

ASYMPTOTIC RESULTS ON THE LENGTH OF COALESCENT TREES

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We give the asymptotic distribution of the length of partial coalescent trees for Beta and related coalescents. This allows us to give the asymptotic distribution of the number of (neutral) mutations in the partial tree. This is a first step to study the asymptotic distribution of a natural estimator of DNA mutation rate for species with large families.

1. Introduction.

1.1. *Motivations.* The Kingman coalescent, see [17, 18], allows to describe the genealogy of n individuals in a Wright–Fisher model, when the size of the whole population is very large and time is well rescaled. In what follows, we consider only neutral DNA mutations and the infinite sites model introduced by Kimura [16], where each mutation occurs at a new site. In particular, if an individual is affected by a mutation, all the descendants of this individual carry this mutation. Notice the total number of mutations observed among n individuals alive today, $S^{(n)}$, corresponds to the number of segregating sites. The Watterson estimator [24] based on $S^{(n)}$ allows to estimate the rate of mutation for the DNA, θ . This estimator is consistent and converges at rate $1/\sqrt{\log(n)}$.

Other models of population where one individual can produce a large number of children give rise to more general coalescent processes than the Kingman coalescent, where multiple collisions appear, see Sagitov [22] and Schweinsberg [23] (such models may be relevant for oysters and some fish species [8, 11]). In Birkner et al. [6] and in Schweinsberg [23] a natural family of one parameter coalescent processes arise to describe the genealogy of such populations: the Beta- $(2 - \alpha, \alpha)$ coalescent with parameter $\alpha \in (1, 2)$. Results from Berestycki et al. [3] give a consistent estimator, based on the observed total number, $S^{(n)}$, of mutations for the rate θ of mutation of DNA. This paper is a first step to study the convergence rate of this estimator or equivalently to the study of the asymptotic distribution of $S^{(n)}$. Results are also known for the asymptotic distribution of $S^{(n)}$ for other coalescent processes, see Drmota et al. [10] and Möhle [19].

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For the Beta coalescent, the asymptotic distribution of $S^{(n)}$ depends on θ but also on the parameter α . In particular, if the mutation rate of the DNA is known, the asymptotic distribution of $S^{(n)}$ allows to deduce an estimation and a confidence interval for α , which in a sense characterize the size of a typical family according to [23].

1.2. *The coalescent tree and mutation rate.* We denote by \mathbb{N}^* the set of positive integers. We consider at time $t = 0$ a number $n \in \mathbb{N}^*$ of individuals, and we look backward in time. Let \mathcal{P}_n be the set of partitions of $\{1, \dots, n\}$. For $t \geq 0$, let $\Pi_t^{(n)}$ be an element of \mathcal{P}_n such that each block of $\Pi_t^{(n)}$ corresponds to the initial individuals which have a common ancestor at time $-t$. We assume that if we consider b blocks, k of them merge into 1 at rate $\lambda_{b,k}$, independently of the current number of blocks. Using this property and the compatibility relation implied when one considers a larger number of initial individuals, Pitman [21], see also Sagitov [22] for a more biological approach, showed the transition rates are given by

$$\lambda_{b,k} = \int_{[0,1]} x^{k-2}(1-x)^{b-k} \Lambda(dx), \quad 2 \leq k \leq b,$$

for some finite measure Λ on $[0, 1]$, and that $\Pi^{(n)}$ is the restriction of the so-called coalescent process defined on the set of partitions of \mathbb{N}^* . The Kingman coalescent corresponds to the case where Λ is the Dirac mass at 0, see [17]. In particular, in the Kingman coalescent, only two blocks merge at a time. The Bolthausen–Sznitman [7] coalescent corresponds to the case where Λ is the Lebesgue measure on $[0, 1]$. The Beta-coalescent introduced in Birkner et al. [6] and in Schweinsberg [23], see also Bertoin and Le Gall [5] and Berestycki et al. [2], corresponds to $\Lambda(dx) = C_0 x^{\alpha-1}(1-x)^{1-\alpha} \mathbf{1}_{(0,1)}(x) dx$ for some constant $C_0 > 0$.

Notice $\Pi^{(n)} = (\Pi_t^{(n)}, t \geq 0)$ is a Markov process starting at the trivial partition of $\{1, \dots, n\}$ into n singletons. We denote by $R_t^{(n)}$ the number of blocks of $\Pi_t^{(n)}$. We have, $R_0^{(n)} = n$, and $R_t^{(n)}$ can be seen as the number of ancestors alive at time $-t$. The apparition time of the most recent common ancestor (MRCA) is $\inf\{t > 0; R_t^{(n)} = 1\}$. We shall omit the superscript (n) when there is no confusion. The process $R = (R_t, t \geq 0)$ is a continuous time Markov process taking values in \mathbb{N}^* . The number of possible choices of $\ell + 1$ blocks among k is $\binom{k}{\ell+1}$ (for $1 \leq \ell \leq k - 1$) and each group of $\ell + 1$ blocks merge at rate $\lambda_{k,\ell+1}$. So the waiting time of R in state k is an exponential random variable with parameter

$$(1) \quad g_k = \sum_{\ell=1}^{k-1} \binom{k}{\ell+1} \lambda_{k,\ell+1} = \int_{(0,1)} (1 - (1-x)^k - kx(1-x)^{k-1}) \frac{\Lambda(dx)}{x^2}$$

and is distributed as E/g_k , where E is an exponential random variable with mean 1.

Let $Y = (Y_k, k \geq 1)$ be the different states of the process R . It is defined by $Y_0 = R_0$ and for $k \geq 1, Y_k = R_{T_k}$, where the sequence of jumping time $(T_k, k \geq 0)$ is defined inductively by $T_0 = 0$ and for $k \geq 1, T_k = \inf\{t > T_{k-1}; R_t \neq R_{T_{k-1}}\}$. We use the convention that $\inf \emptyset = +\infty$ and $Y_k = 1$ for $k \geq \tau_n$, where $\tau_n = \inf\{k; R_{T_k} = 1\}$ is the number of jumps of the process R until it reach the absorbing state 1. The number τ_n is the number of coalescences.

We shall write $Y^{(n)}$ instead of Y when it will be convenient to stress that Y starts at time 0 at point n . Notice Y is an \mathbb{N}^* -valued discrete time Markov chain, with probability transition

$$(2) \quad P(k, k - \ell) = \frac{\binom{k}{\ell+1} \lambda_{k, \ell+1}}{g_k}.$$

The sum of the lengths of all branches in the coalescent tree until the MRCA is distributed as

$$L^{(n)} = \sum_{k=0}^{\tau_n-1} \frac{Y_k^{(n)}}{g_{Y_k^{(n)}}} E_k,$$

where $(E_k, k \geq 0)$ are independent exponential random variables with expectation 1.

In the infinite sites model, one assumes that (neutral) mutations appear in the genealogy at random with rate θ . In particular, conditionally on the length of the coalescent tree $L^{(n)}$, the total number $S^{(n)}$ of mutations is distributed according to a Poisson r.v. with parameter $\theta L^{(n)}$. Therefore, we have that $\frac{S^{(n)} - \theta L^{(n)}}{\sqrt{\theta L^{(n)}}}$ converges in distribution to a standard Gaussian r.v. (with mean 0 and variance 1). If the asymptotic distribution of $L^{(n)}$ is known, one can deduce the asymptotic distribution of $S^{(n)}$.

1.3. Known results.

1.3.1. *Kingman coalescence.* For Kingman coalescence, a coalescence corresponds to the apparition of a common ancestor of only two individuals. In particular, we have for $0 \leq k \leq n - 1, Y_k^{(n)} = n - k$. Thus, we get $\tau_n = n - 1$ as well as $g_{Y_k^{(n)}} = (n - k)(n - k - 1)/2$. We also have

$$\frac{L^{(n)}}{2} = \sum_{k=0}^{n-2} \frac{1}{n - k - 1} E_k = \sum_{k=1}^{n-1} \frac{1}{k} E_{n-k-1}.$$

The r.v. $L^{(n)}/2$ is distributed as the sum of independent exponential r.v. with parameter 1 to $n - 1$, that is as the maximum on $n - 1$ independent exponential r.v. with mean 1; see Feller [12], Section I.6. An easy computation gives that $L^{(n)}/(2 \log(n))$ converges in probability to 1 and that $\frac{L^{(n)}}{2} - \log(n)$ converges in distribution to the Gumbel distribution (with density $e^{-x - \exp -x}$) when n goes to

infinity. It is then easy to deduce that $\frac{S^{(n)} - \theta \mathbb{E}[L^{(n)}]}{\sqrt{\theta \mathbb{E}[L^{(n)}]}}$ converges in distribution to the standard Gaussian distribution. This provides the weak convergence and the asymptotic normality of the Watterson [24] estimator of θ : $\frac{S^{(n)}}{\mathbb{E}[L^{(n)}]} = \frac{S^{(n)}}{\sum_{k=1}^{n-1} 1/k}$. See also the Appendix in [10].

1.3.2. *Bolthausen–Sznitman coalescence.* In Drmota et al. [10], the authors consider the Bolthausen–Sznitman coalescence: Λ is the Lebesgue measure on $[0, 1]$. In this case, they prove that $\frac{1}{n} \log(n)L^{(n)}$ converges in probability to 1 and that $\frac{L^{(n)} - a_n}{b_n}$ converges in distribution to a stable r.v. Z with Laplace transform $\mathbb{E}[e^{-\lambda Z}] = e^{\lambda \log(\lambda)}$ for $\lambda > 0$, where

$$a_n = \frac{n}{\log(n)} + \frac{n \log(\log(n))}{\log(n)^2} \quad \text{and} \quad b_n = \frac{n}{\log(n)^2}.$$

It is then easy to deduce that $\frac{S^{(n)} - \theta a_n}{\theta b_n}$ converges to Z .

1.3.3. *The case $\int_{(0,1]} x^{-1} \Lambda(dx) < \infty$.* In Möhle [19], the author investigates the case where $x^{-1} \Lambda(dx)$ is a finite measure and consider directly the asymptotic distribution of $S^{(n)}$. In particular, he obtains that $S^{(n)}/n\theta$ converges in distribution to a nonnegative r.v. Z uniquely determined by its moments: for $k \geq 1$,

$$\mathbb{E}[Z^k] = \frac{k!}{\prod_{i=1}^k \Phi(i)}, \quad \text{with } \Phi(i) = \int_{[0,1]} (1-x)^i x^{-2} \Lambda(dx).$$

There is an equation in law for Z when Λ is a simple measure, that is when $\int_{(0,1]} x^{-2} \Lambda(dx) < \infty$.

1.3.4. *Beta coalescent.* The Beta- $(2-\alpha, \alpha)$ coalescent corresponds to the case where Λ is the Beta $(2-\alpha, \alpha)$ distribution, with $\alpha \in (1, 2)$:

$$\Lambda(dx) = \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha)} x^{1-\alpha} (1-x)^{\alpha-1} dx.$$

The Kingman coalescent can be viewed as the asymptotic case $\alpha = 2$ and the Bolthausen–Sznitman coalescence as the asymptotic case $\alpha = 1$.

The first-order asymptotic behavior of $L^{(n)}$ is given in [3], Theorem 1.9: $n^{\alpha-2} L^{(n)}$ converges in probability to $\frac{\Gamma(\alpha)\alpha(\alpha-1)}{2-\alpha}$. We shall now investigate the asymptotic distribution of $L^{(n)}$.

1.4. *Main result.* In this paper we shall state a partial result concerning the asymptotic distribution of $L^{(n)}$. We shall only give the asymptotic distribution of the total length of the coalescent tree up to the $\lfloor nt \rfloor$ th coalescence:

$$(3) \quad L_t^{(n)} = \sum_{k=0}^{\lfloor nt \rfloor \wedge (\tau_n - 1)} \frac{Y_k^{(n)}}{g_{Y_k^{(n)}}} E_k,$$

where $\lfloor x \rfloor$ is the largest integer smaller or equal to x for $x \geq 0$.

We say $g = O(f)$, where f is a nonnegative function and g a real valued function defined on a set E (mainly here $E = [0, 1]$ or $E = \mathbb{N}^*$ or $E = \mathbb{N}^* \times [0, 1]$), if there exists a finite constant $C > 0$ such that $|g(x)| \leq Cf(x)$ for all $x \in E$.

Let $v(dx) = x^{-2}\Lambda(dx)$ and $\rho(t) = v((t, 1])$. We assume that $\rho(t) = C_0t^{-\alpha} + O(t^{-\alpha+\zeta})$ for some $\alpha \in (1, 2)$, $C_0 > 0$ and $\zeta > 1 - 1/\alpha$. This includes the Beta($2 - \alpha, \alpha$) distribution for Λ . We have (see Lemma 2.2) that

$$g_n = C_0\Gamma(2 - \alpha)n^\alpha + O(n^{\alpha - \min(\zeta, 1)}).$$

Let $\gamma = \alpha - 1$. Let $V = (V_t, t \geq 0)$ be a α -stable Lévy process with no positive jumps (see Chapter VII in [4]) with Laplace exponent $\psi(u) = u^\alpha/\gamma$: for all $u \geq 0$, $\mathbb{E}[e^{-uV_t}] = e^{tu^\alpha/\gamma}$.

We first give in Proposition 3.1 the asymptotic for the number of coalescences, τ_n :

$$n^{-1/\alpha} \left(n - \frac{\tau_n}{\gamma} \right) \xrightarrow[n \rightarrow \infty]{(d)} V_\gamma.$$

See also Gnedin and Yakubovich [13] and Iksanov and Möhle [14] for different proofs of this results under slightly different or stronger hypothesis. Then we give the asymptotics of $\hat{L}_t^{(n)}$ defined as $C_0\Gamma(2 - \alpha)L_t^{(n)}$ but for the exponential r.v. E_k which are replaced by their mean that is 1 and for $g_{Y_k^{(n)}}$ which is replaced by its equivalent $C_0\Gamma(2 - \alpha)(Y_k^{(n)})^{2-\alpha}$:

$$(4) \quad \hat{L}_t^{(n)} = \sum_{k=0}^{\lfloor nt \rfloor \wedge (\tau_n - 1)} (Y_k^{(n)})^{1-\alpha}.$$

For $t \in [0, \gamma]$, we set

$$v(t) = \int_0^t \left(1 - \frac{r}{\gamma} \right)^{-\gamma} dr.$$

Theorem 5.1 gives that the following convergence in distribution holds for all $t \in (0, \gamma)$

$$(5) \quad n^{-1+\alpha-1/\alpha} (\hat{L}_t^{(n)} - n^{2-\alpha}v(t)) \xrightarrow[n \rightarrow \infty]{(d)} (\alpha - 1) \int_0^t dr \left(1 - \frac{r}{\gamma} \right)^{-\alpha} V_r.$$

Then we deduce our main result, Theorem 6.1. Let $\alpha \in (1, \frac{1+\sqrt{5}}{2})$. Then for all $t \in (0, \gamma)$, we have the following convergence in distribution

$$(6) \quad n^{-1+\alpha-1/\alpha} \left(L_t^{(n)} - n^{2-\alpha} \frac{v(t)}{C_0\Gamma(2 - \alpha)} \right) \xrightarrow[n \rightarrow \infty]{(d)} \frac{\alpha - 1}{C_0\Gamma(2 - \alpha)} \int_0^t dr \left(1 - \frac{r}{\gamma} \right)^{-\alpha} V_r.$$

We also have that $n^{\alpha-2}L_t^{(n)}$ converges in probability to $\frac{v(t)}{C_0\Gamma(2-\alpha)}$ for $\alpha \in (1, 2)$ uniformly on $[0, t_0]$ for any $t_0 \in [0, \gamma)$. See also analogous results in [1] for the Bolthausen–Sznitman coalescent. For $t = \gamma$, intuitively we have $L_\gamma^{(n)}$ close to $L^{(n)}$ as τ_n is close to n/γ . In particular, one expects that $n^{\alpha-2}L^{(n)}$ converges in probability to $\frac{v(\gamma)}{C_0\Gamma(2-\alpha)}$. For the Beta-coalescent,

$$\Lambda(dx) = \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha)}x^{1-\alpha}(1-x)^{\alpha-1}dx,$$

we have $C_0 = 1/\alpha\Gamma(2-\alpha)\Gamma(\alpha)$ and indeed, Theorem 1.9 in [3] gives that $n^{\alpha-2}L^{(n)}$ converges in probability to

$$\frac{\Gamma(\alpha)\alpha(\alpha-1)}{2-\alpha} = \frac{v(\gamma)}{C_0\Gamma(2-\alpha)}.$$

Notice Theorem 1.9 in [3] is stated for more general coalescents than the Beta-coalescent.

In Corollary 6.2, we give the asymptotic distribution of the total number $S_t^{(n)}$ of mutations on the coalescent tree up to the $\lfloor nt \rfloor$ th coalescent for $\alpha \in (1, 2)$. In particular, for $\alpha > \sqrt{2}$, the approximations of the exponential r.v. by their mean are more important than the fluctuations of $\hat{L}^{(n)}$, and the asymptotic distribution is Gaussian.

1.5. Organization of the paper. In Section 2 we give estimates (distribution, Laplace transform) for the number of individuals involved in the first coalescence in a population of n individuals. We prove the asymptotic distribution of the number of collisions, τ_n , in Section 3, as well as an invariance principle for the coalescent process $Y^{(n)}$, see Corollary 3.5. In Section 4, we give error bounds on the approximation of $L_t^{(n)}$ by $\hat{L}_t^{(n)}/C_0\Gamma(2-\alpha)$. Section 5 is devoted to the asymptotic distribution of $\hat{L}_t^{(n)}$. Eventually, our main result, Theorem 6.1, on the asymptotic distribution of $L_t^{(n)}$, and Corollary 6.2, on the asymptotic distribution of the number of mutations $S_t^{(n)}$, and their proofs are given in Section 6.

In what follows, c is a nonimportant constant which value may vary from line to line.

2. Law of the first jump. Let Y be a discrete time Markov chain on \mathbb{N}^* with transition kernel P given by (2) and started at $Y_0 = n$. Let $\mathcal{Y} = (\mathcal{Y}_k, k \geq 0)$ be the filtration generated by Y . We set $X_k^{(n)} = Y_{k-1} - Y_k$ for $k \geq 1$. We give some estimates on the moment of $X_1^{(n)}$ and its Laplace transform.

For $n \geq 1, x \in (0, 1)$, let $B_{n,x}$ be a binomial r.v. with parameter (n, x) . Recall that for $1 \leq k \leq n$, we have

$$(7) \quad \mathbb{P}(B_{n,x} \geq k) = \frac{n!}{(k-1)!(n-k)!} \int_0^x t^{k-1}(1-t)^{n-k} dt.$$

Recall that $\nu(dx) = x^{-2}\Lambda(dx)$ and $\rho(t) = \nu((t, 1])$. Use the first equality in (1) and (7) to get

$$\begin{aligned}
 g_n &= \int_0^1 \sum_{k=2}^n \binom{n}{k} x^k (1-x)^{n-k} \nu(dx) \\
 (8) \qquad &= \int_0^1 \mathbb{P}(B_{n,x} \geq 2) \nu(dx) \\
 &= n(n-1) \int_0^1 (1-t)^{n-2} t \rho(t) dt.
 \end{aligned}$$

Notice also that

$$\mathbb{P}(X_1^{(n)} = k) = P(n, n-k) = \frac{1}{g_n} \int_0^1 \mathbb{P}(B_{n,x} = k+1) \nu(dx)$$

and thus

$$\begin{aligned}
 (9) \qquad \mathbb{P}(X_1^{(n)} \geq k) &= \frac{\int_0^1 \mathbb{P}(B_{n,x} \geq k+1) \nu(dx)}{g_n} \\
 &= \frac{(n-2)!}{k!(n-k-1)!} \frac{\int_0^1 (1-t)^{n-k-1} t^k \rho(t) dt}{\int_0^1 (1-t)^{n-2} t \rho(t) dt}.
 \end{aligned}$$

Let $\alpha \in (1, 2)$ and $\gamma = \alpha - 1$. The following result on the asymptotic distribution of $(X_1^{(n)}, n \geq 2)$ is essentially in [5], Lemma 4.

LEMMA 2.1. *Assume that $\rho(t) = t^{-\alpha}L(t)$, where $L(t), t \in (0, 1]$ is slowly varying at 0. Then $(X_1^{(n)}, n \geq 2)$ converges in distribution to the r.v. X taking values in \mathbb{N}^* and such that for all $k \geq 1$,*

$$\mathbb{P}(X \geq k) = \frac{1}{\Gamma(2-\alpha)} \frac{\Gamma(k+1-\alpha)}{k!}.$$

We have $\mathbb{E}[X] = 1/\gamma, \mathbb{E}[X^2] = +\infty$ and its Laplace transform ϕ is given by: for $u \geq 0$,

$$\phi(u) = \mathbb{E}[e^{-uX}] = 1 + \frac{e^u - 1}{\alpha - 1} [(1 - e^{-u})^{\alpha-1} - 1].$$

We shall use repeatedly the identity of the Beta distribution: for $a > 0$ and $b > 0$, we have

$$(10) \qquad \int_0^1 t^{a-1} (1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

PROOF OF LEMMA 2.1. Following Lemma 4 from [5], it is easy to get that for fixed $k \geq 1$, as n goes to infinity, we have

$$\lim_{n \rightarrow \infty} n^{k+1-\alpha} L(1/n)^{-1} \int_0^1 (1-t)^{n-k-1} t^k \rho(t) dt = \Gamma(k+1-\alpha).$$

Therefore, we get that, for $k \in \mathbb{N}^*$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}(X_1^{(n)} \geq k) &= \lim_{n \rightarrow \infty} \frac{(n-2)!}{k!(n-k-1)!} \frac{\int_0^1 (1-t)^{n-k-1} t^k \rho(t) dt}{\int_0^1 (1-t)^{n-2} t \rho(t) dt} \\ &= \frac{1}{\Gamma(2-\alpha)} \frac{\Gamma(k+1-\alpha)}{k!}. \end{aligned}$$

This ends the first part of the lemma. Since

$$\mathbb{P}(X \geq k) = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \int_0^1 t^{k-\alpha} (1-t)^{\alpha-1} dt,$$

we deduce that

$$\mathbb{E}[X] = \sum_{k \geq 1} \mathbb{P}(X \geq k) = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \int_0^1 \sum_{k \geq 1} t^{k-\alpha} (1-t)^{\alpha-1} dt = \frac{1}{\alpha-1}.$$

Notice that $\mathbb{P}(X = k) = \mathbb{P}(X \geq k) - \mathbb{P}(X \geq k+1)$ and thus

$$\begin{aligned} (11) \quad \mathbb{P}(X = k) &= \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \int_0^1 t^{k-\alpha} (1-t)^\alpha dt \\ &= \frac{\alpha}{\Gamma(2-\alpha)} \frac{\Gamma(k+1-\alpha)}{(k+1)!}. \end{aligned}$$

The asymptotic expansion

$$(12) \quad \Gamma(z) = \sqrt{2\pi} z^{z-1/2} e^{-z} \left(1 + \frac{1}{12z} + o\left(\frac{1}{z}\right) \right)$$

implies $\mathbb{P}(X = k) \sim_{+\infty} \frac{\alpha}{\Gamma(2-\alpha)} k^{-\alpha-1}$. Therefore, we have $\mathbb{E}[X^2] = +\infty$. We compute the Laplace transform of X . Let $u \geq 0$, we have

$$\begin{aligned} \phi(u) = \mathbb{E}[e^{-uX}] &= \frac{\alpha}{\Gamma(2-\alpha)} \sum_{k \geq 1} \frac{1}{(k+1)!} e^{-ku} \int_0^\infty x^{k-\alpha} e^{-x} dx \\ &= \frac{\alpha e^u}{\Gamma(2-\alpha)} \int_0^\infty \sum_{k \geq 2} \frac{1}{k!} e^{-ku} x^{k-1-\alpha} e^{-x} dx \\ &= \frac{\alpha e^u}{\Gamma(2-\alpha)} \int_0^\infty x^{-1-\alpha} e^{-x} (e^{xe^{-u}} - xe^{-u} - 1) dx \\ &= 1 + \frac{e^u - 1}{\alpha - 1} [(1 - e^{-u})^{\alpha-1} - 1], \end{aligned}$$

where we used (11) with

$$\Gamma(k+1-\alpha) = \int_0^\infty x^{k-\alpha} e^{-x} dx$$

for the first equality and two integrations by parts for the last. \square

We give bounds on g_n .

LEMMA 2.2. Assume that $\rho(t) = C_0t^{-\alpha} + O(t^{-\alpha+\zeta})$ for some $C_0 > 0$ and $\zeta > 0$. Then we have, for $n \geq 2$,

$$(13) \quad g_n = C_0\Gamma(2 - \alpha)n^\alpha + O(n^{\alpha - \min(\zeta, 1)}).$$

PROOF. Notice that

$$\begin{aligned} g_n &= n(n - 1) \int_0^1 (1 - t)^{n-2} t (C_0t^{-\alpha} + O(t^{-\alpha+\zeta})) dt \\ &= C_0n(n - 1) \frac{\Gamma(2 - \alpha)\Gamma(n - 1)}{\Gamma(n + 1 - \alpha)} + h_n, \end{aligned}$$

where

$$h_n = n(n - 1) \int_0^1 (1 - t)^{n-2} t^{-\alpha+\zeta+1} O(1) dt.$$

In particular, using (12), we have for $n \geq 2$

$$\begin{aligned} |h_n| &\leq cn(n - 1) \int_0^1 (1 - t)^{n-2} t^{-\alpha+\zeta+1} \\ &= cn(n - 1) \frac{\Gamma(2 - \alpha + \zeta)\Gamma(n - 1)}{\Gamma(n + 1 - \alpha + \zeta)} \leq cn^{\alpha-\zeta}. \end{aligned}$$

Using (12) again, we get that $\Gamma(n - 1) / \Gamma(n + 1 - \alpha) = n^{\alpha-2} + O(n^{\alpha-3})$. This implies that

$$g_n = C_0\Gamma(2 - \alpha)n^\alpha + O(n^{\max(\alpha-1, \alpha-\zeta)}). \quad \square$$

We give an expansion of the first moment of $X_1^{(n)}$.

LEMMA 2.3. Assume that $\rho(t) = C_0t^{-\alpha} + O(t^{-\alpha+\zeta})$ for some $C_0 > 0$ and $\zeta > 0$. Let $\varepsilon_0 > 0$. We set

$$(14) \quad \varphi_n = \begin{cases} n^{-\zeta}, & \text{if } \zeta < \alpha - 1, \\ n^{1-\alpha+\varepsilon_0}, & \text{if } \zeta = \alpha - 1, \\ n^{1-\alpha}, & \text{if } \zeta > \alpha - 1. \end{cases}$$

There exists a constant C_{15} s.t. for all $n \geq 2$, we have

$$(15) \quad \left| \mathbb{E}[X_1^{(n)}] - \frac{1}{\gamma} \right| \leq C_{15}\varphi_n.$$

PROOF. We have

$$\begin{aligned}
 \mathbb{E}[X_1^{(n)}] &= \sum_{k \geq 1} \mathbb{P}(X_1^{(n)} \geq k) = \frac{\int_0^1 \sum_{k \geq 1} \mathbb{P}(B_{n,x} \geq k + 1) \nu(dx)}{g_n} \\
 (16) \quad &= \frac{\int_0^1 (\mathbb{E}[B_{n,x}] - \mathbb{P}(B_{n,x} \geq 1)) \nu(dx)}{g_n} \\
 &= \frac{\int_0^1 nx \nu(dx) - \int_0^1 (1 - (1-x)^n) \nu(dx)}{g_n}
 \end{aligned}$$

$$\begin{aligned}
 (17) \quad &= \frac{n \int_0^1 [1 - (1-t)^{n-1}] \rho(t) dt}{g_n} \\
 &= \frac{\int_0^1 (1-t)^{n-2} (\int_t^1 \rho(r) dr) dt}{\int_0^1 (1-t)^{n-2} t \rho(t) dt},
 \end{aligned}$$

using (9) for the first equality and (8) for the last. Notice that

$$\begin{aligned}
 \int_t^1 \rho(r) dr &= \frac{1}{\gamma} t \rho(t) + O(1) + \int_t^1 O(r^{-\alpha+\zeta}) dr + O(t^{-\alpha+\zeta+1}) \\
 &= \frac{1}{\gamma} t \rho(t) + O(t^{\min(-\alpha+\zeta+1, 0)}) + O(|\log(t)|) \mathbf{1}_{\{\alpha-\zeta=1\}} \\
 &= \frac{1}{\gamma} t \rho(t) + O(t^{\min(-\alpha+\zeta+1, 0)}) + O(t^{-\varepsilon_0}) \mathbf{1}_{\{\alpha-\zeta=1\}}.
 \end{aligned}$$

This implies that

$$\begin{aligned}
 \mathbb{E}[X_1^{(n)}] &= \frac{1}{\gamma} + \frac{n(n-1)}{g_n} \int_0^1 (1-t)^{n-2} (O(t^{\min(-\alpha+\zeta+1, 0)}) \\
 &\quad + O(t^{-\varepsilon_0}) \mathbf{1}_{\{\alpha-\zeta=1\}}) dt.
 \end{aligned}$$

Using (10), (12) and Lemma 2.2, we get

$$\begin{aligned}
 \left| \mathbb{E}[X_1^{(n)}] - \frac{1}{\gamma} \right| &\leq c \frac{n(n-1)}{g_n} \int_0^1 (1-t)^{n-2} (t^{\min(-\alpha+\zeta+1, 0)} + t^{-\varepsilon_0} \mathbf{1}_{\{\alpha-\zeta=1\}}) dt \\
 &\leq cn^{2-\alpha} (n^{-1-\min(-\alpha+\zeta+1, 0)} + n^{-1+\varepsilon_0} \mathbf{1}_{\{\alpha-\zeta=1\}}) \\
 &\leq c\varphi_n. \quad \square
 \end{aligned}$$

We give an upper bound for the second moment of $X_1^{(n)}$.

LEMMA 2.4. Assume that $\rho(t) = O(t^{-\alpha})$. Then there exists a constant C_{18} s.t. for all $n \geq 2$, we have

$$(18) \quad \mathbb{E}[(X_1^{(n)})^2] \leq C_{18} \frac{n^2}{g_n}.$$

PROOF. Using the identity $\mathbb{E}[Y^2] = \sum_{k \geq 1} (2k - 1)\mathbb{P}(Y \geq k)$ for \mathbb{N} -valued random variables, we get

$$\begin{aligned} \mathbb{E}[(X_1^{(n)})^2] &= \frac{\int_0^1 \sum_{k \geq 1} (2k - 1)\mathbb{P}(B_{n,x} \geq k + 1)v(dx)}{g_n} \\ &= \frac{\int_0^1 (\sum_{k \geq 1} (2(k + 1) - 1)\mathbb{P}(B_{n,x} \geq k + 1) - 2 \sum_{k \geq 1} \mathbb{P}(B_{n,x} \geq k + 1))v(dx)}{g_n} \\ &= \frac{\int_0^1 (\mathbb{E}[B_{n,x}^2] - 2\mathbb{E}[B_{n,x}] + \mathbb{P}(B_{n,x} \geq 1))v(dx)}{g_n} \\ &= \frac{\int_0^1 (\mathbb{E}[B_{n,x}^2] - \mathbb{E}[B_{n,x}])v(dx)}{g_n} - \mathbb{E}[X_1^{(n)}] \\ &= \frac{\int_0^1 n(n - 1)x^2v(dx)}{g_n} - \mathbb{E}[X_1^{(n)}] \\ &= 2n(n - 1) \frac{\int_0^1 t\rho(t) dt}{g_n} - \mathbb{E}[X_1^{(n)}], \end{aligned}$$

where we have used (16) for the fourth equality. Use $\int_0^1 t\rho(t) dt < \infty$ and $\mathbb{E}[X_1^{(n)}] \geq 0$ to conclude. \square

We consider ϕ_n the Laplace transform of $X_1^{(n)}$: for $u \geq 0$, $\phi_n(u) = \mathbb{E}[e^{-uX_1^{(n)}}]$.

LEMMA 2.5. Assume that $\rho(t) = C_0t^{-\alpha} + O(t^{-\alpha+\zeta})$ for some $C_0 > 0$ and $\zeta > 0$. Let $\varepsilon_0 > 0$. Recall φ_n given by (14). Then we have, for $n \geq 2$,

$$(19) \quad \phi_n(u) = 1 - \frac{u}{\gamma} + \frac{u^\alpha}{\gamma} + R(n, u),$$

where $R(n, u) = (u\varphi_n + u^2)h(n, u)$ with $\sup_{u \in [0, K], n \geq 2} |h(n, u)| < \infty$ for all $K > 0$.

PROOF. We have

$$\begin{aligned} \phi_n(u) = \mathbb{E}[e^{-uX_1^{(n)}}] &= \sum_{k=1}^{n-1} e^{-uk}\mathbb{P}(X_1^{(n)} = k) \\ &= \sum_{k=1}^{n-1} e^{-uk}\mathbb{P}(X_1^{(n)} \geq k) - \sum_{k=2}^n e^{-u(k-1)}\mathbb{P}(X_1^{(n)} \geq k) \end{aligned}$$

$$\begin{aligned}
 &= e^{-u} + \sum_{k=2}^{n-1} e^{-uk} (1 - e^u) \mathbb{P}(X_1^{(n)} \geq k) \\
 &= e^{-u} + (1 - e^u) \sum_{k=2}^{n-1} \frac{e^{-uk}}{g_n} \int_0^1 \frac{n!}{k!(n-k-1)!} t^k (1-t)^{n-k-1} \rho(t) dt \\
 &= e^{-u} + (1 - e^u) \frac{n}{g_n} \int_0^1 [(1-t(1-e^{-u}))^{n-1} \\
 &\quad - (1-t)^{n-1} - (n-1)e^{-u}t(1-t)^{n-2}] \rho(t) dt \\
 &= 1 + (1 - e^u) \frac{n}{g_n} \int_0^1 [(1-t(1-e^{-u}))^{n-1} - (1-t)^{n-1}] \rho(t) dt,
 \end{aligned}$$

where we used (8) for the last equality. Using (17), this implies

$$(20) \quad \phi_n(u) = 1 + (1 - e^u) \frac{n}{g_n} A + (1 - e^u) \mathbb{E}[X_1^{(n)}].$$

with

$$A = \int_0^1 [(1-t(1-e^{-u}))^{n-1} - 1] \rho(t) dt.$$

Thanks to Lemma 2.3, we have that

$$(21) \quad (1 - e^u) \mathbb{E}[X_1^{(n)}] = -\frac{u}{\gamma} + (u^2 + u\varphi_n)h_1(n, u),$$

where, for all $K > 0$, $\sup_{u \in [0, K], n \geq 2} |h_1(n, u)| < \infty$.

To compute A , we set $a = (1 - e^{-u})$ and $f(t) = t^{-\max(\alpha-1-\zeta, 0)} + t^{-\varepsilon_0} \mathbf{1}_{\{\alpha-\zeta=1\}}$. An integration by part gives

$$\begin{aligned}
 A &= -a(n-1) \int_0^1 (1-at)^{n-2} \left(\int_t^1 \rho(r) dr \right) dt \\
 &= -a(n-1) C_0 \int_0^1 (1-at)^{n-2} \left(\frac{t^{1-\alpha}}{\gamma} + O(f(t)) \right) dt \\
 &= -A_1 + A_2,
 \end{aligned}$$

with

$$A_1 = \frac{a(n-1)}{\gamma} C_0 \int_0^1 (1-at)^{n-2} t^{1-\alpha} dt$$

and

$$A_2 = a(n-1) \int_0^1 (1-at)^{n-2} O(f(t)) dt.$$

We have

$$\begin{aligned} A_1 &= \frac{a^{\alpha-1}(n-1)}{\gamma} C_0 \int_0^a (1-t)^{n-2} t^{1-\alpha} dt \\ &= \frac{a^{\alpha-1}(n-1)}{\gamma} C_0 \int_0^1 (1-t)^{n-2} t^{1-\alpha} dt \\ &\quad - \frac{a^{\alpha-1}(n-1)}{\gamma} C_0 \int_a^1 (1-t)^{n-2} t^{1-\alpha} dt \\ &= \frac{a^{\alpha-1}(n-1)}{\gamma} C_0 \frac{\Gamma(n-1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \\ &\quad - \frac{a^{\alpha-1}(n-1)}{\gamma} C_0 \int_a^1 (1-t)^{n-2} t^{1-\alpha} dt. \end{aligned}$$

Since $a \geq 0$, we have for $u \in [0, K]$ and $n \geq 2$

$$0 \leq \frac{a^{\alpha-1}(n-1)}{\gamma} \int_a^1 (1-t)^{n-2} t^{1-\alpha} dt \leq \frac{(n-1)}{\gamma} \int_a^1 (1-t)^{n-2} dt \leq \frac{1}{\gamma}.$$

Using (12) and Lemma 2.2, we get

$$\left| A_1 - \frac{a^{\alpha-1}}{\gamma} \frac{g_n}{n} \right| \leq c(1 + n^{\alpha-1-\min(\zeta,1)}) \leq cn^{\max(\alpha-1-\zeta,0)},$$

where c does not depend on n and $u \geq 0$. We also have, using (10) and (12),

$$|A_2| \leq ca(n-1) \int_0^1 (1-at)^{n-2} f(t) dt \leq c(n^{\max(\alpha-1-\zeta,0)} + n^{\varepsilon_0} \mathbf{1}_{\{\alpha-\zeta=1\}}).$$

We deduce, using Lemma 2.2 twice, that

$$\left| A + \frac{a^{\alpha-1}}{\gamma} \frac{g_n}{n} \right| \leq c(n^{\max(\alpha-1-\zeta,0)} + n^{\varepsilon_0} \mathbf{1}_{\{\alpha-\zeta=1\}}) \leq c \frac{g_n}{n} \varphi_n.$$

We deduce that

$$\begin{aligned} (22) \quad (1-e^u) \frac{n}{g_n} A &= (1-e^u) \left(-\frac{(1-e^{-u})^{\alpha-1}}{\gamma} + \varphi_n O(1) \right) \\ &= \frac{u^\alpha}{\gamma} + (u^{\alpha+1} + u\varphi_n)h_2(n, u), \end{aligned}$$

where $\sup_{u \in [0, K], n \geq 2} |h_2(n, u)| < \infty$ for all $K > 0$. Then use the expression of ϕ_n given by (20) as well as (21) and (22) to end the proof. \square

3. Asymptotics for the number of jumps. Let $\alpha \in (1, 2)$. We assume that $\rho(t) = C_0 t^{-\alpha} + O(t^{-\alpha+\zeta})$ for some $C_0 > 0$ and $\zeta > 1 - 1/\alpha$.

Let $V = (V_t, t \geq 0)$ be a α -stable Lévy process with no positive jumps (see Chapter VII in [4]) with Laplace exponent $\psi(u) = u^\alpha/\gamma$: for all $u \geq 0$, $\mathbb{E}[e^{-uV_t}] = e^{tu^\alpha/\gamma}$.

Lemma 2.1 implies that $(X_1^{(n)}, \dots, X_k^{(n)})$ converges in distribution to (X_1, \dots, X_k) where $(X_k, k \geq 1)$ is a sequence of independent random variables distributed as X . Using Lemma 2.1 and (12), we get that $\mathbb{P}(X \geq k) \sim_{+\infty} \frac{1}{\Gamma(2-\alpha)} k^{-\alpha}$. Hence, Proposition 9.39 in [9] implies that the law of X is in the domain of attraction of the α -stable distribution. We set

$$W_t^{(n)} = n^{-1/\alpha} \sum_{k=1}^{\lfloor nt \rfloor} \left(X_k - \frac{1}{\gamma} \right) \quad \text{for } t \in [0, \gamma].$$

An easy calculation using the Laplace transform of X shows that for fixed t the sequence $W_t^{(n)}$ converges in distribution to V_t . Then using Theorem 16.14 in [15], we get that the process $(W_t^{(n)}, t \in [0, \gamma])$ converges in distribution to $V = (V_t, t \in [0, \gamma])$. We shall give in Corollary 3.5 a similar result with X_k replaced by $X_k^{(n)}$.

We first give a proof of the convergence of τ_n ; see also [13] and [14] for a different proof. We will use that

$$\sum_{i=1}^{\tau_n} \left(X_i^{(n)} - \frac{1}{\gamma} \right) = n - 1 - \frac{\tau_n}{\gamma}.$$

PROPOSITION 3.1. *We assume that $\zeta > 1 - 1/\alpha$. We have the following convergence in distribution:*

$$n^{-1/\alpha} \left(n - \frac{\tau_n}{\gamma} \right) \xrightarrow[n \rightarrow \infty]{(d)} V_\gamma.$$

PROOF. Using [20], it is enough to prove that $\lim_{n \rightarrow \infty} \mathbb{E}[e^{-un^{-1/\alpha}(n-\tau_n/\gamma)}] = e^{u^\alpha}$ for all $u \geq 0$. Recall $\mathcal{Y} = (\mathcal{Y}_k, k \geq 0)$ is the filtration generated by Y . Notice τ_n is an \mathcal{Y} -stopping time. Recall that for $m \geq 1$, ϕ_m denotes the Laplace transform of $X_1^{(m)}$. For fixed n , and for any $v \geq 0$, the process $(M_{v,k}, k \geq 0)$ defined by

$$M_{v,k} = \prod_{i=1}^k \exp(-vX_i^{(n)} - \log \phi_{Y_{i-1}^{(n)}}(v))$$

is a bounded martingale w.r.t. the filtration \mathcal{Y} . Notice that $\mathbb{E}[M_{v,k}] = 1$. As $X_i = 0$ for $i > \tau_n$, we also have

$$(23) \quad M_{v,k} = \prod_{i=1}^{k \wedge \tau_n} \exp(-vX_i^{(n)} - \log \phi_{Y_{i-1}^{(n)}}(v)).$$

Let $u \geq 0$ and consider a nonnegative sequence $(a_n, n \geq 1)$ which converges to 0. Using (19), we get that

$$M_{ua_n, k} = \exp\left(-ua_n \sum_{i=1}^{k \wedge \tau_n} X_i^{(n)} - \sum_{i=1}^{k \wedge \tau_n} \left(-\frac{ua_n}{\gamma} + \frac{u^\alpha a_n^\alpha}{\gamma} + R(Y_{i-1}^{(n)}, ua_n)\right)\right).$$

In particular, we have

$$(24) \quad M_{ua_n, \tau_n} = \exp\left(-ua_n \left(n - 1 - \frac{\tau_n}{\gamma}\right) - \frac{u^\alpha \tau_n a_n^\alpha}{\gamma} - \sum_{i=1}^{\tau_n} R(Y_{i-1}^{(n)}, ua_n)\right).$$

We first give an upper bound for $\sum_{i=1}^{\tau_n} R(Y_{i-1}^{(n)}, ua_n)$.

LEMMA 3.2. *We assume that $\zeta > 1 - 1/\alpha$. Let $K > 0$. Let $\eta \geq \frac{1}{\alpha}$. There exist $\varepsilon_1 > 0$ and $C_{25}(K)$ a finite constant such that for all $n \geq 1$ and $u \in [0, K]$, a.s. with $a_n = n^{-\eta}$,*

$$(25) \quad \sum_{i=1}^{\tau_n} |R(Y_{i-1}^{(n)}, ua_n)| \leq C_{25}(K)n^{-\varepsilon_1}.$$

PROOF. Notice that $\tau_n \leq n - 1$. We have seen in Lemma 2.5 that $R(n, u) = (u\varphi_n + u^2)h(n, u)$ with $\bar{h}(K) = \sup_{u \in [0, K], n \geq 2} |h(n, u)| < \infty$ and φ_n given by (14). We have $2 - \alpha - \frac{1}{\alpha} = -\alpha(1 - 1/\alpha)^2 < 0$. As $\varepsilon_0 > 0$ is arbitrary in (14), we can take ε_0 small enough so that $1 - \alpha + \varepsilon_0 < 0$ and $2 - \alpha + \varepsilon_0 - 1/\alpha < 0$. We have

$$a_n \sum_{i=1}^{\tau_n} \varphi_{Y_{i-1}^{(n)}} \leq n^{-1/\alpha} \sum_{j=1}^n \varphi_j \leq c \begin{cases} n^{1-\zeta-1/\alpha}, & \text{if } \zeta < \alpha - 1, \\ n^{2-\alpha+\varepsilon_0-1/\alpha}, & \text{if } \zeta = \alpha - 1, \\ n^{2-\alpha-1/\alpha}, & \text{if } \zeta > \alpha - 1. \end{cases}$$

For $\varepsilon_1 > 0$ less than the two positive quantities $-1 + \zeta + \frac{1}{\alpha}$ and $-2 + \alpha - \varepsilon_0 + \frac{1}{\alpha}$, we have $a_n \sum_{i=1}^{\tau_n} \varphi_{Y_{i-1}^{(n)}} \leq cn^{-\varepsilon_1}$. We deduce that, for $u \in [0, K]$,

$$\begin{aligned} \sum_{i=1}^{\tau_n} |R(Y_{i-1}^{(n)}, ua_n)| &\leq \bar{h}(K) \sum_{i=1}^{\tau_n} (\varphi_{Y_{i-1}^{(n)}} ua_n + (ua_n)^2) \\ &\leq \bar{h}(K) \sum_{j=1}^n (\varphi_j K a_n + (K a_n)^2) \\ &\leq c\bar{h}(K)(K n^{-\varepsilon_1} + K^2 n^{1-2/\alpha}), \end{aligned}$$

for some constant c independent of n, u and K . Taking $\varepsilon_1 > 0$ small enough so that $\varepsilon_1 < \frac{2}{\alpha} - 1$, we then get (25). \square

Next we prove the following lemma.

LEMMA 3.3. *We assume that $\zeta > 1 - 1/\alpha$. Let $\varepsilon > 0$. The sequence $(n^{-(1/\alpha)-\varepsilon}(n - 1 - \frac{\tau_n}{\gamma}), n \geq 1)$ converges in probability to 0.*

PROOF. We set $a_n = n^{-1/\alpha-\varepsilon}$. Notice that

$$e^{-ua_n(n-1-\tau_n/\gamma)} = M_{ua_n, \tau_n} e^{u^\alpha \tau_n a_n / \gamma + \sum_{i=1}^{\tau_n} R(Y_{i-1}^{(n)}, ua_n)}.$$

As $\tau_n \leq n - 1$, we have $0 \leq \tau_n a_n^\alpha \leq n^{-\alpha\varepsilon}$. Using (25), we get for $u \geq 0$

$$\begin{aligned} \mathbb{E}[M_{ua_n, \tau_n}] e^{-C25(u)n^{-\varepsilon_1}} &\leq \mathbb{E}[e^{-ua_n(n-1-\tau_n/\gamma)}] \\ &\leq \mathbb{E}[M_{ua_n, \tau_n}] e^{C25(u)n^{-\varepsilon_1} + u^\alpha n^{-\alpha\varepsilon/\gamma}}. \end{aligned}$$

As τ_n is bounded, the stopping time theorem gives $\mathbb{E}[M_{ua_n, \tau_n}] = 1$. We deduce that, for all $u \geq 0$, $\lim_{n \rightarrow \infty} \mathbb{E}[e^{-ua_n(n-1-\tau_n/\gamma)}] = 1$. Using [20], we get the convergence in law of $a_n(n - 1 - \frac{\tau_n}{\gamma})$ to 0, and then in probability as the limit is constant. \square

Let $a_n = n^{-1/\alpha}$ and $u \geq 0$. We have

$$\begin{aligned} \mathbb{E}[e^{-ua_n(n-1-\tau_n/\gamma)}] &= \mathbb{E}[e^{-ua_n(n-1-\tau_n/\gamma)}(1 - e^{-u^\alpha a_n^\alpha (\tau_n/\gamma - n)})] \\ (26) \qquad \qquad \qquad &+ \mathbb{E}[e^{-ua_n(n-1-\tau_n/\gamma)} e^{-u^\alpha a_n^\alpha (\tau_n/\gamma - n)}] \\ &= I_1 + I_2, \end{aligned}$$

with

$$I_1 = \mathbb{E}[e^{-ua_n(n-1-\tau_n/\gamma)}(1 - e^{-u^\alpha a_n^\alpha (\tau_n/\gamma - n)})]$$

and

$$I_2 = \mathbb{E}[M_{ua_n, \tau_n} e^{u^\alpha + \sum_{i=1}^{\tau_n} R(Y_{i-1}^{(n)}, ua_n)}].$$

Using (25) and $\mathbb{E}[M_{ua_n, \tau_n}] = 1$, we get

$$e^{u^\alpha - C25(u)n^{-\varepsilon_1}} \leq I_2 \leq e^{u^\alpha + C25(u)n^{-\varepsilon_1}}.$$

This implies that $\lim_{n \rightarrow \infty} I_2 = e^{u^\alpha}$.

We now prove that $\lim_{n \rightarrow \infty} I_1 = 0$. Recall that $\tau_n \leq n - 1$ so that $\tau_n a_n^\alpha \leq 1$ and thanks to (25), we get

$$\begin{aligned} \mathbb{E}[e^{-ua_n(n-1-\tau_n/\gamma)}] &= \mathbb{E}[M_{ua_n, \tau_n} e^{u^\alpha \tau_n a_n^\alpha / \gamma + \sum_{i=1}^{\tau_n} R(Y_{i-1}^{(n)}, ua_n)}] \\ &\leq M(u) \mathbb{E}[M_{ua_n, \tau_n}] = M(u), \end{aligned}$$

where $M(u)$ is a constant which does not depend on n . By the Cauchy–Schwarz inequality, we get that

$$\begin{aligned} I_1^2 &= \mathbb{E}[e^{-2ua_n(n-1-\tau_n/\gamma)}(1 - e^{-u^\alpha a_n^\alpha (\tau_n/\gamma - n)})^2] \\ &\leq \mathbb{E}[e^{-2ua_n(n-1-\tau_n/\gamma)}] \mathbb{E}[(1 - e^{-u^\alpha a_n^\alpha (\tau_n/\gamma - n)})^2] \\ &\leq M(2u) \mathbb{E}[(1 - e^{-u^\alpha (1/n)(\tau_n/\gamma - n)})^2]. \end{aligned}$$

Notice $(\frac{1}{n}(\frac{\tau_n}{\gamma} - n), n \geq 1)$ is bounded from below and above by finite constants, and thanks to Lemma 3.3 it converges to 0 in probability. Hence, we deduce that

$$\lim_{n \rightarrow \infty} \mathbb{E}[(1 - e^{-u^\alpha 1/n(\tau_n/\gamma - n)})^2] = 0.$$

This implies that $\lim_{n \rightarrow \infty} I_1 = 0$.

From the convergence of I_1 and I_2 , we deduce from (26) that

$$\lim_{n \rightarrow \infty} \mathbb{E}[e^{-u^\alpha n(n-1-\tau_n/\gamma)}] = e^{u^\alpha}.$$

This ends the proof of the proposition. \square

We now give a general result.

PROPOSITION 3.4. *We assume that $\zeta > 1 - 1/\alpha$. Let $f_n : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be uniformly bounded functions such that*

$$\kappa = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{\lfloor n\gamma \rfloor} f_n(k/n)^\alpha$$

exists. Then we have the following convergence in distribution:

$$(27) \quad V^{(n)}(f_n) := n^{-1/\alpha} \sum_{k=1}^{\tau_n} f_n(k/n) \left(X_k^{(n)} - \frac{1}{\gamma} \right) \xrightarrow[n \rightarrow \infty]{(d)} \kappa^{1/\alpha} V_1.$$

In particular, if $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a bounded locally Riemann integrable function, then

$$(28) \quad V^{(n)}(f) = n^{-1/\alpha} \sum_{k=1}^{\tau_n} f(k/n) \left(X_k^{(n)} - \frac{1}{\gamma} \right) \xrightarrow[n \rightarrow \infty]{(d)} \int_0^\gamma f(t) dV_t,$$

where the distribution of $\int_0^\gamma f(t) dV_t$ is characterized by its Laplace transform: for $u \geq 0$,

$$(29) \quad \mathbb{E} \left[\exp \left(-u \int_0^\gamma f(t) dV_t \right) \right] = \exp \left(\frac{u^\alpha}{\gamma} \int_0^\gamma f^\alpha(t) dt \right).$$

If we apply this Proposition with step functions, we deduce the following result.

COROLLARY 3.5. *We assume that $\zeta > 1 - 1/\alpha$. Let $V_t^{(n)} = V^{(n)}(\mathbf{1}_{[0,t]}) = n^{-1/\alpha} \sum_{k=1}^{\lfloor nt \rfloor \wedge \tau_n} (X_k^{(n)} - \frac{1}{\gamma})$ for $t \in [0, \gamma)$, and $V_\gamma^{(n)} = V^{(n)}(\mathbf{1}) = n^{-1/\alpha} (n - 1 - \frac{\tau_n}{\gamma})$. The finite-dimensional marginals of the process $(V_t^{(n)}, t \in [0, \gamma])$ converges in law to those of the process $(V_t, t \in [0, \gamma])$.*

PROOF. Thanks to [20], it is enough to prove that

$$\mathbb{E}[\exp(-uV^{(n)}(f_n))] \xrightarrow{n \rightarrow \infty} e^{\kappa u^\alpha / \gamma}.$$

Taking uf_n as f_n , we shall only consider the case $u = 1$.

We set $a = \sup_{n \geq 1, x \geq 0} |f_n(x)|$ and for any bounded function g ,

$$A_n(g) = \exp \sum_{k=1}^{\tau_n} (-n^{-1/\alpha} g(k/n) X_k^{(n)} - \log \phi_{Y_{k-1}^{(n)}}(n^{-1/\alpha} g(k/n))).$$

A martingale argument provides that $\mathbb{E}[A_n(g)] = 1$. Using (19), we get that

$$\begin{aligned} A_n(g) &= \exp \left(-n^{-1/\alpha} \sum_{k=1}^{\tau_n} g(k/n) \left(X_k^{(n)} - \frac{1}{\gamma} \right) \right. \\ &\quad \left. - n^{-1} \sum_{k=1}^{\tau_n} \frac{g^\alpha(k/n)}{\gamma} - \sum_{k=1}^{\tau_n} R(Y_{k-1}^{(n)}, n^{-1/\alpha} g(k/n)) \right) \\ &= \exp \left(-V^{(n)}(g) - n^{-1} \sum_{k=1}^{\tau_n} \frac{g^\alpha(k/n)}{\gamma} - \sum_{k=1}^{\tau_n} R(Y_{k-1}^{(n)}, n^{-1/\alpha} g(k/n)) \right). \end{aligned}$$

Let

$$\Lambda_n = n^{-1} \sum_{k=1}^{\lfloor n\gamma \rfloor} \frac{f_n^\alpha(k/n)}{\gamma} - n^{-1} \sum_{k=1}^{\tau_n} \frac{f_n^\alpha(k/n)}{\gamma}$$

and write

$$\mathbb{E}[e^{-V^{(n)}(f_n)}] = I_1 + I_2$$

with

$$I_1 = \mathbb{E}[e^{-V^{(n)}(f_n)}(1 - e^{\Lambda_n})] \quad \text{and} \quad I_2 = \mathbb{E}[e^{-V^{(n)}(f_n)}e^{\Lambda_n}].$$

First of all, let us prove that I_1 converges to 0 when n tends to ∞ . Recall that the functions f_n are uniformly bounded by a . Thanks to (25), we have

$$\begin{aligned} \mathbb{E}[e^{-2V^{(n)}(f_n)}] &= \mathbb{E}[e^{-V^{(n)}(2f_n)}] \\ &= \mathbb{E}[A_n(2f_n)e^{n^{-1} \sum_{k=1}^{\tau_n} (2^\alpha f_n^\alpha(k/n))/\gamma + \sum_{k=1}^{\tau_n} R(Y_{k-1}^{(n)}, n^{-1/\alpha} 2f_n(k))}] \leq M, \end{aligned}$$

where M is a finite constant which does not depend on n . By Cauchy–Schwarz’ inequality, we get that

$$\begin{aligned} (I_1)^2 &\leq (\mathbb{E}[e^{-V^{(n)}(f_n)}|1 - e^{\Lambda_n}|])^2 \\ &\leq \mathbb{E}[e^{-V^{(n)}(2f_n)}] \mathbb{E}[(1 - e^{\Lambda_n})^2] \\ &\leq M \mathbb{E}[(1 - e^{\Lambda_n})^2]. \end{aligned}$$

Moreover, as $|1 - e^x| \leq e^{|x|} - 1$ and $\Lambda_n \leq \frac{a^\alpha}{n^\gamma} |\lfloor n\gamma \rfloor - \tau_n|$, we get

$$(30) \quad \mathbb{E}[(1 - e^{\Lambda_n})^2] \leq \mathbb{E}\left[\left(1 - e^{|\lfloor n\gamma \rfloor - \tau_n| a^\alpha / (n^\gamma)}\right)^2\right].$$

The quantity $\frac{|\lfloor n\gamma \rfloor - \tau_n| a^\alpha}{n^\gamma}$ is bounded and goes to 0 in probability when n goes to infinity. Therefore, the right-hand side of (30) converges to 0. This implies that $\lim_{n \rightarrow \infty} I_1 = 0$.

Let us now consider the convergence of I_2 . Remark that

$$I_2 = \mathbb{E}\left[A_n(f_n) \exp\left(n^{-1} \sum_{k=1}^{\lfloor n\gamma \rfloor} \frac{f_n^\alpha(k/n)}{\gamma} + \sum_{k=1}^{\tau_n} R(Y_{k-1}^{(n)}, n^{-1/\alpha} f_n(k))\right)\right].$$

Recall that f_n is bounded by a and that $\mathbb{E}[A_n(f_n)] = 1$. Using Lemma 3.2, we get for some $\varepsilon > 0$

$$(31) \quad \begin{aligned} & \exp\left(-C_{25}(a)n^{-\varepsilon_1} - n^{-1} \sum_{k=1}^{\lfloor n\gamma \rfloor} \frac{f_n^\alpha(k/n)}{\gamma}\right) \\ & \leq \mathbb{E}\left[A_n(f_n) \exp\left(n^{-1} \sum_{k=1}^{\lfloor n\gamma \rfloor} \frac{f_n^\alpha(k/n)}{\gamma} + \sum_{k=1}^{\tau_n} R(Y_{k-1}^{(n)}, n^{-1/\alpha} f_n(k))\right)\right] \\ & \leq \exp\left(C_{25}(a)n^{-\varepsilon_1} + n^{-1} \sum_{k=1}^{\lfloor n\gamma \rfloor} \frac{f_n^\alpha(k/n)}{\gamma}\right). \end{aligned}$$

As

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{\lfloor n\gamma \rfloor} f_n^\alpha(k/n) = \kappa,$$

we get that $\lim_{n \rightarrow \infty} I_2 = e^{\kappa/\gamma}$, which achieves the proof of (27). To get (28), notice that

$$\kappa = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{\lfloor n\gamma \rfloor} f(k/n)^\alpha = \int_0^\gamma f(t)^\alpha dt. \quad \square$$

4. First approximation of the length of the coalescent tree. Let $\alpha \in (1, 2)$. We assume that $\rho(t) = C_0 t^{-\alpha} + O(t^{-\alpha+\zeta})$ for some $C_0 > 0$ and $\zeta > 1 - 1/\alpha$.

Recall that the length of the coalescent tree up to the $\lfloor nt \rfloor$ th coalescence is, for $t \geq 0$, given by (3). The next lemma gives an upper bound on the error when one replaces the exponential random variables by their mean.

LEMMA 4.1. For $t \geq 0$, let

$$\tilde{L}_t^{(n)} = \sum_{k=0}^{\lfloor nt \rfloor \wedge (\tau_n - 1)} \frac{Y_k^{(n)}}{g_{Y_k^{(n)}}}.$$

There exists a finite constant C_{32} such that, we have

$$(32) \quad \mathbb{E} \left[\sup_{t \geq 0} (L_t^{(n)} - \tilde{L}_t^{(n)})^2 \right] \leq C_{32} \begin{cases} n^{3-2\alpha}, & \text{if } \alpha < 3/2, \\ \log(n), & \text{if } \alpha = 3/2, \\ 1, & \text{if } \alpha > 3/2. \end{cases}$$

PROOF. Recall that $\mathcal{Y} = (\mathcal{Y}_k, k \geq 0)$ denotes the filtration generated by Y . Conditionally on \mathcal{Y} , the random variables $\frac{Y_k^{(n)}}{g_{Y_k^{(n)}}}(E_k - 1)$ are independent with zero mean. We deduce that

$$\begin{aligned} \mathbb{E} \left[\sup_{t \geq 0} (L_t^{(n)} - \tilde{L}_t^{(n)})^2 | \mathcal{Y} \right] &= \mathbb{E} \left[\sup_{t \geq 0} \left(\sum_{k=0}^{\lfloor nt \rfloor \wedge (\tau_n - 1)} \frac{Y_k^{(n)}}{g_{Y_k^{(n)}}}(E_k - 1) \right)^2 | \mathcal{Y} \right] \\ &\leq 4 \sum_{k=0}^{\tau_n - 1} \left(\frac{Y_k^{(n)}}{g_{Y_k^{(n)}}} \right)^2 \\ &\leq 4 \sum_{\ell=1}^n \left(\frac{\ell}{g_\ell} \right)^2, \end{aligned}$$

where we used Doob inequality for martingale in the first inequality. Thanks to (13), we get

$$\mathbb{E} \left[\sup_{t \geq 0} (L_t^n - \tilde{L}_t^{(n)})^2 | \mathcal{Y} \right] \leq c \sum_{\ell=1}^n \ell^{2-2\alpha} \leq c \begin{cases} n^{3-2\alpha}, & \text{if } \alpha < 3/2, \\ \log(n), & \text{if } \alpha = 3/2, \\ 1, & \text{if } \alpha > 3/2, \end{cases}$$

where c is nonrandom. This implies the result. \square

LEMMA 4.2. For $t \geq 0$, let

$$\hat{L}_t^{(n)} = \sum_{k=0}^{\lfloor nt \rfloor \wedge (\tau_n - 1)} (Y_k^{(n)})^{-\gamma}.$$

There exists a finite constant C_{33} such that for all $t \geq 0$, we have

$$(33) \quad \left| \tilde{L}_t^{(n)} - \frac{\hat{L}_t^{(n)}}{C_0 \Gamma(2 - \alpha)} \right| \leq C_{33} \begin{cases} n^{2-\alpha-\zeta}, & \text{if } \zeta < 2 - \alpha, \\ \log(n), & \text{if } \zeta = 2 - \alpha, \\ 1, & \text{if } \zeta > 2 - \alpha. \end{cases}$$

PROOF. Use (13) to get that

$$\tilde{L}_t^{(n)} - \frac{\hat{L}_t^{(n)}}{C_0 \Gamma(2 - \alpha)} = \sum_{k=0}^{\lfloor nt \rfloor \wedge (\tau_n - 1)} (Y_k^{(n)})^{-\gamma} O((Y_k^{(n)})^{-\min(\zeta, 1)}).$$

We deduce that

$$\begin{aligned} \left| \tilde{L}_t^{(n)} - \frac{\hat{L}_t^{(n)}}{C_0 \Gamma(2 - \alpha)} \right| &\leq c \sum_{\ell=1}^n \ell^{-\alpha+1-\min(\zeta, 1)} \\ &\leq c \begin{cases} n^{2-\alpha-\zeta}, & \text{if } \zeta < 2 - \alpha, \\ \log(n), & \text{if } \zeta = 2 - \alpha, \\ 1, & \text{if } \zeta > 2 - \alpha. \end{cases} \quad \square \end{aligned}$$

5. Limit distribution of $\hat{L}_t^{(n)}$. Let $\alpha \in (1, 2)$ and $\gamma = \alpha - 1$. For $t \in [0, \gamma]$, we set

$$v(t) = \int_0^t \left(1 - \frac{r}{\gamma}\right)^{-\gamma} dr.$$

THEOREM 5.1. We assume that $\rho(t) = C_0 t^{-\alpha} + O(t^{-\alpha+\zeta})$ for some $C_0 > 0$ and $\zeta > 1 - 1/\alpha$.

(i) Let $t_0 \in [0, \gamma)$ and $\delta > 0$. The following convergence in probability holds:

$$(34) \quad n^{(\alpha-1)/2-\delta} \sup_{0 \leq t \leq t_0} |n^{-2+\alpha} \hat{L}_t^{(n)} - v(t)| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

(ii) Let $t \in [0, \gamma)$. The following convergence in distribution holds:

$$(35) \quad n^{-1+\alpha-1/\alpha} (\hat{L}_t^{(n)} - n^{2-\alpha} v(t)) \xrightarrow[n \rightarrow \infty]{(d)} (\alpha - 1) \int_0^t dr \left(1 - \frac{r}{\gamma}\right)^{-\alpha} V_r.$$

PROOF. Let $\varepsilon_2 \in (0, \gamma)$, $t_0 = \gamma - \varepsilon_2$ and $t \in [0, t_0]$. We use a Taylor expansion to get

$$\begin{aligned} \hat{L}_t^{(n)} &= \sum_{k=0}^{\lfloor nt \rfloor \wedge (\tau_n - 1)} \left(n - \sum_{i=1}^k X_i^{(n)} \right)^{-\gamma} \\ &= \sum_{k=0}^{\lfloor nt \rfloor \wedge (\tau_n - 1)} \left(n - \frac{k}{\gamma} - \sum_{i=1}^k \left(X_i^{(n)} - \frac{1}{\gamma} \right) \right)^{-\gamma} \\ (36) \quad &= \sum_{k=0}^{\lfloor nt \rfloor \wedge (\tau_n - 1)} \left(n - \frac{k}{\gamma} \right)^{-\gamma} (1 - \Delta_{n,k})^{-\gamma} \\ &= I_n(t) + \gamma J_n(t) + \gamma(\gamma + 1) R_n(t) \end{aligned}$$

with $\Delta_{n,k} = \frac{\sum_{i=1}^k (X_i^{(n)} - 1/\gamma)}{n - k/\gamma}$ and

$$I_n(t) = \sum_{k=0}^{\lfloor nt \rfloor \wedge (\tau_n - 1)} \left(n - \frac{k}{\gamma} \right)^{-\gamma},$$

$$J_n(t) = \sum_{k=1}^{\lfloor nt \wedge (\tau_n - 1) \rfloor} \left(n - \frac{k}{\gamma} \right)^{-\gamma-1} \sum_{i=1}^k \left(X_i^{(n)} - \frac{1}{\gamma} \right),$$

$$R_n(t) = \sum_{k=1}^{\lfloor nt \wedge (\tau_n - 1) \rfloor} \left(n - \frac{k}{\gamma} \right)^{-\gamma} \int_0^{\Delta_{n,k}} (\Delta_{n,k} - s)(1 - s)^{-\gamma-2} ds.$$

Notice that a.s. $\Delta_{n,k} < 1$, so that $R_n(t)$ is well defined.

Convergence of $I_n(t)$. We write

$$I_n(t) = n^{2-\alpha} I_{n,1}(t) \mathbf{1}_{\{nt < \tau_n\}} + I_n(t) \mathbf{1}_{\{nt \geq \tau_n\}}$$

$$\text{with } I_{n,1}(t) = \frac{1}{n} \sum_{k=0}^{\lfloor nt \rfloor} \left(1 - \frac{k}{n\gamma} \right)^{-\gamma}.$$

Standard computation yields

$$I_{n,1}(t) = v(t) + \frac{1}{n} h_3(n, t),$$

where $\sup_{t \in [0, t_0], n \geq 1} |h_3(n, t)| < \infty$. Hence, we have for $\varepsilon > 0$

$$\begin{aligned} & \mathbb{P} \left(n^{-1+\alpha-1/\alpha} \sup_{0 \leq t \leq t_0} |I_n(t) - n^{2-\alpha} v(t)| \geq \varepsilon \right) \\ &= \mathbb{P} \left(n^{1-1/\alpha} \sup_{0 \leq t \leq t_0} |I_{n,1}(t) - v(t)| \geq \varepsilon, nt_0 < \tau_n \right) \\ & \quad + \mathbb{P} \left(n^{-1+\alpha-1/\alpha} \sup_{0 \leq t \leq t_0} |I_n(t) - n^{2-\alpha} v(t)| \geq \varepsilon, nt_0 \geq \tau_n \right) \\ & \leq \mathbb{P} \left(n^{-1/\alpha} \sup_{0 \leq t \leq t_0} |h_3(n, t)| \geq \varepsilon \right) + \mathbb{P}(nt_0 \geq \tau_n). \end{aligned}$$

According to Lemma 3.3, τ_n/n converges in probability to γ . This implies that for all $t \in [0, \gamma)$

$$(37) \quad \lim_{n \rightarrow \infty} \mathbb{P}(nt \geq \tau_n) = 0.$$

As $n^{-1/\alpha} \sup_{0 \leq t \leq t_0} |h_3(n, t)| < \varepsilon$ for n large enough, we deduce the following convergence in probability:

$$(38) \quad n^{-1+\alpha-1/\alpha} \sup_{0 \leq t \leq t_0} |I_n(t) - n^{2-\alpha} v(t)| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

Convergence of $J_n(t)$. Let $t \in [0, t_0]$. To get the convergence of $J_n(t)$, notice that

$$(39) \quad \begin{aligned} J_n(t) &= \sum_{i=1}^{\lfloor nt \wedge (\tau_n - 1) \rfloor} \left(X_i^{(n)} - \frac{1}{\gamma} \right) \sum_{k=i}^{\lfloor nt \wedge (\tau_n - 1) \rfloor} \left(n - \frac{k}{\gamma} \right)^{-\alpha} \\ &= n^{1-\alpha} J_{n,1} \mathbf{1}_{\{nt < \tau_n\}} + J_n(t) \mathbf{1}_{\{nt \geq \tau_n\}}, \end{aligned}$$

with

$$J_{n,1} = \sum_{i=1}^{\lfloor nt \rfloor \wedge (\tau_n - 1)} f_n(i/n) \left(X_i^{(n)} - \frac{1}{\gamma} \right)$$

and

$$f_n(r) = \frac{1}{n} \sum_{j=\lfloor nr \rfloor}^{\lfloor nt \rfloor} \left(1 - \frac{j}{n\gamma} \right)^{-\alpha}.$$

The functions f_n are finite and uniformly bounded as for $n \geq 2/\varepsilon_2$,

$$0 \leq f_n(r) \leq f_n(0) = \frac{1}{n} \sum_{k=0}^{\lfloor nt \rfloor} \left(1 - \frac{k}{n\gamma} \right)^{-\alpha} \leq \int_0^{\gamma - \varepsilon_2/2} \left(1 - \frac{s}{\gamma} \right)^{-\alpha} ds < \infty.$$

Notice that

$$\kappa = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{\lfloor n\gamma \rfloor} f_n(k/n)^\alpha = \int_0^t dr \left(\int_r^t \left(1 - \frac{s}{\gamma} \right)^{-\alpha} ds \right)^\alpha.$$

We deduce from Proposition 3.4 that $(n^{-1/\alpha} J_{n,1}, n \geq 2)$ converges in distribution to $\kappa^{1/\alpha} V_1$. For $\varepsilon' > 0$, we have $\mathbb{P}(\mathbf{1}_{\{nt \geq \tau_n\}} | J_n(t) | \geq \varepsilon') \leq \mathbb{P}(nt \geq \tau_n)$. Then we use (39) and (37) to conclude that the following convergence in distribution holds:

$$(40) \quad n^{-1+\alpha-1/\alpha} J_n(t) \xrightarrow[n \rightarrow \infty]{(d)} \kappa^{1/\alpha} V_1.$$

Convergence of $R_n(t)$. Let $t \in [0, t_0]$. We shall now prove that $n^{-1+\alpha-1/\alpha} R_n(t)$ converges to 0 in probability. Let $\varepsilon \in (0, \gamma)$. We have $R_n(t) = R_{n,1} + R_{n,2}$, with

$$R_{n,1} = \sum_{k=1}^{\lfloor nt \rfloor} \left(n - \frac{k}{\gamma} \right)^{-\gamma} \mathbf{1}_{\{k < \tau_n\}} R_{n,1,k},$$

$$R_{n,1,k} = \mathbf{1}_{\{\Delta_{n,k} < 1-\varepsilon\}} \int_0^{\Delta_{n,k}} (\Delta_{n,k} - s)(1-s)^{-\gamma-2} ds,$$

$$R_{n,2} = \sum_{k=1}^{\lfloor nt \rfloor} \left(n - \frac{k}{\gamma} \right)^{-\gamma} \mathbf{1}_{\{k < \tau_n\}} \mathbf{1}_{\{\Delta_{n,k} \geq 1-\varepsilon\}} \int_0^{\Delta_{n,k}} (\Delta_{n,k} - s)(1-s)^{-\gamma-2} ds.$$

We have

$$\mathbb{E}[|R_{n,1,k}|] \leq \frac{\varepsilon^{-\gamma-2}}{2} \mathbb{E}[(\Delta_{n,k})^2] \leq \frac{c}{n^2} \mathbb{E} \left[\left(\sum_{i=1}^k \left(X_i^{(n)} - \frac{1}{\gamma} \right) \right)^2 \right],$$

where we used that $k \leq n(\gamma - \varepsilon_2)$ for the last inequality and c depends on ε and ε_2 .

Recall $\mathcal{Y} = (\mathcal{Y}_k, k \geq 0)$ is the filtration generated by Y . We consider the \mathcal{Y} -martingale $N_r = \sum_{j=1}^r \Delta N_r$, with $\Delta N_r = X_r^{(n)} - \mathbb{E}[X_r^{(n)} | \mathcal{Y}_{r-1}]$. We have

$$\mathbb{E} \left[\left(\sum_{i=1}^k \left(X_i^{(n)} - \frac{1}{\gamma} \right) \right)^2 \right] \leq 2\mathbb{E}[N_k^2] + 2\mathbb{E} \left[\left(\sum_{i=1}^k \left(\mathbb{E}[X_i^{(n)} | \mathcal{Y}_{i-1}] - \frac{1}{\gamma} \right) \right)^2 \right].$$

Notice that

$$\mathbb{E}[N_k^2] = \mathbb{E} \left[\sum_{i=1}^k (\Delta N_i)^2 \right] \leq \mathbb{E} \left[\sum_{i=1}^k \mathbb{E}[(X_i^{(n)})^2 | \mathcal{Y}_{i-1}] \right] \leq \mathbb{E} \left[\sum_{i=1}^k (X_i^{(n)})^2 \right].$$

Using that, conditionally on \mathcal{Y}_{i-1} , $X_i^{(n)}$ and $X_1^{(Y_{i-1})}$ have the same distribution, we get that

$$\mathbb{E}[N_k^2] \leq \sum_{j=1}^n \mathbb{E}[(X_1^{(j)})^2].$$

Thanks to (18) and (13), we deduce that

$$\mathbb{E}[N_k^2] \leq C_{18} \sum_{j=1}^n \frac{j^2}{g_j} \leq c \sum_{j=1}^n j^{2-\alpha} \leq cn^{3-\alpha}.$$

Using (15) and (13), we get

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{i=1}^k \left(\mathbb{E}[X_i^{(n)} | \mathcal{Y}_{i-1}] - \frac{1}{\gamma} \right) \right)^2 \right] &\leq \mathbb{E} \left[\left(\sum_{i=1}^k \left| \mathbb{E}[X_i^{(n)} | \mathcal{Y}_{i-1}] - \frac{1}{\gamma} \right| \right)^2 \right] \\ &\leq \mathbb{E} \left[\left(\sum_{i=1}^k C_{15} \varphi_{Y_{i-1}} \right)^2 \right] \\ &\leq c \left(\sum_{j=1}^n \varphi_j \right)^2 \leq cn^{3-\alpha}, \end{aligned}$$

where for the last inequality we used (14) with $\varepsilon_0 > 0$ small enough (such that $1 + 2\varepsilon_0 < \alpha$) and the fact that $\zeta > 1 - 1/\alpha$ implies $2 - 2\zeta \leq 3 - \alpha$ as $\alpha \in (1, 2)$. This implies that for $k \leq n(\gamma - \varepsilon_2) = nt_0$

$$(41) \quad \mathbb{E} \left[\left(\sum_{i=1}^k \left(X_i^{(n)} - \frac{1}{\gamma} \right) \right)^2 \right] \leq cn^{3-\alpha}$$

therefore $\mathbb{E}[|R_{n,1,k}|] \leq cn^{1-\alpha}$ and $\mathbb{E}[|R_{n,1}|] \leq cn^{3-2\alpha}$. Thus, we get that $(n^{-1+\alpha-1/\alpha} R_{n,1}, n \geq 1)$ converges in probability to 0 since $-1 + \alpha - 1/\alpha + 3 - 2\alpha = -(\alpha - 1)^2/\alpha < 0$ for $\alpha > 1$.

We now consider $R_{n,2}$. Suppose that $k \leq \lfloor nt \rfloor - 1$ satisfies $\Delta_{n,k} \geq 1 - \varepsilon$ on $\{nt < \tau_n\}$. Then on $\{nt < \tau_n\}$, we have

$$\begin{aligned} \Delta_{n,k+1} &= \Delta_{n,k} + \frac{X_{k+1}^{(n)} - 1/\gamma + \Delta_{n,k}/\gamma}{n - (k + 1)/\gamma} \\ &\geq \Delta_{n,k} + \frac{X_{k+1}^{(n)} - \varepsilon/\gamma}{n - (k + 1)/\gamma} \geq \Delta_{n,k}, \end{aligned}$$

where we used that $\gamma > \varepsilon$ for the first inequality and $X_{k+1}^{(n)} \geq 1$ for the last. In particular, on $\{nt < \tau_n\}$, if $\Delta_{n,k} \geq 1 - \varepsilon$ for some $k \leq \lfloor nt \rfloor$, then we have $\Delta_{n,\lfloor nt \rfloor} \geq 1 - \varepsilon$. This implies that $\mathbf{1}_{\{nt < \tau_n\}} R_{n,2} = \mathbf{1}_{\{\Delta_{n,\lfloor nt \rfloor} \geq 1 - \varepsilon\}} \mathbf{1}_{\{nt < \tau_n\}} R_{n,2}$. With the notation of Corollary 3.5, we have

$$\begin{aligned} \{nt < \tau_n\} \cap \{\Delta_{n,\lfloor nt \rfloor} \geq 1 - \varepsilon\} &\subset \left\{ V_t^{(n)} \geq (1 - \varepsilon) \left(n - \frac{\lfloor nt \rfloor}{\gamma} \right) n^{-1/\alpha} \right\} \\ &\subset \{n^{-1+1/\alpha} V_t^{(n)} \geq c\}, \end{aligned}$$

and then for any $\varepsilon' > 0$

$$\begin{aligned} \mathbb{P}(n^{-1+\alpha-1/\alpha} |R_{n,2}| \geq \varepsilon', nt < \tau_n) &= \mathbb{P}(\mathbf{1}_{\{\Delta_{n,\lfloor nt \rfloor} \geq 1 - \varepsilon\}} n^{-1+\alpha-1/\alpha} |R_{n,2}| \geq \varepsilon', nt < \tau_n) \\ &\leq \mathbb{P}(\Delta_{n,\lfloor nt \rfloor} \geq 1 - \varepsilon, nt < \tau_n) \\ &\leq \mathbb{P}(n^{-1+1/\alpha} V_t^{(n)} \geq c). \end{aligned}$$

Use the convergence of $V_t^{(n)}$, see Corollary 3.5, to get that the right-hand side of the last inequality converges to 0 as n goes to infinity. Then notice that $\mathbb{P}(n^{-1+\alpha-1/\alpha} |R_{n,2}| \geq \varepsilon', nt \geq \tau_n) \leq \mathbb{P}(nt \geq \tau_n)$ which converges to 0 thanks to (37).

Thus $n^{-1+\alpha-1/\alpha} R_n(t)$ converges in probability to 0. As $t \mapsto R_n(t)$ is nonnegative and nondecreasing, we conclude that

$$(42) \quad n^{-1+\alpha-1/\alpha} \sup_{0 \leq t \leq t_0} R_n(t) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

We deduce from (36), (38), (40) and (42) that

$$(43) \quad n^{-1+\alpha-1/\alpha} (\hat{L}_t^{(n)} - n^{2-\alpha} v(t)) \xrightarrow[n \rightarrow \infty]{(d)} \gamma \left[\int_0^t dr \left(\int_r^t \left(1 - \frac{s}{\gamma} \right)^{-\alpha} ds \right)^\alpha \right]^{1/\alpha} V_1.$$

To obtain (35), use (29) to get that $\gamma [\int_0^t dr (\int_r^t (1 - \frac{s}{\gamma})^{-\alpha} ds)^\alpha]^{1/\alpha} V_1$ is distributed as $\gamma \int_0^t dV_r \int_r^t (1 - \frac{s}{\gamma})^{-\alpha} ds$ which in turn is equal to $\int_0^t dr (1 - \frac{r}{\gamma})^{-\alpha} V_r$.

To get (34), thanks to (36), (38) and (42), we have to check that $n^{(\alpha-1)/2-\delta-2+\alpha} \sup_{0 \leq t \leq t_0} |J_n(t)|$ converges to 0 in probability for any $\delta > 0$. Notice that for $t \in [0, t_0]$,

$$\begin{aligned} |J_n(t)| &\leq \sum_{k=1}^{\lfloor nt \rfloor \wedge (\tau_n - 1)} \left(n - \frac{k}{\gamma}\right)^{-\gamma-1} \left| \sum_{i=1}^k \left(X_i^{(n)} - \frac{1}{\gamma}\right) \right| \\ &\leq \sum_{k=1}^{\lfloor nt_0 \rfloor \wedge (\tau_n - 1)} \left(n - \frac{k}{\gamma}\right)^{-\gamma-1} \left| \sum_{i=1}^k \left(X_i^{(n)} - \frac{1}{\gamma}\right) \right|. \end{aligned}$$

Use (41) to deduce that

$$\mathbb{E} \left[\sup_{0 \leq t \leq t_0} |J_n(t)|^2 \right]^{1/2} \leq cn^{(5-3\alpha)/2}.$$

This implies that $n^{(\alpha-1)/2-\delta-2+\alpha} \sup_{0 \leq t \leq t_0} |J_n(t)|$ converges to 0 in L^2 and thus in probability. This ends the proof of (34). \square

6. Proof of the main result. Let

$$\alpha_0 = \frac{1 + \sqrt{5}}{2}.$$

Notice that for $\alpha \in (1, \alpha_0)$, we have $-1 + \alpha - 1/\alpha < 0$, whereas for $\alpha \geq \alpha_0$, $-1 + \alpha - 1/\alpha \geq 0$. Recall $\gamma = \alpha - 1$. We define $a(t)$ for $t \in [0, \gamma]$ by

$$a(t) = \frac{v(t)}{C_0 \Gamma(2 - \alpha)} \quad \text{where } v(t) = \int_0^t \left(1 - \frac{r}{\gamma}\right)^{-\gamma} dr.$$

We also set

$$V_t^* = \frac{\alpha - 1}{C_0 \Gamma(2 - \alpha)} \int_0^t \left(1 - \frac{r}{\gamma}\right)^{-\alpha} V_r dr \quad \text{for } t \in (0, \gamma).$$

Let $x_+ = \max(x, 0)$ denote the positive part of x .

THEOREM 6.1. *Let $\alpha \in (1, 2)$. We assume that $\rho(t) = C_0 t^{-\alpha} + O(t^{-\alpha+\zeta})$ for some $C_0 > 0$ and $\zeta > 1 - 1/\alpha$.*

(i) *Let $t_0 \in [0, \gamma)$ and $\delta > 0$. We have the following convergence in probability:*

$$(44) \quad n^{-((5-3\alpha)_+/2)-\delta} \sup_{0 \leq t \leq t_0} |L_t^{(n)} - n^{2-\alpha} a(t)| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

In particular, we have $n^{-2+\alpha} L_t^{(n)} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} a(t)$ for all $t \in [0, \gamma)$.

(ii) If $\alpha \in (1, \alpha_0)$, for $t \in (0, \gamma)$, the following convergence in distribution holds:

$$(45) \quad n^{-1+\alpha-1/\alpha} (L_t^{(n)} - a(t)n^{2-\alpha}) \xrightarrow[n \rightarrow \infty]{(d)} V_t^*.$$

PROOF. First of all, let us consider the case $\alpha \in (1, \alpha_0)$. Lemma 4.1 and Chebyshev inequality imply that for $\alpha \in (1, \alpha_0)$, we have the following convergence in probability:

$$\lim_{n \rightarrow \infty} n^{-1+\alpha-1/\alpha} \sup_{t \geq 0} |L_t^{(n)} - \tilde{L}_t^{(n)}| = 0.$$

This and Lemma 4.2 imply that for $\alpha \in (1, \alpha_0)$, we have the following convergence in probability

$$\lim_{n \rightarrow \infty} n^{-1+\alpha-1/\alpha} \sup_{t \geq 0} \left| L_t^{(n)} - \frac{\hat{L}_t^{(n)}}{C_0 \Gamma(2-\alpha)} \right| = 0.$$

Then (45) and (44) for $\alpha \in (1, \alpha_0)$ are a direct consequence of Theorem 5.1.

For $\alpha \in [\alpha_0, 2)$, note that $\alpha > 3/2$ and $-1 + \alpha - 1/\alpha \geq 0$. As $\zeta > 1 - 1/\alpha$ and $\alpha \geq \alpha_0$, that is, $1 - 1/\alpha \geq 2 - \alpha$, we get $\zeta > 2 - \alpha$. We then use Lemma 4.1, Lemma 4.2 (only with $\zeta > 2 - \alpha$) and Theorem 5.1 to get (44) for $\alpha \in [\alpha_0, 2)$. \square

Let $S_t^{(n)}$ be the total number of mutations up to the $[nt]$ th coalescence, for $t \in (0, \gamma)$. Conditionally on $L_t^{(n)}$, $S_t^{(n)}$ is a Poisson r.v. with parameter $\theta L_t^{(n)}$. The next corollary is a consequence of Theorem 6.1.

COROLLARY 6.2. We assume that $\rho(t) = C_0 t^{-\alpha} + O(t^{-\alpha+\zeta})$ for some $C_0 > 0$ and $\zeta > 1 - 1/\alpha$. Let $t \in (0, \gamma)$ and G be a standard Gaussian r.v., independent of V .

(i) For $\alpha \in (1, \sqrt{2})$, we have

$$n^{-1+\alpha-1/\alpha} (S_t^{(n)} - \theta a(t)n^{2-\alpha}) \xrightarrow[n \rightarrow \infty]{(d)} \theta V_t^*.$$

(ii) For $\alpha \in (\sqrt{2}, 2)$, we have

$$n^{-1+\alpha/2} (S_t^{(n)} - \theta a(t)n^{2-\alpha}) \xrightarrow[n \rightarrow \infty]{(d)} \sqrt{\theta a(t)} G.$$

(iii) For $\alpha = \sqrt{2}$, we have $-1 + \alpha - \frac{1}{\alpha} = 1 - \frac{\alpha}{2}$ and

$$n^{-1+\alpha-1/\alpha} (S_t^{(n)} - \theta a(t)n^{2-\alpha}) \xrightarrow[n \rightarrow \infty]{(d)} \theta V_t^* + \sqrt{\theta a(t)} G.$$

PROOF. Let us compute the characteristic function $\psi_n(u, v)$ of the 2-dimensional r.v. (G_n, H_n) with

$$G_n = \frac{S_t^{(n)} - \theta L_t^{(n)}}{\sqrt{\theta a(t)n^{2-\alpha}}} \quad \text{and} \quad H_n = n^{-1+\alpha-1/\alpha}(L_t^{(n)} - a(t)n^{2-\alpha}).$$

Using that, conditionally on $L_t^{(n)}$, the law of $S_t^{(n)}$ is a Poisson distribution with parameter $\theta L_t^{(n)}$, we have

$$\psi_n(u, v) = \mathbb{E}[e^{iuG_n} e^{ivH_n}] = \mathbb{E}[e^{-\theta L_t^{(n)}(1 - e^{iu/\sqrt{\theta a(t)n^{2-\alpha}}} + iv/\sqrt{\theta a(t)n^{2-\alpha}})} e^{ivH_n}].$$

Using (i) of Theorem 6.1, we get that

$$-\theta L_t^{(n)}(1 - e^{iu/\sqrt{\theta a(t)n^{2-\alpha}}} + iv/\sqrt{\theta a(t)n^{2-\alpha}})$$

tends to $-u^2/2$ in probability and has a nonnegative real part.

We first consider the case $\alpha \in (1, \sqrt{2}]$. We have $\sqrt{2} < \alpha_0$. Hence, applying (ii) of Theorem 6.1, we get that (G_n, H_n) converges in distribution to (G, V_t^*) , where G is a standard Gaussian r.v. independent of V . Notice that

$$S_t^{(n)} = \theta a(t)n^{2-\alpha} + \theta n^{1-\alpha+1/\alpha} H_n + \sqrt{\theta a(t)}n^{1-\alpha/2} G_n$$

and then

$$n^{-1+\alpha-1/\alpha}(S_t^{(n)} - \theta a(t)n^{2-\alpha}) = \theta H_n + \sqrt{\theta a(t)}n^{\alpha/2-1/\alpha} G_n.$$

When $\alpha \leq \sqrt{2}$, notice that $\alpha/2 - 1/\alpha < 0$ (resp. = 0) if $\alpha < \sqrt{2}$ (resp. $\alpha = \sqrt{2}$). This gives (i) and (iii) of the corollary.

Now we consider the case $\alpha \in (\sqrt{2}, 2)$. We write

$$n^{-1+\alpha/2}(S_t^{(n)} - \theta a(t)n^{2-\alpha}) = \sqrt{\theta a(t)}G_n + n^{-1+\alpha/2}(L_t^{(n)} - a(t)n^{2-\alpha}).$$

We still have that G_n converges in law to G . Moreover, the convergences (45) and (44) imply that $n^{-1+\alpha/2}(L_t^{(n)} - a(t)n^{2-\alpha})$ converges to 0 in probability. This gives (ii). \square

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