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Total length of the genealogical tree for quadratic stationary continuous-state branching processes

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Abstract. We prove the existence of the total length process for the genealogical tree of a population model with random size given by quadratic stationary continuous-state branching processes. We also give, for the one-dimensional marginal, its Laplace transform as well as the fluctuation of the corresponding convergence. This result is to be compared with the one obtained by Pfaffelhuber and Wakolbinger for a constant size population associated to the Kingman coalescent. We also give a time reversal property of the number of ancestors process at all times, and a description of the so-called lineage tree in this model.

Résumé. Nous démonstrons l'existence du processus de longueur totale renormalisée pour l'arbre généalogique dans un modèle de population dont la taille évolue suivant un processus de branchement continu quadratique (diffusion de Feller). Nous donnons également la loi unidimensionnelle de la longueur totale de l'arbre généalogique ainsi que les fluctuations associées à la renormalisation. Ce résultat est à rapprocher de ceux obtenus par Pfaffelhuber et Wakolbinger dans le cadre d'une population de taille constante associée au processus de coalescence de Kingman. Nous établissons également une propriété d'invariance par retournement du temps pour le processus du nombre des ancêtres qui permet d'obtenir en particulier une description du processus ancestral dans ce modèle.

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1. Introduction

1.1. *The model*

Stochastic models for the evolution of a stationary population goes back to the Wright–Fisher model, which is for a finite fixed size population in discrete generations. Fleming–Viot processes extend those models to the infinite size population (with infinitesimal individuals) in continuous time, see Donnelly and Kurtz [7]. On the other hand, Galton–Watson processes model the evolution of a discrete random-size population in discrete generations based on the branching property: descendants of two individuals in the same generation behaves independently. Continuous state branching (CB) processes extend those models to the infinite size population (with infinitesimal individuals) in continuous time. The description of the genealogy of the CB processes is done using historical Dawson–Watanabe processes, see Donnelly and Kurtz [8], or Lévy trees, see Duquesne and Le Gall [9]. In order to consider Galton–Watson processes or CB processes in stationary regime, one has to condition them on non-explosion and non-extinction. Then one gets the Galton–Watson processes or CB processes with an immortal individual, see Delmas and Hénard [6] in

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this direction for non-homogeneous models and the references therein. These can also be seen as Galton–Watson processes or CB processes with immigration if one removes the immortal individual. We shall consider one of the simplest models considered in Chen and Delmas [4] of the CB processes with an immortal individual which corresponds to a quadratic sub-critical branching mechanism, in this direction Evans [11] first developed this model in the case of superprocesses. The results we present concern the neutral population.

1.2. *The genealogy*

Describing the genealogy of a large population is a key issue in population genetics. A well-established model in this direction is the Kingman coalescent which describes the genealogy of a Fleming–Viot process. Intuitively we may think of the Kingman coalescent at some fixed time s as a random tree with infinitely many leaves (corresponding to the individuals alive at time s), when backwards in time any two lineages coalesce independently at rate 1. See Pitman [22] and Sagitov [24] for a general description of the coalescent processes.

The study of the evolution in t of the genealogical tree of the population at time t, or of some of its functionals has recently attracted some interest in mathematical population genetics. In this direction for the quadratic Fleming–Viot process (associated to the Kingman coalescent), see Greven, Pfaffelhuber and Winter [14]. The functionals of the genealogical tree of interest are:

- The time to the most recent common ancestor (TMRCA) at time t is the distance between any leaf (which are all living individuals at time t) of the genealogical tree and its root. Pfaffelhuber and Wakolbinger [20] studied the evolving Kingman coalescent case and Evans and Ralph [12] the large branching population case.
- The number of mutations observed in a population in a neutral model is distributed as a Poisson random variable with mean the rate of mutation times the total tree length of the genealogical tree (other similar quantities of interest are the number of mutations which appear only once; this is distributed as a Poisson random variable with mean the rate of mutation times the total length of the external branches of the genealogical tree). This motivated the study of the rescaled total length of the coalescent trees which converges in distribution to the Gumbel distribution at a given fixed time for the Kingman coalescent (see Janson and Kersting [15] for the external length asymptotics). The corresponding limit process has been studied in Pfaffelhuber, Wakolbinger and Weisshaupt [21] as well as Dahmer, Knobloch and Wakolbinger [5] where it is proved that the limit process is not a semi-martingale.

  An extension has been provided for other Λ-coalescents, see Kersting, Schweinsberg and Wakolbinger [16] for Beta-coalescents and Schweinsberg [25] for the Bolthausen–Sznitman coalescent.

Our main objective is to study the limit process of the renormalized total length of the genealogical tree in a population with random size given by a quadratic stationary CB process.

1.3. *Main results*

We model the random size of the population at time t by Z_t with $(Z_t, t \in \mathbb{R})$ a stationary CB (or CB process with immigration) process with sub-critical quadratic branching mechanism. This model, see [4] or Section 2.3 for a precise definition, is characterized by two positive parameters θ and β, which describe the mean size of the population and a time scale:

- The random size of the population, Z_t, is distributed as the sum of two independent exponential random variables with mean $1/(2\theta)$.
- The TMRCA of the population living at time t, A_t, is distributed as the maximum of two independent exponential random variables with mean $1/(2\beta\theta)$.

In particular, we have:

$$\mathbb{E}[Z_t] = \frac{1}{\theta} \quad \text{and} \quad \mathbb{E}[A_t] = \frac{3}{4\beta}\mathbb{E}[Z_t].$$

For $s < t$ let M_s^t be the number of ancestors at time s of the population living at time t, the immortal individual being excluded, see (17) for a precise definition. The following time reversal property for the number of ancestors process $(M_s^{s+r}, s \in \mathbb{R}, r > 0)$, see Theorem 4.3, is similar to the time reversal property of the look-down process in

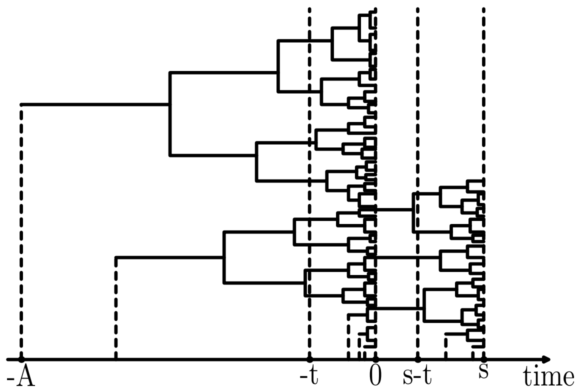

Fig. 1. We represent the genealogical tree of the population living at time 0 and s ($s > 0$) with x axis as the line of the immortal individual. At $t > 0$ units of time before present, the number of ancestors is $M^0_{-t} = 8$ for the population living at time 0 and $M^s_{s-t} = 4$ for the population living at time s. The TMRCA of the population living at time 0 is A.

the Kingman case, see also Lemma 8 from Aldous and Popovic [2] in a critical branching process setting at a fixed time. The proof of the next theorem does not rely on discrete approximation as in [2].

Theorem 1 (Time reversal property). *The process* $(M^s_{s-r}, s \in \mathbb{R}, r > 0)$ *is distributed as* $(M^{s+r}_s, s \in \mathbb{R}, r > 0)$.

We define for $r > 0$ the "probability" of an infinitesimal individual to have descendants r units of time forward, see definition (5), as:

$$c(r) := \frac{2\theta}{\mathrm{e}^{2\beta\theta r} - 1}.$$

The point process of lineage tree $\mathcal{A}_s$ of the population at time s is defined by Popovic [23] (see also [2]) in a critical branching setting (see also the references in Remark 4.2), and it corresponds in our setting to the jumping times of the process $(M^s_{s-r}, r > 0)$:

$$\varrho(\mathcal{A}_s) := \{r > 0; M^s_{s-r} - M^s_{(s-r)_-} = 1\}.$$

The lineage tree of Z_s at some current time s is depicted in Figure 1. Using the time reversal property, we deduce in Remark 4.2 the following corollary.

Corollary 2. *The point process* $\varrho(\mathcal{A}_s)$ *has the same distribution as the set* $\{\zeta_j; x_j < Z_s\}$ *where* $\sum_{j\in J} \delta_{x_j,\zeta_j}(dx, dz)$ *is a Poisson point measure on* $(0, +\infty)^2$ *with intensity* $dx|c'(z)|\,dz$ *and independent of* Z_s.

The process $(M^s_{s-r}, r > 0)$ is a (forward) death process and a (backward) birth process whose intensities are given in Propositions 3.2 and 3.3. We also give in Proposition 3.4 a reconstruction result of the process $(Z_{s-t}, t > 0)$ from the process $(M^s_{s-r}, r > 0)$ by grafting CB processes, and we then deduce a formula on the weighted integral of the ancestor process, see Corollary 3.5. For the reconstruction of the CB processes from backbones instead of the genealogical tree see also Duquesne and Winkel [10].

The total length of the genealogical tree for the population living at time s, up to time $s - \varepsilon$ (with $\varepsilon > 0$) is given by:

$$L^s_\varepsilon = \int_\varepsilon^\infty M^s_{s-r}\, dr,$$

and we consider the normalized total length up to time $s - \varepsilon$ defined by:

$$\mathcal{L}^s_\varepsilon = L^s_\varepsilon - Z_s \int_\varepsilon^\infty c(r)\, dr.$$

We have the following result, see Theorems 5.2 and 5.8 as well as Lemma 5.4.

Theorem 3. *There exists a càdlàg stationary process* $(W_s, s \in \mathbb{R})$, *such that for all* $s \in \mathbb{R}$, *the compensated tree length* $(\mathcal{L}_\varepsilon^s, \varepsilon > 0)$ *converges a.s. and in* L^2 *to* W_s.

Furthermore we have for $\lambda > 0$:

$$\mathrm{E}\left[\mathrm{e}^{-2\beta\theta\lambda W_0}\middle| Z_0 = \frac{z}{2\theta}\right] = \mathrm{e}^{-z\varphi(\lambda)} \quad \textit{and} \quad \mathrm{E}\left[\mathrm{e}^{-2\lambda\beta\theta W_0}\right] = \left(1+\varphi(\lambda)\right)^{-2}$$

with

$$\varphi(\lambda) = -\lambda \int_0^1 dv \frac{1-v^\lambda}{1-v}.$$

Proposition 5.5 gives the fluctuation: $\sqrt{\beta}(\mathcal{L}_\varepsilon - W_0)/\sqrt{\varepsilon}$ converges in distribution as ε goes down to 0 towards $\sqrt{2Z_0}G_1$, with $G_1 \sim \mathcal{N}(0,1)$ a standard Gaussian random variable independent of Z_0.

Notice the process $(\mathcal{L}_\varepsilon^s, s \in \mathbb{R})$ is not continuous, and this implies that W is not continuous. We also provide the covariance of W, see Proposition 5.6, and get, see Remark 5.7, that there exists some finite positive constant C such that:

$$\mathrm{E}\left[(W_s - W_0)^2\right] \sim_{0+} Cs\left[\log(s)\right]^2, \tag{1}$$

i.e. $\lim_{s\to 0+} \mathrm{E}[(W_s - W_0)^2]/(s[\log(s)]^2) = C$.

Remark 4. *We present here some open problems*:

I. *In the context of Fleming–Viot processes the tightness of the approximating total length process has been shown in* [21] *and the convergence on the path space is derived. However we only have convergence at a fixed time s for* $\mathcal{L}_\varepsilon^s$ *in Theorem* 3. *The convergence on the path space is still open*.

II. *Following* [21], *it is proved in* [5] *that the tree length process has infinite quadratic variation. The fact that W has infinite quadratic variation*, *which is suggested by* (1), *is still open*.

III. *Proposition* 5.5 *identifies the Gaussian fluctuation limit of* $\mathcal{L}_\varepsilon$ *at a fixed time. An interesting problem is the covariance structure of this process for different times. The result will be presented in forthcoming paper.*

2. Population model

Let $\beta > 0$ and $\theta > 0$ be fixed scale parameters.

2.1. *Sub-critical quadratic CB process*

Consider a sub-critical branching mechanism $\psi(\lambda) = \beta\lambda^2 + 2\beta\theta\lambda$, let $\mathbb{P}_x$ be the law of a CB process $Y = (Y_t, t \geq 0)$ started at mass x with branching mechanism ψ. We extend Y on $\mathbb{R}$ by setting $Y_t = 0$ for $t < 0$. Let $\mathbb{E}_x$ and $\mathbb{N}$ be respectively the corresponding expectation and canonical measure (excursion measure) associated to Y. Recall that Y is Markovian under $\mathbb{P}_x$ and $\mathbb{N}$. We have for every $t > 0$:

$$\mathbb{E}_x\left[\mathrm{e}^{-\lambda Y_t}\right] = \mathrm{e}^{-xu(\lambda,t)} \quad \text{for } \lambda > -\frac{2\theta}{1-\mathrm{e}^{-2\beta\theta t}}$$

with

$$u(\lambda,t) = \mathbb{N}\left[1-\mathrm{e}^{-\lambda Y_t}\right] = \frac{2\theta\lambda}{(2\theta+\lambda)\mathrm{e}^{2\beta\theta t} - \lambda} \tag{2}$$

satisfying the backward and forward equations:

$$\partial_t u(\lambda,t) = -\psi\left(u(\lambda,t)\right) \quad \text{and} \quad \partial_t u(\lambda,t) = -\psi(\lambda)\partial_\lambda u(\lambda,t). \tag{3}$$

Then it is easy to derive that for $t > 0$:

$$\beta \int_0^t u(\lambda, r)\, dr = \log\left(1 + \lambda \frac{1 - e^{-2\beta\theta t}}{2\theta}\right) \quad \text{and} \quad \beta \int_0^\infty u(\lambda, t)\, dt = \log\left(1 + \frac{\lambda}{2\theta}\right). \tag{4}$$

Let $c(t) = \lim_{\lambda\to\infty} u(\lambda, t)$ and denote by $\zeta = \inf\{t > 0; Y_t = 0\}$ the lifetime of Y under $\mathbb{N}$. Then we have for $t > 0$:

$$c(t) = \mathbb{N}[\zeta > t] = \frac{2\theta}{e^{2\beta\theta t} - 1}. \tag{5}$$

From the Markov property of Y, we deduce that for $s > 0$ and $t, \lambda \geq 0$:

$$u\big(u(\lambda, s), t\big) = u(\lambda, t + s) \quad \text{and} \quad u\big(c(s), t\big) = c(t + s). \tag{6}$$

We deduce from (4) that for $s > t > 0$:

$$\beta \int_s^{+\infty} c(r)\, dr = \beta \int_0^{+\infty} u\big(c(s), r\big)\, dr = \log\left(1 + \frac{c(s)}{2\theta}\right) = -\log\big(1 - e^{-2\beta\theta s}\big) \tag{7}$$

as well as

$$\beta \int_t^s c(r)\, dr = \log\left(\frac{1 - e^{-2\beta\theta s}}{1 - e^{-2\beta\theta t}}\right). \tag{8}$$

We easily get the following results for $t > 0$:

$$\mathbb{N}[Y_t] = e^{-2\beta\theta t} \tag{9}$$

as well as

$$\mathbb{N}\big[e^{-\lambda Y_t} \mathbf{1}_{\{\zeta > t\}}\big] = c(t) - u(\lambda, t), \tag{10}$$

and, thanks to the Markov property of Y and (3) for $s > 0$, $t > 0$:

$$\mathbb{N}[Y_s \mathbf{1}_{\{Y_{s+t}=0\}}] = \mathbb{N}\big[Y_s e^{-c(t) Y_s}\big] = \frac{\psi(c(s+t))}{\psi(c(t))} = e^{2\beta\theta s} \left(\frac{c(s+t)}{c(t)}\right)^2. \tag{11}$$

2.2. *Genealogy of the CB process Y*

We will recall the genealogical tree for the CB process which is studied in Le Gall [19] or Duquesne and Le Gall [9]. Since the branching mechanism is quadratic, the corresponding Lévy process is just the Brownian motion with drift. Let $B = (B_t, t \in \mathbb{R}_+)$ be a standard Brownian motion. We consider the Brownian motion $B^\theta = (B^\theta_t, t \in \mathbb{R}_+)$ with negative drift and the corresponding reflected process above its minimum $H = (H(t), t \in \mathbb{R}_+)$:

$$B^\theta_t = \sqrt{\frac{2}{\beta}} B_t - 2\theta t \quad \text{and} \quad H(t) = B^\theta_t - \inf_{s\in[0,t]} B^\theta_s.$$

We deduce from equation (1.7) in [9] that H is the height process associated to the branching mechanism ψ. For a function h defined on $\mathbb{R}_+$, we set:

$$\max(h) = \max\big(h(t), t \in \mathbb{R}_+\big).$$

Let $\mathrm{N}[dH]$ be the excursion measure of H above 0 normalized such that $\mathrm{N}[\max(H) \geq r] = c(r)$. Let $(\ell^x_t(H), t \in \mathbb{R}_+, x \in \mathbb{R}_+)$ be the local time of H at time t and level x. Let $\zeta = \inf\{t > 0; H(t) = 0\}$ be the duration of the excursion H under $\mathrm{N}[dH]$. We recall that $(\ell^r_\zeta(H), r \in \mathbb{R}_+)$ under N is distributed as Y under $\mathbb{N}$, see Theorem 1.4.1

in [9] for details. From now on we shall identify Y with $(\ell^r_\zeta(H), r \in \mathbb{R}_+)$ and write $\mathbb{N}$ for N. We now recall the construction of the genealogical tree of the CB process Y from H.

Let f be a continuous non-negative function defined on $[0, +\infty)$, such that $f(0) = 0$, with compact support. We set $\zeta^f = \sup\{t; f(t) > 0\}$, with the convention that $\sup \varnothing = 0$. Let d^f be the non-negative function defined by:

$$d^f(s,t) = f(s) + f(t) - 2 \inf_{u \in [s\wedge t, s\vee t]} f(u).$$

It can be easily checked that d^f is a semi-metric on $[0, \zeta^f]$. One can define the equivalence relation associated to d^f by $s \sim t$ if and only if $d^f(s,t) = 0$. Moreover, when we consider the quotient space $T^f = [0, \zeta^f]/\sim$ and, noting again d^f the induced metric on T^f and rooting T^f at $\varnothing^f$, the equivalence class of 0, it can be checked that the space $(T^f, d^f, \varnothing^f)$ is a compact rooted real tree.

The so-called genealogical tree of the CB process Y is the real tree $\mathcal{T} = (T^H, d^H, \varnothing^H)$. In what follows, we shall mainly present the result using the height process H instead of the genealogical tree $\mathcal{T}$, and say that H codes for the genealogy of Y.

Let $a > 0$ and $(H_k, k \in \mathcal{K}_a)$ be the excursions of H above level a. It is well known that $\sum_{k\in\mathcal{K}_a} \delta_{H_k}(dH)$ is under $\mathbb{N}$ and conditionally on $(Y_r, r \in [0,a])$, a Poisson point measure with intensity $Y_a \mathbb{N}[dH]$. We define the number of ancestors at time a of the population living at time b as the number of excursions above level a which reach level $b > a$ by:

$$R_a^b(H) := \sum_{k\in\mathcal{K}_a} \mathbf{1}_{\{\max(H_k) \geq b-a\}}.$$

When there is no confusion, we shall write R_a^b for $R_a^b(H)$. Notice that R_a^b is conditionally on Y_a a Poisson random variable with mean $c(b-a)Y_a$.

We compute functionals of R in Section A.1.

2.3. *The population model*

We model the population using a stationary CB process. Let $\mathbb{D}$ be the space of càdlàg paths having 0 as a trap. Consider under P a Poisson point measure

$$\mathcal{N}(dt, dY) = \sum_{i\in I} \delta_{(t_i, Y^i)}(dt, dY) \tag{12}$$

on $\mathbb{R} \times \mathbb{D}$ with intensity $2\beta\, dt \mathbb{N}[dY]$. We shall consider the process $Z = (Z_t, t \in \mathbb{R})$ defined by

$$Z_t := \sum_{t_i \leq t} Y^i_{t-t_i}. \tag{13}$$

Let E be the expectation with respect to P. According to [4], Z is a CB process with branching mechanism ψ, conditionally on non-extinction. Notice the process Z is a.s. finite, a.s. positive and stationary. We shall model a population with random size by the process Z. The process Z can be seen as a CB process with immigration or a population with an infinite lineage (or immortal individual).

Using the property of the Poisson point measure, we have:

$$\mathrm{E}\big[\mathrm{e}^{-\lambda Z_t}\big] = \left(1 + \frac{\lambda}{2\theta}\right)^{-2}, \tag{14}$$

which also gives:

$$\mathrm{E}[Z_t] = \frac{1}{\theta} \quad \text{and} \quad \mathrm{E}\big[Z_t^2\big] = \frac{3}{2\theta^2}. \tag{15}$$

Using the branching property of Y, it is easy to get for $s \geq 0$:

$$\mathbb{E}[Z_0 Z_s] = \frac{2 + \mathrm{e}^{-2\beta\theta s}}{2\theta^2}. \tag{16}$$

3. The number of ancestors process

3.1. *Definition*

We describe the genealogy of Z using the framework developed in Section 2.2. Let

$$\mathcal{N}'(dt, dH) = \sum_{i \in I} \delta_{(t_i, H^i)}(dt, dH)$$

be a Poisson point measure with intensity $2\beta\, dt \mathbb{N}(dH)$. We will write Y_a^i for $\ell^a(H^i)$ for $i \in I$ and use (13) for the definition of Z. Let $\mathcal{T}^i$ be the genealogical tree associated to H^i. Consider the real line as an infinite spine, and for all $i \in I$, graft the tree $\mathcal{T}^i$ at level t_i on the infinite spine. This defines a tree which we call the genealogical tree of the process Z. Thus $\sum_{i \in I} \delta_{(t_i, H^i)}$ allows to code (on an enlarged space) the genealogy of Z defined by (13).

Let $r < t$. We define the number of ancestors, excluding the immortal individual, at time r of the population living at time t, M_r^t, by:

$$M_r^t := \sum_{i \in I} \mathbf{1}_{\{t_i < r\}} R_{r-t_i}^{t-t_i}\left(H^i\right). \tag{17}$$

We shall identify M_{-r} with M_{-r}^0 for $r > 0$, when there is no risk of confusion. The time to the most recent common ancestor (TMRCA) of the population living at time 0 is defined as $\inf\{r > 0; M_{-r} = 0\}$. We shall call $(M_r^t, -\infty < r < t < +\infty)$ the number of ancestors process.

Remark 3.1. *Notice the time order on H^i allows to define an order structure on $\mathcal{T}^i$, which could then be described as a planar tree. Then grafting $\mathcal{T}^i$ at level t_i either on the left or on the right of the infinite spine would define a planar genealogical tree of the process Z. Since this order structure is of no use to the study of the length of the genealogical tree, we decide to omit it and concentrate on the number of ancestors process instead.*

Recall from [4], Section 6, that conditionally on $(Z_{-u}, u \geq r)$, M_{-r} is a Poisson random variable with intensity $c(r)Z_{-r}$. This implies, using (15) and (14) that for $t > 0$:

$$\mathrm{E}[M_{-t}] = \frac{c(t)}{\theta}, \qquad \mathrm{E}\left[M_{-t}^2\right] = \frac{c(t)}{\theta}\left(1 + \frac{3}{2}\frac{c(t)}{\theta}\right) = 2\frac{\mathrm{e}^{2\beta\theta t} + 2}{(\mathrm{e}^{2\beta\theta t} - 1)^2}, \tag{18}$$

$$\mathrm{E}\left[\mathrm{e}^{-\lambda M_{-r}}\right] = \mathrm{E}\left[\mathrm{e}^{-(1-\mathrm{e}^{-\lambda})c(r)Z_{-r}}\right] = \left(1 + \frac{c(r)}{2\theta}\left(1 - \mathrm{e}^{-\lambda}\right)\right)^{-2}, \tag{19}$$

and moreover by Corollary 6.5 in [4] a.s.

$$\lim_{r \to 0+} \frac{M_{-r}}{c(r)} = Z_0. \tag{20}$$

3.2. *Associated birth and death process*

Thanks to the branching property, we get that the process $(M_t, t < 0)$ is a birth process starting from 0 at $-\infty$. The birth rate is the sum of two terms: the first one is the contribution of the immortal individual and it is equal to $2\beta c(-t)\, dt$; the second one is the contribution of the current ancestors and is equal to $\beta c(-t) M_{t-}\, dt$, see Proposition A.3. We deduce the following result.

Proposition 3.2. *The process* $(M_t, t<0)$ *is a càdlàg birth process starting from* 0 *at* $-\infty$ *with rate* $\beta c(|t|)(M_t+2)$ *at time* $t<0$. *Equivalently, the process* $(\tilde{M}_t, t<0)$ *defined by* $\lim_{t\to-\infty}\tilde{M}_t=0$ *and*

$$d\tilde{M}_t = dM_t - \beta c\big(|t|\big)(M_t+2)\,dt$$

is a martingale (*with respect to its natural filtration*) *whose jumps are equal to* 1.

Similarly, we can check the following result.

Proposition 3.3. *The process* $(M_{(-t)_-}, t>0)$ *is a càdlàg death process with rate*

$$M_{-t}\big|c'(t)\big|/c(t) = \beta M_{-t}\big(2\theta + c(t)\big).$$

We can also recover the process $(Z_t, t<0)$ from $(M_t, t<0)$ by grafting CB processes on the number of ancestors process. Notice there is a contribution from the immortal individual with rate $2\beta\mathbb{N}[dY;\zeta<|t|]\,dt$ (as we do not take into account the contributions which reach the current time 0) and from the genealogical tree, according to Proposition A.3, we have the contributions of $Y^{(\mathrm{g}),i}$ and we only keep the contributions of $Y^{(\mathrm{d}),i}$ which do not reach the current time 0; this gives a contribution with rate $2\beta M_t\mathbb{N}[dY;\zeta<|t|]\,dt$. Therefore, we have the following result.

Proposition 3.4. *Let* $\sum_{i\in I}\delta_{t_i,\tilde{Y}_i}(dt, d\tilde{Y})$ *be, conditionally on* $(M_t, t<0)$, *a Poisson point measure on* $(-\infty,0)\times\mathbb{D}$ *with intensity*

$$2\beta(M_t+1)\mathbb{N}\big[dY;\zeta<|t|\big]\,dt.$$

Then, conditionally on $(M_t, t<0)$, *the process* $(\tilde{Z}_t, t<0)$ *is distributed as* $(Z_t, t<0)$ *where for all* $t<0$:

$$\tilde{Z}_t = \sum_{t_i\le t}\tilde{Y}^i_{t-t_i}.$$

Moments for the process $(M_t, t<0)$ are given in Section A.2.

We deduce from Proposition 3.4, the following remarkable formula on the weighted integral of the number of ancestors process.

Corollary 3.5. *Let* $t>0$. *We have*:

$$\mathrm{E}\big[\mathrm{e}^{-2\beta\int_t^{+\infty}dr\,(c(r-t)-c(r))M_{-r}}\big] = \left(\frac{2\theta}{2\theta+c(t)}\right)^2 = \mathrm{E}\big[\mathrm{e}^{-c(t)Z_0}\big].$$

Proof. According to Proposition 3.4 and Lemma 3.1 in [4], we have:

$$\mathrm{E}\big[\mathrm{e}^{-\lambda Z_t}\big] = \mathrm{E}\left[\exp\left(-2\beta\int_t^{+\infty}dr\,(M_{-r}+1)\mathbb{N}\big[\big(1-\mathrm{e}^{-\lambda Y_{r-t}}\big)\mathbf{1}_{\{\zeta<r\}}\big]\right)\right].$$

Notice that

$$\mathbb{N}\big[\big(1-\mathrm{e}^{-\lambda Y_{r-t}}\big)\mathbf{1}_{\{\zeta<r\}}\big] = u\big(\lambda+c(t), r-t\big) - u\big(c(t), r-t\big) = u\big(\lambda+c(t), r-t\big) - c(r).$$

Thanks to (4) and (7), we get:

$$2\beta\int_t^{+\infty}dr\,\big(u\big(\lambda+c(t), r-t\big)-c(r)\big) = 2\log\left(1+\frac{\lambda+c(t)}{2\theta}\right) - 2\log\left(1+\frac{c(t)}{2\theta}\right).$$

Then use (14) to get:

$$\mathrm{E}\left[\exp\left(-2\beta\int_t^{+\infty}dr\,\big(u\big(\lambda+c(t), r-t\big)-c(r)\big)M_{-r}\right)\right] = \left(\frac{2\theta}{2\theta+\lambda}\right)^2\left(\frac{2\theta+\lambda+c(t)}{2\theta+c(t)}\right)^2.$$

Letting λ goes to infinity and (14) give the result. $\square$

4. Time reversal of the number of ancestors process

The next result is in a sense a consequence of the time reversibility of the process Y with respect to its lifetime ζ.

Lemma 4.1. *The random variable* $(Z_0, (M_{-t}, t > 0))$ *and* $(Z_0, (M_0^t, t > 0))$ *have the same distribution.*

This result will be generalized in Theorem 4.3.

Remark 4.2. *Up to a random labeling of the individuals, see Remark 3.1 in our setting, the point process ϱ of the lineage tree $\mathcal{A}$ defined in [2] or [23] of the population living at time 0 is given by the coalescent times of the genealogical tree or equivalently by the jumping times of the process* $(M_t, t < 0)$:

$$\varrho(\mathcal{A}) = \big\{|t|; t < 0 \text{ s.t. } M_t - M_{t\_} = 1\big\}.$$

Let $\sum_{j\in J} \delta_{x_j, \hat{Y}^j}$ be a Poisson point measure on $(0, +\infty) \times \mathbb{D}$ with intensity $dx\mathbb{N}[dY]$ and independent of Z_0. Let $\hat{\zeta}_j$ denote the lifetime of $\hat{Y}^j$. By considering the genealogies and using the branching property, we get:

$$\big(M_0^t, t > 0\big) \overset{(\mathrm{d})}{=} \left(\sum_{x_j < Z_0} \mathbf{1}_{\{\hat{\zeta}_j \geq t\}}, t > 0\right). \tag{21}$$

Then, thanks to Lemma 4.1, we deduce that the coalescent times $\varrho(\mathcal{A})$ are distributed as the family of lifetimes:

$$\varrho(\mathcal{A}) \overset{(\mathrm{d})}{=} \{\hat{\zeta}_j; j \in J \text{ s.t. } x_j < Z_0\}.$$

Notice that by construction, $\sum_{j\in J} \delta_{x_j, \hat{\zeta}_j}$ is a Poisson point measure on $(0, +\infty)^2$ with intensity $dx|c'(t)|\,dt$ and is independent of Z_0.

This result is similar to the one in [2] or [23] for a critical CB process (corresponding to $\theta = 0$ in our framework) born in the past according to the Lebesgue measure on $(-\infty, 0)$, with the intensity of the corresponding Poisson point measure on $(0, 1) \times (0, +\infty)$ given by $dx t^{-2}\,dt$; see also [13] for extensions concerning the model developed in [23]. (Notice the two intensities are similar near 0 as $|c'(t)| \sim_{0+} 1/(\beta t^2)$.) Similar results are given for other models, see [18] for non-quadratic CB processes, and [17] for Crump–Mode–Jagers processes.

Proof of Lemma 4.1. By (20), a.s. $Z_0 = \lim_{t\to 0+} M_{-t}/c(t)$. Since $c(t) = \mathbb{N}[\zeta \geq t]$, we can deduce from (21), using standard results on Poisson point measures, that a.s. $Z_0 = \lim_{t\to 0+} M_0^t/c(t)$. This and the fact that $(M_{-t}, t > 0)$ and $(M_0^t, t > 0)$ are Markov processes, imply that it is enough to check that (M_{-t}, M_{-r}) and (M_0^t, M_0^r) have the same distribution for $r > t > 0$ to prove the lemma.

Let $r > t > 0$. On one hand, notice that each of the M_0^t ancestors at time 0 of the population living at time t generate independently a population (at time 0) distributed according to $\mathbb{N}[dY|\zeta > t]$. This implies that

$$M_0^r \overset{(\mathrm{d})}{=} \sum_{i=1}^{M_0^t} \mathbf{1}_{\{\tilde{\zeta}_i > r\}}, \tag{22}$$

where $(\tilde{Y}^i, i \in \mathbb{N}^*)$ are independent, independent of M_0^t and distributed according to $\mathbb{N}[dY|\zeta > t]$. This readily implies that M_0^r is, conditionally on M_0^t, binomial with parameter $(M_0^t, \frac{c(r)}{c(t)})$. (This could have been deduced from Corollary A.2.) Thus using (19), we have for $\lambda > 0$ and $\mu > 0$:

$$\begin{aligned} \mathrm{E}\big[\mathrm{e}^{-\lambda M_0^r - \mu M_0^t}\big] &= \mathrm{E}\left[\left(\mathrm{e}^{-\mu}\left(1 - \frac{c(r)}{c(t)}\big(1 - \mathrm{e}^{-\lambda}\big)\right)\right)^{M_0^t}\right] = \left(1 + \frac{c(t)}{2\theta}\left(1 - \mathrm{e}^{-\mu}\left(1 - \frac{c(r)}{c(t)} + \frac{c(r)}{c(t)}\mathrm{e}^{-\lambda}\right)\right)\right)^{-2} \\ &= \left(1 + \frac{c(t)}{2\theta}\big(1 - \mathrm{e}^{-\mu}\big) + \frac{c(r)}{2\theta}\big(1 - \mathrm{e}^{-\lambda}\big)\mathrm{e}^{-\mu}\right)^{-2}. \end{aligned} \tag{23}$$

On the other hand, given M_{-r}, M_{-t} can be decomposed into two parts:

$$M_{-t} = M^{I[-r,-t]} + \sum_{j=1}^{M_{-r}} \tilde{M}^{r,j}_{r-t}, \tag{24}$$

where

(i) $M^{I[-r,-t]}$ is the number of ancestors at time $-t$ of the population living at time 0 corresponding to a population Y^i (see definition (12)) immigrating at time t_i belonging to $(-r,-t)$ and $M^{I[-r,-t]}$ is independent of M_{-r}.

(ii) $(\tilde{M}^{r,j}_{r-t}, j \in \mathbb{N}^*)$ are independent, independent of M_{-r} and each one represents the number of ancestors at time $-t$ generated by one of the ancestors at time $-r$. By construction, $(\tilde{M}^{r,j}_{r-t}, j \in \mathbb{N}^*)$ are distributed as R^r_{r-t} under $\mathbb{N}[dY|\zeta > r]$.

We get for $\lambda > 0$ and $\mu > 0$:

$$\begin{aligned} \mathrm{E}\big[\mathrm{e}^{-\lambda M_{-r} - \mu M_{-t}}\big] &= \mathrm{E}\big[\mathrm{e}^{-\lambda M_{-r}} \mathrm{E}\big[\mathrm{e}^{-\mu M_{-t}} | M_{-r}\big]\big] \\ &= \mathrm{E}\big[\mathrm{e}^{-\lambda M_{-r}} \mathbb{N}\big[\mathrm{e}^{-\mu R^r_{r-t}} | \zeta > r\big]^{M_{-r}}\big] \mathbb{E}\big[\mathrm{e}^{-\mu M^{I[-r,-t]}}\big]. \end{aligned} \tag{25}$$

Using (47), we obtain:

$$\mathbb{N}\big[\mathrm{e}^{-\mu R^r_{r-t}} | \zeta > r\big] = \frac{\mathbb{N}[\zeta > r] - \mathbb{N}[1 - \mathrm{e}^{-\mu R^r_{r-t}}]}{\mathbb{N}[\zeta > r]} = \frac{c(r) - u((1 - \mathrm{e}^{-\mu})c(t), r - t)}{c(r)}.$$

By the exponential formula for Poisson point measures and (4), we have:

$$\mathrm{E}\big[\mathrm{e}^{-\mu M^{I[-r,-t]}}\big] = \mathrm{e}^{-2\beta \int_0^{r-t} ds \mathbb{N}[1 - \mathrm{e}^{-\mu R^{t+s}_s}]} = \left(1 + \big(1 - \mathrm{e}^{-\mu}\big)\big(1 - \mathrm{e}^{-2\beta\theta(r-t)}\big)\frac{c(t)}{2\theta}\right)^{-2}.$$

Plugging the above computations in (25), and using (19), we get:

$$\mathrm{E}\big[\mathrm{e}^{-\lambda M_{-r} - \mu M_{-t}}\big] = \left[1 + \frac{c(t)}{2\theta}\big(1 - \mathrm{e}^{-\mu}\big) + \frac{c(r)}{2\theta}\big(1 - \mathrm{e}^{-\lambda}\big)\mathrm{e}^{-\mu}\right]^{-2}.$$

This and (23) imply that (M_{-t}, M_{-r}) and (M^t_0, M^r_0) have the same distribution. □

We now give the main theorem of this section on the time reversal of the number of ancestors process.

Theorem 4.3. *The process $(M^s_{s-r}, s \in \mathbb{R}, r > 0)$ is distributed as $(M^{s+r}_s, s \in \mathbb{R}, r > 0)$.*

Remark 4.4. *Using stationarity, we deduce that the truncated total length process of the genealogical tree, see (35):*

$$\left(L^s_\varepsilon = \int_\varepsilon^\infty M^s_{s-r}\, dr, s \in \mathbb{R}, \varepsilon > 0\right)$$

is distributed as the (backward) truncated total lifetime process of the population

$$\left(\int_\varepsilon^\infty M^{-s+r}_{-s}\, dr, s \in \mathbb{R}, \varepsilon > 0\right),$$

which in a sense is illustrated in Figure 2.

In particular, we deduce from Remark 4.2 the following distribution equality:

$$\big(L^0_\varepsilon, \varepsilon > 0\big) \overset{(\mathrm{d})}{=} \left(\sum_{x_j < Z_0} (\zeta_j - \varepsilon)_+, \varepsilon > 0\right), \tag{26}$$

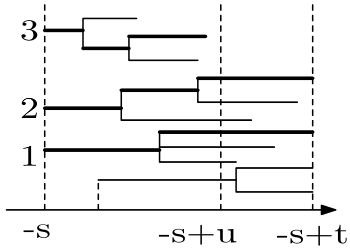

Fig. 2. At a fixed time $-s$, three families are present, they will all die out at time $-s+t$. M_{-s}^{-s+u} equals the number of bold lines present at any time $-s+u$. Then the total lifetime of the population at $-s$ is to add the length of the bold parts together.

where $x_+ = \max(x,0)$ *and* $\sum_{j\in J}\delta_{x_j,\zeta_j}$ *is a Poisson point measure on* $(0,+\infty)^2$ *with intensity* $dx|c'(t)|\,dt$ *independent of* Z_0.

Before giving the proof of Theorem 4.3 which is postponed at the end of this section, we first give a preliminary lemma.

We define the forward process for the individuals living at time s (which relies on their life-time) $\mathcal{M}_s^{(\mathrm{f})} = (M_s^{r+s}, r>0)$ and the backward process for the ancestors of the population living at time s, $\mathcal{M}_s^{(\mathrm{b})} = (M_{s-r}^{s}, r>0)$.

Lemma 4.5. *We have for* $t>r>0$, $s\ge 0$, $\lambda>0$ *and* $\mu>0$:

$$\mathbb{E}\big[\mathrm{e}^{-\lambda M_{-s-r}^{-s}-\mu M_{-s-t}^{-s}}\big|\mathcal{M}_0^{(\mathrm{b})}\big] = \mathcal{K}_{\lambda,\mu}\big(\mathcal{M}_0^{(\mathrm{b})}\big), \tag{27}$$

$$\mathbb{E}\big[\mathrm{e}^{-\lambda M_s^{s+r}-\mu M_s^{s+t}}\big|\mathcal{M}_0^{(\mathrm{f})}\big] = \mathcal{K}_{\lambda,\mu}\big(\mathcal{M}_0^{(\mathrm{f})}\big), \tag{28}$$

with $\mathcal{K}_{\lambda,\mu}$ *some measurable deterministic function depending on* λ *and* μ.

Proof. We first prove (27). Using Proposition 3.4, the quantity $(M_{-s-r}^{-s}, M_{-s-t}^{-s})$, conditionally on $\mathcal{M}_0^{(\mathrm{b})}$, consists of three parts:

(i) The ancestors at time $-s-t$ of the current population at 0, that is M_{-s-t}.
(ii) The ancestors coming from the immortal individual (or from the immigration) over $v\in(-\infty,-s-t)$, whose intensity is $2\beta\,dv(M_v+1)\mathbb{N}[dY;\zeta<-v]$.
(iii) The ancestors coming from the immortal individual (or from the immigration) over $v\in(-s-t,-s-r)$, whose intensity is $2\beta du(M_v+1)\mathbb{N}[dY;\zeta<-v]$. Notice that in this case, we have $M_{-s-t}^{-s}=0$.

This implies:

$$\begin{aligned}
&\mathbb{E}\big[\mathrm{e}^{-\lambda M_{-s-r}^{-s}-\mu M_{-s-t}^{-s}}\big|\mathcal{M}_0^{(\mathrm{b})}\big]\\
&\quad = \mathrm{e}^{-(\mu+\lambda)M_{-s-t}}\exp\left(-2\beta\int_{s+r}^{s+t} dv(M_{-v}+1)\mathbb{N}\big[\big(1-\mathrm{e}^{-\lambda R_{v-s-r}^{v-s}}\big)\mathbf{1}_{\{\zeta<v\}}\big]\right)\\
&\qquad \times\exp\left(-2\beta\int_{s+t}^{\infty} dv(M_{-v}+1)\mathbb{N}\big[\big(1-\mathrm{e}^{-\lambda R_{v-s-r}^{v-s}-\mu R_{v-s-t}^{v-s}}\big)\mathbf{1}_{\{\zeta<v\}}\big]\right).
\end{aligned}$$

Lemma A.5 implies

$$\mathbb{N}\big[\big(1-\mathrm{e}^{-\lambda R_{v-s-r}^{v-s}}\big)\mathbf{1}_{\{\zeta<v\}}\big] = u(\delta_1, v-s-r)-c(v), \tag{29}$$

$$\mathbb{N}\big[\big(1-\mathrm{e}^{-\lambda R_{v-s-r}^{v-s}-\mu R_{v-s-t}^{v-s}}\big)\mathbf{1}_{\{\zeta<v\}}\big] = u(\delta_3, v-s-t)-c(v), \tag{30}$$

with

$$\delta_1 = \big(1-\mathrm{e}^{-\lambda}\big)c(r)+\mathrm{e}^{-\lambda}c(r+s) \quad\text{and}\quad \delta_3 = \big(1-\mathrm{e}^{-\mu}\big)c(t)+\mathrm{e}^{-\mu}u(\delta_1, t-r). \tag{31}$$

Thus, we deduce (27) with:

$$\mathcal{K}_{\lambda,\mu}\big(\mathcal{M}_0^{(\mathrm{b})}\big) = \mathrm{e}^{-(\lambda+\mu)M_{-s-t}-2\beta\int_{s+r}^{+\infty} g(v)(M_{-v}+1)\,dv}, \tag{32}$$

with g defined by:

$$g(v) = \mathbf{1}_{(s+r,s+t)}(v)u(\delta_1, v-s-r) + \mathbf{1}_{(s+t,+\infty)}(v)u(\delta_3, v-s-t) - c(v). \tag{33}$$

On the other hand, using the Williams' decomposition given in Abraham and Delmas [1], the quantity (M_s^{s+r}, M_s^{s+t}), conditionally on $\mathcal{M}_0^{(\mathrm{f})}$, consists of three parts:

(i) The individuals of the current populations which have descendants at time $s+t$, that is, M_0^{s+t}.
(ii) The part coming from the immigration over the time interval $[0,s]$ which arrives with rate $2\beta\,dv\mathbb{N}[dY]$.
(iii) The individuals $i \in I$, for some index set I, living at time 0 with lifetime $\zeta_i > s+r$ generate individuals, with intensity $2\beta\,dv\mathbb{N}[dY; \zeta < \zeta_i - v]$, over the time interval $[0,s]$ and some of them are still alive at time $s+r$ and $s+t$.

Notice the set $\{\zeta_i, i \in I\} = \{t; M_0^t = M_0^{t-}+1\}$ is measurable with respect to the σ-field generated by $\mathcal{M}_0^{(\mathrm{f})}$. Therefore, we have:

$$\begin{aligned}
&\mathbb{E}\big[\mathrm{e}^{-\lambda M_s^{s+r}-\mu M_s^{s+t}}\big|\mathcal{M}_0^{(\mathrm{f})}\big]\\
&\quad= \mathrm{e}^{-(\mu+\lambda)M_0^{s+t}}\exp\bigg(-2\beta\int_0^s dv\mathbb{N}\big[\big(1-\mathrm{e}^{-\lambda R_{s-v}^{s+r-v}-\mu R_{s-v}^{s+t-v}}\big)\big]\bigg)\\
&\qquad\times\exp\bigg(-2\beta\sum_{\zeta_i>s+r}\int_0^s dv\mathbb{N}\big[\big(1-\mathrm{e}^{-\lambda R_{s-v}^{s+r-v}-\mu R_{s-v}^{s+t-v}}\big)\mathbf{1}_{\{\zeta<\zeta_i-v\}}\big]\bigg).
\end{aligned}$$

We set for $q \geq s+r$:

$$G(q) = \int_0^s dv\mathbb{N}\big[\big(1-\mathrm{e}^{-\lambda R_{s-v}^{s+r-v}-\mu R_{s-v}^{s+t-v}}\big)\mathbf{1}_{\{\zeta<q-v\}}\big].$$

Notice that $G(s+r)=0$. With this notation we can write

$$\mathbb{E}\big[\mathrm{e}^{-\lambda M_s^{s+r}-\mu M_s^{s+t}}\big|\mathcal{M}_0^{(\mathrm{f})}\big] = \mathrm{e}^{-(\mu+\lambda)M_0^{s+t}-2\beta G(+\infty)-2\beta\sum_{\zeta_i>s+r}G(\zeta_i)}. \tag{34}$$

First, we compute the derivative of G on $(s+r,s+t)$. Thanks to Lemma A.5, see (49), we have for $q\in[s+r,s+t]$:

$$G(q) = \int_0^s dv\mathbb{N}\big[\big(1-\mathrm{e}^{-\lambda R_{s-v}^{s+r-v}}\big)\mathbf{1}_{\{\zeta<q-v\}}\big] = \int_0^s dv\big(u\big(\gamma_1(q), s-v\big) - c(q-v)\big)$$

with

$$\gamma_1(q) = \big(1-\mathrm{e}^{-\lambda}\big)c(r) + \mathrm{e}^{-\lambda}c(q-s).$$

Notice that for $q\in(s+r,s+t)$,

$$\begin{aligned}
\partial_q\int_0^s dv u\big(\gamma_1(q), s-v\big) &= \partial_q\gamma_1(q)\int_0^s dv\partial_\lambda u\big(\gamma_1(q), s-v\big)\\
&= -\partial_q\gamma_1(q)\int_0^s dv\frac{\partial_t u(\gamma_1(q), s-v)}{\psi(\gamma_1(q))}\\
&= \partial_q\gamma_1(q)\frac{\gamma_1(q)-u(\gamma_1(q),s)}{\psi(\gamma_1(q))},
\end{aligned}$$

where we used (3) for the second equality. We also have:

$$-\partial_q \int_0^s dv c(q-v) = c(q-s) - c(q).$$

Recall δ_1 defined in (31). Elementary computations yield for $q \in (s+r, s+t)$:

$$\begin{aligned}
c(q) + \partial_q G(q) &= \partial_q \gamma_1(q) \frac{\gamma_1(q) - u(\gamma_1(q), s)}{\psi(\gamma_1(q))} + c(q-s) \\
&= 2\theta \frac{(\mathrm{e}^{2\beta\theta(r+s)} - 1) - \mathrm{e}^{-\lambda}\mathrm{e}^{2\beta\theta r}(\mathrm{e}^{2\beta\theta s} - 1)}{(\mathrm{e}^{2\beta\theta(r+s)} - 1)(\mathrm{e}^{2\beta\theta(q-s)} - 1) - \mathrm{e}^{-\lambda}(\mathrm{e}^{2\beta\theta s} - 1)(\mathrm{e}^{2\beta\theta(q-s)} - \mathrm{e}^{2\beta\theta r})} \\
&= u(\delta_1, q-s-r).
\end{aligned}$$

Thanks to (33), we deduce that for $q \in (s+r, s+t)$, $\partial_q G(q) = g(q)$.

Second, we compute the derivative of G on $(s+t, +\infty)$. Thanks to Lemma A.5, see (50), we have for $q > s+t$:

$$G(q) = \int_0^s dv \big(u\big(\gamma_2(q), s-v\big) - c(q-v)\big),$$

with

$$\gamma_2(q) = \big(1 - \mathrm{e}^{-\lambda}\big)c(r) + \mathrm{e}^{-\lambda}\big(1 - \mathrm{e}^{-\mu}\big)c(t) + \mathrm{e}^{-(\lambda+\mu)}c(q-s).$$

Similar arguments as in the first part give for $q > s+t$:

$$c(q) + \partial_q G(q) = \partial_q \gamma_2(q) \frac{\gamma_2(q) - u(\gamma_2(q), s)}{\psi(\gamma_2(q))} + c(q-s).$$

Recall δ_3 defined in (31). Elementary (but tedious) computations yield for $q > s+t$:

$$c(q) + \partial_q G(q) = u(\delta_3, q-s-t),$$

so that for $q > s+t$, $\partial_q G(q) = g(q)$.

For all $0 < v < s$, $0 < r < t$, we have $\mathbb{N}$-a.e. $\lim_{q \downarrow s+t} R_{s-v}^{s+t-v} \mathbf{1}_{\{\zeta < q-v\}} = 0$. This implies that G is continuous at $s+t$. Since $G(s+r) = 0$, we deduce that for all $q \geq s+r$,

$$G(q) = \int_{s+r}^q g(v)\, dv.$$

In particular, we get $G(+\infty) = \int_{s+r}^{+\infty} g(v)\, dv$ as well as:

$$\sum_{i;\zeta_i > s+r} G(\zeta_i) = \int_{s+r}^{+\infty} dv g(v) \sum_{i;\zeta_i > s+r} \mathbf{1}_{\{v \leq \zeta_i\}} = \int_{s+r}^{+\infty} dv g(v) M_0^v.$$

This, (34) and (32) imply:

$$\mathbb{E}\big[\mathrm{e}^{-\lambda M_s^{s+r} - \mu M_s^{s+t}} \big| \mathcal{M}_0^{(\mathrm{f})}\big] = \mathrm{e}^{-(\mu+\lambda)M_0^{s+t} - 2\beta \int_{s+r}^{+\infty} dv g(v)(M_0^v + 1)} = \mathcal{K}_{\lambda,\mu}\big(\mathcal{M}_0^{(\mathrm{f})}\big).$$

This ends the proof of the lemma. □

Proof of Theorem 4.3. Notice the (stationary) processes $\mathcal{M}^{(\mathrm{b})} = (\mathcal{M}_s^{(\mathrm{b})}, s \in \mathbb{R})$ and $\mathcal{M}^{(\mathrm{f})} = (\mathcal{M}_s^{(\mathrm{f})}, s \in \mathbb{R})$ are Markovian. Since the process $M_{-s-\bullet}^{-s}$ (resp. $M_s^{s+\bullet}$) conditionally on $\mathcal{M}_0^{(\mathrm{b})}$ (resp. $\mathcal{M}_0^{(\mathrm{f})}$) is Markovian, we deduce from Lemma 4.5 that the transition kernels of $\mathcal{M}^{(\mathrm{b})}$ and $\mathcal{M}^{(\mathrm{f})}$ are equal. Then use Lemma 4.1 to conclude. □

5. The total length process

5.1. *Total length process*

We define the total length of the genealogical tree for the population living at time s, up to time $s-\varepsilon$ (with $\varepsilon>0$) by:

$$L_\varepsilon^s := \int_\varepsilon^\infty M_{s-r}^s\, dr. \tag{35}$$

In order to study the asymptotic of L_ε^s as ε goes to 0, we consider the normalized total length

$$\mathcal{L}_\varepsilon^s := L_\varepsilon^s - Z_s \int_\varepsilon^\infty c(r)\, dr = L_\varepsilon^s + \frac{Z_s}{\beta} \log\bigl(1-\mathrm{e}^{-2\beta\theta\varepsilon}\bigr), \tag{36}$$

where we used (7) for the last equality. When $s=0$, we write L_ε (resp. $\mathcal{L}_\varepsilon$) for L_ε^s (resp. $\mathcal{L}_\varepsilon^s$). By stationarity, the distributions of $(L_\varepsilon^s, \varepsilon>0)$ and $(\mathcal{L}_\varepsilon^s, \varepsilon>0)$ do not depend on s. We have:

$$\mathrm{E}[L_\varepsilon] = \frac{1}{\theta}\int_\varepsilon^\infty c(r)\, dr = -\frac{1}{\beta\theta}\log\bigl(1-\mathrm{e}^{-2\beta\theta\varepsilon}\bigr) \quad \text{and} \quad \mathrm{E}[\mathcal{L}_\varepsilon]=0. \tag{37}$$

In order to study the convergence of $\mathcal{L}_\varepsilon$, we first give an elementary lemma. Recall the dilogarithm function defined for $0\leq t\leq 1$ by:

$$\mathrm{Li}_2(t) = -\int_0^t \frac{\log(1-x)}{x}\, dx$$

and $\mathrm{Li}_2(0)=0$, $\mathrm{Li}_2(1)=\pi^2/6$.

Lemma 5.1. *For $\eta>\varepsilon>0$, we have*:

$$\mathrm{E}\bigl[(\mathcal{L}_\eta-\mathcal{L}_\varepsilon)^2\bigr] = \frac{1}{\beta^2\theta^2}\bigl[\mathrm{Li}_2\bigl(1-\mathrm{e}^{-2\beta\theta\eta}\bigr) - \mathrm{Li}_2\bigl(1-\mathrm{e}^{-2\beta\theta\varepsilon}\bigr)\bigr] + \frac{2\varepsilon}{\beta\theta}\log\left(\frac{1-\mathrm{e}^{-2\beta\theta\varepsilon}}{1-\mathrm{e}^{-2\beta\theta\eta}}\right). \tag{38}$$

In particular, *we have* $\lim_{\varepsilon\to 0}\mathrm{E}[(\mathcal{L}_\varepsilon)^2] = \frac{1}{\beta^2\theta^2}\frac{\pi^2}{6}$.

Notice that the last term in the right-hand side of (38) is negative.

Proof of Lemma 5.1. Notice that

$$\mathrm{E}\bigl[(\mathcal{L}_\eta-\mathcal{L}_\varepsilon)^2\bigr] = \mathrm{E}\left[\left(\int_\varepsilon^\eta \bigl(M_{-r}-c(r)Z_0\bigr)\, dr\right)^2\right].$$

We have:

$$\mathrm{E}\left[\left(\int_\varepsilon^\eta \bigl(M_{-r}-c(r)Z_0\bigr)\, dr\right)^2\right] = 2\int_{[\varepsilon,\eta]^2} dt\, dr\, \mathbf{1}_{\{t<r\}}\mathrm{E}\bigl[\bigl(M_{-r}-c(r)Z_0\bigr)\bigl(M_{-t}-c(t)Z_0\bigr)\bigr].$$

It is easy to derive that for $r>t>0$:

$$\mathrm{E}\bigl[c(r)c(t)Z_0^2\bigr] = \frac{3}{2}\frac{c(r)c(t)}{\theta^2},$$

$$\mathrm{E}[M_{-r}M_{-t}] = \frac{c(r)}{\theta}\left(1+\frac{3}{2}\frac{c(t)}{\theta}\right),$$

$$\mathrm{E}\big[M_{-r}c(t)Z_0\big] = \mathrm{E}\big[M_0^r c(t)Z_0\big] = c(r)c(t)\mathrm{E}\big[Z_0^2\big] = \frac{3}{2}\frac{c(r)c(t)}{\theta^2},$$

$$\mathrm{E}\big[M_{-t}c(r)Z_0\big] = \frac{3}{2}\frac{c(r)c(t)}{\theta^2},$$

where we used (15) for the first equality, (54) for the second, Lemma 4.1 for the third, and the fact that conditionally on Z_0, M_0^r is Poisson with parameter $c(t)Z_0$ for the fourth. We deduce that for $r > t > 0$:

$$\mathrm{E}\big[\big(M_{-r} - c(r)Z_0\big)\big(M_{-t} - c(t)Z_0\big)\big] = \frac{c(r)}{\theta},$$

and thus:

$$\begin{aligned}
&\mathrm{E}\bigg[\bigg(\int_\varepsilon^\eta \big(M_{-r} - c(r)Z_0\big)\,dr\bigg)^2\bigg] \\
&\quad = \frac{2}{\theta}\int_\varepsilon^\eta (r-\varepsilon)c(r)\,dr \\
&\quad = -\frac{1}{\beta^2\theta^2}\int_{\mathrm{e}^{-2\beta\theta\eta}}^{\mathrm{e}^{-2\beta\theta\varepsilon}} \frac{\log(y)}{1-y}\,dy + \frac{2\varepsilon}{\beta\theta}\int_{\mathrm{e}^{-2\beta\theta\eta}}^{\mathrm{e}^{-2\beta\theta\varepsilon}} \frac{1}{y-1}\,dy \\
&\quad = \frac{1}{\beta^2\theta^2}\big[\mathrm{Li}_2\big(1-\mathrm{e}^{-2\beta\theta\eta}\big) - \mathrm{Li}_2\big(1-\mathrm{e}^{-2\beta\theta\varepsilon}\big)\big] + \frac{2\varepsilon}{\beta\theta}\log\bigg(\frac{1-\mathrm{e}^{-2\beta\theta\varepsilon}}{1-\mathrm{e}^{-2\beta\theta\eta}}\bigg).
\end{aligned}$$

The second assertion is immediate. □

We have the a.s. and L^2 convergence of $(\mathcal{L}_\varepsilon^s, \varepsilon > 0)$ as ε goes down to 0.

Theorem 5.2. *Let $s \in \mathbb{R}$. The compensated tree length $(\mathcal{L}_\varepsilon^s, \varepsilon > 0)$ converges a.s. and in L^2: there exists a random variable $W_s \in L^2$ such that*

$$\mathcal{L}_\varepsilon^s \xrightarrow[{[\varepsilon\to 0]}]{L^2 \text{ and a.s.}} W_s.$$

We have:

$$\mathrm{E}[W_s] = 0 \quad \textit{and} \quad \mathrm{E}\big[W_s^2\big] = \frac{1}{\beta^2\theta^2}\frac{\pi^2}{6}.$$

By stationarity, we deduce that the distribution of W_s does not depend on s. Furthermore, we have the convergence a.s. and in L^2 of the finite dimensional marginals of the process $(\mathcal{L}_\varepsilon^s, s \in \mathbb{R})$ towards those of the process $W = (W_s, s \in \mathbb{R})$ as ε goes down to 0.

Proof of Theorem 5.2. By stationarity, we only need to consider the case $s = 0$. We deduce from Lemma 5.1 the L^2 convergence of $(\mathcal{L}_\varepsilon, \varepsilon > 0)$ as ε goes down to 0 towards a limit W_0 as well as the first and second moment of W_0.

We now prove the a.s. convergence. We deduce from Lemma 5.1 that for $\eta > 0$ small enough:

$$\mathrm{E}\big[(\mathcal{L}_\eta - W_0)^2\big] = \frac{1}{\beta^2\theta^2}\mathrm{Li}_2\big(1-\mathrm{e}^{-2\beta\theta\eta}\big) \le \frac{4\eta}{\beta\theta}.$$

Set $a_n = 1/n^2$ for $n \in \mathbb{N}^*$. We deduce that $(\mathcal{L}_{a_n}, n \in \mathbb{N}^*)$ converges a.s. to W_0. For $\varepsilon \in [a_{n+1}, a_n]$, we have:

$$\mathcal{L}_{a_n} - Z_0\int_{a_{n+1}}^{a_n} c(r)\,dr \le \mathcal{L}_\varepsilon \le \mathcal{L}_{a_{n+1}} + Z_0\int_{a_{n+1}}^{a_n} c(r)\,dr.$$

Since for n large enough:

$$\int_{a_{n+1}}^{a_n} c(r)\,dr \le c(a_{n+1})(a_n - a_{n+1}) \le \frac{5}{\beta n},$$

we deduce that $(\mathcal{L}_\varepsilon, \varepsilon > 0)$ converges a.s. to W_0 as ε goes down to 0. □

Remark 5.3. *We deduce from Lemma 5.1 and (15) that*:

$$\mathrm{E}\big[(L_\varepsilon)^2\big] = \mathrm{E}\big[(\mathcal{L}_\varepsilon)^2\big] + \frac{2}{\beta}\log\big(1-\mathrm{e}^{-2\beta\theta\varepsilon}\big)\mathrm{E}[Z_0 L_\varepsilon] - \frac{3}{2\beta^2\theta^2}\log\big(1-\mathrm{e}^{-2\beta\theta\varepsilon}\big)^2.$$

Arguing as the proof of Lemma 5.1 we get:

$$\mathrm{E}[Z_0 L_\varepsilon] = \int_\varepsilon^\infty c(r)\,dr\,\mathrm{E}\big[Z_0^2\big] = \frac{3}{2\beta\theta^2}\log\big(1-\mathrm{e}^{-2\beta\theta\varepsilon}\big).$$

This gives:

$$\mathrm{E}\big[(L_\varepsilon)^2\big] = \mathrm{E}\big[(\mathcal{L}_\varepsilon)^2\big] + \frac{3}{2\beta^2\theta^2}\log\big(1-\mathrm{e}^{-2\beta\theta\varepsilon}\big)^2.$$

We get the following equivalence for the expectation and variance of L_ε as ε goes down to 0:

$$\mathrm{E}[L_\varepsilon] \sim_{0+} \frac{1}{\beta\theta}\log(1/\varepsilon) \quad \textit{and} \quad \mathrm{Var}(L_\varepsilon) \sim_{0+} \frac{1}{2\beta^2\theta^2}\log(1/\varepsilon)^2.$$

5.2. *Distribution and fluctuation for the* 1*-dimensional marginal*

We provide the distribution of W_0 via its Laplace transform.

Lemma 5.4. *For $\lambda > 0$ and $z > 0$, we have*:

$$\mathrm{E}\left[\mathrm{e}^{-2\beta\theta\lambda W_0}\,\middle|\, Z_0 = \frac{z}{2\theta}\right] = \mathrm{e}^{-z\varphi(\lambda)} \quad \textit{and} \quad \mathrm{E}\big[\mathrm{e}^{-2\lambda\beta\theta W_0}\big] = \big(1+\varphi(\lambda)\big)^{-2},$$

with

$$\varphi(\lambda) = -\lambda\int_0^1 dv\,\frac{1-v^\lambda}{1-v}.$$

From the proof below, we get that the distribution of W_0 is infinitely divisible (conditionally on Z_0 or not). Notice that for $\lambda = n$, we get $\varphi(n) = -nH_n$, where H_n is the harmonic number.

Proof of Lemma 5.4. We use the notations from Remark 4.4. According to Remark 4.4, see also (26), and since $c(t) = \mathbb{N}[\zeta > t]$, we get that $(Z_0, (\mathcal{L}_\varepsilon, \varepsilon > 0))$ is distributed as $(Z_0, (\tilde{\mathcal{L}}_\varepsilon, \varepsilon > 0))$, with

$$\tilde{\mathcal{L}}_\varepsilon = \sum_{x_j < Z_0} (\zeta_j - \varepsilon)_+ - Z_0\mathbb{N}\big[(\zeta-\varepsilon)_+\big].$$

In particular W_0 is distributed as $\tilde{W}_0 = \lim_{\varepsilon\downarrow 0}\tilde{\mathcal{L}}_\varepsilon$.

The exponential formula for Poisson point measures gives for any $\lambda > 0$:

$$\mathrm{E}\left[\mathrm{e}^{-\lambda\tilde{\mathcal{L}}_\varepsilon}\,\middle|\, Z_0 = \frac{z}{2\theta}\right] = \exp\left(-\frac{z}{2\theta}\mathbb{N}\big[1-\mathrm{e}^{-\lambda(\zeta-\varepsilon)_+}\big] + \frac{\lambda z}{2\theta}\mathbb{N}\big[(\zeta-\varepsilon)_+\big]\right) = \exp\big(-zK_\varepsilon(\lambda)\big)$$

with

$$
\begin{aligned}
K_\varepsilon(\lambda) &= \frac{1}{2\theta}\mathbb{N}\big[1-\mathrm{e}^{-\lambda(\zeta-\varepsilon)_+}-\lambda(\zeta-\varepsilon)_+\big] \\
&= 2\beta\theta\int_\varepsilon^\infty dt\,\frac{\mathrm{e}^{2\beta\theta t}}{(\mathrm{e}^{2\beta\theta t}-1)^2}\big(1-\mathrm{e}^{-\lambda(t-\varepsilon)}-\lambda(t-\varepsilon)\big) \\
&= 2\beta\theta\mathrm{e}^{2\beta\theta\varepsilon}\int_0^\infty dt\,\frac{\mathrm{e}^{2\beta\theta t}}{(\mathrm{e}^{2\beta\theta(t+\varepsilon)}-1)^2}\big(1-\mathrm{e}^{-\lambda t}-\lambda t\big),
\end{aligned}
$$

where we used that $\mathbb{N}[d\zeta]_{|\zeta=t}=-c'(t)\,dt=4\beta\theta^2\frac{\mathrm{e}^{2\beta\theta t}}{(\mathrm{e}^{2\beta\theta t}-1)^2}\,dt$ for the second equality. Notice that by dominated convergence:

$$
\lim_{\varepsilon\downarrow 0}K_\varepsilon(\lambda)=\vartheta(\lambda)
$$

with

$$
\vartheta(u)=2\beta\theta\int_0^\infty dt\,\frac{\mathrm{e}^{2\beta\theta t}}{(\mathrm{e}^{2\beta\theta t}-1)^2}\big(1-\mathrm{e}^{-\lambda t}-\lambda t\big).
$$

Letting ε goes down to 0, we deduce the Laplace transform of W_0, for $\lambda>0$:

$$
\mathrm{E}\left[\mathrm{e}^{-2\beta\theta\lambda W_0}\middle|Z_0=\frac{z}{2\theta}\right]=\mathrm{e}^{-z\varphi(\lambda)}
$$

with

$$
\begin{aligned}
\varphi(\lambda)=\vartheta(2\beta\theta\lambda) &= \int_0^\infty dt\,\frac{\mathrm{e}^t}{(\mathrm{e}^t-1)^2}\big(1-\mathrm{e}^{-\lambda t}-\lambda t\big) \\
&= -2\lambda\int_0^\infty dt\,\frac{1-\mathrm{e}^{-\lambda t}}{\mathrm{e}^t-1} \\
&= -\lambda\int_0^1 dv\,\frac{1-v^\lambda}{1-v},
\end{aligned}
$$

where we used integration by parts in the third equality. Notice that, conditionally on Z_0, W_0 is infinitely divisible with Lévy measure $|c'(t)|\,dt$.

We also have:

$$
\begin{aligned}
\mathrm{E}\big[\mathrm{e}^{-2\beta\theta\lambda W_0-\mu Z_0}\big] &= \mathrm{E}\big[\mathrm{e}^{-\mu Z_0-2\theta\varphi(\lambda)Z_0}\big] \\
&= \left(1+\varphi(\lambda)+\frac{\mu}{2\theta}\right)^{-2}.
\end{aligned}
$$

The second result of the lemma follows by taking $\mu=0$. □

We also give the following result on the fluctuation of $\mathcal{L}_\varepsilon$.

Proposition 5.5. *We have the following convergence in distribution*:

$$
\big(Z_0,\sqrt{\beta/\varepsilon}(\mathcal{L}_\varepsilon-W_0)\big)\xrightarrow[{[\varepsilon\to 0]}]{(\mathrm{d})}(Z_0,\sqrt{2Z_0}G_1),
$$

with $G_1\sim\mathcal{N}(0,1)$ a standard Gaussian random variable independent of Z_0.

Proof. We keep the notations from the proof of Lemma 5.4. Mimicking the proof of Lemma 5.4, we get for $\lambda \in \mathbb{R}$, $\varepsilon > \eta > 0$:

$$\mathrm{E}\big[\mathrm{e}^{i\lambda(\mathcal{L}_\varepsilon - \mathcal{L}_\eta)} | Z_0\big] = \mathrm{e}^{-Z_0 f_{\varepsilon,\eta}(\lambda)},$$

with

$$f_{\varepsilon,\eta}(\lambda) = \mathbb{N}\big[1 - \mathrm{e}^{i\lambda((\zeta-\varepsilon)_+ - (\zeta-\eta)_+)} + i\lambda\big((\zeta-\varepsilon)_+ - (\zeta-\eta)_+\big)\big].$$

Notice that $0 \le (x-\eta)_+ - (x-\varepsilon)_+ \le x \wedge \varepsilon$ for $x > 0$. Since $\mathbb{N}[(\zeta\wedge\varepsilon)^2]$ is finite, we deduce by dominated convergence that $\lim_{\eta\to 0} f_{\varepsilon,\eta}(\lambda) = f_\varepsilon(\lambda)$, with

$$\begin{aligned} f_\varepsilon(\lambda) &= \mathbb{N}\big[1 - \mathrm{e}^{-i\lambda(\zeta\wedge\varepsilon)} - i\lambda(\zeta\wedge\varepsilon)\big] \\ &= 4\beta\theta^2 \int_0^\varepsilon dt \frac{\mathrm{e}^{2\beta\theta t}}{(\mathrm{e}^{2\beta\theta t}-1)^2}\big(1 - i\lambda t - \mathrm{e}^{-i\lambda t}\big) + \frac{2\theta}{\mathrm{e}^{2\beta\theta\varepsilon}-1}\big(1 - i\lambda\varepsilon - \mathrm{e}^{-i\lambda\varepsilon}\big). \end{aligned}$$

We deduce that:

$$\mathrm{E}\big[\mathrm{e}^{i\lambda(\mathcal{L}_\varepsilon - W_0)} | Z_0\big] = \mathrm{e}^{-Z_0 f_\varepsilon(\lambda)}$$

and thus for $\mu \in \mathbb{R}$:

$$\mathrm{E}\bigg[\exp\bigg(i\lambda\frac{\mathcal{L}_\varepsilon - W_0}{\sqrt{\varepsilon}} + i\mu Z_0\bigg)\bigg] = \mathrm{E}\big[\exp\big(Z_0\big(i\mu - f_\varepsilon(\lambda/\sqrt{\varepsilon})\big)\big)\big].$$

Dominated convergence yields:

$$\lim_{\varepsilon\to 0} f_\varepsilon(\lambda/\sqrt{\varepsilon}) = \frac{\lambda^2}{\beta}.$$

We deduce that:

$$\lim_{\varepsilon\to 0} \mathrm{E}\bigg[\exp\bigg(i\lambda\frac{\mathcal{L}_\varepsilon - W_0}{\sqrt{\varepsilon}} + i\mu Z_0\bigg)\bigg] = \mathrm{E}\big[\mathrm{e}^{Z_0(i\mu - \lambda^2/\beta)}\big] = \mathrm{E}\bigg[\exp\bigg(i\lambda\sqrt{\frac{2Z_0}{\beta}}G_1 + i\mu Z_0\bigg)\bigg].$$

This gives the result. □

5.3. *Path properties of the process W*

We first give the covariance of the process W, whose proof is given in Section 5.4.

Proposition 5.6. *Let $s \in \mathbb{R}_*$. We have*:

$$\begin{aligned} \mathrm{E}[W_0 W_s] = \frac{1}{2\beta^2\theta^2}\bigg[&\frac{\pi^2}{6}\mathrm{e}^{-2\beta\theta s} + \mathrm{e}^{2\beta\theta s}\operatorname{Li}_2\big(\mathrm{e}^{-2\beta\theta s}\big)\bigg] \\ &- 2\big(\mathrm{e}^{2\beta\theta s} - \mathrm{e}^{-2\beta\theta s}\big)\int_0^\infty \frac{dr}{\mathrm{e}^{2\beta\theta r}-1}\int_s^{s+r}\frac{dq}{\mathrm{e}^{2\beta\theta q}-1}. \end{aligned}$$

Remark 5.7. *From this a short calculus calculation shows that there is a constant C such that*:

$$\mathrm{E}\big[(W_s - W_0)^2\big] \sim_{0+} Cs\big[\log(s)\big]^2.$$

This suggests that the process W is not continuous. Indeed, recall definition (12) and notice the process L_ε^s, for fixed ε, has jumps at least at any time $\zeta_i - t_i$ for any $i \in I$ such that ζ_i, the death time of Y^i, is larger than ε. The same holds for W.

We have the following result on the existence of a càdlàg version of W, whose proof is given in Section 5.5.

Theorem 5.8. *There exists a càdlàg $\mathbb{R}$-valued process $W' = (W'_s, s \in \mathbb{R})$ having the same finite dimensional marginals as W.*

5.4. *Proof of Proposition* 5.6

By Theorem 5.2, we have:

$$\mathrm{E}[W_0 W_s] = \lim_{\varepsilon\to 0+} \mathrm{E}\big[\mathcal{L}_\varepsilon \mathcal{L}_\varepsilon^s\big].$$

We turn to the calculation of $\mathrm{E}[\mathcal{L}_\varepsilon \mathcal{L}_\varepsilon^s]$ with ε small enough. We have:

$$\mathrm{E}\big[\mathcal{L}_\varepsilon \mathcal{L}_\varepsilon^s\big] = \mathrm{E}\bigg[\bigg(L_\varepsilon - Z_0\int_\varepsilon^\infty c(r)\,dr\bigg)\bigg(L_\varepsilon^s - Z_s\int_\varepsilon^\infty c(r)\,dr\bigg)\bigg] = B_1 - B_2 - B_3 + B_4,$$

with $B_1 = \mathrm{E}[L_\varepsilon L_\varepsilon^s]$,

$$B_2 = \mathrm{E}[L_\varepsilon Z_s]\int_\varepsilon^\infty c(r)\,dr, \qquad B_3 = \mathrm{E}\big[L_\varepsilon^s Z_0\big]\int_\varepsilon^\infty c(r)\,dr, \qquad B_4 = \bigg(\int_\varepsilon^\infty c(r)\,dr\bigg)^2 \mathrm{E}[Z_0 Z_s].$$

We first compute B_4. Using (16) and (7) we get:

$$B_4 = \frac{2+\mathrm{e}^{-2\beta\theta s}}{2\beta^2\theta^2}\big(\log\big(1-\mathrm{e}^{-2\beta\theta\varepsilon}\big)\big)^2.$$

We compute B_2. For $-r < 0 < s$, we have, using Proposition 3.4:

$$\begin{aligned}
\mathrm{E}[M_{-r} Z_s] &= \mathrm{E}\big[M_{-r}\mathrm{E}\big[Z_s|\sigma(M_u, u\le -r)\big]\big] \\
&= \mathrm{E}\Bigg[M_{-r}\Bigg(\sum_{i=1}^{M_{-r}} \tilde{Y}^i_{s+r} + \sum_{-r<t_i<s} Y^i_{s-t_i}\Bigg)\Bigg] \\
&= \mathrm{E}\big[M_{-r}^2\big]\mathbb{N}[Y_{s+r}|\zeta > r] + 2\beta\mathrm{E}[M_{-r}]\int_{-r}^s \mathbb{N}[Y_{s-t}]\,dt.
\end{aligned}$$

Since by (5) and (9),

$$\mathbb{N}[Y_{s+r}|\zeta > r] = \frac{\mathbb{N}[Y_{s+r}\mathbf{1}_{\{\zeta>r\}}]}{\mathbb{N}[\zeta > r]} = \frac{\mathrm{e}^{-2\beta\theta(s+r)}}{c(r)},$$

and by (18) and again (9), we have:

$$\mathrm{E}[M_{-r} Z_s] = \frac{2+\mathrm{e}^{-2\beta\theta s}}{\theta(\mathrm{e}^{2\beta\theta r}-1)}.$$

Hence, we have by (7):

$$B_2 = \mathrm{E}[L_\varepsilon Z_s]\int_\varepsilon^\infty c(r)\,dr = \frac{2+\mathrm{e}^{-2\beta\theta s}}{2\beta^2\theta^2}\big(\log\big(1-\mathrm{e}^{-2\beta\theta\varepsilon}\big)\big)^2,$$

that is, $B_2 = B_4$.

For B_3, we first compute $\mathrm{E}[Z_0 M^s_{s-q}]$. By stationary, we have $\mathrm{E}[Z_0 M^s_{s-q}] = \mathrm{E}[Z_{-s} M_{-q}]$. For $\varepsilon < q < s$, using (16), we get:

$$\mathrm{E}[Z_{-s} M_{-q}] = c(q)\mathrm{E}[Z_{-q} Z_{-s}] = \frac{2 + \mathrm{e}^{2\beta\theta(q-s)}}{\theta(\mathrm{e}^{2\beta\theta q} - 1)}.$$

For $q > s$, a decomposition similar to the one used to compute B_2 gives:

$$\mathrm{E}[Z_{-s} M_{-q}] = \frac{\mathrm{e}^{2\beta\theta s} + 2}{\theta(\mathrm{e}^{2\beta\theta q} - 1)} - \frac{(\mathrm{e}^{2\beta\theta s} - 1)^2}{\theta(\mathrm{e}^{2\beta\theta q} - 1)^2} \mathrm{e}^{2\beta\theta(q-s)}.$$

Thus we have:

$$\begin{aligned} B_3 &= \int_\varepsilon^\infty c(r)\,dr \int_\varepsilon^\infty dq\,\mathrm{E}\big[Z_0 M^s_{s-q}\big] \\ &= \int_\varepsilon^\infty \frac{c(r)}{\theta}\,dr \left[\int_\varepsilon^s dq \frac{\mathrm{e}^{2\beta\theta(q-s)} + 2}{(\mathrm{e}^{2\beta\theta q} - 1)} + \int_s^\infty dq \left[\frac{\mathrm{e}^{2\beta\theta s} + 2}{(\mathrm{e}^{2\beta\theta q} - 1)} - \frac{(\mathrm{e}^{2\beta\theta s} - 1)^2}{(\mathrm{e}^{2\beta\theta q} - 1)^2} \mathrm{e}^{2\beta\theta(q-s)}\right]\right]. \end{aligned}$$

For B_1, the integrand is computed in Lemma A.7. We have:

$$\begin{aligned} B_1 &= 2\int_\varepsilon^\infty dr \int_\varepsilon^{s+r} dq \frac{\mathrm{e}^{2\beta\theta(q-s)} + 2}{(\mathrm{e}^{2\beta\theta r} - 1)(\mathrm{e}^{2\beta\theta q} - 1)} \\ &\quad + 2\int_\varepsilon^\infty dr \int_{s+r}^\infty dq \left[\frac{\mathrm{e}^{2\beta\theta(r+s)} + 2}{(\mathrm{e}^{2\beta\theta r} - 1)(\mathrm{e}^{2\beta\theta q} - 1)} - \frac{\mathrm{e}^{2\beta\theta q}(1 - \mathrm{e}^{-2\beta\theta s})(\mathrm{e}^{2\beta\theta(r+s)} - 1)}{(\mathrm{e}^{2\beta\theta r} - 1)(\mathrm{e}^{2\beta\theta q} - 1)^2}\right]. \end{aligned}$$

Since $B_2 = B_4$, this gives:

$$\begin{aligned} \mathrm{E}\big[\mathcal{L}_\varepsilon \mathcal{L}^s_\varepsilon\big] &= B_1 - B_3 \\ &= 2\mathrm{e}^{-2\beta\theta s} \int_\varepsilon^\infty dr \int_s^{s+r} dq \frac{\mathrm{e}^{2\beta\theta q}}{(\mathrm{e}^{2\beta\theta r} - 1)(\mathrm{e}^{2\beta\theta q} - 1)} \\ &\quad + 2\mathrm{e}^{2\beta\theta s} \int_\varepsilon^\infty dr \int_{s+r}^\infty dq \frac{\mathrm{e}^{2\beta\theta r}}{(\mathrm{e}^{2\beta\theta r} - 1)(\mathrm{e}^{2\beta\theta q} - 1)} \\ &\quad - 2\mathrm{e}^{2\beta\theta s} \int_\varepsilon^\infty dr \int_s^\infty dq \frac{1}{(\mathrm{e}^{2\beta\theta q} - 1)(\mathrm{e}^{2\beta\theta r} - 1)}. \end{aligned}$$

Basic calculations yield:

$$\begin{aligned} \mathrm{E}[W_0 W_s] &= \frac{1}{2\beta^2\theta^2}\left[\frac{\pi^2}{6}\mathrm{e}^{-2\beta\theta s} + \mathrm{e}^{2\beta\theta s}\,\mathrm{Li}_2\big(\mathrm{e}^{-2\beta\theta s}\big)\right] \\ &\quad - 2\big(\mathrm{e}^{2\beta\theta s} - \mathrm{e}^{-2\beta\theta s}\big)\int_0^\infty dr \int_s^{s+r} \frac{dq}{(\mathrm{e}^{2\beta\theta r} - 1)(\mathrm{e}^{2\beta\theta q} - 1)}. \end{aligned}$$

The proof is then complete.

5.5. *Proof of Theorem* 5.8

According to Billingsley [3], Theorem 3.16, and thanks to the stationary property, it is enough to check the following two conditions:

(i) Right continuity in probability: for all $\lambda > 0$,

$$\lim_{h\downarrow 0} \mathrm{P}\big(|W_h - W_0| > \lambda\big) = 0. \tag{39}$$

(ii) Control of the jumps: there exists $\gamma > 0$, $\delta > 0$ such that for some constant $C > 0$ and all $\lambda > 0$, $s, t \in (0, 1/8)$,

$$\mathrm{P}\big(|W_{-t} - W_0| \wedge |W_s - W_0| \geq 6\lambda\big) \leq C\lambda^{-4\gamma}(s+t)^{1+\delta}. \tag{40}$$

Notice that Proposition 5.6 (see also Remark 5.7) implies the L^2-continuity of W. This in turn implies (39) and thus (i) is satisfied.

We shall now focus on (ii) and (40). In this section C denotes any finite positive constants which may vary from line to line. For notational convenience, we shall write for $\varepsilon > 0$, $s \in \mathbb{R}$:

$$\int_0^\varepsilon dr\big(M^s_{s-r} - c(r)Z_s\big) = W_s - \mathcal{L}^s_\varepsilon.$$

We define for $h > |u| > 0$:

$$\begin{aligned}
A_1(u,h) &= -\int_h^\infty \big(M_{-r} - M^u_{-r}\big)\,dr,\\
A_2(u,h) &= Z_0\int_h^\infty c(r)\,dr - Z_u\int_{h+u}^\infty c(r)\,dr,\\
A_3(u,h) &= -\int_0^h \big(M_{-r} - c(r)Z_0\big)\,dr,\\
A_4(u,h) &= \int_0^{h+u} \big(M^u_{u-r} - c(r)Z_u\big)\,dr.
\end{aligned}$$

For $s > 0$, $t > 0$ and $h > s + t$, we have:

$$W_s - W_0 = \sum_{i=1}^4 A_i(s,h) \quad \text{and} \quad W_0 - W_{-t} = -\sum_{i=1}^4 A_i(-t,h). \tag{41}$$

In a first step we give upper bounds for the probability of A_i to be large in the following lemmas.

Lemma 5.9. *There exists a finite constant C_1 such that for all $s, t \in (0, 1/8)$, $h > 2(s+t) > 0$ and $\lambda > 0$, we have*:

$$\mathrm{P}\big(\big|A_1(s,h)\big| \wedge \big|A_1(-t,h)\big| > \lambda\big) \leq C_1\frac{(s+t)^2}{h^4}.$$

Proof. Notice that $A_1(u,h) \neq 0$ implies $|M_{-h} - M^u_{-h}| \geq 1$. Therefore, we have:

$$\begin{aligned}
&\mathrm{P}\big(\big|A_1(s,h)\big| \wedge \big|A_1(-t,h)\big| > \lambda\big)\\
&\quad \leq \mathrm{P}\big(M_{-h} - M^s_{-h} \neq 0, M_{-h} - M^{-t}_{-h} \neq 0\big)\\
&\quad = 1 - \mathrm{P}\big(M_{-h} = M^s_{-h}\big) - \mathrm{P}\big(M_{-h} = M^{-t}_{-h}\big) + \mathrm{P}\big(M_{-h} = M^s_{-h}, M_{-h} = M^{-t}_{-h}\big)\\
&\quad = 1 - \mathrm{P}\big(M_{-h} = M^s_{-h}\big) - \mathrm{P}\big(M_{-h} = M^{-t}_{-h}\big) + \mathrm{P}\big(M^s_{-h} = M^{-t}_{-h}\big),
\end{aligned} \tag{42}$$

where we used for the last equality that the sequence $(M^u_{-h}, u > -h)$ is non-increasing.

Let $r > t > 0$. According to representation (22), we have that M^r_0 is, conditionally on M^t_0, binomial with parameter $(M^t_0, \frac{c(r)}{c(t)})$. This implies:

$$\mathrm{P}\big(M^r_0 = M^t_0\big) = \mathrm{E}\bigg[\bigg(\frac{c(r)}{c(t)}\bigg)^{M^t_0}\bigg] = \bigg(1 + \frac{c(t) - c(r)}{2\theta}\bigg)^{-2},$$

where we used (19) for the last equality. By stationarity, we deduce from (42) that:

$$\begin{aligned}\mathrm{P}\big(\big|A_1(s,h)\big| \wedge \big|A_1(-t,h)\big| > \lambda\big) &\le 1 - \frac{1}{(1+x)^2} - \frac{1}{(1+y)^2} + \frac{1}{(1+x+y)^2} \\ &= \int_0^x dv \int_0^y dz \frac{6}{(1+v+z)^4} \\ &\le 6xy,\end{aligned}$$

with

$$x = \frac{c(h-t) - c(h)}{2\theta} \quad \text{and} \quad y = \frac{c(h) - c(h+s)}{2\theta}.$$

Notice that:

$$\big(1 - \mathrm{e}^{-2\beta\theta h}\big)\big(\mathrm{e}^{2\beta\theta(h+s)} - 1\big) \ge \mathrm{e}^{2\beta\theta h} + \mathrm{e}^{-2\beta\theta h} - 2 \ge (2\beta\theta)^2 h^2.$$

Since for $s \in (0, 1/4)$, we have $\mathrm{e}^{2\beta\theta s} - 1 \le Cs$, we deduce that $y \le Cs/h^2$. Similarly, and using $h - t \ge h/2$, we also get $x \le Ct/h^2$. This implies:

$$\mathrm{P}\big(\big|A_1(s,h)\big| \wedge \big|A_1(-t,h)\big| > \lambda\big) \le C\frac{st}{h^4} \le C\frac{(s+t)^2}{h^4}.$$

□

Lemma 5.10. *There exists a finite constant C_2 such that for all $h > 2|u| > 0$, with $u \in [-1/8, 1/8]$, $\lambda > 0$, we have*:

$$\mathrm{P}\big(\big|A_2(u,h)\big| > \lambda\big) \le C_2 \frac{u^2}{\lambda^4 h^4}.$$

Proof. We write $A_2(u,h) = A_{2,1} + A_{2,2}$ with

$$A_{2,1} = (Z_0 - Z_u)\int_{u+h}^{+\infty} c(r)\,dr = \frac{Z_u - Z_0}{\beta}\log\big(1 - \mathrm{e}^{-2\beta\theta(h+u)}\big) \quad \text{and} \quad A_{2,2} = Z_0 \int_h^{h+u} c(r)\,dr.$$

We have:

$$\mathrm{P}\big(\big|A_2(u,h)\big| > \lambda\big) \le \mathrm{P}\big(|A_{2,1}| > \lambda/2\big) + \mathrm{P}\big(|A_{2,2}| > \lambda/2\big). \tag{43}$$

Tchebychev's inequality gives:

$$\mathrm{P}\big(|A_{2,1}| > \lambda/2\big) \le \frac{2^4}{(\lambda\beta)^4}\log\big(1 - \mathrm{e}^{-2\beta\theta(h+u)}\big)^4 \mathrm{E}\big[|Z_u - Z_0|^4\big].$$

Since Z is a Feller diffusion, see Section 7.1 in [4], we have $\mathrm{E}[|Z_u - Z_0|^4] \le Cu^2$. For $x > 0$, we have $0 \le -\log(1 - \mathrm{e}^{-x}) \le 1/x$. Using that $h > 2|u|$, we get:

$$\mathrm{P}\big(|A_{2,1}| > \lambda/2\big) \le C\frac{u^2}{\lambda^4 (h+u)^4} \le C\frac{u^2}{\lambda^4 h^4}. \tag{44}$$

Since c is decreasing and $h > 2|u|$, we have:

$$\int_h^{h+u} c(r)\,dr \le |u| c\big(h - |u|\big) \le C\frac{|u|}{h} \le C\frac{\sqrt{|u|}}{h}.$$

Then, using Tchebychev's inequality, we get:

$$\mathrm{P}\big(|A_{2,2}| > \lambda/2\big) \le C\frac{u^2}{\lambda^4 h^4}. \tag{45}$$

Then use (43) with (44) and (45) to conclude. □

Lemma 5.11. *Let $p \in \mathbb{N}^*$. There exists a finite constant C_3 such that for all $h > 0$ and $\lambda > 0$, we have*:

$$\mathrm{P}\big(\big|A_3(u,h)\big| > \lambda\big) \le C_3 \frac{h^p}{\lambda^{2p}}.$$

Proof. Using $\mathrm{E}[\prod_{k=1}^{2p} |X_k|] \le \prod_{k=1}^{2p} \mathrm{E}[X_k^{2p}]^{1/2p}$ and Fubini's theorem, we get:

$$\mathrm{E}\big[A_3(u,h)^{2p}\big] \le \left(\int_0^h dr \mathrm{E}\big[\big(M_{-r} - c(r)Z_0\big)^{2p}\big]^{1/2p}\right)^{2p}.$$

For a Poisson random variable X with mean m, we have:

$$\mathrm{E}\big[(X-m)^{2p}\big] \le C'\big(m^p + m\big),$$

where the constant C' doesn't depend on m. Thanks to Lemma 4.1, we have that M_{-r} is, conditionally on Z_0, a Poisson random variable with mean $c(r)Z_0$. This implies that:

$$\mathrm{E}\big[\big(M_{-r} - c(r)Z_0\big)^{2p}\big]^{1/2p} \le C\big(\sqrt{c(r)} + c(r)^{1/2p}\big).$$

We deduce that:

$$\int_0^h dr \mathrm{E}\big[\big(M_{-r} - c(r)Z_0\big)^{2p}\big]^{1/2p} \le C(\sqrt{h} \wedge 1) \le C\sqrt{h}.$$

Therefore, we have $\mathrm{E}[A_3(u,h)^{2p}] \le Ch^p$ and we conclude using Tchebychev's inequality. □

Lemma 5.12. *Let $p \in \mathbb{N}^*$. There exists a finite constant C_4 such that for all $h > 2|u|$ and $\lambda > 0$, we have*:

$$\mathrm{P}\big(\big|A_4(u,h)\big| > \lambda\big) \le C_4 \frac{h^p}{\lambda^{2p}}.$$

Proof. By stationarity, we have that $A_4(u,h)$ is distributed as $-A_3(u, h+u)$. Then use Lemma 5.11 to conclude. □

We complete the proof of Theorem 5.8 by proving (40). Using

$$\big\{|x+y| \wedge \big|x'+y'\big| > 6\lambda\big\} \subset \big\{|x| \wedge \big|x'\big| > 3\lambda\big\} \cup \big\{|y| > 3\lambda\big\} \cup \big\{\big|y'\big| > 3\lambda\big\},$$

we get:

$$\begin{aligned}
&\mathrm{P}\big(|W_{-t} - W_0| \wedge |W_s - W_0| \ge 6\lambda\big) \\
&\quad \le \mathrm{P}\big(\big|A_1(s,h)\big| \wedge \big|A_1(-t,h)\big| > 3\lambda\big) + \sum_{i=2}^{4} \mathrm{P}\big(\big|A_i(s,h)\big| > \lambda\big) + \sum_{i=2}^{4} \mathrm{P}\big(\big|A_i(-t,h)\big| > \lambda\big).
\end{aligned} \tag{46}$$

Let $\delta \in (0, 1/3)$, $3(1+\delta)/2 < \gamma < 2$ and $p \in \mathbb{N}^*$ such that $2p/(p+4) > \gamma$. Notice that $(1+\delta)/2\gamma < 1/3$. Set:

$$x = \frac{\lambda^{4\gamma}}{(s+t)^{1+\delta}} \quad \text{and} \quad h = (s+t)^{(1+\delta)/2\gamma} x^{1/4}.$$

If $x < 1$, then we have that (40) holds trivially with $C = 1$. So we shall assume that $x \ge 1$. For $s, t \in (0, 1/8)$, we have:

$$h \ge (s+t)^{(1+\delta)/2\gamma} \ge (s+t)^{1/3} > 2(s+t).$$

So the hypothesis of the previous lemmas are satisfied for $s, t \in (0, 1/8)$. Using $\gamma - (1+\delta) > 0$, Lemma 5.9 implies:

$$\mathrm{P}\big(\big|A_1(s,h)\big| \wedge \big|A_1(-t,h)\big| > 3\lambda\big) \le C_1 \frac{(s+t)^2}{h^4} = C_1 \frac{(s+t)^{(2/\gamma)(\gamma-(1+\delta))}}{x} \le \frac{C_1}{x}.$$

For $u \in \{-t, s\}$, using $2\gamma - 3(1+\delta) > 0$, Lemma 5.10 implies:

$$\mathrm{P}\big(\big|A_2(u,h)\big| > \lambda\big) \le C_2 \frac{u^2}{\lambda^4 h^4} \le C_2 \frac{(s+t)^2}{\lambda^4 h^4} = C_2 \frac{(s+t)^{2-3(1+\delta)/\gamma}}{x^{1+1/\gamma}} \le \frac{C_2}{x}.$$

For $u \in \{-t, s\}$, using $p(2-\gamma) \ge 8p/(p+4) > 4\gamma$ together with Lemma 5.11 resp. Lemma 5.12, we get:

$$\mathrm{P}\big(\big|A_3(u,h)\big| > \lambda\big) \le C_3 \frac{h^p}{\lambda^{2p}} = \frac{C_3}{x^{(p/(4\gamma))(2-\gamma)}} \le \frac{C_3}{x}$$

resp. for $u \in \{-t, s\}$:

$$\mathrm{P}\big(\big|A_4(u,h)\big| > \lambda\big) \le C_4 \frac{h^p}{\lambda^{2p}} \le \frac{C_4}{x}.$$

We deduce that for $s, t \in (0, 1/8)$, and $x \ge 1$.

$$\mathrm{P}\big(|W_{-t} - W_0| \wedge |W_s - W_0| \ge 6\lambda\big) \le \frac{C}{x}.$$

This ends the proof of (40) and thus (ii).

Appendix

A.1. Functionals of the number of ancestors for the process Y

We have the following results.

Lemma A.1. *Let $\lambda, v, q \in (0, +\infty)$. We have*:

$$\mathbb{N}\big[1 - \mathrm{e}^{-\lambda R_v^{v+q}}\big] = u\big(c(q)\big(1 - \mathrm{e}^{-\lambda}\big), v\big), \tag{47}$$

$$\mathbb{N}\big[R_v^{v+q}\big] = c(q)\mathrm{e}^{-2\beta\theta v} \quad \textit{and} \quad \mathbb{N}\big[R_v^{v+q} | \zeta > v+q\big] = \frac{c(q)}{c(v+q)} \mathrm{e}^{-2\beta\theta v}. \tag{48}$$

Proof. Since R_v^{v+q} is, conditionally on Y_v, a Poisson random variable with parameter $c(q)Y_v$, we get thanks to (2):

$$\mathbb{N}\big[1 - \mathrm{e}^{-\lambda R_v^{v+q}}\big] = \mathbb{N}\big[1 - \mathrm{e}^{-c(q)(1-\mathrm{e}^{-\lambda})Y_v}\big] = u\big(c(q)\big(1 - \mathrm{e}^{-\lambda}\big), v\big).$$

The equalities (48) are a consequence of (9) and (5) and the equality $\mathbb{N}[R_v^{v+q}] = \mathbb{N}[Y_v]\mathbb{N}[\zeta \ge q]$. □

We shall need later on other closed formulas for the joint distribution of the number of ancestors at different times. We first give (in a slightly more general statement) the conditional distribution of R_v^{v+q+s} knowing R_v^{v+q}.

Let $v, q, s \in (0, +\infty)$. Notice that an ancestor at time v of the population at time $v+q$ is also an ancestor of the population at time $v+q+s$ with probability $c(q+s)/c(q)$ and this happens independently of the other ancestors of the population at times before $v+q$. We deduce the following corollary.

Corollary A.2. *Let $v, q, s \in (0, +\infty)$. Conditionally on $(R_u^h; u \in (0, v], h \in (u, v+q])$, the random variable R_v^{v+q+s} has under $\mathbb{N}$ a binomial distribution with parameter $(R_v^{v+q}, c(q+s)/c(q))$.*

We recall the decomposition of H before and after $T_b = \inf\{t; H(t) = b\}$ under $\mathbb{N}$ in order to give the conditional distribution of R_{v+r}^{v+q} knowing R_v^{v+q} with $r < q$.

On $\{T_b < +\infty\}$, let $((\alpha_i^{(g)}, \beta_i^{(g)}), i \in I^{(g)})$ (resp. $((\alpha_i^{(d)}, \beta_i^{(d)}), i \in I^{(d)})$) be the excursion intervals of H above its minimum backward on the left of T_b (resp. forward on the right of T_b). Define $H_i^{(g)}$ for $i \in I^{(g)}$ as follows:

$$H_i^{(g)}(t) = H\big(\big(t + \alpha_i^{(g)}\big) \wedge \beta_i^{(g)}\big) - H\big(\alpha_i^{(g)}\big), \quad t \geq 0,$$

and $H_i^{(d)}$ similarly for $i \in I^{(d)}$. It is well known, see [9] or [1], that under $\mathbb{N}[\cdot|T_b < +\infty]$ the measures

$$\sum_{i \in I^{(g)}} \delta_{(H(\alpha_i^{(g)}), H_i^{(g)})}(dt, dH) \quad \text{and} \quad \sum_{i \in I^{(d)}} \delta_{(H(\alpha_i^{(d)}), H_i^{(d)})}(dt, dH)$$

are independent Poisson point measures with respective intensity:

$$\mathbf{1}_{[0,b]}(t)\beta\, dt \mathbb{N}\big[dH; \max(H) < b - t\big] \quad \text{and} \quad \mathbf{1}_{[0,b]}(t)\beta\, dt \mathbb{N}[dH].$$

Let $v, q \in (0, +\infty)$. By considering the R_v^{v+q} excursions of H above level v which reach level $v + q$ and the previous representation for each of those excursions (with $b = q$), we easily deduce the following result.

Proposition A.3. *Let $v, q \in (0, +\infty)$. Conditionally on (R_v^{v+q}, Y_v), the process $(Y_{t+v}, t \in [0, q])$ is distributed under $\mathbb{N}$ as $(\tilde{Y}_t, t \in [0, q])$ with*:

$$\tilde{Y}_t = Y'_t + \sum_{t_i^{(g)} \leq t} Y^{(g),i}_{t - t_i^{(g)}} + \sum_{t_i^{(d)} \leq t} Y^{(d),i}_{t - t_i^{(d)}},$$

where Y' is distributed according to $\mathbb{P}_{Y_v}(\cdot|\zeta < q)$, $\sum_{i \in I^{(g)}} \delta_{(t_i^{(g)}, Y^{(g),i})}$ and $\sum_{i \in I^{(d)}} \delta_{(t_i^{(d)}, Y^{(d),i})}$ are independent Poisson point measures independent of Y' with respective intensity:

$$\mathbf{1}_{[0,q]}(t)\beta R_v^{v+q} \mathbb{N}[dY; \zeta < q - t]\, dt \quad \textit{and} \quad \mathbf{1}_{[0,q]}(t)\beta R_v^{v+q} \mathbb{N}[dY]\, dt.$$

We deduce the following corollary on the conditional distribution of R_{v+r}^{v+q} knowing R_v^{v+q}.

Corollary A.4. *Let $\lambda, v, q, r \in (0, +\infty)$ with $q > r$. We have*:

$$\mathbb{N}\big[\exp\big(-\lambda R_{v+r}^{v+q}\big)\big|R_v^{v+q}\big] = h(\lambda)^{R_v^{v+q}},$$

with

$$h(\lambda) = \mathrm{e}^{-\lambda}\left(1 - \frac{u(c(q-r)(1 - \mathrm{e}^{-\lambda}), r)}{c(r)}\right).$$

Proof. We have:

$$\begin{aligned}
\mathbb{N}\big[\exp\big(-\lambda R_{v+r}^{v+q}\big)\big|R_v^{v+q}\big] &= \exp\left(-\lambda R_v^{v+q} - \beta R_v^{v+q} \int_0^r ds \mathbb{N}\big[1 - \mathrm{e}^{-\lambda R_s^{s+q-r}}\big]\right) \\
&= \exp\left(-\lambda R_v^{v+q} - \beta R_v^{v+q} \int_0^r ds u\big(c(q-r)\big(1 - \mathrm{e}^{-\lambda}\big), s\big)\right) \\
&= \mathrm{e}^{-\lambda R_v^{v+q}}\left(1 + c(q-r)\big(1 - \mathrm{e}^{-\lambda}\big)\frac{1 - \mathrm{e}^{-2\beta\theta r}}{2\theta}\right)^{-R_v^{v+q}} \\
&= h(\lambda)^{R_v^{v+q}},
\end{aligned}$$

where we used Proposition A.3 for the first equality, (47) for the second and (4) for the third and some elementary computations for the last. □

We give the following elementary results which are used in Section 4.

Lemma A.5. *Let $\lambda, \mu, v, q, s \in (0, +\infty)$. We have with $\kappa_1 = (1 - \mathrm{e}^{-\lambda})c(q) + \mathrm{e}^{-\lambda}c(q+s)$:*

$$\mathbb{N}\big[\big(1 - \mathrm{e}^{-\lambda R_v^{v+q}}\big)\mathbf{1}_{\{\zeta < v+q+s\}}\big] = u(\kappa_1, v) - c(v+q+s); \tag{49}$$

for $0 < v' < q$, with $\kappa_2 = (1 - \mathrm{e}^{-\mu})c(q - v') + \mathrm{e}^{-\mu}\kappa_1$:

$$\mathbb{N}\big[\big(1 - \mathrm{e}^{-\lambda R_v^{v+q} - \mu R_v^{v+q-v'}}\big)\mathbf{1}_{\{\zeta < v+q+s\}}\big] = u(\kappa_2, v) - c(v+q+s); \tag{50}$$

for $0 < v' < v$ with $\kappa_3 = (1 - \mathrm{e}^{-\mu})c(q + v') + \mathrm{e}^{-\mu}u(\kappa_1, v')$:

$$\mathbb{N}\big[\big(1 - \mathrm{e}^{-\lambda R_v^{v+q} - \mu R_{v-v'}^{v+q}}\big)\mathbf{1}_{\{\zeta < v+q+s\}}\big] = u\big(\kappa_3, v - v'\big) - c(v+q+s); \tag{51}$$

and

$$\mathbb{N}\big[R_v^{v+q}\mathbf{1}_{\{\zeta < v+q+s\}}\big] = \big(c(q) - c(q+s)\big)\mathrm{e}^{2\beta\theta v}\left(\frac{c(v+q+s)}{c(q+s)}\right)^2. \tag{52}$$

Proof. We have:

$$\begin{aligned}
\mathbb{N}\big[\big(1 - \mathrm{e}^{-\lambda R_v^{v+q}}\big)\mathbf{1}_{\{\zeta < v+q+s\}}\big] &= \mathbb{N}\left[\big(1 - \mathrm{e}^{-\lambda R_v^{v+q}}\big)\left(1 - \frac{c(q+s)}{c(q)}\right)^{R_v^{v+q}}\right] \\
&= u\left(c(q)\left(1 - \mathrm{e}^{-\lambda}\left(1 - \frac{c(q+s)}{c(q)}\right)\right), v\right) - u\big(c(q+s), v\big),
\end{aligned}$$

where we used $\{\zeta < v+q+s\} = \{R_v^{v+q+s} = 0\}$ and Corollary A.2 for the first equality and (47) twice for the second. Then use (6) to get (49).

The proof of (50) relies on the same type of arguments and is left to the reader.

Taking the derivative with respect to λ at $\lambda = 0$ in (49) gives (52).

We prove (51). Set $\mathrm{e}^{-\tilde{\lambda}} = \mathrm{e}^{-\lambda}(1 - \frac{c(q+s)}{c(q)}) = 1 - \frac{\kappa_1}{c(q)}$. We have:

$$\begin{aligned}
\mathbb{N}\big[\big(1 - \mathrm{e}^{-\lambda R_v^{v+q} - \mu R_{v-v'}^{v+q}}\big)\mathbf{1}_{\{\zeta < v+q+s\}}\big] &= \mathbb{N}\left[\big(1 - \mathrm{e}^{-\lambda R_v^{v+q} - \mu R_{v-v'}^{v+q}}\big)\left(1 - \frac{c(q+s)}{c(q)}\right)^{R_v^{v+q}}\right] \\
&= \mathbb{N}\big[1 - \mathrm{e}^{-\tilde{\lambda} R_v^{v+q} - \mu R_{v-v'}^{v+q}}\big] - u\big(c(q+s), v\big) \\
&= \mathbb{N}\big[1 - \big(h_1(\lambda)\mathrm{e}^{-\mu}\big)^{R_{v-v'}^{v+q}}\big] - c(v+q+s),
\end{aligned}$$

where we used $\{\zeta < v+q+s\} = \{R_v^{v+q+s} = 0\}$ and Corollary A.2 for the first equality, (47) for the second, (6) and Corollary A.4 for the third with

$$h_1(\lambda) = \mathrm{e}^{-\tilde{\lambda}}\left(1 - \frac{u(c(q)(1 - \mathrm{e}^{-\tilde{\lambda}}), v')}{c(v')}\right) = \left(1 - \frac{\kappa_1}{c(q)}\right)\left(1 - \frac{u(\kappa_1, v')}{c(v')}\right).$$

Then use (47) to get $\mathbb{N}[1 - (h_1(\lambda)\mathrm{e}^{-\mu})^{R_{v-v'}^{v+q}}] = u(h_2(\mu), v - v')$ with

$$h_2(\mu) = c\big(q + v'\big)\big(1 - \mathrm{e}^{-\mu}h_1(\lambda)\big).$$

Taking $\mu = 0$, we get using (47) and (6):

$$\begin{aligned}u\big(h_2(0), v - v'\big) &= \mathbb{N}\big[\big(1 - \mathrm{e}^{-\lambda R_v^{v+q}}\big)\mathbf{1}_{\{\zeta < v+q+s\}}\big] + c(v + q + s)\\ &= u(\kappa_1, v)\\ &= u\big(u\big(\kappa_1, v'\big), v - v'\big).\end{aligned}$$

We deduce that $h_2(0) = u(\kappa_1, v')$ and since

$$h_2(\mu) = c\big(q + v'\big)\big(1 - \mathrm{e}^{-\mu}\big) + \mathrm{e}^{-\mu} h_2(0),$$

we get (51). □

A.2. Moments of the number of ancestors for the process Z

We easily get the following result using Proposition 3.2.

Corollary A.6. *Let $r \geq t > 0$. We have*:

$$\mathrm{E}[M_{-t}|M_{-r}] = \frac{c(t)}{\theta}\big(1 - \mathrm{e}^{-2\beta\theta(r-t)}\big) + \frac{c(t)}{c(r)}\mathrm{e}^{-2\beta\theta(r-t)} M_{-r} \tag{53}$$

and

$$\mathrm{E}[M_{-t} M_{-r}] = \frac{c(r)}{\theta}\left(1 + \frac{3}{2}\frac{c(t)}{\theta}\right) = 2\frac{\mathrm{e}^{2\beta\theta t} + 2}{(\mathrm{e}^{2\beta\theta t} - 1)(\mathrm{e}^{2\beta\theta r} - 1)}. \tag{54}$$

Proof. Let $g(t) = \mathrm{E}[M_{-t}|M_{-r}]$ and $h(t) = \mathrm{E}[M_{-t}] = c(t)/\theta$. Using Proposition 3.2, we get that for $r \geq t > 0$:

$$g(t) = M_{-r} + \int_t^r \beta c(s)\big(g(s) + 2\big)\, ds \quad \text{and} \quad h(t) = h(r) + \int_t^r \beta c(s)\big(h(s) + 2\big)\, ds.$$

This implies that $g'(t) - h'(t) = -\beta c(t)(g(t) - h(t))$ and thus:

$$g(t) - h(t) = \big(M_{-r} - \mathrm{E}[M_{-r}]\big)\mathrm{e}^{\beta \int_t^r c(s)\, ds}.$$

Then use (8) and (18) to get (53).

Taking the expectation in (53) and using the second part of (18), we get:

$$\begin{aligned}\mathrm{E}[M_{-t} M_{-r}] &= \frac{c(t)c(r)}{\theta^2}\big(1 - \mathrm{e}^{2\beta\theta(t-r)}\big) + \frac{c(t)c(r)}{\theta^2}\mathrm{e}^{2\beta\theta(t-r)}\left(\frac{\theta}{c(r)} + \frac{3}{2}\right)\\ &= \frac{1}{2}\frac{c(t)c(r)}{\theta^2}\big(\mathrm{e}^{2\beta\theta t} + 2\big)\\ &= \frac{c(r)}{\theta}\left(1 + \frac{3}{2}\frac{c(t)}{\theta}\right).\end{aligned}$$

□

The following lemma generalizes (54), and is used in the proof of Proposition 5.6.

Lemma A.7. *Let $r > 0$, $s > 0$ and $q > 0$. For $s + r \geq q$, we have*:

$$\mathrm{E}\big[M_{-r} M_{s-q}^s\big] = \frac{c(r)}{\theta}\frac{c(q)}{\theta}\left(\frac{\theta}{c(q - s)} + \frac{3}{2}\right). \tag{55}$$

For $q \geq s+r$, we have:

$$\mathrm{E}\big[M_{-r}M^s_{s-q}\big]=\frac{c(r)}{\theta}\frac{c(q)}{\theta}\bigg(\frac{\theta}{c(q-s)}\frac{c(q)}{c(r+s)}+\frac{3}{2}\bigg). \tag{56}$$

Proof. First, we consider the case $s+r\geq q$. Given M_{-r}, M^s_{s-q} can be decomposed in two parts:

$$M^s_{s-q}=M^{I[-r,s-q]}+\sum_{j=1}^{M_{-r}}\tilde{M}^{(s+r),j}_{s+r-q},$$

where $M^{I[-r,s-q]}$ is the number of ancestors coming from the immortal individual on the interval $(-r,s-q)$, and $\tilde{M}^{(s+r),j}_{s+r-q}$ represents the number of ancestors generated by one of the ancestors at time $-r$. More precisely, we have

(i) $M^{I[-r,s-q]}$ is the number of ancestors at time $s-q$ of the population living at time s corresponding to all the populations Y^i (see definition (12)) with immigration time t_i belonging to $(-r,s-q)$ and $M^{I[-r,s-q]}$ is independent of M_{-r}.

(ii) $(\tilde{M}^{(s+r),j}_{s+r-q}, j\in\mathbb{N}^*)$ are independent, independent of M_{-r} and are distributed as R^{s+r}_{s+r-q} under $\mathbb{N}[dY|\zeta>r]$.

We deduce that:

$$\mathrm{E}\big[M^s_{s-q}|M_{-r}\big]=2\beta\int_0^{s+r-q}\mathbb{N}\big[R_t^{t+q}\big]\,dt+M_{-r}\mathbb{N}\big[R^{s+r}_{s+r-q}|\zeta>r\big].$$

Using (48) and (54), elementary computations give:

$$\mathrm{E}\big[M_{-r}M^s_{s-q}\big]=\mathrm{E}\big[M_{-r}\mathrm{E}\big[M^s_{s-q}|M_{-r}\big]\big]=\frac{2(\mathrm{e}^{2\beta\theta(q-s)}+2)}{(\mathrm{e}^{2\beta\theta q}-1)(\mathrm{e}^{2\beta\theta r}-1)}.$$

This gives (55).

Second, we consider the case $q\geq s+r$. Given $(M^s_\ell, Z_\ell, \ell\leq s-q)$, the number of ancestors M_{-r} can be decomposed in three parts:

$$M_{-r}=M^{I[s-q,-r]}+\sum_{j=1}^{M^s_{s-q}}\tilde{M}^{(q-s),j}_{q-s-r}+\sum_{i'}R^{(q-s),i'}_{q-s-r}\big(\hat{Y}^{i'}\big),$$

where $M^{I[s-q,-r]}$ is the number of ancestor coming from the immigration on the interval $(s-q,-r)$, $\tilde{M}^{(q-s),j}_{q-s-r}$ represents the number of ancestors generated by one of the M^s_{s-q} ancestors at time $s-q$, and $\hat{Y}^{i'}$ is a population generated from one of the individuals at time $s-q$ (among the population of size Z_{s-q}) which dies before time s (that is with lifetime less than q). More precisely, we have

(i) $M^{I[s-q,-r]}$ is the number of ancestors at time $-q$ of the population living at time 0 corresponding to all the populations Y^i (see definition (12)) with immigration time t_i belonging to $(s-q,-r)$ and $M^{I[s-q,-r]}$ is independent of $(M^s_\ell, Z_\ell, \ell\leq s-q)$.

(ii) $(\tilde{M}^{(q-s),j}_{q-s-r}, j\in\mathbb{N}^*)$ are independent, independent of $(M^s_\ell, Z_\ell, \ell\leq s-q)$ and are distributed as R^{q-s}_{q-s-r} under $\mathbb{N}[dY|\zeta>q]$.

(iii) Conditionally on $(M^s_\ell, Z_\ell, \ell\leq s-q)$, $\sum_{i'}\delta_{\hat{Y}^{i'}}$ is a Poisson point measure with intensity $Z_{s-q}\mathbb{N}[dY,\zeta<q]$.

We deduce that:

$$\mathrm{E}\big[M_{-r}|\sigma\big(M^s_\ell, Z_\ell, \ell\leq s-q\big)\big]=\mathrm{E}\big[M^{I[s-q,-r]}\big]+M^s_{s-q}\mathbb{N}\big[R^{q-s}_{q-s-r}|\zeta>q\big]+Z_{s-q}\mathbb{N}\big[R^{q-s}_{q-s-r}\mathbf{1}_{\{\zeta<q\}}\big].$$

We have:

$$\mathrm{E}\big[M^{I[s-q,-r]}\big]=2\beta\int_{s-q}^{-r}dv\,\mathbb{N}\big[R^{-v}_{-r-v}\big]=\frac{c(r)}{\theta}\big(1-\mathrm{e}^{-2\beta\theta(q-s-r)}\big).$$

Using (52), we get:

$$\mathbb{N}\big[R^{q-s}_{q-s-r}\mathbf{1}_{\{\zeta<q\}}\big]=\big(c(r)-c(r+s)\big)\frac{\psi(c(q))}{\psi(c(s+r))},$$

as well as:

$$\begin{aligned}\mathbb{N}\big[R^{q-s}_{q-s-r}|\zeta>q\big]&=c(q)^{-1}\mathbb{N}\big[R^{q-s}_{q-s-r}\mathbf{1}_{\{\zeta>q\}}\big]\\&=\frac{c(r)}{c(q)}\mathrm{e}^{-2\beta\theta(q-s-r)}-\frac{c(r)-c(r+s)}{c(q)}\frac{\psi(c(q))}{\psi(c(s+r))}.\end{aligned}$$

Then elementary computation yields:

$$\begin{aligned}\mathrm{E}\big[M_{-r}M^s_{s-q}\big]&=\mathrm{E}\big[M^s_{s-q}\mathrm{E}\big[M_{-r}|\sigma\big(M^s_\ell,Z_\ell,\ell\le s-q\big)\big]\big]\\&=\frac{2(\mathrm{e}^{2\beta\theta(r+s)}+2)}{(\mathrm{e}^{2\beta\theta r}-1)(\mathrm{e}^{2\beta\theta q}-1)}-\frac{c(r)-c(r+s)}{\theta}\frac{\psi(c(q))}{\psi(c(s+r))}.\end{aligned}$$

Then it is straightforward to get the desired result. □

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