar{W}^{i,j_{k+1}}_s,\ s \geqslant 0)$. This last formula coincides with definition (5), with k replaced by $k+1$. And we set for $s \in [a^{i,j_k}, b^{i,j_k}]$, using (6)

$$\begin{aligned}V^i_s((k+1)\varepsilon) &= U^{i,j_k}_{s-a^{i,j_k}}(u^{i,j_k}_k)\\ &= V^i_s(k\varepsilon) + \sigma(X^{\varepsilon}_{k\varepsilon}, V^i_s(k\varepsilon))[W^i_s(\phi^{i,\varepsilon}_s((k+1)\varepsilon)) - W^i_s(\phi^{i,\varepsilon}_s(k\varepsilon))]\\ &\quad + b(X^{\varepsilon}_{k\varepsilon}, V^i_s(k\varepsilon))[\phi^{i,\varepsilon}_s((k+1)\varepsilon) - \phi^{i,\varepsilon}_s(k\varepsilon)].\end{aligned}$$

The above definition is the discrete version of (2).

Notice the function $V^i_\cdot((k+1)\varepsilon)$ is constant over each excursion interval $[a^{i,j_{k+1}}, b^{i,j_{k+1}}]$, and let $V^{i,j_{k+1}}$ be its value. This proves the second sentence of property (B), with k replaced by $k+1$.

Step 4: Again from the Markov property of the Brownian snake (see the second part of Proposition 13 in the appendix), the random measure $\sum_{j_{k+1}\in J^{i,j_k}_{k+1}} \delta_{(V^{i,j_{k+1}},\bar{W}^{i,j_{k+1}})}$ is, conditionally on the σ-field $\mathscr{G}^\varepsilon_{k\varepsilon}$ and $\mathscr{E}^{i,j_k}$, distributed according to a Poisson point measure with intensity $X^{i,j_k}_{(k+1)\varepsilon}(\mathrm{d}x)\mathbb{N}_{0,\Delta/2}[\mathrm{d}W]$.

Recall the σ-field $\mathscr{G}^\varepsilon_{(k+1)\varepsilon}$ is the σ-field generated by $\mathscr{G}^\varepsilon_{k\varepsilon}$ and the family $(\mathscr{E}^{i,j_k},\ i\in I, j_k\in J^i_k)$. Let $J^i_{k+1}=\bigcup_{j_k\in J^i_k} J^{i,j_k}_{k+1}$. We then deduce that the random measure

$$\sum_{i\in I}\sum_{j_{k+1}\in J^i_{k+1}} \delta_{(V^{i,j_{k+1}},\bar{W}^{i,j_{k+1}})}$$

is, conditionally on the σ-field $\mathscr{G}^\varepsilon_{(k+1)\varepsilon}$, distributed according to a Poisson point measure with intensity $X^\varepsilon_{(k+1)\varepsilon}(\mathrm{d}x)\mathbb{N}_{0,\Delta/2}[\mathrm{d}W]$ (recall that $X^\varepsilon_{(k+1)\varepsilon}=\sum_{i\in I,\ j_k\in J_k} X^{i,j_k}_{(k+1)\varepsilon}$). Hence property (B) is fulfilled for k replaced by $k+1$.

2.3. Results

Let $X^\varepsilon=(X^\varepsilon_t,\ t\geqslant 0)$ be the right continuous step function which is the extension of $(X^\varepsilon_{k\varepsilon},\ k\in\mathbb{N})$. Let $D=D(\mathbb{R}^+,\mathscr{M}_\mathrm{f})$ be the Polish space of càdlàg paths from $\mathbb{R}^+$ to $\mathscr{M}_\mathrm{f}$, with the Skorokhod topology. Let θ, b and σ be bounded continuous functions defined on $\mathscr{M}_\mathrm{f}\times\mathbb{R}^d$ taking values respectively in $\mathbb{R},\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$. We also assume the functions θ and σ are positive. We write $A(\mu)$ for the infinitesimal generator of the d-dimensional diffusion with drift $b(\mu,\cdot)$ and diffusion coefficient $\sigma(\mu,\cdot)$ $(\mu\in\mathscr{M}_\mathrm{f})$.

Theorem 1. *The family of law of measure valued processes X^ε, for $\varepsilon\in(0,1]$ is C-tight in D as ε decreases to* 0. *Any limiting measure valued process $Y=(Y_t,\ t\geqslant 0)$ satisfies the martingale problem* (*MP*): *for any bounded C^2 function*, φ, *with bounded derivatives*,

$$Y_0=\mu_0,$$

$$(Y_t,\varphi)=(Y_0,\varphi)+\int_0^t (Y_s,\theta(Y_s)A(Y_s)\varphi)\,\mathrm{d}s+M(\varphi)_t, \tag{11}$$

where $M(\varphi)$ is a continuous martingale with quadratic variation

$$\langle M(\varphi)\rangle_t=\int_0^t (Y_s,\theta(Y_s)\varphi^2)\,\mathrm{d}s.$$

Furthermore, *any limiting measure valued process Y has a continuous version.*

We will follow an idea due to Perkins (2002) to prove this theorem.

Unfortunately, we were unable to prove the uniqueness of the martingale problem, even for the historical process (see in Perkins, 1995 why it is more convenient to look at the historical processes for uniqueness of the martingale problem). Uniqueness is trivially proved in the very particular cases of the next two remarks, where in fact the interaction disappears.

Remark. The particular cases $\sigma = \sigma(x)$, $b = b(x)$ and $\theta = \theta(x)$ with the additional condition $\theta(x) \geqslant \theta_0 > 0$ correspond to the usual superprocess with underlying process a diffusion with diffusion coefficient $\sigma(x)$ and drift $b(x)$, and branching mechanism $2\theta(x)z^2$ (see Dynkin, 1991). In this case the martingale problem (MP) has a unique solution.

Remark. One can also consider the other particular case $\sigma=\sigma(x)$, $b=b(x)$ and $\theta=\theta(\mu)$ with the additional condition $\theta(\mu) \geqslant \theta_0 > 0$. Then we consider the superprocess X with underlying process a diffusion coefficient $\sigma(x)$ and drift $b(x)$ and branching mechanism $2z^2$. We define the continuous additive functional of X by: $Q_t = \int_0^t \mathrm{d}u/\theta(X_u)$ and its continuous inverse $R_t = Q_t^{-1}$. It is easy to check that the process $Y=(Y_t=X_{R_t},\ t \geqslant 0)$ is a solution to the martingale problem (MP). To prove that the solution of (MP) is unique, consider $\tilde{Y}$ another solution to (MP). Set $R_t = \int_0^t \theta(\tilde{Y}_u)\,\mathrm{d}u$ and consider its continuous inverse $\tilde{Q}_t = \tilde{R}_t^{-1}$. It is then easy to check that the process $\tilde{X} = (\tilde{X}_t = \tilde{Y}_{\tilde{Q}_t},\ t \geqslant 0)$ is the solution to the martingale problem (MP) with $\theta = 1$. Since this martingale problem has a unique solution, we get that X and $\tilde{X}$ are equally distributed. And so Y and $\tilde{Y}$ have the same law. In this case the martingale problem has also a unique solution in law.

3. Intermediate results

Before giving the proof of Theorem 1, we set five lemmas. Let c denote a constant which may vary from line to line. For $t \in [0, +\infty)$ we set $[t]$ the unique integer such that $[t] \leqslant t < [t]+1$. For $f \in \mathscr{B}(\mathbb{R}^d)$ we will consider the following norms: $\|f\|_\infty = \sup_{x\in\mathbb{R}^d} |f(x)|$,

$$\|f\|_{\mathrm{Lip}} = \sup_{x,y\in\mathbb{R}^d;\ x\neq y} \frac{|f(x)-f(y)|}{|x-y|},$$

as well as for $\varphi \in C^2$,

$$\|\varphi\|_* = \|\varphi\|_{\mathrm{Lip}} + \sum_{l=1}^d \left(\left\| \frac{\partial\varphi}{\partial x_l} \right\|_\infty + \left\| \frac{\partial\varphi}{\partial x_l} \right\|_{\mathrm{Lip}} + \sum_{k=1}^d \left(\left\| \frac{\partial^2\varphi}{\partial x_l \partial x_k} \right\|_\infty + \left\| \frac{\partial^2\varphi}{\partial x_l \partial x_k} \right\|_{\mathrm{Lip}} \right) \right).$$

Let φ be a C^2 real function defined on $\mathbb{R}^d$, bounded with bounded Lipschitz derivatives. Let $T > 0$ be fixed.

We want to use the structure of the snake excursion U^{i,j_k} (see (6)) to express $(X_{k\varepsilon}^{i,j_k}, \varphi)$ as a sum of martingales and a process of finite variation. Recall the functions

$\phi_\cdot^{i,\varepsilon}(k\varepsilon)$ and $\phi_\cdot^{i,\varepsilon}((k+1)\varepsilon)$ are constant on the time indices where the excursion U^{i,j_k} is defined. We set for $i\in I,\ j_k\in J_k^i$,

$$\Delta M^{i,j_k}(\varphi)=(X^{i,j_k}_{(k+1)\varepsilon},\varphi)-\int_{\phi_\cdot^{i,\varepsilon}(k\varepsilon)}^{\phi_\cdot^{i,\varepsilon}((k+1)\varepsilon)}(\tilde X^{i,j_k}_u,A(X^\varepsilon_{k\varepsilon},V^{i,j_k})\varphi)\,\mathrm{d}u,$$

where $\tilde X^{i,j_k}$ has been defined in (7). And now we define what will be a $\mathscr{G}^\varepsilon$-martingale: $M(\varphi)_0=0$ and for $k\geqslant 0$,

$$M(\varphi)_{(k+1)\varepsilon}=M(\varphi)_{k\varepsilon}-(X^\varepsilon_{k\varepsilon},\varphi)+\sum_{i\in I,\ j_k\in J_k^i}\Delta M^{i,j_k}(\varphi).$$

In particular, we have

$$\begin{aligned}(X^\varepsilon_{(k+1)\varepsilon},\varphi)&-M(\varphi)_{(k+1)\varepsilon}\\ &=(X^\varepsilon_{k\varepsilon},\varphi)-M(\varphi)_{k\varepsilon}\\ &\quad+\sum_{i\in I,\ j_k\in J_k^i}\int_{\phi_\cdot^{i,\varepsilon}(k\varepsilon)}^{\phi_\cdot^{i,\varepsilon}((k+1)\varepsilon)}(\tilde X^{i,j_k}_u,A(X^\varepsilon_{k\varepsilon},V^{i,j_k})\varphi)\,\mathrm{d}u.\end{aligned}\tag{12}$$

We rewrite this as

$$\begin{aligned}(X^\varepsilon_{(k+1)\varepsilon},\varphi)&=(X^\varepsilon_{k\varepsilon},\varphi)+M(\varphi)_{(k+1)\varepsilon}-M(\varphi)_{k\varepsilon}\\ &\quad+\varepsilon(X^\varepsilon_{k\varepsilon},\theta(X^\varepsilon_{k\varepsilon})A(X^\varepsilon_{k\varepsilon})\varphi)+\eta^\varepsilon_{k+1},\end{aligned}\tag{13}$$

where

$$\begin{aligned}\eta^\varepsilon_{k+1}=&\sum_{i\in I,\ j_k\in J_k^i}\int_{\phi_\cdot^{i,\varepsilon}(k\varepsilon)}^{\phi_\cdot^{i,\varepsilon}((k+1)\varepsilon)}(\tilde X^{i,j_k}_u,A(X^\varepsilon_{k\varepsilon},V^{i,j_k})\varphi)\,\mathrm{d}u\\ &-\varepsilon(X^\varepsilon_{k\varepsilon},\theta(X^\varepsilon_{k\varepsilon})A(X^\varepsilon_{k\varepsilon})\varphi).\end{aligned}\tag{14}$$

From (9), and property (B), we get that u_k^{i,j_k} is $\mathscr{G}^\varepsilon_{k\varepsilon}$-measurable. Since $\sum_{i\in I,\ j_k\in J_k^i}\delta_{U^{i,j_k}}$ is conditionally on $\mathscr{G}^\varepsilon_{k\varepsilon}$ distributed according to a Poisson point measure with intensity $\int X^\varepsilon_{k\varepsilon}(\mathrm{d}x)\mathbb{N}_{x,A(X^\varepsilon_{k\varepsilon},x)}[\mathrm{d}W]$, we get from the definition of $X^\varepsilon_{(k+1)\varepsilon}$, formula (8) and (10), that

$$\mathbb{E}[(X^\varepsilon_{(k+1)\varepsilon},\varphi)|\mathscr{G}^\varepsilon_{k\varepsilon}]=\int\mu(\mathrm{d}x)\mathbb{N}_{x,A(\mu,x)}[(X_{\varepsilon\theta(\mu,x)},\varphi)],\tag{15}$$

with $\mu=X^\varepsilon_{k\varepsilon}$.

Lemma 2. *The process $((X^\varepsilon_{k\varepsilon},\mathbf{1}),\ k\in\mathbb{N})$ is an L^2 $\mathscr{G}^\varepsilon_{k\varepsilon}$-martingale. Moreover, we have*

$$\mathbb{E}\left[\sup_{k\leqslant T/\varepsilon}(X^\varepsilon_{k\varepsilon},\mathbf{1})^2\right]\leqslant 4T\|\theta\|_\infty(\mu_0,\mathbf{1})+4(\mu_0,\mathbf{1})^2.\tag{16}$$

Proof. We use the notation $\mu = X^\varepsilon_{k\varepsilon}$. From (15), we get

$$\mathbb{E}[(X^\varepsilon_{(k+1)\varepsilon}, \mathbf{1}) | \mathscr{G}^\varepsilon_{k\varepsilon}] = \int \mu(\mathrm{d}x)\ \mathbb{N}_{x,A(\mu,x)}[(X_{\varepsilon\theta(\mu,x)}, \mathbf{1})] = (\mu, \mathbf{1}).$$

Hence the process $((X^\varepsilon_{k\varepsilon}, \mathbf{1}), k \in \mathbb{N})$ is a non-negative $\mathscr{G}^\varepsilon_{k\varepsilon}$-martingale. Using the second moment formula for a Poisson point measure and (A.3) in the appendix, we get

$$\begin{aligned}\mathbb{E}[(X^\varepsilon_{(k+1)\varepsilon}, \mathbf{1})^2 | \mathscr{G}^\varepsilon_{k\varepsilon}] &= (\mu, \mathbf{1})^2 + \int \mu(\mathrm{d}x)\ \mathbb{N}_{x,A(\mu,x)}[(X_{\varepsilon\theta(\mu,x)}, \mathbf{1})^2] \\ &= (\mu, \mathbf{1})^2 + 4 \int \mu(\mathrm{d}x)\ \varepsilon\theta(\mu, x).\end{aligned}$$

We set $M_k = (X^\varepsilon_{k\varepsilon}, \mathbf{1})$, and $\langle M \rangle_k$ for its square function with $\langle M \rangle_0 = 0$. We deduce from the previous equality that

$$\begin{aligned}\langle M \rangle_{k+1} - \langle M \rangle_k &= \mathbb{E}[(X^\varepsilon_{(k+1)\varepsilon}, \mathbf{1})^2 - (X^\varepsilon_{k\varepsilon}, \mathbf{1})^2 | \mathscr{G}^\varepsilon_{k\varepsilon}] \\ &= 4 \int \mu(\mathrm{d}x)\ \varepsilon\theta(\mu, x) \\ &\leqslant 4\varepsilon \|\theta\|_\infty (X^\varepsilon_{k\varepsilon}, \mathbf{1}).\end{aligned}$$

Hence, we have

$$\mathbb{E}[\langle M \rangle_k] \leqslant 4\varepsilon \|\theta\|_\infty \sum_{l=0}^{k-1} \mathbb{E}[(X^\varepsilon_{l\varepsilon}, \mathbf{1})] \leqslant 4k\varepsilon \|\theta\|_\infty (\mu_0, \mathbf{1}).$$

Using Doob's inequality, we get

$$\begin{aligned}\mathbb{E}\left[\sup_{k \leqslant T/\varepsilon} (X^\varepsilon_{k\varepsilon}, \mathbf{1})^2\right] &\leqslant 4\mathbb{E}[(X^\varepsilon_{[T/\varepsilon]\varepsilon}, \mathbf{1})^2] = 4\mathbb{E}[\langle M \rangle_{[T/\varepsilon]}] + 4(\mu_0, 1)^2 \\ &\leqslant 4T \|\theta\|_\infty (\mu_0, \mathbf{1}) + 4(\mu_0, \mathbf{1})^2. \qquad \square\end{aligned}$$

Lemma 3. *The process $(M(\varphi)_{k\varepsilon},\ k \in \mathbb{N})$ is an L^2 $\mathscr{G}^\varepsilon_{k\varepsilon}$-martingale. We also have*:

1. *For $k \leqslant T/\varepsilon$,*

$$\mathbb{E}[(M(\varphi)_{(k+1)\varepsilon} - M(\varphi)_{k\varepsilon})^2 | \mathscr{G}^\varepsilon_{k\varepsilon}] \leqslant c\, \varepsilon \|\varphi\|^2_\infty (X^\varepsilon_{k\varepsilon}, \mathbf{1}), \tag{17}$$

   and

$$\mathbb{E}[(M(\varphi)_{(k+1)\varepsilon} - M(\varphi)_{k\varepsilon})^4 | \mathscr{G}^\varepsilon_{k\varepsilon}] \leqslant c\varepsilon^2 \|\varphi\|^4_\infty (1 + (X^\varepsilon_{k\varepsilon}, \mathbf{1})^2), \tag{18}$$

   where the constant c is independent of φ, k and ε.

2. *For $t = k\varepsilon$, $s = l\varepsilon$, where $l, k \in \mathbb{N}$, and $0 \leqslant s \leqslant t \leqslant T$,*

$$\langle M(\varphi) \rangle_t - \langle M(\varphi) \rangle_s \leqslant c(t - s) \|\varphi\|^2_\infty \sup_{k' \leqslant T/\varepsilon} (X^\varepsilon_{k'\varepsilon}, \mathbf{1}), \tag{19}$$

   where the constant c is independent of φ, t, s and ε.

3. *We have*

$$\mathbb{E}[\langle M(\varphi)\rangle^2_{[T/\varepsilon]\varepsilon}] \leqslant c\|\varphi\|^4_\infty, \tag{20}$$

where the constant c is independent of φ *and* ε.

Proof. We still use the notation $\mu = X^\varepsilon_{k\varepsilon}$.

From (13), it is easy to prove by induction that $M(\varphi)_{k\varepsilon}$ is integrable. Let us now prove that $(M(\varphi)_{k\varepsilon},\ k\in\mathbb{N})$ is a martingale. From (12), using (7) the definition of $\tilde{X}^{i,j_k}$, the fact that $\sum_{i\in I,\ j_k\in J^i_k}\delta_{(V^{i,j_k},U^{i,j_k})}$ is conditionally on $\mathscr{G}^\varepsilon_{k\varepsilon}$ distributed according to a Poisson point measure with intensity $\mu(\mathrm{d}x)\mathbb{N}_{x,A(\mu,x)}[\mathrm{d}W]$ and eventually (15) we get

$$\begin{aligned}
&\mathbb{E}[M(\varphi)_{(k+1)\varepsilon} - M(\varphi)_{k\varepsilon}|\mathscr{G}^\varepsilon_{k\varepsilon}]\\
&\quad= \mathbb{E}[(X^\varepsilon_{(k+1)\varepsilon},\varphi)|\mathscr{G}^\varepsilon_{k\varepsilon}] - (\mu,\varphi)\\
&\qquad- \mathbb{E}\left[\sum_{i\in I,\ j_k\in J^i_k}\int_{\phi^{i,\varepsilon}_\cdot(k\varepsilon)}^{\phi^{i,\varepsilon}_\cdot((k+1)\varepsilon)} (\tilde{X}^{i,j_k}_u, A(X^\varepsilon_{k\varepsilon}, V^{i,j_k})\varphi)\,\mathrm{d}u|\mathscr{G}^\varepsilon_{k\varepsilon}\right]\\
&\quad= \int \mu(\mathrm{d}x)\ \mathbb{N}_{x,A(\mu,x)}[(X_{\varepsilon\theta(\mu,x)},\varphi)] - (\mu,\varphi)\\
&\qquad- \int \mu(\mathrm{d}x)\ \mathbb{N}_{x,A(\mu,x)}\left[\int_0^{\varepsilon\theta(x,\mu)} \mathrm{d}u(X_u, A(\mu,x)\varphi)\right]\\
&\quad= \int \mu(\mathrm{d}x)\ \left(\mathbb{E}_x\left[\varphi(Z_{\varepsilon\theta(x,\mu)}) - \varphi(x) - \int_0^{\varepsilon\theta(x,\mu)} \mathrm{d}u\,A(\mu,x)\varphi(Z_u)\right]\right)\\
&\quad= 0.
\end{aligned}$$

For the third equality, we introduced the process $(Z_s,\ s\geqslant 0)$ which is under $\mathbb{E}_x$ a diffusion with infinitesimal generator $A(\mu,x)$ started at point x, and we used the first moment formula (A.1) for the Brownian snake.

Hence $(M(\varphi)_{k\varepsilon},\ k\in\mathbb{N})$ is a martingale.

1. Let us now compute $\mathbb{E}[(M(\varphi)_{(k+1)\varepsilon} - M(\varphi)_{k\varepsilon})^p|\mathscr{G}^\varepsilon_{k\varepsilon}]$ for $p=2,4$. We have

$$\begin{aligned}
&\mathbb{E}[(M(\varphi)_{(k+1)\varepsilon} - M(\varphi)_{k\varepsilon})^p|\mathscr{G}^\varepsilon_{k\varepsilon}]\\
&\quad= \mathbb{E}\left[\left((\mu,\varphi) - \sum_{i\in I,\ j_k\in J^i_k}\Delta M^{i,j_k}(\varphi)\right)^p\Bigg|\mathscr{G}^\varepsilon_{k\varepsilon}\right]\\
&\quad= \sum_{m=0}^p (-1)^m\binom{p}{m}(\mu,\varphi)^{p-m}\mathbb{E}\left[\left(\sum_{i\in I,\ j_k\in J^i_k}\Delta M^{i,j_k}(\varphi)\right)^m\Bigg|\mathscr{G}^\varepsilon_{k\varepsilon}\right],
\end{aligned}\tag{21}$$

with

$$\Delta M^{i,j_k}(\varphi) = (X^{i,j_k}_{(k+1)\varepsilon}, \varphi) - \int_{\phi^{i,\varepsilon}_{\cdot}(k\varepsilon)}^{\phi^{i,\varepsilon}_{\cdot}((k+1)\varepsilon)} (\tilde{X}^{i,j_k}_u, A(X^\varepsilon_{k\varepsilon}, V^{i,j_k})\varphi)\,\mathrm{d}u.$$

First of all, let us compute the Laplace transform

$$\mathscr{A} = \mathbb{E}\left[\exp - \left\{\lambda \sum_{i\in I,\ j_k\in J^i_k} \left((X^{i,j_k}_{(k+1)\varepsilon}, \varphi) \right.\right.\right.$$

$$\left.\left.\left. + \int_{\phi^{i,\varepsilon}_{\cdot}(k\varepsilon)}^{\phi^{i,\varepsilon}_{\cdot}((k+1)\varepsilon)} \mathrm{d}u (\tilde{X}^{i,j_k}_u, \psi(V^{i,j_k}, \cdot)) \right) \right\} \middle| \mathscr{G}^\varepsilon_{k\varepsilon} \right], \tag{22}$$

where φ and ψ are non-negative bounded measurable functions defined, respectively, on $\mathbb{R}^d$ and $\mathbb{R}^d \times \mathbb{R}^d$, and $\lambda \geqslant 0$. Using the Laplace transform for Poisson point measure, we have

$$\mathscr{A} = \exp - \int \mu(\mathrm{d}x)\ \mathbb{N}_{x,A(\mu,x)} \left[1 - \exp - \lambda \left((X_{\varepsilon\theta(\mu,x)}, \varphi) \right.\right.$$

$$\left.\left. + \int_0^{\varepsilon\theta(\mu,x)} \mathrm{d}u (X_u, \psi(x, \cdot)) \right) \right].$$

Let us introduce $(P_t,\ t \geqslant 0)$ the transition kernel of the diffusion with infinitesimal generator $A(\mu, x_0)$ for $x_0 \in \mathbb{R}^d$ fixed. If we define

$$v_{\lambda,x_0}(t,x) = \mathbb{N}_{x,A(\mu,x_0)} \left[1 - \exp - \lambda \left((X_t, \varphi) + \int_0^t \mathrm{d}u (X_u, \psi(x_0, \cdot)) \right) \right],$$

then v_{λ,x_0} solves the equation

$$v_{\lambda,x_0}(t,x) + 2 \int_0^t P_{t-s}(v_{\lambda,x_0}(s)^2)(x)\,\mathrm{d}s$$

$$= \lambda \mathbb{N}_{x,A(\mu,x_0)} \left[(X_t, \varphi) + \int_0^t \mathrm{d}u (X_u, \psi(x_0, \cdot)) \right].$$

For λ small enough, the function v_{λ,x_0} can be developed as a power series in λ. In particular,

$$v_{\lambda,x_0}(t,x) = \lambda\alpha_{1,x_0}(t,x) + \lambda^2\alpha_{2,x_0}(t,x) + \lambda^3\alpha_{3,x_0}(t,x) + \lambda^4\alpha_{4,x_0}(t,x) + \lambda^5 g_{\lambda,x_0}(t,x),$$

where g is uniformly bounded in $(t, x, x_0, \lambda) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times [0, 1]$. Using the previous integral equation, we have

$$\alpha_{1,x_0}(t,x) = \mathbb{N}_{x,A(\mu,x_0)} \left[(X_t, \varphi) + \int_0^t \mathrm{d}u (X_u, \psi(x_0, \cdot)) \right],$$

$$\alpha_{2,x_0}(t,x) = -2 \int_0^t P_{t-s}(\alpha_{1,x_0}(s)^2)(x)\,\mathrm{d}s,$$

$$\alpha_{3,x_0}(t,x) = -4\int_0^t P_{t-s}(\alpha_{1,x_0}(s)\alpha_{2,x_0}(s))(x)\,\mathrm{d}s,$$

$$\alpha_{4,x_0}(t,x) = -2\int_0^t P_{t-s}(\alpha_{2,x_0}(s)^2 + 2\alpha_{1,x_0}(s)\alpha_{3,x_0}(s))(x)\,\mathrm{d}s.$$

So we have, with the notation $\alpha_i = \alpha_{i,\cdot}(\varepsilon\theta(\mu,\cdot),\cdot)$,

$$\begin{aligned}
\mathscr{A} &= \exp - \int \mu(\mathrm{d}x)\ \mathbb{N}_{x,A(\mu,x)}\left[1 - \exp\left[-\lambda\left((X_{\varepsilon\theta(\mu,x)},\varphi) \right.\right.\right.\\
&\quad \left.\left.\left. + \int_0^{\varepsilon\theta(\mu,x)} \mathrm{d}u(X_u,\psi(x,\cdot))\right)\right]\right]\\
&= 1 - \lambda(\mu,\alpha_1) + \lambda^2\left[-(\mu,\alpha_2) + \tfrac{1}{2}(\mu,\alpha_1)^2\right]\\
&\quad + \lambda^3\left[-(\mu,\alpha_3) + (\mu,\alpha_2)(\mu,\alpha_1) - \tfrac{1}{6}(\mu,\alpha_1)^3\right]\\
&\quad + \lambda^4\Big[-(\mu,\alpha_4) + (\mu,\alpha_3)(\mu,\alpha_1)\\
&\quad + \tfrac{1}{2}(\mu,\alpha_2)^2 - \tfrac{1}{2}(\mu,\alpha_2)(\mu,\alpha_1)^2 + \tfrac{1}{24}(\mu,\alpha_1)^4\Big] + o(\lambda^4).
\end{aligned}$$

We deduce from (22) that

$$\mathbb{E}\left[\left(\sum_{i\in I,\ j_k\in J_k^i} X^{i,j_k}_{(k+1)\varepsilon},\varphi\right) + \int_{\phi^{i,\varepsilon}_{\cdot}(k\varepsilon)}^{\phi^{i,\varepsilon}_{\cdot}((k+1)\varepsilon)} (\tilde{X}^{i,j_k}_u,\psi(V^{i,j_k},\cdot))\,\mathrm{d}u\,\middle|\,\mathscr{G}^{\varepsilon}_{k\varepsilon}\right] = (\mu,\alpha_1),$$

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i\in I,\ j_k\in J_k^i} (X^{i,j_k}_{(k+1)\varepsilon},\varphi) + \int_{\phi^{i,\varepsilon}_{\cdot}(k\varepsilon)}^{\phi^{i,\varepsilon}_{\cdot}((k+1)\varepsilon)} (\tilde{X}^{i,j_k}_u,\psi(V^{i,j_k},\cdot))\,\mathrm{d}u\right)^2\,\middle|\,\mathscr{G}^{\varepsilon}_{k\varepsilon}\right]\\
&\quad = (\mu,\alpha_1)^2 - 2(\mu,\alpha_2),
\end{aligned}\tag{23}$$

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i\in I,\ j_k\in J_k^i} (X^{i,j_k}_{(k+1)\varepsilon},\varphi) + \int_{\phi^{i,\varepsilon}_{\cdot}(k\varepsilon)}^{\phi^{i,\varepsilon}_{\cdot}((k+1)\varepsilon)} (\tilde{X}^{i,j_k}_u,\psi(V^{i,j_k},\cdot))\,\mathrm{d}u\right)^3\,\middle|\,\mathscr{G}^{\varepsilon}_{k\varepsilon}\right]\\
&\quad = (\mu,\alpha_1)^3 - 6(\mu,\alpha_1)(\mu,\alpha_2) + 6(\mu,\alpha_3),\\
&\mathbb{E}\left[\left(\sum_{i\in I,\ j_k\in J_k^i} (X^{i,j_k}_{(k+1)\varepsilon},\varphi) + \int_{\phi^{i,\varepsilon}_{\cdot}(k\varepsilon)}^{\phi^{i,\varepsilon}_{\cdot}((k+1)\varepsilon)} (\tilde{X}^{i,j_k}_u,\psi(V^{i,j_k},\cdot))\,\mathrm{d}u\right)^4\,\middle|\,\mathscr{G}^{\varepsilon}_{k\varepsilon}\right]\\
&\quad = (\mu,\alpha_1)^4 - 12(\mu,\alpha_1)^2(\mu,\alpha_2) + 12(\mu,\alpha_2)^2 + 24(\mu,\alpha_1)(\mu,\alpha_3) - 24(\mu,\alpha_4).
\end{aligned}$$

Using a polarization argument, we have the same result for any bounded measurable function φ and ψ. In particular, we can take $\psi(x_0,\cdot) = -A(\mu,x_0)\varphi$. Moreover, in that case,

$$\begin{aligned}\alpha_{1,x_0}(t,x) &= \mathbb{N}_{x,A(\mu,x_0)}\left[(X_t,\varphi) + \int_0^t \mathrm{d}u(X_u,\psi(x_0,\cdot))\right]\\ &= \mathbb{E}_x\left[\varphi(Z_t) - \int_0^t A(\mu,x_0)\varphi(Z_s)\,\mathrm{d}s\right]\\ &= \varphi(x),\end{aligned}$$

where $(Z_t,\ t \geqslant 0)$ is a diffusion with infinitesimal generator $A(\mu,x_0)$ started at x.

We also have upper bounds for the others α_i:

$$|\alpha_{2,x_0}(t,x)| = \left|-2\int_0^t P_{t-s}(\alpha_{1,x_0}(s)^2)(x)\,\mathrm{d}s\right| \leqslant 2t\|\varphi\|_\infty^2$$

$$|\alpha_{3,x_0}(t,x)| = \left|-4\int_0^t P_{t-s}(\alpha_{1,x_0}(s)\alpha_{2,x_0}(s))(x)\,\mathrm{d}s\right| \leqslant 4t^2\|\varphi\|_\infty^3$$

$$|\alpha_{4,x_0}(t,x)| = \left|-2\int_0^t P_{t-s}(\alpha_{2,x_0}(s)^2 + 2\alpha_{1,x_0}(s)\alpha_{3,x_0}(s))(x)\,\mathrm{d}s\right| \leqslant 8t^3\|\varphi\|_\infty^4.$$

Since θ is bounded from above, we get using the previous computations and formula (21)

$$\begin{aligned}\mathbb{E}[(M(\varphi)_{(k+1)\varepsilon} - M(\varphi)_{k\varepsilon})^2|\mathscr{G}_{k\varepsilon}^\varepsilon] &= -2(\mu,\alpha_2)\\ &\leqslant c\varepsilon(\mu,1)\|\varphi\|_\infty^2\\ &= c\varepsilon\|\varphi\|_\infty^2(X_{k\varepsilon}^\varepsilon,\mathbf{1}),\end{aligned}\tag{24}$$

and

$$\begin{aligned}\mathbb{E}[(M(\varphi)_{(k+1)\varepsilon} - M(\varphi)_{k\varepsilon})^4|\mathscr{G}_{k\varepsilon}^\varepsilon] &= 12(\mu,\alpha_2)^2 - 24(\mu,\alpha_4)\\ &\leqslant c\varepsilon^2\|\varphi\|_\infty^4((X_{k\varepsilon}^\varepsilon,\mathbf{1})^2 + (X_{k\varepsilon}^\varepsilon,\mathbf{1}))\\ &\leqslant c\varepsilon^2\|\varphi\|_\infty^4(1 + (X_{k\varepsilon}^\varepsilon,\mathbf{1})^2).\end{aligned}$$

2. It is a direct consequence of (17).

3. Recall c denotes a constant which value may vary from line to line. From (19), we deduce that for $k \leqslant T/\varepsilon$,

$$\begin{aligned}\mathbb{E}[\langle M(\varphi)\rangle_{k\varepsilon}^2] &\leqslant c\varepsilon^2k^2\|\varphi\|_\infty^4\,\mathbb{E}\left[\sup_{l\leqslant T/\varepsilon}(X_{l\varepsilon}^\varepsilon,\mathbf{1})^2\right]\\ &\leqslant c\varepsilon^2k^2\|\varphi\|_\infty^4(T(\mu_0,\mathbf{1}) + (\mu_0,\mathbf{1})^2),\end{aligned}$$

where we used (16). We deduce (20). $\square$

Recall the definition of η_k^ε:

$$\eta_{k+1}^\varepsilon = \sum_{i\in I,\ j_k\in J_k^i} \int_{\phi_\cdot^{i,\varepsilon}(k\varepsilon)}^{\phi_\cdot^{i,\varepsilon}((k+1)\varepsilon)} (\tilde{X}_u^{i,j_k}, A(X_{k\varepsilon}^\varepsilon, V^{i,j_k})\varphi)\,\mathrm{d}u - \varepsilon(X_{k\varepsilon}^\varepsilon, \theta(X_{k\varepsilon}^\varepsilon)A(X_{k\varepsilon}^\varepsilon)\varphi).$$

Lemma 4. *We have the convergence of* $\sup_{0\leqslant l\leqslant[T/\varepsilon]}|\sum_{k=1}^l \eta_k^\varepsilon|$ *to* 0 *in* L^1 *as* ε *decreases to* 0.

Proof. We still use the notation $\mu = X_{k\varepsilon}^\varepsilon$. We first prove that $\mathbb{E}[(\eta_{k+1}^\varepsilon)^2|\mathcal{G}_{k\varepsilon}^\varepsilon]$ can be bounded from above by $c\varepsilon^3(1+(\mu,\mathbf{1})^2)$. Applying (23) with the function 0 for φ and the function $A(\mu,x)\varphi(\cdot)$ for $\psi(x,\cdot)$, we get

$$\mathbb{E}\left[\left(\sum_{i\in I,\ j_k\in J_k^i} \int_{\phi_\cdot^{i,\varepsilon}(k\varepsilon)}^{\phi_\cdot^{i,\varepsilon}((k+1)\varepsilon)} (\tilde{X}_u^{i,j_k}, A(X_{k\varepsilon}^\varepsilon, V^{i,j_k})\varphi)\,\mathrm{d}u\right)^2 \middle| \mathcal{G}_{k\varepsilon}^\varepsilon\right] = (\mu,\alpha_1)^2 - 2(\mu,\alpha_2),$$

where, thanks to (A.1),

$$\alpha_1(x) = \mathbb{N}_{x,A(\mu,x)}\left[\int_0^{\varepsilon\theta(\mu,x)} \mathrm{d}u(X_u, A(\mu,x)\varphi)\right] = \int_0^{\varepsilon\theta(\mu,x)} \mathrm{d}u\ \mathrm{P}_u(A(\mu,x)\varphi)(x),$$

and $(\mathrm{P}_u,\ u\geqslant 0)$ denotes the transition semi-group with infinitesimal generator $A(\mu,x)$. By definition, we have $\alpha_2(x) = \alpha_{2,x}(\varepsilon\theta(\mu,x),x)$, where, thanks to (A.1),

$$\begin{aligned}\alpha_{2,x_0}(t,x) &= -2\int_0^t P_{t-s}(\alpha_{1,x_0}(s)^2)(x)\,\mathrm{d}s\\ &= -2\int_0^t P_{t-s}\left(\left(\int_0^s P_u(A(\mu,x_0)\varphi)\right)^2\right)(x)\,\mathrm{d}s.\end{aligned}$$

So we have

$$\begin{aligned}&\mathbb{E}[(\eta_{k+1}^\varepsilon)^2|\mathcal{G}_{k\varepsilon}^\varepsilon]\\ &\quad= (\mu,\alpha_1)^2 - 2(\mu,\alpha_2) + \varepsilon^2\left(\int \mu(\mathrm{d}x)\ \theta(\mu,x)A(\mu,x)\varphi(x)\right)^2\\ &\qquad - 2\varepsilon(\mu,\alpha_1)\int \mu(\mathrm{d}x)\theta(\mu,x)A(\mu,x)\varphi(x).\end{aligned}$$

We deduce that

$$\begin{aligned}\mathbb{E}[(\eta_{k+1}^\varepsilon)^2|\mathcal{G}_{k\varepsilon}^\varepsilon] &= -2(\mu,\alpha_2)\\ &\quad + \left(\int \mu(\mathrm{d}x)\int_0^{\varepsilon\theta(\mu,x)} \mathrm{d}u[\mathrm{P}_u(A(\mu,x)\varphi) - A(\mu,x)\varphi](x)\right)^2.\end{aligned}$$

Since the coefficients of $A(\mu,x)$ are uniformly bounded, we deduce that $|\alpha_{2,x_0}(t,x)| \leqslant c\|\varphi\|_*^2 t^3$, that is $\|\alpha_2\|_\infty \leqslant c\|\varphi\|_*^2 \varepsilon^3$, where c depends on θ, b and σ. Since the coefficients of $A(\mu,x)$ are uniformly bounded, we deduce that for $0 \leqslant u \leqslant \|\theta\|_\infty$,

$$\|\mathrm{P}_u(A(\mu,x)\varphi) - A(\mu,x)\varphi\|_\infty \leqslant c\|\varphi\|_* \sqrt{u}.$$

We get that for $\varepsilon \in (0,1]$,

$$\mathbb{E}[(\eta_{k+1}^\varepsilon)^2 | \mathscr{G}_{k\varepsilon}^\varepsilon] \leqslant c\varepsilon^3(\mu,\mathbf{1})\|\varphi\|_*^2 + c\varepsilon^3(\mu,\mathbf{1})^2\|\varphi_*^2.$$

In particular, we have for $\varepsilon \in (0,1]$,

$$\mathbb{E}[|\eta_{k+1}^\varepsilon| \mathscr{G}_{k\varepsilon}^\varepsilon] \leqslant \mathbb{E}[(\eta_{k+1}^\varepsilon)^2 | \mathscr{G}_{k\varepsilon}^\varepsilon]^{1/2} \leqslant c\varepsilon^{3/2}\|\varphi\|_*((X_{k\varepsilon}^\varepsilon,\mathbf{1}) + 1),$$

where the constant c depends only on the bounds of θ, b and σ. Therefore, we deduce that for $T > 0$,

$$\mathbb{E}\left[\sum_{k=1}^{[T/\varepsilon]} |\eta_k|\right] \leqslant c\sqrt{\varepsilon}T\|\varphi\|_* \left(1 + \mathbb{E}\left[\sup_{0\leqslant k\leqslant [T/\varepsilon]} (X_{k\varepsilon}^\varepsilon,\mathbf{1})\right]\right).$$

From Lemma 2, we deduce that

$$\mathbb{E}\left[\sum_{k=1}^{[T/\varepsilon]} |\eta_k|\right] \leqslant c\sqrt{\varepsilon}T(1+T)\|\varphi\|_*, \tag{25}$$

where c depends only on the bounds of θ, b and σ. Therefore, we have the convergence of $\sup_{0\leqslant l\leqslant [T/\varepsilon]}|\sum_{k=1}^{l} \eta_k|$ to 0 in L^1 as ε decreases to 0. □

Lemma 5. *We have*

$$\langle M(\varphi)\rangle_{(k+1)\varepsilon} = \langle M(\varphi)\rangle_{k\varepsilon} + 4\varepsilon(X_{k\varepsilon}^\varepsilon, \theta(X_{k\varepsilon}^\varepsilon)\varphi^2) + \kappa_k,$$

where $\sum_{k=0}^{[T/\varepsilon]} \kappa_k$ *converge in* L^2 *to* 0 *as* ε *decreases to* 0.

Proof. We still write μ for $X_{k\varepsilon}^\varepsilon$. Recall from (24) that

$$\langle M(\varphi)\rangle_{(k+1)\varepsilon} - \langle M(\varphi)\rangle_{k\varepsilon} = 4\int \mu(\mathrm{d}x)\int_0^{\varepsilon\theta(\mu,x)} \mathrm{d}s\ \mathbb{E}_x[\varphi(Z_s)^2],$$

where $(Z_s,\ s\geqslant 0)$ is under $\mathbb{E}_x$ a diffusion with infinitesimal generator $A(\mu,x)$ started at point x. In particular, for $s \in [0,\|\theta\|_\infty]$,

$$\begin{aligned}\mathbb{E}_x[|\varphi(Z_s)^2 - \varphi(Z_0)^2|] &\leqslant 2\|\varphi\|_\infty\|\varphi\|_{\mathrm{Lip}}\mathbb{E}_x[|Z_s - Z_0|]\\ &\leqslant c\|\varphi\|_\infty\|\varphi\|_*\sqrt{s},\end{aligned}$$

where the constant c depends only on θ, b, σ and T. Therefore, we have for $\varepsilon \in (0,1]$,

$$\begin{aligned}|\kappa_k| &\leqslant 4\int \mu(\mathrm{d}x)\int_0^{\varepsilon\theta(\mu,x)} \mathrm{d}s\ \mathbb{E}_x[|\varphi(Z_s)^2 - \varphi(Z_0)^2|]\\ &\leqslant c\varepsilon^{3/2}\|\varphi\|_\infty\|\varphi\|_*(\mu,\mathbf{1}),\end{aligned}$$

We deduce that for $T \geqslant 0$, $\varepsilon \in (0,1]$,

$$\sum_{k=0}^{[T/\varepsilon]} |\kappa_k| \leqslant c\sqrt{\varepsilon} T \|\varphi\|_\infty \|\varphi\|_* \sup_{0 \leqslant k \leqslant [T/\varepsilon]} (X^\varepsilon_{k\varepsilon}, \mathbf{1}). \tag{26}$$

We deduce from Lemma 2, that $\sum_{k=0}^{[T/\varepsilon]} \kappa_k$ converges to 0 in L^2 as ε decreases to 0. □

Lemma 6. *For every $\rho, T > 0$, there is a compact set $K_{\rho,T}$ in $\mathbb{R}^d$ such that*

$$\sup_{0<\varepsilon\leqslant 1} \mathbb{P}\left(\sup_{0\leqslant k\leqslant T/\varepsilon} (X^\varepsilon_{k\varepsilon}, \mathbf{1}_{K^c_{\rho,T}}) > \rho \right) < \rho.$$

Proof. Let B_R denote the centered ball of $\mathbb{R}^d$ with radius R. Let g be a non-negative function of class C^2 with bounded Lipschitz derivatives, defined on $\mathbb{R}^d$ such that $g = 0$ on B_1, and $g = 1$ outside B_2. Set $g_R(x) = g(x/R)$, with $R \geqslant 1$. We want to check that $\mathbb{E}[\sup_{0\leqslant k\leqslant [T/\varepsilon]} (X^\varepsilon_{k\varepsilon}, g_R)]$ converges to 0 as R increases to ∞ uniformly in $\varepsilon \in (0,1]$. From

$$(X^\varepsilon_{k\varepsilon}, g_R) = M(g_R)_{k\varepsilon} + \sum_{l=1}^{k} \eta_l + \varepsilon \sum_{l=1}^{k} (X^\varepsilon_{l\varepsilon}, \theta(X^\varepsilon_{l\varepsilon}) A(X^\varepsilon_{l\varepsilon}) g_R) + (\mu_0, g_R),$$

we deduce that

$$\sup_{0\leqslant k\leqslant [T/\varepsilon]} (X^\varepsilon_{k\varepsilon}, g_R) \leqslant \sup_{0\leqslant k\leqslant [T/\varepsilon]} M(g_R)_{k\varepsilon} + \sum_{k=1}^{[T/\varepsilon]} |\eta_k|$$
$$+ c\,\frac{T}{R} \sup_{0\leqslant k\leqslant [T/\varepsilon]} (X^\varepsilon_{k\varepsilon}, \mathbf{1}) + (\mu_0, g_R), \tag{27}$$

where c depends only on θ, b and σ. We used that

$$\|g_R\|_* \leqslant \frac{1}{R} \|g\|_* \leqslant c/R.$$

Using this inequality again, we deduce from (25), with φ replaced by g_R, and (16) respectively that the second and third terms of the right-hand member converge to 0 in L^1 as R increases to $+\infty$ uniformly in $\varepsilon \in (0,1]$. Notice also that the last term, (μ_0, g_R), converges to 0 as R increases to $+\infty$.

From Doob's inequality and the definition of κ (in Lemma 5), we get

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leqslant k\leqslant [T/\varepsilon]} M(g_R)^2_{k\varepsilon} \right] &\leqslant 4\mathbb{E}[M(g_R)^2_{[T/\varepsilon]\varepsilon}] \\
&= 4\mathbb{E}[\langle M(g_R) \rangle_{[T/\varepsilon]\varepsilon}] \\
&\leqslant 4\mathbb{E}\left[\sum_{l=1}^{[T/\varepsilon]} \kappa_l \right] + 4\varepsilon \sum_{k=0}^{[T/\varepsilon]} \mathbb{E}[(X^\varepsilon_{k\varepsilon}, \theta(X^\varepsilon_{k\varepsilon}) g_R^2)].
\end{aligned} \tag{28}$$

We deduce from (26) with φ replaced by g_R, and (16), that $\mathbb{E}[\sum_{l=1}^{[T/\varepsilon]} \kappa_l]$ converges to 0 as R increases to $+\infty$ uniformly in $\varepsilon \in (0,1]$. Since θ is bounded from above, we deduce from (13) and then (25) that, for $k \leqslant [T/\varepsilon]$,

$$
\begin{aligned}
\mathbb{E}[(X^\varepsilon_{k\varepsilon}, \theta(X^\varepsilon_{k\varepsilon}) g_R^2)] &\leqslant c\mathbb{E}[(X^\varepsilon_{k\varepsilon}, g_R^2)] \\
&\leqslant c\mathbb{E}\left[\sum_{l=1}^{k} |\eta_l^\varepsilon|\right] + c\varepsilon\sum_{l=0}^{k-1} \mathbb{E}[(X^\varepsilon_{l\varepsilon}, \theta(X^\varepsilon_{l\varepsilon}) A(X^\varepsilon_{l\varepsilon}) g_R^2)] + c(\mu_0, g_R^2) \\
&\leqslant c\sqrt{\varepsilon}\|g_R^2\|_* + c\varepsilon\frac{1}{R}\sum_{l=0}^{k-1} \mathbb{E}[(X^\varepsilon_{l\varepsilon}, \mathbf{1})] + c(\mu_0, g_R^2) \\
&\leqslant c\sqrt{\varepsilon}\,\frac{1}{R} + c\varepsilon\,\frac{1}{R}\,k\mathbb{E}\left[\sup_{0\leqslant l\leqslant [T/\varepsilon]} (X^\varepsilon_{l\varepsilon}, \mathbf{1})\right] + c(\mu_0, g_R^2),
\end{aligned}
$$

where the constant c depends only on θ, b, σ and T. Since $\lim_{R\to\infty} (\mu_0, g_R^2) = 0$, we deduce that $\mathbb{E}[(X^\varepsilon_{k\varepsilon}, \theta(X^\varepsilon_{k\varepsilon}) g_R^2)]$ converges to 0 as R increases to $+\infty$ uniformly in $\varepsilon \in (0,1]$.

We deduce from those results and the upper bound in (28), that $\mathbb{E}[\sup_{0\leqslant k\leqslant [T/\varepsilon]} M(g_R)^2_{k\varepsilon}]$ decreases to 0 as R increases to $+\infty$ uniformly in $\varepsilon \in (0,1]$. This implies, thanks to (27), that $\mathbb{E}[\sup_{0\leqslant k\leqslant T/\varepsilon} (X^\varepsilon_{k\varepsilon}, \mathbf{1}_{B^c_{2R}})]$ decreases to 0 as R increases to ∞ uniformly in $\varepsilon \in (0,1]$. In particular, for every $\rho, T > 0$, there exist $R > 1$ such that

$$
\sup_{0<\varepsilon\leqslant 1} \mathbb{P}\left(\sup_{0\leqslant k\leqslant T/\varepsilon} (X^\varepsilon_{k\varepsilon}, \mathbf{1}_{B^c_{2R}}) > \rho\right) < \rho. \qquad \square
$$

4. Proof of Theorem 1

The proof will be done in five lemmas and follows (Perkins, 2002, Section II.4). Theorem 1 is a direct consequence of Lemmas 10 and 12. Let $\varphi \in C^2$ be such that $\|\varphi\|_* < \infty$. To remember that $M(\varphi)$ depends on ε, we will write now $M^\varepsilon(\varphi)$ instead of $M(\varphi)$. Let $M^\varepsilon(\varphi) = (M^\varepsilon(\varphi)_t,\ t \geqslant 0)$ be the right continuous step function which is the extension of $(M^\varepsilon(\varphi)_{k\varepsilon},\ k \in \mathbb{N})$.

Lemma 7. *The process $(\langle M^\varepsilon(\varphi)\rangle,\ \varepsilon \in (0,1])$ is C-tight as ε decreases to 0.*

Proof. Thanks to Proposition VI.3.26 of Jacod and Shiryaev (1987), it is enough to check that for all $T > 0$, $\alpha > 0$ and $\eta > 0$, there exist $K > 0$ and $h > 0$, $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$,

$$
\mathbb{P}\left(\sup_{t\leqslant T} \langle M^\varepsilon(\varphi)\rangle_t \geqslant K\right) \leqslant \alpha, \tag{29}
$$

$$
\mathbb{P}\left(\sup_{s\leqslant t\leqslant T, |t-s|\leqslant h} \langle M^\varepsilon(\varphi)\rangle_t - \langle M^\varepsilon(\varphi)\rangle_s \geqslant \eta\right) \leqslant \alpha. \tag{30}
$$

Using (19) in Lemma 3 with $s=0$ and $t=T$, we have

$$\mathbb{P}\left(\sup_{t\leqslant T}\langle M^\varepsilon(\varphi)\rangle_t \geqslant K\right) \leqslant \frac{1}{K}\,\mathbb{E}[\langle M^\varepsilon(\varphi)\rangle_T]$$

$$\leqslant \frac{1}{K}\,c\|\varphi\|_\infty^2 T\,\mathbb{E}\left[\sup_{k\leqslant T/\varepsilon}(X^\varepsilon_{k\varepsilon},1)\right]$$

$$\leqslant \frac{1}{K}\,c\|\varphi\|_\infty^2 T\,\mathbb{E}\left[\sup_{k\leqslant T/\varepsilon}(X^\varepsilon_{k\varepsilon},1)^2\right]^{1/2}.$$

Then (29) can be deduced from Lemma 2.

Notice that if $|t-s|\leqslant h$, then $|[t/\varepsilon]\varepsilon-[s/\varepsilon]\varepsilon|\leqslant h+\varepsilon$. Using again (19) in Lemma 3, we have

$$\mathbb{P}\left(\sup_{s\leqslant t\leqslant T, |t-s|\leqslant h}\langle M^\varepsilon(\varphi)\rangle_t - \langle M^\varepsilon(\varphi)\rangle_s \geqslant \eta\right)$$

$$\leqslant \mathbb{P}\left(c(h+\varepsilon)\|\varphi\|_\infty^2 \sup_{k\leqslant T/\varepsilon}(X^\varepsilon_{k\varepsilon},\mathbf{1})\geqslant\eta\right)$$

$$\leqslant \frac{c^2(h+\varepsilon)^2\|\varphi\|_\infty^4}{\eta^2}\,\mathbb{E}\left[\sup_{k\leqslant T/\varepsilon}(X^\varepsilon_{k\varepsilon},\mathbf{1})^2\right].$$

And (30) can be deduced from Lemma 2. □

Lemma 8. *The process* $(M^\varepsilon(\varphi),\ \varepsilon\in(0,1])$ *is C-tight as* ε *decreases to* 0.

Proof. We have already proved the C-tightness of $(\langle M^\varepsilon(\varphi)\rangle,\ \varepsilon\in(0,1])$. From Theorem VI.4.13 of Jacod and Shiryaev (1987), we get that $(M^\varepsilon(\varphi),\ \varepsilon\in(0,1])$ is tight. To get the C-tightness, it is enough to check (see Proposition VI.3.26 of Jacod and Shiryaev, 1987) that for all $T>0$ and all $\eta>0$,

$$\lim_{\varepsilon\to 0}\mathbb{P}\left(\sup_{k\leqslant T/\varepsilon}|M^\varepsilon(\varphi)_{(k+1)\varepsilon}-M^\varepsilon(\varphi)_{k\varepsilon}|\geqslant\eta\right)=0.$$

We have:

$$\mathbb{P}\left(\sup_{k\leqslant T/\varepsilon}|M^\varepsilon(\varphi)_{(k+1)\varepsilon}-M^\varepsilon(\varphi)_{k\varepsilon}|\geqslant\eta\right)$$

$$\leqslant \frac{1}{\eta^4}\,\mathbb{E}\left[\sup_{k\leqslant T/\varepsilon}(M^\varepsilon(\varphi)_{(k+1)\varepsilon}-M^\varepsilon(\varphi)_{k\varepsilon})^4\right]$$

$$\leqslant \frac{1}{\eta^4}\,\mathbb{E}\left[\sum_{k\leqslant T/\varepsilon}(M^\varepsilon(\varphi)_{(k+1)\varepsilon}-M^\varepsilon(\varphi)_{k\varepsilon})^4\right]$$

$$\leqslant \frac{1}{\eta^4}\,c\|\varphi\|_\infty^4\varepsilon^2\,\frac{T}{\varepsilon}\,\mathbb{E}\left[1+\sup_{k\leqslant T/\varepsilon}(X^\varepsilon_{k\varepsilon},\mathbf{1})^2\right],$$

where we used (18) of Lemma 3 for the last inequality. We conclude using Lemma 2. □

Lemma 9. *The process* $(X^\varepsilon(\varphi),\ \varepsilon\in(0,1])$ *is* C*-tight as* ε *decreases to* 0.

Proof. From (13) we get, for $k\varepsilon\leqslant t<(k+1)\varepsilon$,

$$\begin{aligned}(X^\varepsilon_t,\varphi) &= (X^\varepsilon_{k\varepsilon},\varphi)\\ &= (\mu_0,\varphi)+M^\varepsilon(\varphi)_{k\varepsilon}+\varepsilon\sum_{l<k}(X^\varepsilon_{l\varepsilon},\theta(X^\varepsilon_{l\varepsilon})A(X^\varepsilon_{l\varepsilon})\varphi)+\sum_{l\leqslant k}\eta^\varepsilon_k\\ &= (\mu_0,\varphi)+M^\varepsilon(\varphi)_{k\varepsilon}+\Lambda^\varepsilon_t+Z^\varepsilon_t,\end{aligned}$$

where

$$\Lambda^\varepsilon_t=\int_0^{[t/\varepsilon]\varepsilon}(X^\varepsilon_u,\theta(X^\varepsilon_u)A(X^\varepsilon_u)\varphi)\,\mathrm{d}u \quad\text{and}\quad Z^\varepsilon_t=\sum_{l\leqslant k}\eta^\varepsilon_k.$$

Let us check that $(\Lambda^\varepsilon,\ \varepsilon\in(0,1])$ is C-tight as ε decreases to 0. Since $\Lambda^\varepsilon_0=0$, we have

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leqslant s\leqslant T}|\Lambda^\varepsilon_s|\geqslant K\right) &\leqslant \frac{1}{K^2}\,\mathbb{E}\left[\sup_{0\leqslant s\leqslant T}\left(\int_0^{[s/\varepsilon]\varepsilon}(X^\varepsilon_u,\theta(X^\varepsilon_u)A(X^\varepsilon_u)\varphi)\,\mathrm{d}u\right)^2\right]\\ &\leqslant \frac{c}{K^2}\|\varphi\|_*^2\,\mathbb{E}\left[\left(\int_0^{[T/\varepsilon]\varepsilon}(X^\varepsilon_u,\mathbf{1})\,\mathrm{d}u\right)^2\right]\\ &\leqslant \frac{c}{K^2}\|\varphi\|_*^2,\end{aligned}$$

thanks to Lemma 2. We also have for $0\leqslant s\leqslant t\leqslant T,\ h\geqslant 0$,

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\leqslant s\leqslant t\leqslant T,|t-s|\leqslant h}|\Lambda^\varepsilon_t-\Lambda^\varepsilon_s|\geqslant\eta\right)\\ &\quad=\mathbb{P}\left(\sup_{0\leqslant s\leqslant t\leqslant T,|t-s|\leqslant h}\left|\int_{[s/\varepsilon]\varepsilon}^{[t/\varepsilon]\varepsilon}(X^\varepsilon_u,\theta(X^\varepsilon_u)A(X^\varepsilon_u)\varphi)\,\mathrm{d}u\right|\geqslant\eta\right)\end{aligned}$$

$$\leqslant \frac{c}{\eta^2}\,\|\varphi\|_*^2(h+\varepsilon)^2\mathbb{E}\left[\sup_{k\leqslant T/\varepsilon}(X^\varepsilon_{k\varepsilon},1)^2\right]$$

$$\leqslant \frac{c}{\eta^2}\,\|\varphi\|_*^2(h+\varepsilon)^2,$$

thanks to Lemma 2. Thanks to Proposition VI.3.26 of Jacod and Shiryaev (1987), those two inequalities imply that $(A^\varepsilon,\ \varepsilon\in(0,1])$ is C-tight. From Lemma 4, we get that $\sup_{t\leqslant T} Z^\varepsilon_t$ converges to 0 in L^1 as ε decreases to 0. In particular, it is C-tight. As a sum of C-tight processes, the family $(X^\varepsilon(\varphi),\ \varepsilon\in(0,1])$ is C-tight as ε decreases to 0. □

Lemma 10. *The process family of process $(X^\varepsilon,\ \varepsilon\in(0,1])$ is C-tight as ε decreases to* 0.

This result is a consequence of the next theorem which is stated in (Perkins, 2002, Theorem II.4.1), Lemmas 9 and 6.

Let $C_b(\mathbb{R}^d)=\{f:\mathbb{R}^d\to\mathbb{R},\ f$ bounded and continuous$\}$. Let D_0 be a separating class in C_b in $\mathscr{M}_{\mathrm{f}}$ (that is if μ and ν belongs to $\mathscr{M}_{\mathrm{f}}$, if $\mu(f)=\nu(f)$ for all $\varphi\in D_0$, then $\mu=\nu$) containing 1 and which is closed under addition.

Theorem 11. *A sequence of càdlàg $\mathscr{M}_{\mathrm{f}}$-valued process $(X^\varepsilon,\ \varepsilon\in(0,1])$ is C-tight as ε decreases to* 0, *in $D(\mathbb{R}+,\mathscr{M}_{\mathrm{f}})$ if and only if the following conditions hold*:

(1) $\forall\varphi\in D_0$, *the process $(X^\varepsilon(\varphi),\ \varepsilon\in(0,1])$ is C-tight in $D(\mathbb{R}_+,\mathbb{R})$ as ε decreases to* 0.
(2) *For every $\rho,T>0$, there is a compact set $K_{\rho,T}$ in $\mathbb{R}^d$ such that*

$$\sup_{\varepsilon\in(0,1]}\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}(X^\varepsilon_t,\mathbf{1}_{K^c_{\rho,T}})>\rho\right)<\rho.$$

Lemma 12. *Any limiting measure valued process $Y=(Y_t,\ t\geqslant 0)$ of $(X^\varepsilon,\ \varepsilon\in(0,1])$ as ε decreases to* 0, *satisfies the martingale problem* (*MP*) *and has a continuous version.*

Proof. Let $(\varepsilon_n,\ n\in\mathbb{N})$ be a sequence decreasing to 0 such that $(X^{\varepsilon_n},\ n\in\mathbb{N})$ converges in law to Y. Using Skorokhod's representation theorem, we may suppose that we have an a.s. convergence. Recall from (13) that

$$(X^{\varepsilon_n}_t,\varphi)=(\mu_0,\varphi)+M^{\varepsilon_n}(\varphi)_t+\int_0^{[t/\varepsilon_n]\varepsilon_n}(X^{\varepsilon_n}_u,\theta(X^{\varepsilon_n}_u)A(X^{\varepsilon_n}_u)\varphi)\,\mathrm{d}u+\sum_{l\leqslant t/\varepsilon_n}\eta^{\varepsilon_n}_l, \tag{31}$$

for any φ such that $\|\varphi\|_*<\infty$.

Using (25) as well as (26), and Lemma 2, we deduce that $\sup_{t\leqslant T}\sum_{l\leqslant t/\varepsilon_n}\eta^{\varepsilon_n}_l$ (resp. $\sup_{t\leqslant T}\sum_{l\leqslant t/\varepsilon_n}\kappa^{\varepsilon_n}_l$) converges to 0 in L^1 (resp. L^2) as $n\to\infty$. There exists a subsequence of $(\varepsilon_n,\ n\geqslant 0)$ such that those two convergences hold a.s. We still write $(\varepsilon_n,\ n\geqslant 0)$ for this subsequence. Since X^{ε_n} is C-tight, we get that Y is continuous and that a.s. for all $t\geqslant 0$, $X^{\varepsilon_n}_t$ converges to Y_t. In particular, since $\|\varphi\|_*$ is finite, this implies that a.s. for all $t \geqslant 0$, $(X_t^{\varepsilon_n}, \varphi)$ converges to (Y_t, φ) and $\int_0^{[t/\varepsilon_n]\varepsilon_n} (X_u^{\varepsilon_n}, \theta(X_u^{\varepsilon_n})A(X_u^{\varepsilon_n})\varphi)\, \mathrm{d}u$ converges to $\int_0^t (Y_u, \theta(Y_u)A(Y_u)\varphi)\, \mathrm{d}u$. From (31) we deduce that $(M^{\varepsilon_n}(\varphi)_t,\ t \geqslant 0)$ converges a.s. to a continuous process say $(M(\varphi)_t,\ t \geqslant 0)$. And we have

$$(Y_t, \varphi) = (\mu_0, \varphi) + M(\varphi)_t + \int_0^t (Y_u, \theta(Y_u)A(Y_u)\varphi)\, \mathrm{d}u. \tag{32}$$

From Lemma 5, we have

$$\langle M^{\varepsilon_n}(\varphi)\rangle_t = 4\int_0^{[t/\varepsilon_n]\varepsilon_n} (X_u^{\varepsilon_n}, \theta(X_u^{\varepsilon_n})\varphi^2)\, \mathrm{d}u + \sum_{k<[t/\varepsilon_n]\varepsilon_n} \kappa_k^{\varepsilon_n}.$$

In particular, $(\langle M^{\varepsilon_n}(\varphi)\rangle_t,\ t \geqslant 0)$ converge a.s. to

$$Q = \left(4\int_0^t (Y_u, \theta(Y_u)\varphi^2)\, \mathrm{d}u,\ t \geqslant 0\right).$$

From (20) and Doob's inequality, we deduce that for any $T \geqslant 0$, the family of martingales $(M^{\varepsilon_n}(\varphi)_t,\ t \in [0,T])$ (resp. $(M^{\varepsilon_n}(\varphi)_t^2 - \langle M^{\varepsilon_n}(\varphi)\rangle_t,\ t \in [0,T])$) is uniformly bounded in L^4 (resp. L^2). This implies in particular that $M(\varphi)$ is an L^2 martingale and $M(\varphi)^2 - Q$ is an L^1 martingale (with respect to the filtration generated by $M(\varphi)$ and Q). Since $M(\varphi)$ and Q are continuous, we also get that $\langle M(\varphi)\rangle = Q$. To end the proof, we need to check that $M(\varphi)$ is a martingale with respect to the filtration generated by Y. Let $m \geqslant 1$, f be a bounded continuous function defined on $\mathscr{M}_\mathrm{f}^m$. Let $0 \leqslant t_1 \leqslant \cdots \leqslant t_m \leqslant t \leqslant s$. Because of the uniform integrability of $M^{\varepsilon_n}(\varphi)$, we have that $\mathbb{E}[f(X_{t_1}^{\varepsilon_n}, \ldots, X_{t_m}^{\varepsilon_n})(M^{\varepsilon_n}(\varphi)_s - M^{\varepsilon_n}(\varphi)_t)]$ converges to $\mathbb{E}[f(Y_{t_1}, \ldots, Y_{t_m})(M(\varphi)_s - M(\varphi)_t)]$ as $n \to \infty$. Since $\mathbb{E}[f(X_{t_1}^{\varepsilon_n}, \ldots, X_{t_m}^{\varepsilon_n})(M^{\varepsilon_n}(\varphi)_s - M^{\varepsilon_n}(\varphi)_t)] = 0$, we deduce that $\mathbb{E}[f(Y_{t_1}, \ldots, Y_{t_m})(M(\varphi)_s - M(\varphi)_t)] = 0$. As this equality holds for any m, $0 \leqslant t_1 \leqslant \cdots \leqslant t_m \leqslant t \leqslant s$ and any bounded continuous function f, and since $M(\varphi)$ is adapted to the filtration generated by Y (thanks to formula (32)), we deduce that $M(\varphi)$ is a martingale with respect to the filtration generated by Y and that Q is its quadratic variation. □

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Appendix A

A.1. Moment formula for the Brownian snake

We recall some moment formula for superprocesses under the excursion measure $\mathbb{N}$ (see e.g., Delmas, 1999, Section 6.1). Recall notations from Section 2.1. Let φ and ψ be bounded measurable functions defined on $\mathbb{R}^d$.

Let Z be a diffusion with infinitesimal generator A started at point x under $\mathbb{E}_x$. Let $(\mathrm{P}_v,\ v \geqslant 0)$ denote the transition kernel of the diffusion Z. We have for $u > 0$,

$$\mathbb{N}_{x,A}[(X_u, \varphi)] = \mathbb{E}_x[\varphi(Z_u)] = \mathrm{P}_u(\varphi)(x). \tag{A.1}$$

We have for $u \geqslant v > 0$,

$$\mathbb{N}_{x,A}[(X_u, \varphi)(X_v, \psi)] = \mathbb{N}_{x,A}[(X_v, \varphi)(X_v, \mathrm{P}_{u-v}\psi)], \tag{A.2}$$

and

$$\mathbb{N}_{x,A}[(X_u, \varphi)(X_v, \psi)] = 4 \int_0^v \mathrm{d}r \; \mathbb{E}[\mathrm{P}_{u-r}\varphi(Z_r)\mathrm{P}_{v-r}\psi(Z_r)]. \tag{A.3}$$

A.2. Markov property for the Brownian snake

Recall notations from Section 2.1. We refer to (Le Gall, 1995, Section 2), for the proof of the following statements.

We consider the process $(X_t(W),\ t > 0)$ under $\mathbb{N}_{x,A}[\mathrm{d}W]$. Recall $(W_s,\ s \geqslant 0)$ is a continuous C-valued process with lifetime process $(\zeta_s,\ s \in [0,\sigma])$, where σ is the length of the life-time excursion above 0 under $\mathbb{N}_{x,A}[\mathrm{d}W]$. Let $t_0 > 0$ be fixed. We consider the inverse of the time spent by the snake under level t_0:

$$\kappa(s) = \inf \left\{ r \in [0,\sigma];\ \int_0^r \mathbf{1}_{\{t_0 > \zeta_u\}}\,\mathrm{d}u > s \right\}.$$

And we define the continuous process $W'_s = W_{\kappa_s}$ which make sense $\mathbb{N}_{x,A}$-a.e. Let $\mathscr{E}_{t_0}$ be the σ-field generated by the process $(W'_s,\ s \geqslant 0)$.

By the continuity of the life-time process, the set $\{s \in (0,\sigma);\ t < \zeta_s\}$ can be written as a countable union of disjoint open intervals (a^j, b^j), $j \in J$. Notice J might be empty if the lifetime process does not reach level t_0. From the snake property, the paths $W_s(t)$ coincide for $t \in [0, t_0]$ and for $s \in (a^j, b^j)$. We set $V^j = W_s(t_0)$. For $s > 0$, we define

$$\zeta^j_s = \zeta_{(a^j+s)\wedge b^j} - t_0$$

$$W^j_s(t) = W_{(a^j+s)\wedge b^j}(t - t_0) \quad \text{for } t \in [0, \zeta^j_s],$$

and $W^j = (W^j_s(t),\ s \geqslant 0, t \geqslant 0)$. The family $(W^j,\ j \in J)$ are the excursions of the Brownian snake above level t_0. We have the following result.

Proposition 13.

- *The random measure X_{t_0} is $\mathscr{E}_{t_0}$-measurable.*
- *The random measure $\sum_{j \in J} \delta_{(V^j, W^j)}$ is conditionally on $\mathscr{E}_{t_0}$ a Poisson measure with intensity $X_{t_0}(\mathrm{d}x)\mathbb{N}_{x,A}[\mathrm{d}W]$.*

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