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Solutions of $\Delta u = 4u^2$ with Neumann's conditions using the Brownian snake

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Abstract. We consider a Brownian snake $(W_s, s \geq 0)$ with underlying process a reflected Brownian motion in a bounded domain D. We construct a continuous additive functional $(L_s, s \geq 0)$ of the Brownian snake which counts the time spent by the end points $\hat{W}_s$ of the Brownian snake paths on ∂D. The random measure $Z = \int \delta_{\hat{W}_s} dL_s$ is supported by ∂D. Then we represent the solution v of $\Delta u = 4u^2$ in D with weak Neumann boundary condition $\varphi \geq 0$ by using exponential moment of (Z, φ) under the excursion measure of the Brownian snake. We then derive an integral equation for v. For small φ it is then possible to describe negative solution of $\Delta u = 4u^2$ in D with weak Neumann boundary condition φ.

In contrast to the exit measure of the Brownian snake out of D, the measure Z is more regular. In particular we show it is absolutely continuous with respect to the surface measure on ∂D for dimension 2 and 3.

1. Introduction

The Dirichlet problem associated to the equation $\Delta u = 4u^2$ has led to a considerable amount of work by many authors and the Brownian snake introduced by Le Gall in [13] has proved to be a powerful tool for this study. For example, in [14], Le Gall prove a representation formula for all nonnegative solutions in dimension 2, using the continuity of the density of the exit measure. The Brownian snake is a path-valued Markov process which, loosely speaking, represents a cloud of branching Brownian particles and the exit measure of the Brownian snake is a measure supported by the particles when they leave D for the first time.

In this paper we give a probabilistic representation formula for the nonnegative solution of the non linear Neumann problem in a bounded smooth domain D:

$$\begin{aligned} \Delta u &= 4u^2 \quad \text{in } D,\\ \frac{\partial u}{\partial n} + \kappa u &= \varphi \quad \text{on } \partial D, \end{aligned} \tag{1}$$

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where κ is a nonnegative continuous function on ∂D, $\partial u/\partial n(x)$ is the outward normal derivative of u at $x \in \partial D$, and φ is a nonnegative measurable function defined on ∂D.

As for the Dirichlet problem associated to $\Delta u = 4u^2$ (see [13]), we will prove that solutions of (1) can be represented using a random measure, Z, built from the Brownian snake. However, for the Neumann problem, the underlying motion will be a reflected Brownian motion in D. We expect the measure Z will play the same role as the exit measure for the Dirichlet problem and that this probabilistic representation will lead to new results for the Neumann problem, such as a trace boundary representation. With this goal in mind, it might be interesting to study the properties and regularity of Z. We give here some results in this direction (see section 8), but many interesting problems (as the regularity of the density in dimension 2 or 3) are still open.

In [1], the author considered for underlying motion of the Brownian snake a reflected Brownian motion in D killed when it reaches a fixed subset F of ∂D. Then, using a random measure built from this Brownian snake, the author represented nonnegative solution of $\Delta u = 4u^2$ with mixed Neumann-Dirichlet boundary conditions:

$$\begin{cases} \dfrac{\partial u}{\partial n} = f & \text{on } \partial D \backslash F \\ u = g & \text{on } F. \end{cases}$$

However, for technical reasons, it was not possible to consider the case $F = \emptyset$ of Neumann's boundary conditions.

Let us now present our results. We consider a Brownian snake $(W_s, s \geq 0)$ with underlying process a reflected Brownian motion in D (see [10] for a definition and properties of the Brownian snake). Let us recall that W_s is a path stopped at its lifetime ζ_s, and that for a fixed s, it is distributed according to a reflected Brownian motion in D. We define in section 3 the following continuous additive functional (CAF) of the Brownian snake:

$$L_s^\varepsilon = \int_0^s \frac{1}{\varepsilon} \mathbf{1}_{\{\hat{W}_u \in D_\varepsilon\}} \, du,$$

where $\hat{W}_s = W_s(\zeta_s)$ is the end of the path W_s and D_ε is the ε-neighborhood of ∂D in $\bar{D}$. Intuitively, as $\varepsilon \downarrow 0$, dL_s^ε converges to, say dL_s, the infinitesimal increment of the local time of the path W_s on ∂D at time ζ_s. In particular, the CAF L increases at times s such that $\hat{W}_s \in \partial D$. See lemma 3.1 for the precise statement.

Then we define the random measure Z by the formula

$$Z(dy) = \frac{1}{2} \int_0^\infty \delta_{\hat{W}_s}(dy) \, dL_s,$$

where δ_a is the Dirac mass at point a. In particular the support of Z is a subset of ∂D. Under the excursion measure, $\mathbb{N}_x$, of the Brownian snake started at point $x \in \bar{D}$, Z

is finite, but its total mass is not integrable under $\mathbb{N}_x$. We prove in proposition 6.3 that the function

$$v(x) = \mathbb{N}_x \left[1 - \mathrm{e}^{-(Z,\varphi)} \right],$$

where $(Z, \varphi) = \int \varphi(y)\ Z(dy)$ and $\varphi \geq 0$, is a nonnegative solution of (1) with $\kappa = 0$.

In section 8, we consider the properties of the measure Z. In particular, we prove in proposition 8.4 that this measure is absolutely continuous with respect to the surface measure on ∂D if the dimension of the space is 2 or 3. Let us note that the measure Z is more regular than the so-called exit measure which is singular for $d \geq 3$ (see [2]).

To study the function v, it is necessary to introduce a family of measures Z_θ which increases to Z as θ decreases to 0, and which have an integrable total mass under $\mathbb{N}_x$. The idea is to kill the underlying reflected Brownian motion $(B_t, t \geq 0)$ at time $\tau(\theta)$, where $(\tau(\theta), \theta > 0)$ is a family of random variables increasing to $+\infty$ as θ decreases to 0. The random variables $\tau(\theta)$ are independent of B and exponential with parameter θ. Let R be the right continuous inverse of $\tau(\cdot)$. R is build in such a way that it is a Markov process. Then we may consider the Brownian snake (W_s, R_s) associated to the spatial motion (B, R). Then we consider formally the measure

$$Z_\theta(dy) = \frac{1}{2} \int_0^\infty \delta_{\hat{W}_s}(dy) \mathbf{1}_{\{\theta \leq R_s(\zeta_s)\}}\ dL_s.$$

The precise definition is given by formula (5). Then it is easier to study the function

$$v_\theta(x) = \mathbb{N}_x \left[1 - \mathrm{e}^{-(Z_\theta,\varphi)} \right],$$

and deduce the properties of v since v_θ increases to v as θ decreases to 0.

In particular, using the special Markov property introduced by Le Gall in [12], we prove in section 4, proposition 4.1, that $v \in C^2(D)$ and solves

$$\Delta v = 4v^2 \quad \text{in} \quad D.$$

Section 5 is devoted to the proof of proposition 5.6, which states that v_θ is a weak solution of

$$\begin{aligned} \Delta u - 2\theta u &= 4u^2 \quad \text{in } D, \\ \frac{\partial u}{\partial n} &= \varphi \quad \text{on } \partial D. \end{aligned}$$

By letting θ decreases to 0, we get in section 6, proposition 6.3, that v is a weak solution of (1) with $\kappa = 0$.

Let l be the local time of B on ∂D. By considering a reflected Brownian motion killed when the continuous additive functional $\frac{1}{2} \int_0^t \kappa(B_u)\ dl_u$ reaches the value of an independent exponential random variable of parameter 1, instead of the initial reflected Brownian motion, the previous results can be extended to the general case κ continuous and nonnegative.

The next sections deal with the particular case $\kappa = 0$. More precisely, we prove in section 7, lemma 7.4, that a bounded function u is a weak nonnegative solution of (1) (with $\kappa = 0$), if and only if it solves the two integral equations

$$4 \int_D u(y)^2 \, dy = \int_{\partial D} \varphi(y) \, \sigma(dy),$$

where σ is the surface measure on ∂D, and

$$u(x) + 2 \int_D g(x, y) u(y)^2 \, dy - a_D \int_D u(y) \, dy = \frac{1}{2} \int_{\partial D} g(x, y) \varphi(y) \, \sigma(dy),$$

where $a_D^{-1} = \int_D dy$, and $g(x, y)$ is the green function of the reflected Brownian motion:

$$g(x, y) = \int_0^{+\infty} [p_t(x, y) - a_D] \, dt,$$

with $p_t(x, y)$ the density transition kernel of the reflected Brownian motion. Furthermore, there is a unique nonnegative weak solution of (1) (with $\kappa = 0$), thanks to corollary 7.5. Notice however, there might exist other weak solutions to (1), for example negative solutions as stated in proposition 7.2.2.

Eventually, in section 9 we recall some useful facts on reflected Brownian motion and on probabilistic representation of linear partial differential equations. This section also includes the proof of the convergence of the approximating scheme of the CAF L.

2. Notations

Let D be a bounded domain (connected open subset of $\mathbb{R}^d$) with C^3 boundary. Let $\bar{D}$ be the closure of D.

First we consider a reflected Brownian motion B in $\bar{D}$. For every $x_0 \in \bar{D}$, we denote by $\mathbb{P}_{x_0}$ its law when starting at point x_0 at time 0. Some facts on this process are recalled in the appendix.

Let us now construct a process that allow us to stop the paths according to exponential independent times of parameter θ, which must increase to $+\infty$ as θ decreases to 0. We first consider a Poisson measure $\mathcal{N}$ on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dx\, dt$, independent of B. We denote by $(x_i, t_i)_{i \in I}$ the atoms of this measure and we set

$$R(t) = \inf\{x_i; t_i \le t\},$$

with the usual convention $\inf \emptyset = +\infty$.

We set $\bar{\mathbb{R}}_+ = [0, +\infty) \cup \{+\infty\}$. The path $(R(t), t \ge 0)$ is a càdlàg decreasing $\bar{\mathbb{R}}_+$-valued process starting from $+\infty$. We have, for every $t \ge 0$ and every $\theta \ge 0$,

$$\mathbb{P}(R(t) > \theta) = \mathbb{P}(\mathcal{N}([0, \theta] \times [0, t]) = 0) = \mathrm{e}^{-\theta t}.$$

So, for every t, $R(t)$ is distributed as an exponential random variable of parameter t. Notice that $\tau(\theta) = \inf\{t \geq 0; R(t) < \theta\}$ is distributed as an exponential random variable with parameter θ. And the family of random variables $(\tau(\theta), \theta > 0)$ increases as θ decreases to 0.

Moreover, we have, for every $0 \leq s < t$,

$$R(t) = \min\left\{R(s), \inf\{x_i; s < t_i \leq t\}\right\} \overset{(d)}{=} R(s) \wedge \tilde{R}(t-s), \tag{2}$$

where $\tilde{R}$ is an independent copy of R. Consequently, R is an homogeneous Markov process.

Finally, let τ be an exponential random variable of parameter 1 independent of B and R. We denote by l the local time of B on ∂D and we set

$$(\kappa \cdot l)_t = \int_0^t \kappa(B_s) dl_s.$$

Then, the process $(\Theta_t, t \geq 0)$ defined by

$$\Theta_t = \left(B_t, R(t), \mathbf{1}_{\{\frac{1}{2}(\kappa \cdot l)_t \leq \tau\}}, (\kappa \cdot l)_t\right)$$

is an homogeneous $E = \mathbb{R}^d \times \bar{\mathbb{R}}_+ \times \{0, 1\} \times \mathbb{R}_+$-valued Markov process. Let $\mathbb{P}_{\tilde{x}_0}$ denote its law, when started at $\tilde{x}_0 \in E$ at time 0.

Let $\| \ \|$ be the Euclidean norm on $\mathbb{R}^d$. For every $r, r' \in \bar{\mathbb{R}}_+$ we set

$$\bar{d}(r, r') = \left|\arctan r - \arctan r'\right|$$

with the convention $\arctan(+\infty) = \frac{\pi}{2}$. We denote by $\delta(j, j')$ the discrete distance on $\{0, 1\}$. Eventually, for $\tilde{x} = (x, r, j, k)$ and $\tilde{y} = (y, r', j', k')$ in E, we set

$$d_E(\tilde{x}, \tilde{y}) = \|x - y\| + \bar{d}(r, r') + \delta(j, j') + |k - k'|.$$

d_E is a distance on E and (E, d_E) is a Polish space.

We now describe the Brownian snake with underlying motion Θ (see [4]). The spatial motion will correspond to the underlying reflected Brownian motion. The other three components are only used to kill the reflected paths at nice random times.

A killed path in E is a càdlàg E-valued function $\tilde{w} = (\tilde{w}(u), u \in [0, \zeta))$ where ζ is called the lifetime of $\tilde{w}$. We will denote $\tilde{w}(u) = (W(u), R(u), J(u), K(u))$ for $u \in [0, \zeta)$, and we assume that W and K are continuous. Let $\mathcal{W}$ be the set of killed paths in E. For $\tilde{x}_0 = (x_0, r_0, j_0, k_0) \in E$, let $\mathcal{W}_{\tilde{x}_0}$ be the set of killed paths starting at point $\tilde{x}_0$. For $\tilde{w} \in \mathcal{W}_{\tilde{x}_0}$, we set the end point of the path $\tilde{w}$: $(\hat{W}, \hat{R}, \hat{J}, \hat{K}) = \tilde{w}(\zeta-)$ if the limit exists, ∂ otherwise where ∂ is an isolated cemetery point added to E.

For $\tilde{w} \in \mathcal{W}$, we define the exit time of an open set $O \subset \mathbb{R}^d$ by

$$\tau_O(\tilde{w}) = \inf\{u \geq 0, W(u) \notin O\},$$

with the usual convention $\inf \emptyset = +\infty$. Notice we just consider the spatial motion W to define the exit time.

For $t \geq 0$ let $\bar{d}_t$ (resp. δ_t) be the Skorokhod distance on the space $\mathbb{D}([0,t],\bar{\mathbb{R}}_+)$ (resp. $\mathbb{D}([0,t],\{0,1\})$) of $\bar{\mathbb{R}}_+$-valued (resp. $\{0,1\}$-valued) càdlàg functions defined on $[0,t]$. Then, for $\tilde{w}$ and $\tilde{w}'$ in $\mathcal{W}$, we set

$$\begin{aligned} d(\tilde{w},\tilde{w}') = {} & d_E(w(0),w'(0)) + |\zeta - \zeta'| \\ & + \sup_{0\leq t<(\zeta\wedge\zeta')} \Big(\big\| W(t) - W'(t) \big\| + \big| K(t) - K'(t) \big| \Big) \\ & + \int_0^{\zeta\wedge\zeta'} \Big(\bar{d}_t(R_{\leq t}, R'_{\leq t}) \wedge 1 + \delta_t(J_{\leq t}, J'_{\leq t}) \wedge 1 \Big) dt, \end{aligned}$$

where $R_{\leq t}$ (resp. $J_{\leq t}$) for instance stands for the restriction of R (resp. J) to $[0,t]$. It is easy to check that d is a distance on $\mathcal{W}$ and that $(\mathcal{W},d)$ is a Polish space. We agree that very point $\tilde{x} \in E$ can be considered as a trivial killed path with lifetime $\zeta = 0$.

Let $\big(\tilde{W}_s, s \geq 0\big) = \big((W_s, R_s, J_s, K_s), s \geq 0\big)$ be the canonical process on $C(\mathbb{R}_+, \mathcal{W}_{\tilde{x}_0})$, the set of continuous functions on $[0,+\infty)$ into $\mathcal{W}_{\tilde{x}_0}$. We will denote by ζ_s the lifetime of $\tilde{W}_s$. For $\tilde{w} \in \mathcal{W}_{\tilde{x}_0}$, let $\mathrm{P}^*_{\tilde{w}}$ be the probability on $C(\mathbb{R}_+, \mathcal{W}_{\tilde{x}_0})$ under which the canonical process is a Brownian snake with underlying Markov process Θ starting at $\tilde{w}$ and constant after $\sigma = \inf\{s \geq 0; \zeta_s = 0\}$ (see [4] section 4.1). We denote by $\mathbb{N}_{\tilde{x}_0}$ the excursion measure of the Brownian snake away from the trivial path $\tilde{x}_0$ in $\mathcal{W}_{\tilde{x}_0}$ and $\sigma = \inf\{s > 0; \zeta_s = 0\}$ its duration. Recall that $(\hat{W}_s, \hat{R}_s, \hat{J}_s, \hat{K}_s)$ denote the end path of $\tilde{W}_s$ when it exists and ∂ otherwise. Eventually, we write $\mathbb{N}_{x_0} = \mathbb{N}_{(x_0,+\infty,1,0)}$ as well as $\mathcal{W}_x = \mathcal{W}_{(x,+\infty,1,0)}$.

We recall the formula for the first moment of the Brownian snake ([4]).

Let F be a nonnegative measurable function defined on $\mathcal{W}_{\tilde{x}_0}$. We have

$$\mathbb{N}_{\tilde{x}_0}\left[\int_0^\sigma F(\tilde{W}_s)\,ds\right] = \int_0^\infty ds\ \mathbb{E}_{\tilde{x}_0}[F(\Theta^{(s)})], \tag{3}$$

where $\Theta^{(s)}$ is distributed under $\mathbb{P}_{\tilde{x}_0}$ as Θ but killed at time s.

3. The additive functional L

Let us consider the continuous additive functional (CAF) of the Brownian snake defined for $\alpha \geq 0$, $\varepsilon > 0$ by: for $s \geq 0$,

$$L_s^{\alpha,\varepsilon} = \int_0^s \frac{1}{\varepsilon}\,\mathbf{1}_{\{\hat{W}_u \in D_\varepsilon\}}\,\mathrm{e}^{-\alpha\zeta_u}\ du,$$

where $D_\varepsilon = \{x \in D; \text{dist}\ (x,\partial D) < \varepsilon\}$, and dist $(x,\partial D)$ denote the Euclidean distance from x to the boundary of D.

Intuitively, as $\varepsilon \downarrow 0$, $\frac{1}{\varepsilon}\mathbf{1}_{\{\hat{W}_u\in D_\varepsilon\}}du$ converge to the infinitesimal increment of the local time on ∂D of W_u at its lifetime. The term $\mathrm{e}^{-\alpha\zeta_u}$, with $\alpha > 0$, is introduced in order to get a CAF with finite L^2 moments.

The next lemma gives the convergence of the CAF $L^{\alpha,\varepsilon}$. Let $\tilde{x}_0 = (x_0, r_0, j_0, k_0) \in E$ such that $x_0 \in D$. Recall that l is the local time of the reflected Brownian motion B on ∂D.

Lemma 3.1. *There exists a sequence* $(\varepsilon_k, k \geq 0)$ *decreasing to* 0 *such that* $\mathbb{N}_{\tilde{x}_0}$*-a.e. for all* $s \geq 0$, L_s^{0,ε_k} *converge to a limit say* L_s *as* $k \to \infty$. *The process* $(L_s, s \geq 0)$ *is a continuous additive functional of the Brownian snake. The Revuz measure of the continuous additive functional* L, μ, *defined on* $\mathcal{W}_{\tilde{x}_0}$ *is given by: for any nonnegative measurable function defined on* $\mathcal{W}_{\tilde{x}_0}$,

$$(\mu, F) = \mathbb{E}_{\tilde{x}_0}\left[\int_0^\infty F(\Theta^{(u)})\, dl_u\right],$$

where $\Theta^{(u)} = (\Theta(u'), u' \in [0, u))$. *We also have the formula*

$$\mathbb{N}_{\tilde{x}_0}\left[\int_0^\sigma F(\tilde{W}_u)\, dL_u\right] = (\mu, F). \tag{4}$$

The proof of this lemma is postponed to the appendix 9.4.

Remark. For $\alpha > 0$, the continuous additive functional $L_s^\alpha = \int_0^s \mathrm{e}^{-\alpha\zeta_u}\ dL_u$, is the limit of $L_s^{\alpha,\varepsilon_k}$ for all $s \geq 0$ $\mathbb{N}_{\tilde{x}_0}$-a.e. (see the proof of the above lemma). Its Revuz measure defined on $\mathcal{W}_{\tilde{x}_0}$ is given by $\mu_\alpha(d\tilde{w}) = \mathrm{e}^{-\alpha\zeta}\ \mu(d\tilde{w})$, where ζ is the lifetime of $\tilde{w}$. Notice that μ is not finite since $(\mu, \mathbf{1}) = +\infty$, whereas μ_α is finite (thanks to (22), (23) and (25)), and we have

$$(\mu_\alpha, \mathbf{1}) = \mathbb{E}_{x_0}\left[\int_0^\infty \mathrm{e}^{-\alpha u}\ dl_u\right] = \int_0^\infty \mathrm{e}^{-\alpha u}\ du \int_{\partial D} p_u(x_0, y)\, \sigma(dy),$$

the α-potential of the local time l, with $\sigma(dx)$ as the surface measure on ∂D. Following the terminology of [11], μ_α is of finite energy and is the measure associated to L^α. From (33), by letting ε decreases to 0, we get its energy $\mathcal{E}(\mu_\alpha)$:

$$\mathcal{E}(\mu_\alpha) = \frac{1}{2}\,\mathbb{N}_{\tilde{x}_0}\left[(L_\sigma^\alpha)^2\right] = 2\mathbb{E}_{x_0}\left[\int_0^\infty dt\ \mathrm{e}^{-\alpha t}\, u^\alpha(B_t)^2\right],$$

where $u^\alpha(y) = \mathbb{E}_y\left[\int_0^\infty \mathrm{e}^{-\alpha u}\ dl_u\right]$, for $y \in \bar{D}$, is the α-potential of the local time l. □

4. The measures Z^κ

It is clear that the measure dL increases only when $\hat{W}_u \in \partial D$. For $\theta \geq 0$, we define under $\mathbb{N}_{\tilde{x}_0}$ the random measure Z_θ^κ on ∂D. Let φ be a measurable non negative function defined on ∂D. We extend φ by setting $\varphi(\partial) = 0$. We set

$$(Z_\theta^\kappa, \varphi) = \frac{1}{2}\int_0^\sigma \varphi(\hat{W}_u)\mathbf{1}_{\{\hat{R}_u \geq \theta\}}\mathbf{1}_{\{\hat{J}_u = 1\}} dL_u. \tag{5}$$

Notice that since J_u and R_u are decreasing $\mathbb{N}_{\tilde{x}_0}$-a.e., we have, for $\theta > 0$, $Z_\theta = 0$ if $\tilde{x}_0 = (x_0, r_0, 0, k_0)$ or $\tilde{x}_0 = (x_0, 0, j_0, k_0)$. Therefore, we will be interested only in the nontrivial case where $\tilde{x}_0 = (x_0, r_0, j_0, k_0)$ with $j_0 = 1$ and $r_0 > 0$.

We shall omit the indices κ (resp. θ) in Z^κ_θ when $\kappa = 0$ (resp. $\theta = 0$). For example, we write Z for Z^0_0. Notice that (Z_θ, φ) can be represented as $\frac{1}{2}\int_0^\sigma \varphi(\hat{W}_u)$ $\mathbf{1}_{\{\hat{R}_u \geq \theta\}} dL_u$, since for $\kappa = 0$, $\mathbb{N}_{\tilde{x}_0}$-a.e. $\int_0^\sigma \mathbf{1}_{\{\hat{J}_u = 0\}} dL_u = 0$. This is clear (recall $\tilde{x}_0 = (x_0, r_0, 1, k_0)$, with $r_0 > 0$), since

$$\mathbb{N}_{\tilde{x}_0}\left[\int_0^\sigma \mathbf{1}_{\{\hat{J}_u=0\}} dL_u\right] = \mathbb{E}_{(x_0,r_0,1,k_0)}\left[\int_0^\infty \mathbf{1}_{\{(\kappa\cdot l)_u > 2\tau\}} dl_u\right] = 0,$$

for $\kappa = 0$ and $(\kappa \cdot l)_0 = k_0 \leq 2\tau$.

For any $\kappa \geq 0$, $(Z^\kappa_\theta, \varphi)$ increases to (Z^κ, φ) as θ decreases to 0 since $\int_0^\sigma \mathbf{1}_{\{\hat{R}_u=0\}} dL_u = 0$ $\mathbb{N}_{\tilde{x}_0}$-a.e. (recall $\tilde{x}_0 = (x_0, r_0, 1, k_0)$ with $r_0 > 0$). Therefore for $\varphi \geq 0$, we have

$$(Z, \varphi) \geq (Z^\kappa, \varphi) \geq (Z^\kappa_\theta, \varphi) \geq 0.$$

We consider the function v^κ_θ defined on $\bar{D}$ by:

$$v^\kappa_\theta(x) = \mathbb{N}_x[1 - \mathrm{e}^{-(Z^\kappa_\theta, \varphi)}].$$

We shall omit the indices κ or θ when they are zero. For example, we write

$$v(x) = \mathbb{N}_x[1 - \mathrm{e}^{-(Z,\varphi)}].$$

Since the support of Z^κ_θ is a subset of ∂D, we deduce that $1 - \mathrm{e}^{-(Z_\theta,\varphi)}$ is bounded from above by $\mathbf{1}_{\{\mathcal{R}^D \cap \partial D \neq \emptyset\}}$, where $\mathcal{R}^O$, with O an open subset of D, is the range of (the spatial component of) the Brownian snake in O, that is

$$\mathcal{R}^O = \{W_s(t \wedge \tau_O(\tilde{W}_s)), s \geq 0, t \geq 0\}.$$

In particular for $x \in D$, $v^\kappa_\theta(x)$ is bounded from above by $u_D(x) = \mathbb{N}_x[\mathbf{1}_{\{\mathcal{R}^D \cap \partial D \neq \emptyset\}}]$. Notice that u_D is the maximal nonnegative solution in D of $\Delta u = 4u^2$. This is a consequence of proposition 4.4 in [11] and the fact that the law of B stopped when it first reaches ∂D is the law of a Brownian motion stopped when it first reaches ∂D. From the monotone convergence theorem, we deduce that $v^\kappa_\theta(x) \uparrow v^\kappa(x)$ as $\theta \downarrow 0$ for any $x \in D$.

Proposition 4.1. *Let φ be a measurable nonnegative function defined on ∂D. The function $v^\kappa(x) = \mathbb{N}_x[1 - \mathrm{e}^{-(Z^\kappa,\varphi)}]$ defined on D is a nonnegative solution of $\Delta u = 4u^2$ in D.*

We first recall some results on exit measures.

Let O be an open subset of D. Let $\Omega_O = O \times \bar{\mathbb{R}}_+ \times \{0, 1\} \times \mathbb{R}^+$ and $\tilde{x}_0 \in \Omega_O$. As in [13], let $X^{\Omega_O}(d\tilde{x})$ be the exit measure of the Brownian snake $\tilde{W}$ out of Ω_O under the excursion measure (notice condition (H) is satisfied here). We also define the σ-field $\mathcal{E}^{\Omega_O}$ which is intuitively generated by the paths $\tilde{W}_s$ up to their exit time of O. More precisely, let

$$\eta_s = \inf\left\{s'; \int_0^{s'} \mathbf{1}_{\{\zeta_u \leq \tau_O(\tilde{W}_u)\}}\, du > s\right\}.$$

and define the process $\tilde{W}'_s = \tilde{W}_{\eta_s}$ under $\mathbb{N}_{\tilde{x}_0}$. The σ-field $\mathcal{E}^{\Omega_O}$ is generated by $\tilde{W}'$ and the collection of all $\mathbb{N}_{\tilde{x}_0}$-negligible set of $C(\mathbb{R}^+, \mathcal{W}_{\tilde{x}_0})$.

Now we describe the excursion of $\tilde{W}$ outside Ω_O. The random open set $\{s \in [0, \sigma], \tau_O(\tilde{W}_s) < \zeta_s\}$ can be written as a countable union of disjoint open intervals (a_i, b_i), $i \in I$, where I is a set of indices possibly empty. Because of the property of the Brownian snake, notice that for $s \in [a_i, b_i]$, $\tau_O(\tilde{W}_s)$ and $\tilde{W}_s(\tau_O(\tilde{W}_s))$ are constant equal to $t_i = \zeta_{a_i}$ and $\tilde{x}_i = \hat{\tilde{W}}_{a_i}$. We then define the excursion $\tilde{W}^i$ outside Ω_O as an element of $C(\mathbb{R}^+, \mathcal{W}_{\tilde{x}_i})$ by

$$\tilde{W}^i_s(t) = \tilde{W}_{(a_i+s)\wedge b_i}(t + t_i), \quad t \in [0, \zeta^i_s = \zeta_{(a_i+s)\wedge b_i} - t_i).$$

We recall theorem 2.4 of [13] (see also proposition 7 of [4]):

Theorem 4.2 (Le Gall). *Conditionally on $\mathcal{E}^{\Omega_O}$, the point measure $\sum_{i\in I} \delta_{\tilde{W}^i}$ is under $\mathbb{N}_{\tilde{x}_0}$ a Poisson measure with intensity $\int X^{\Omega_O}(d\tilde{x})\, \mathbb{N}_{\tilde{x}}(\cdot)$.*

Proof of proposition 4.1. Let O and Q be open subsets of $\mathbb{R}^d$ such that $\bar{O} \subset Q$ and $\bar{Q} \subset D$. The necessity of Q will appear later. There exists $\varepsilon_0 > 0$, such that $\bar{Q} \cap D_{\varepsilon_0} = \emptyset$. Let $\tilde{x}_0 \in \Omega_O$. Let φ be a nonnegative continuous function defined on $\bar{D}$. We set

$$(Z^{\kappa,\varepsilon}_\theta, \varphi) = \frac{1}{2}\int_0^\sigma \varphi(\hat{W}_u)\mathbf{1}_{\{\hat{R}_u\geq\theta\}}\mathbf{1}_{\{\hat{J}_u=1\}}dL^{0,\varepsilon}_u,$$

where we recall that $dL^{0,\varepsilon}_u = \frac{1}{\varepsilon}\mathbf{1}_{\{\hat{W}_u\in D_\varepsilon\}}\, du$.

With obvious notations, we have under $\mathbb{N}_{\tilde{x}_0}$: for any $\varepsilon \in (0, \varepsilon_0)$ and $\theta > 0$,

$$(Z^{\kappa,\varepsilon}_\theta, \varphi) = \sum_{i\in I}\frac{1}{2}\int_0^\infty \varphi(\hat{W}^i_s)\mathbf{1}_{\{\hat{R}^i_u\geq\theta\}}\mathbf{1}_{\{\hat{J}^i_u=1\}}\frac{1}{\varepsilon}\mathbf{1}_{\{\hat{W}^i_s\in D_\varepsilon\}}\, ds.$$

We deduce from theorem 4.2 that for any $\tilde{x}_0 \in \Omega_O$,

$$\mathbb{N}_{\tilde{x}_0}\left(\mathrm{e}^{-(Z^{\kappa,\varepsilon}_\theta, \varphi)}\,|\mathcal{E}^{\Omega_O}\right) = \exp\left[-\int X^{\Omega_O}(d\tilde{x})\mathbb{N}_{\tilde{x}}\left(1-\mathrm{e}^{-(Z^{\kappa,\varepsilon}_\theta, \varphi)}\right)\right].$$

We will now prove that the law of $Z^{\kappa,\varepsilon}_\theta$ under $\mathbb{N}_{\tilde{x}}$ is the law of $\mathbf{1}_{\{r>\theta,j=1\}}Z^{\kappa,\varepsilon}_\theta$ under $\mathbb{N}_x$, where $\tilde{x} = (x, r, j, k)$. Notice from the Markov property of Θ that $\tilde{W}_s = (W_s, R_s, J_s, K_s)$ under $\mathbb{N}_{(x,r,j,k)}$ is distributed as $\tilde{W}'_s = (W_s, \min\{R_s, r\}, jJ_s, k + K_s)$ under $\mathbb{N}_{(x,+\infty,1,0)} = \mathbb{N}_x$. In particular,

$$Z^{\kappa,\varepsilon}_\theta(dy) = \frac{1}{2}\int_0^\infty \delta_{\hat{W}_u}(dy)\mathbf{1}_{\{\hat{R}_u\geq\theta\}}\mathbf{1}_{\{\hat{J}_u=1\}}\frac{1}{\varepsilon}\mathbf{1}_{\{\hat{W}_u\in D_\varepsilon\}}\, du,$$

under $\mathbb{N}_{\tilde{x}}$ is distributed as $Z'^{\kappa,\varepsilon}_\theta$ under $\mathbb{N}_x$, where,

$$Z'^{\kappa,\varepsilon}_\theta(dy) = \frac{1}{2}\int_0^\infty \delta_{\hat{W}'_u}(dy)\mathbf{1}_{\{\hat{R}'_u\geq\theta\}}\mathbf{1}_{\{\hat{J}'_u=1\}}\frac{1}{\varepsilon}\mathbf{1}_{\{\hat{W}'_u\in D_\varepsilon\}}\, du,$$

with $(\hat{W}'_u, \hat{R}'_u, \hat{J}'_u, \hat{K}'_u)$ the end point of the path $\tilde{W}'_u$. We have under $\mathbb{N}_x$,

$$Z'^{\kappa,\varepsilon}_\theta = \frac{1}{2}\int_0^\infty \delta_{\hat{W}_u} \mathbf{1}_{\{\min(\hat{R}_u, r)\geq\theta\}} \mathbf{1}_{\{j\hat{J}_u=1\}} \frac{1}{\varepsilon} \mathbf{1}_{\{\hat{W}_u\in D_\varepsilon\}}\, du = \mathbf{1}_{\{r\geq\theta, j=1\}} Z^{\kappa,\varepsilon}_\theta. \quad (6)$$

We deduce that for $\tilde{x} = (x, r, j, k)$, and either $r > 0$ or $\theta > 0$, we have

$$\mathbb{N}_{\tilde{x}}\left[1 - \mathrm{e}^{-(Z^{\kappa,\varepsilon}_\theta,\, \varphi)}\right] = \mathbf{1}_{\{r\geq\theta, j=1\}}\mathbb{N}_x\left[1 - \mathrm{e}^{-(Z^{\kappa,\varepsilon}_\theta,\, \varphi)}\right].$$

Remark 4.3. Notice that $1-\exp[-(Z^{\kappa,\varepsilon}_\theta, \varphi)] \leq \mathbf{1}_{\{\mathcal{R}^Q\cap\partial Q\neq\emptyset\}}$, and $\mathbb{N}_{\tilde{x}}[\mathbf{1}_{\{\mathcal{R}^Q\cap\partial Q\neq\emptyset\}}] = u_Q(x)$ is uniformly bounded on O. In particular we get from dominated convergence as $\varepsilon \downarrow 0$ (along the sequence $(\varepsilon_k, k \geq 1)$ of lemma 3.1), that

$$\mathbb{N}_{\tilde{x}}\left[1 - \mathrm{e}^{-(Z^{\kappa}_\theta,\, \varphi)}\right] = \mathbf{1}_{\{r\geq\theta, j=1\}}\mathbb{N}_x\left[1 - \mathrm{e}^{-(Z^{\kappa}_\theta,\, \varphi)}\right]. \quad (7)$$

Therefore, we get

$$\mathbb{N}_{\tilde{x}_0}\left(\mathrm{e}^{-(Z^{\kappa,\varepsilon}_\theta,\, \varphi)}\,|\mathcal{E}^{\Omega_O}\right) = \exp\left[-\int X^{\Omega_O}(dx, dr, dj, dk)\mathbf{1}_{\{r\geq\theta, j=1\}}\mathbb{N}_x \times\left(1 - \mathrm{e}^{-(Z^{\kappa,\varepsilon}_\theta,\, \varphi)}\right)\right].$$

Arguing as in the above remark, and letting θ decreases to 0, we also have, from dominated convergence, that

$$\mathbb{N}_{\tilde{x}_0}\left(\mathrm{e}^{-(Z^{\kappa},\, \varphi)}\,|\mathcal{E}^{\Omega_O}\right) = \exp\left[-\int X^{\Omega_O}(dx, dr, dj, dk)\mathbf{1}_{\{r>0, j=1\}}\mathbb{N}_x \times\left(1 - \mathrm{e}^{-(Z^{\kappa},\, \varphi)}\right)\right].$$

Using formula (36) from [4], we deduce that

$$\mathbb{N}_{\tilde{x}_0}\left[\int X^{\Omega_O}(dx, dr, dj, dk)(1 - \mathbf{1}_{\{r>0, j=1\}})\right] = \mathbb{E}_{\tilde{x}_0}\left[1 - \mathbf{1}_{\{R(T)>0, (\kappa\cdot l)_T\geq 2\tau\}}\right],$$

where T is the exit time for B of O. Recall that $\tilde{x}_0 = (x_0, r_0, j_0, k_0)$ is such that $r_0 > 0$ and $j_0 = 1$. Since $x_0 \in \bar{O} \subset D$, we deduce that the local time l and also $(\kappa\cdot l)$ didn't increase before T. Therefore a.s. $(\kappa\cdot l)_T = (\kappa\cdot l)_0 = k_0 < 2\tau$, where we use that $j_0 = 1$ for the last inequality. Since T is finite a.s., we deduce that $R(T) > 0$ a.s. Hence we get that $\mathbb{N}_{\tilde{x}_0}$-a.e. $\int X^{\Omega_O}(dx, dr, dj, dk)(1 - \mathbf{1}_{\{r>0, j=1\}}) = 0$. This implies that for any $x_0 \in O$,

$$\mathbb{N}_{\tilde{x}_0}\left(\mathrm{e}^{-(Z^{\kappa},\, \varphi)}\,|\mathcal{E}^{\Omega_O}\right) = \exp\left[-\int X^O(dx)\mathbb{N}_x\left(1 - \mathrm{e}^{-(Z^{\kappa},\, \varphi)}\right)\right],$$

where $X^O(dx) = X^{\Omega_O}(dx, \bar{\mathbb{R}}^+, \{0, 1\}, \mathbb{R}^+)$.

From class monotone theorem, we deduce this equality is true for any measurable nonnegative function φ defined on ∂D. Set $r_0 = +\infty$, $j_0 = 1$ and $k_0 = 0$

and take the expectation with respect to $\mathbb{N}_{x_0}$, to deduce that $v^\kappa(x_0) = \mathbb{N}_{x_0}[1 - \mathrm{e}^{-(Z^\kappa, \varphi)}]$ is bounded in O and satisfies: for any $x_0 \in O$,

$$v^\kappa(x_0) = \mathbb{N}_{x_0}[1 - \mathrm{e}^{-(X^O, v^\kappa)}].$$

But, under $\mathbb{N}_{x_0}$, X^O is distributed as the exit measure of O of the Brownian snake with underlying motion a Brownian notion started at x_0. Since O is arbitrary (but for $\bar{O} \subset D$), we deduce from corollary 4.3 of [12] that v^κ is a nonnegative solution of $\Delta u = 4u^2$ in D. □

5. Properties of v_θ^κ for $\theta > 0$

Let φ be a bounded nonnegative measurable function defined on ∂D. The same ideas as in [1] lead to the equation satisfied by v_θ^κ. We assume in this section that $\theta > 0$.

Proposition 5.1. *The function v_θ^κ is bounded on $\bar{D}$.*

Proof. By definition, we know that v_θ^κ is non negative. To get the upper bound, for every $x \in \bar{D}$, we have from (5) and (4)

$$\begin{aligned} v_\theta^\kappa(x) &\le \mathbb{N}_x\big[(Z_\theta^\kappa, \varphi)\big] \\ &= \frac{1}{2}\mathbb{N}_x\left[\int_0^{+\infty} \varphi(\hat{W}_u)\mathbf{1}_{\{\hat{R}_u \ge \theta\}}\mathbf{1}_{\{\hat{J}_i = 1\}} dL_u\right] \\ &= \frac{1}{2}\mathbb{E}_x\left[\int_0^{+\infty} \varphi(B_u)\mathbf{1}_{\{R(u)\ge\theta\}}\mathbf{1}_{\{\frac{1}{2}(\kappa\cdot l)_u \le \tau\}} dl_u\right] \\ &= \frac{1}{2}\,\mathbb{E}_x\left[\int_0^{+\infty} \varphi(B_u)\,\mathrm{e}^{-\theta u}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_u}\; dl_u\right]. \end{aligned}$$

This last quantity is bounded since φ is bounded on ∂D, $\theta > 0$, and since

$$\sup_{x\in\bar{D}} \mathbb{E}_x\left[\int_0^{+\infty} \mathrm{e}^{-\theta s}\, dl_s\right] < +\infty,$$

thanks to (24). □

Proposition 5.2. *The function v_θ^κ is solution of the integral equation: for all $x \in \bar{D}$,*

$$\begin{aligned} &v_\theta^\kappa(x) + 2\mathbb{E}_x\left[\int_0^{+\infty} v_\theta^\kappa(B_s)^2\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\; ds\right] \\ &\quad = \frac{1}{2}\,\mathbb{E}_x\left[\int_0^{+\infty} \varphi(B_s)\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\; dl_s\right]. \end{aligned} \tag{8}$$

Proof. We follow the proof of theorem 4.2 of [1]. By definition of v_θ^κ and Z_θ^κ, we have, for every $x \in \partial D$,

$$\begin{aligned} v_\theta^\kappa(x) &= \mathbb{N}_x\left[1 - \mathrm{e}^{-(Z_\theta^\kappa,\varphi)}\right] \\ &= \mathbb{N}_x\left[1 - \exp\left(-\frac{1}{2}\int_0^{+\infty} \varphi(\hat{W}_s)\mathbf{1}_{\{\hat{R}_s\ge\theta\}}\mathbf{1}_{\{\hat{J}_s=1\}} dL_s\right)\right] \\ &= \frac{1}{2}\mathbb{N}_x\left[\int_0^{+\infty} dL_s\, \varphi(\hat{W}_s)\mathbf{1}_{\{\hat{R}_s\ge\theta\}}\mathbf{1}_{\{\hat{J}_s=1\}}\right. \\ &\qquad \left.\times \exp\left(-\frac{1}{2}\int_s^{+\infty} dL_u\, \varphi(\hat{W}_u)\mathbf{1}_{\{\hat{R}_u\ge\theta\}}\mathbf{1}_{\{\hat{J}_u=1\}}\right)\right]. \end{aligned}$$

Let us recall $\mathrm{P}^*_{\tilde{w}}$ is the law of the Brownian snake started at $\tilde{w}$ and killed when its lifetime reaches 0 (see the end of section 2). We denote by $\mathrm{E}^*_{\tilde{w}}$ the expectation with respect to $\mathrm{P}^*_{\tilde{w}}$. Now, we replace

$$\exp\left(-\frac{1}{2}\int_s^{+\infty} dL_u\, \varphi(\hat{W}_u)\mathbf{1}_{\{\hat{R}_u\ge\theta\}}\mathbf{1}_{\{\hat{J}_u=1\}}\right)$$

by its predictable projection to get

$$\begin{aligned} v_\theta^\kappa &= \frac{1}{2}\mathbb{N}_x\left[\int_0^{+\infty} dL_s\, \varphi(\hat{W}_s)\mathbf{1}_{\{\hat{R}_s\ge\theta\}}\mathbf{1}_{\{\hat{J}_s=1\}}\right. \\ &\qquad \left.\times\, \mathrm{E}^*_{\tilde{W}_s}\left[\mathrm{e}^{\left(-\frac{1}{2}\int_0^{+\infty} dL_u\, \varphi(\hat{W}_u)\mathbf{1}_{\{\hat{R}_u\ge\theta\}}\mathbf{1}_{\{\hat{J}_u=1\}}\right)}\right]\right]. \end{aligned}$$

Let us now compute, for $\tilde{w} = (W, R, J, K) \in \mathcal{W}_x$,

$$\mathrm{E}^*_{\tilde{W}_s}\left[\mathrm{e}^{-\frac{1}{2}\int_0^{+\infty} dL_u\, \varphi(\hat{W}_u)\mathbf{1}_{\{\hat{R}_u\ge\theta\}}\mathbf{1}_{\{\hat{J}_u=1\}}}\right] = \mathrm{E}^*_{\tilde{W}_s}\left[\mathrm{e}^{-(Z_\theta^\kappa,\varphi)}\right].$$

We consider the Brownian snake under $\mathrm{P}^*_{\tilde{w}}$ and we set $(\alpha_i, \beta_i)_{i\in I}$ the excursion intervals of $\zeta_s - \inf_{[0,s]}\zeta_r$ above 0. For every $i \in I$, we define $\tilde{W}^i \in \mathbb{C}(\mathbb{R}_+, \mathcal{W}_{\zeta_{\alpha_i}, \tilde{w}(\zeta_{\alpha_i})})$ by setting, for every $s \ge 0$,

$$\tilde{W}^i_s(t) = \tilde{W}_{(\alpha_i+s)\wedge\beta_i}(\zeta_{\alpha_i} + t) \quad t \in [0, \zeta^i_s = \zeta_{(\alpha_i+s)\wedge\beta_i} - \zeta_{\alpha_i}).$$

We have

$$(Z_\theta^\kappa, \varphi) = \sum_{i\in I}\frac{1}{2}\int_0^\infty \varphi(\hat{W}^i_u)\mathbf{1}_{\{\hat{R}^i_u\ge\theta\}}\mathbf{1}_{\{\hat{J}^i_u=1\}}\, dL^i_u,$$

where L^i is the CAF of lemma 3.1 for the snake $\tilde{W}^i$. Let us recall the Poissonian representation of the Brownian snake (proposition 2.5 of [12]). For $\tilde{w} \in \mathcal{W}_x$, the point measure

$$\sum_{i\in I}\delta_{(\zeta_{\alpha_i}, \tilde{W}^i)}$$

is under $\mathrm{P}^*_{\tilde{w}}$ a Poisson point measure with intensity

$$2\,\mathbf{1}_{[0,\zeta)}(t)dt\,\mathbb{N}_{\tilde{w}(t)}(dW).$$

Thus we have

$$\begin{aligned}
\mathrm{E}^*_{\tilde{w}}\left[\mathrm{e}^{-(Z^\kappa_\theta,\varphi)}\right] &= \exp\left(-2\int_0^\zeta dt\,\mathbb{N}_{\tilde{w}(t)}\left[1-\mathrm{e}^{-(Z^\kappa_\theta,\varphi)}\right]\right)\\
&= \exp\left(-2\int_0^\zeta dt\,\mathbf{1}_{\{R(t)\geq\theta,\,J(t)=1\}}\mathbb{N}_{W(t)}\left[1-\mathrm{e}^{-(Z^\kappa_\theta,\varphi)}\right]\right)\\
&= \exp\left(-2\int_0^\zeta dt\,\mathbf{1}_{\{R(t)\geq\theta,\,J(t)=1\}}v^\kappa_\theta(W(t))\right),
\end{aligned}$$

with $\tilde{w}(t)=(W(t),R(t),J(t),K(t))$ where we used equation (7) for the second equality. Since the processes R_s and J_s are decreasing, we have

$$v^\kappa_\theta(x)=\frac{1}{2}\mathbb{N}_x\left[\int_0^{+\infty}dL_s\,\varphi(\hat{W}_s)\mathbf{1}_{\{\hat{R}_s\geq\theta\}}\mathbf{1}_{\{\hat{J}_s=1\}}\exp\left(-2\int_0^{\zeta_s}dt\,v^\kappa_\theta\big(W_s(t)\big)\right)\right].$$

Eventually we get, using equation (4), the equation

$$\begin{aligned}
v^\kappa_\theta(x) &= \frac{1}{2}\mathbb{E}_x\left[\int_0^{+\infty}dl_s\,\varphi(B_s)\mathbf{1}_{\{R(s)\geq\theta\}}\mathbf{1}_{\{(\kappa\cdot l)_s\leq 2\tau\}}\exp\left(-2\int_0^s dt\,v^\kappa_\theta\big(B_t\big)\right)\right]\\
&= \frac{1}{2}\mathbb{E}_x\left[\int_0^{+\infty}dl_s\,\varphi(B_s)\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\exp\left(-2\int_0^s dt\,v^\kappa_\theta(B_t)\right)\right] \qquad (9)
\end{aligned}$$

that we will re-use at the end of the proof.

Let us now compute

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^{+\infty}dl_s\,\varphi(B_s)\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\left(1-\exp\left(-2\int_0^s dt\,v^\kappa_\theta(B_t)\right)\right)\right]\\
&= 2\,\mathbb{E}_x\left[\int_0^{+\infty}dl_s\,\varphi(B_s)\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\right.\\
&\qquad\left.\times\int_0^s dt\,v^\kappa_\theta(B_t)\exp\left(-2\int_t^s du\,v^\kappa_\theta(B_u)\right)\right]\\
&= 2\int_0^{+\infty}dt\,\mathbb{E}_x\left[v^\kappa_\theta(B_t)\int_t^{+\infty}dl_s\,\varphi(B_s)\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\right.\\
&\qquad\left.\times\exp\left(-2\int_t^s du\,v^\kappa_\theta(B_u)\right)\right]\\
&= 2\int_0^{+\infty}dt\,\mathbb{E}_x\left[v^\kappa_\theta(B_t)\,\mathrm{e}^{-\theta t}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_t}\,\mathbb{E}_{B_t}\right.\\
&\qquad\left.\times\left[\int_0^{+\infty}dl_s\,\varphi(B_s)\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,\mathrm{e}^{\left(-2\int_0^s du\,v^\kappa_\theta(B_u)\right)}\right]\right]\\
&= 4\int_0^{+\infty}dt\,\mathbb{E}_x\left[v^\kappa_\theta(B_t)^2\,\mathrm{e}^{-\theta t}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_t}\right]
\end{aligned}$$

by equation (9). Now, if we rewrite equation (9) as

$$
\begin{aligned}
v_\theta^\kappa(x) = \frac{1}{2}\mathbb{E}_x\left[\int_0^{+\infty} dl_s\, \varphi(B_s)\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\right] \\
-\frac{1}{2}\mathbb{E}_x\left[\int_0^{+\infty} dl_s\, \varphi(B_s)\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\left(1-\exp\left(-2\int_0^s dt\, v_\theta^\kappa(B_t)\right)\right)\right],
\end{aligned}
$$

the last computation gives the sought-after equation. □

Proposition 5.3. *Let T be a stopping time (with respect to the natural filtration of B), finite a.s. Then v_θ^κ satisfies the equation: for all $x \in \bar{D}$,*

$$
\begin{aligned}
&v_\theta^\kappa(x) + 2\mathbb{E}_x\left[\int_0^T v_\theta^\kappa(B_s)^2\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,ds\right] \\
&= \mathbb{E}_x\left[v_\theta^\kappa(B_T)\,\mathrm{e}^{-\theta T}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_T}\right] + \frac{1}{2}\mathbb{E}_x\left[\int_0^T \varphi(B_s)\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,dl_s\right].
\end{aligned}
$$

Proof. Let us first compute

$$
\begin{aligned}
&\mathbb{E}_x\left[\int_0^{+\infty} v_\theta^\kappa(B_s)^2\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,ds\right] \\
&= \mathbb{E}_x\left[\int_0^T v_\theta^\kappa(B_s)^2\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,ds\right] \\
&\quad +\mathbb{E}_x\left[\int_T^{+\infty} v_\theta^\kappa(B_s)^2\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,ds\right] \\
&= \mathbb{E}_x\left[\int_0^T v_\theta^\kappa(B_s)^2\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,ds\right] \\
&\quad +\mathbb{E}_x\left[\mathrm{e}^{-\theta T}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_T}\,\mathbb{E}_{B_T}\left[\int_0^{+\infty} v_\theta^\kappa(B_s)^2\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,ds\right]\right],
\end{aligned}
$$

by the strong Markov property of B. Now, by proposition 5.2, we have,

$$
\begin{aligned}
&\mathbb{E}_{B_T}\left[\int_0^{+\infty} v_\theta^\kappa(B_s)^2\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,ds\right] \\
&= \frac{1}{4}\mathbb{E}_{B_T}\left[\int_0^{+\infty} \varphi(B_s)\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,dl_s\right] - \frac{1}{2}v_\theta^\kappa(B_T).
\end{aligned}
$$

So, plugging this equality into the previous formula gives

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^{+\infty} v_\theta^\kappa(B_s)^2\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,ds\right]\\
&= \mathbb{E}_x\left[\int_0^{T} v_\theta^\kappa(B_s)^2\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,ds\right]\\
&\quad+\frac{1}{4}\mathbb{E}_x\left[\mathrm{e}^{-\theta T}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_T}\,\mathbb{E}_{B_T}\left[\int_0^{+\infty}\varphi(B_s)\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,dl_s\right]\right]\\
&\quad-\frac{1}{2}\mathbb{E}_x\left[\mathrm{e}^{-\theta T}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_T}\,v_\theta^\kappa(B_T)\right]\\
&= \mathbb{E}_x\left[\int_0^{T} v_\theta^\kappa(B_s)^2\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,ds\right]\\
&\quad+\frac{1}{4}\mathbb{E}_x\left[\int_T^{+\infty}\varphi(B_s)\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,dl_s\right]\\
&\quad-\frac{1}{2}\mathbb{E}_x\left[\mathrm{e}^{-\theta T}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_T}\,v_\theta^\kappa(B_T)\right],
\end{aligned}$$

using the strong Markov property again. Subtracting to (8) two times the last equation gives the result. □

Corollary 5.4. *The function v_θ^κ belongs to $C^2(D)$ and is solution of $\Delta u = 4u^2 + 2\theta u$ on D.*

Proof. Let $x \in D$. As D is an open subset, there exists $\varepsilon > 0$ such that the ball $B(x,\varepsilon)$ centered at x and of radius ε is included in D. Let T be the exit time of B out of this ball. Then, under $\mathbb{P}_x$, $(B_u)_{0\le u\le T}$ is a standard Brownian motion stopped when it leaves $B(x,\varepsilon)$ and $l_T = 0$ $\mathbb{P}_x$-a.s. Proposition 5.3 gives now

$$v_\theta^\kappa(x) + 2\mathbb{E}_x\left[\int_0^T v_\theta^\kappa(B_s)^2\,\mathrm{e}^{-\theta s}\,ds\right] = \mathbb{E}_x\left[v_\theta(B_T)\,\mathrm{e}^{-\theta T}\right]$$

where B is a standard Brownian motion and classical results on the Brownian motion give the proposition. □

Proposition 5.5. *The function v_θ^κ is continuous on $\bar{D}$.*

Proof. We fix a time $t > 0$ and we apply proposition 5.3 to $T = t$. We have

$$\begin{aligned}
v_\theta^\kappa(x) = \mathbb{E}_x\left[v_\theta^\kappa(B_t)\right]\mathrm{e}^{-\theta t} &-\mathbb{E}_x\left[v_\theta^\kappa(B_t)\left(1-\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_t}\right)\right]\mathrm{e}^{-\theta t}\\
&+\frac{1}{2}\mathbb{E}_x\left[\int_0^t\varphi(B_s)\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,dl_s\right]\\
&-2\mathbb{E}_x\left[\int_0^t v_\theta^\kappa(B_s)^2\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,ds\right].
\end{aligned}$$

As φ and v_θ^κ are bounded, the three last terms converge to 0 uniformly in x and, as t decreases to 0 thanks to (26) with $n = 1$. Furthermore for fixed $t > 0$, the application $x \mapsto \mathbb{E}_x\left[v_\theta^\kappa(B_t)\right]$ is continuous on $\bar{D}$. This implies that v_θ^κ is continuous on $\bar{D}$. □

Let $C_b^2(D)$ be the set of bounded functions defined on D which are of class C^2 with bounded derivatives of order 1 and 2.

Proposition 5.6. *If φ is continuous on ∂D, then, for every test function $\phi \in C_b^2(D) \cap C^1(\bar{D})$, we have*

$$\begin{aligned}&\int_D \Delta\phi(x) v_\theta^\kappa(x)dx - 4\int_D \phi(x)v_\theta^\kappa(x)^2dx - 2\theta\int_D \phi(x)v_\theta^\kappa(x)dx\\ &= \int_{\partial D}\frac{\partial\phi}{\partial n}(y)v_\theta^\kappa(y)\sigma(dy) - \int_{\partial D}\phi(y)\varphi(y)\sigma(dy) - \int_{\partial D}\phi(y)v_\theta^\kappa(y)\kappa(y)\sigma(dy).\end{aligned}$$

In particular, for $\phi = 1$, we have

$$4\int_D v_\theta^\kappa(x)^2dx + 2\theta\int_D v_\theta^\kappa(x)dx = \int_{\partial D}\varphi(y)\sigma(dy) + \int_{\partial D}\kappa(y)v_\theta^\kappa(y)\sigma(dy).$$

Proof. The proof is similar to the proof of theorem 4.10 of [1]. First, we use the definition of the reflected Brownian motion via a martingale problem (see [9]). This gives that, for every $x \in \bar{D}t > 0$, and $\phi \in C_b^2(D) \cap C^1(\bar{D})$,

$$\mathbb{E}_x\big[\phi(B_t) - \phi(x)\big] = \frac{1}{2}\mathbb{E}_x\left[\int_0^t \Delta\phi(B_s)ds\right] - \frac{1}{2}\mathbb{E}_x\left[\int_0^t \frac{\partial\phi}{\partial n}(B_s)dl_s\right].$$

So, multiplying by $v_\theta^\kappa(x)$ and integrating on D leads to, for every $t > 0$,

$$\begin{aligned}&\int_D v_\theta^\kappa(x)\mathbb{E}_x\big[\phi(B_t) - \phi(x)\big]dx\\ &= \frac{1}{2}\int_0^t ds\int_D v_\theta^\kappa(x)\mathbb{E}_x\big[\Delta\phi(B_s)\big]dx - \frac{1}{2}\int_D v_\theta^\kappa(x)\mathbb{E}_x\left[\int_0^t \frac{\partial\phi}{\partial n}(B_s)dl_s\right]dx\\ &= \frac{1}{2}\int_0^t ds\int_D \Delta\phi(x)\mathbb{E}_x\big[v_\theta^\kappa(B_s)\big]dx - \frac{1}{2}\int_D v_\theta^\kappa(x)\mathbb{E}_x\left[\int_0^t \frac{\partial\phi}{\partial n}(B_s)dl_s\right]dx\end{aligned}$$

because of the symmetry of the density of B_s. Using the symmetry again and then proposition 5.3 with $T = t$, we have

$$\begin{aligned}&\int_D v_\theta^\kappa(x)\mathbb{E}_x\big[\phi(B_t) - \phi(x)\big]dx\\ &\quad= \int_D \phi(x)\mathbb{E}_x\big[v_\theta^\kappa(B_t) - v_\theta^\kappa(x)\big]dx\\ &\quad= \int_D \phi(x)\mathbb{E}_x\left[v_\theta^\kappa(B_t)\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_t} - v_\theta^\kappa(x)\right]dx\\ &\qquad - \int_D \phi(x)\mathbb{E}_x\left[v_\theta^\kappa(B_t)\left(1 - \mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_t}\right)\right]dx\\ &\quad= \int_D \phi(x)\big(v_\theta^\kappa(x)\,\mathrm{e}^{\theta t} - v_\theta^\kappa(x)\big)dx\\ &\qquad + 2\int_D \phi(x)\,\mathrm{e}^{\theta t}\,\mathbb{E}_x\left[\int_0^t v_\theta^\kappa(B_s)^2\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,ds\right]dx\end{aligned}$$

$$-\frac{1}{2}\int_D \phi(x)\,\mathrm{e}^{\theta t}\, E_x\left[\int_0^t \varphi(B_s)\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,dl_s\right]dx$$
$$-\int_D \phi(x)\mathbb{E}_x\left[v_\theta^\kappa(B_t)\left(1-\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_t}\right)\right]dx.$$

So, we have, for every $t > 0$,

$$\frac{1}{2}\int_D dx\ \Delta\phi(x)\frac{1}{t}\int_0^t ds\,\mathbb{E}_x\big[v_\theta^\kappa(B_s)\big]-\frac{1}{2}\int_D dx\ v_\theta^\kappa(x)\frac{1}{t}\mathbb{E}_x\left[\int_0^t \frac{\partial\phi}{\partial n}(B_s)dl_s\right]$$
$$=\int_D \phi(x)v_\theta^\kappa(x)\frac{1}{t}(\mathrm{e}^{\theta t}-1)dx$$
$$+2\int_D \phi(x)\frac{1}{t}\,\mathrm{e}^{\theta t}\,\mathbb{E}_x\left[\int_0^t v_\theta^\kappa(B_s)^2\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,ds\right]dx$$
$$-\frac{1}{2}\int_D \phi(x)\frac{1}{t}\,\mathrm{e}^{\theta t}\,\mathbb{E}_x\left[\int_0^t \varphi(B_s)\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,dl_s\right]dx$$
$$-\int_D \phi(x)\frac{1}{t}\mathbb{E}_x\left[v_\theta^\kappa(B_t)\left(1-\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_t}\right)\right]dx.$$

Now we let t goes to 0 and use the continuity B as well as the continuity of v_θ^κ on $\bar{D}$, the lemmas 9.2, 9.3, 9.4 and 9.5 to get the equation of the proposition. □

For $k \in \mathbb{N}$, $\alpha \in (0, 1]$, let $C^{k,\alpha}(\Omega)$ be the set of functions defined on Ω which are k times differentiable such that their k^{th} derivative is Hölder with parameter α.

Proposition 5.7 (Recall that $\theta > 0$). *Let $\varphi \in C^{1,\alpha}(\partial D)$ be nonnegative. The function v_θ belongs to $C^2(D)\cap C^1(\bar{D})$ and it is the unique nonnegative solution of the Neumann problem*

$$\begin{aligned}\Delta u &= 4u^2 + 2\theta u \quad \text{in } D\\ \frac{\partial u}{\partial n} &= \varphi \quad \text{on } \partial D.\end{aligned}\tag{10}$$

Furthermore, v_θ belongs to $C^{2,\alpha}(\bar{D})$.

Proof. Since $\varphi \in C^{1,\alpha}(\partial D)$, we deduce from propositions 9.6 and 9.9, that the function $\frac{1}{2}\,\mathbb{E}_x\left[\int_0^{+\infty}\varphi(B_s)\,\mathrm{e}^{-\theta s}\ dl_s\right]$ belongs to $C^{2,\alpha}(\bar{D})$. Since v_θ is bounded, we deduce from proposition 9.8, that the function $\mathbb{E}_x\left[\int_0^{+\infty} v_\theta(B_s)^2\,\mathrm{e}^{-\theta s}\ ds\right]$ belongs to $C^{0,1}(\bar{D})$. Thanks to (8), this implies that $v_\theta \in C^{0,1}(\bar{D})$. Using propositions 9.7 and 9.9, we deduce again from (8), that $v_\theta \in C^{2,\alpha}(\bar{D})$. From proposition 9.7, we get that v_θ is a solution of (10).

Let us check the uniqueness of solutions to (10). Let $u \in C^2(D)\cap C^1(\bar{D})$ be another nonnegative solution of (10). Set $w = u - v_\theta$. The function w solves

$$\begin{aligned}\Delta w - kw &= 0 \quad \text{in } D\\ \frac{\partial w}{\partial n} &= 0 \quad \text{on } \partial D,\end{aligned}$$

where $k = 4(u + v_\theta) + 2\theta > 0$ belongs to $C^1(\bar{D})$. From the maximum principle (see theorem 8 in [15]), we get that either that $w \leq 0$ or $w > 0$ is constant in D.

Using $-w$ instead of w, we deduce that w is constant in D. Therefore we have $u = v_\theta + c$. Subtracting (10) applied to u and v_θ, we get that $c(4v_\theta + 2c + \theta) = 0$. Either $c = 0$ or v_θ is constant. If v_θ is constant, from (10) we get that $\varphi = 0$ and by construction $v_\theta = 0$. This in turn implies that $c(2c + \theta) = 0$. Since $u = c$ is nonnegative, we get that $u = 0$. In any case $c = 0$ and thus we have $u = v_\theta$. □

6. Properties of v^κ

Let φ be a bounded nonnegative measurable function defined on ∂D. Recall that for $\kappa \geq 0$ and $x \in \bar{D}$, $v^\kappa(x) = \mathbb{N}_x\left[1 - \mathrm{e}^{-(Z^\kappa, \varphi)}\right]$ and $v^\kappa \leq v^0 = v$.

Proposition 6.1. *The function v^κ is bounded on $\bar{D}$ for $\kappa \geq 0$.*

The proof of this theorem is at the end of this section.

Proposition 6.2. *The function v^κ is continuous on $\bar{D}$.*

Proof. By construction, we have that v^κ_θ increases to v^κ as θ decreases to 0. From proposition 5.3, we get by dominated convergence: for $x \in \bar{D}$,

$$v^\kappa(x) + \mathbb{E}_x\left[\int_0^t v^\kappa(B_s)^2\, \mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\; ds\right] = \mathbb{E}_x\left[v^\kappa(B_t)\, \mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_t}\right] + \frac{1}{2}\mathbb{E}_x\left[\int_0^t \varphi(B_s)\, \mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\; dl_s\right].$$

Now, as φ and v^κ are bounded, we conclude as in the proof of proposition 5.5. □

By dominated convergence, we deduce from proposition 5.6 the next result as θ decreases to 0.

Proposition 6.3. *Assume φ is a continuous nonnegative function on ∂D. For any test function $\phi \in C^2_b(D) \cap C^1(\bar{D})$, we have:*

$$\int_D \Delta\phi(x) v^\kappa(x)\; dx - 4\int_D \phi(x) v^\kappa(x)^2\; dx = \int_{\partial D} \frac{\partial\phi}{\partial n}(y) v^\kappa(y)\; \sigma(dy) - \int_{\partial D} \phi(y)\varphi(y)\; \sigma(dy) - \int_{\partial D} \phi(y)\kappa(y) v^\kappa(y)\sigma(dy). \tag{11}$$

In particular, for $\phi = 1$, we have

$$4\int_D v^\kappa(x)^2 dx = \int_{\partial D} \varphi(y)\sigma(dy) + \int_{\partial D} \kappa(y) v^\kappa(y)\sigma(dy).$$

Notice that any function $v \in C^2(D) \cap C^1(\bar{D})$ solution of the Neumann problem (1) satisfies the integral equation (11), for any test function.

Definition 6.4. *We say that a bounded measurable function v which satisfies (11) for any test function is a weak solution of the Neumann problem (1).*

We will mainly consider weak solutions that are continuous on $\bar{D}$.

Proof of proposition 6.1. Because $v^\kappa \le v = v^0$, it is enough to prove the proposition for v.

Let g be a continuous nonnegative function defined on ∂D. Consider the Dirichlet problem in D:

$$\begin{aligned} \Delta u - 2\theta u &= 4u^2 \quad \text{in } D, \\ u &= g \quad \text{on } \partial D. \end{aligned} \tag{12}$$

From [7], we know there exists only one nonnegative solution to this equation u_θ, and u_θ belongs to $C^2(D) \cap C^0(\bar{D})$. Since $L = \frac{\Delta}{2} - \theta$ is the infinitesimal generator of the Brownian motion killed at an independent exponential time with parameter $\theta > 0$, we also have the following integral equation:

$$u_\theta(x) + 2\mathbb{E}_x \left[\int_0^{\tau_D} u_\theta(B_s)^2 \, \mathrm{e}^{-\theta s} \; ds \right] = \mathbb{E}_x \left[g(B_{\tau_D}) \, \mathrm{e}^{-\theta \tau_D} \right], \tag{13}$$

where $\tau_D = \inf\{t \ge 0; B_t \notin D\}$. This integral representation is also valid for $\theta = 0$. The next lemma give a regularity result on u_θ when g is smooth. Recall that D has a C^3 boundary.

Lemma 6.5. *Let $\theta > 0$. If $g \in C^{2,\alpha}(\partial D)$, then the nonnegative solution u_θ of (13) lies in $C^{2,\alpha'}(\bar{D})$, where $\alpha' = \min(\alpha, 1/2)$.*

This result doesn't seem optimal since α' might be less than α. The proof of this lemma is in section 9.5.

From now on, we assume that $\theta > 0$. Notice that under the hypothesis of lemma 6.5, the normal derivative of u_θ is continuous and well defined. However this normal derivative can be negative at some point of ∂D. We can't represent u_θ as $\mathbb{N}_x \left[1 - \mathrm{e}^{-(Z_\theta, \varphi)}\right]$, with φ the normal derivative of u_θ in general. For our purpose it will be sufficient to consider $u_{N,\theta}$ solution of (13) with g constant equal to $N > 0$. From (13), we have $u_{N,\theta} < N$ in D. Therefore the normal derivative of $u_{N,\theta}$, say $\varphi_{N,\theta}$ is nonnegative.

Let us find a lower bound for $\varphi_{N,\theta}$ independent of $\theta > 0$. Since D is bounded with C^3 boundary, there exists $r_0 > 0$, such that for any $x_0 \in \partial D$, the open ball $B(x_0 - r_0 n_{x_0}, r_0)$ with radius r_0 centered at $x_0 - r_0 n_{x_0}$, where n_{x_0} is the outward normal of D at point x_0, lies in D. Let w_N be the unique nonnegative solution of $\Delta u = 4u^2$ in $B(x_0 - r_0 n_{x_0}, r_0)$ with boundary condition $w_N = N$ on $\partial B(x_0 - r_0 n_{x_0}, r_0)$. In particular $w_N \in C^2(\bar{D})$, thanks to lemma 6.5. Since $u_{N,\theta} < N$ in D, we deduce that $u_{N,\theta} \le w_N$ on $\partial B(x_0 - r_0 n_{x_0}, r_0)$. Let $z = u_{N,\theta} - w_N$. The function z satisfies $\Delta z - kz \ge 0$ in D with $k = 4(u_{N,\theta} + w_N) \ge 0$ and $z \le 0$ on $\partial B(x_0 - r_0 n_{x_0}, r_0)$. From the maximum principle (theorem 6 in [15]), we get that $z \le 0$ in $B(x_0 - r_0 n_{x_0}, r_0)$, hence

$$u_{N,\theta} \le w_N \quad \text{in } B(x_0 - r_0 n_{x_0}, r_0).$$

Since $w_N(x_0) = u_{N,\theta}(x_0) = N$, we have for $\varepsilon > 0$ small enough

$$w_N(x_0) - w_N(x_0 - \varepsilon n_{x_0}) \le u_{N,\theta}(x_0) - u_{N,\theta}(x_0 - \varepsilon n_{x_0}).$$

This implies that $\phi_N(x_0) \le \varphi_{N,\theta}(x_0)$, where ϕ_N is the normal derivative of w_N.

Lemma 6.6. *There exists a constant c_0 depending only on r_0 and the dimension d, such that $\phi_N(x_0) \geq (N - c_0)/r_0$ for all $N > 0$.*

For the clarity of the exposition, the proof of this lemma is postponned to section 9.5.

Let $\varphi \geq 0$ measurable defined on ∂D. Let $N \geq r_0 \sup_{x \in \partial D} |\varphi(x)| + c_0$. Notice N is independent of $\theta > 0$. Since $u_{N,\theta}$ is in $C^{2,\alpha}(\bar{D})$ for some $\alpha > 0$, we get that $u_{N,\theta}$ is a strong solution of (10) with boundary condition $\varphi_{N,\theta} = \dfrac{\partial u_{N,\theta}}{\partial n}$ on ∂D. We deduce from proposition 5.7, that $u_{N,\theta}(x) = \mathbb{N}_x\left[1 - \mathrm{e}^{-(Z_\theta, \varphi_{N,\theta})}\right]$. From lemma 6.6, we deduce that $\varphi \leq \varphi_{N,\theta}$ and thus $v_\theta \leq u_{N,\theta} \leq N$. Since this upper bound is uniform in $\theta > 0$, we deduce that v itself is bounded from above by N. □

7. An integral equation for v

From (23), we see the green kernel

$$G(x, y) = \int_0^\infty [p_t(x, y) - a_D]dt, \quad \text{where } a_D = 1/\int_D dy,$$

is well defined a.e. in $\bar{D} \times \bar{D}$. If h is a measurable bounded function defined on D, we set $Gh(x) = \int_D G(x, y)h(y)\, dy$. If φ is a measurable bounded function defined on ∂D, we set $G\varphi\sigma(x) = \int_{\partial D} G(x, y)\varphi(y)\, \sigma(dy)$.

From now on, let φ be a bounded measurable nonnegative function on ∂D.

Proposition 7.1. *Let $v(x) = \mathbb{N}_x\left[1 - \mathrm{e}^{-(Z, \varphi)}\right]$. Then v satisfies the integral equation: for $x \in \bar{D}$,*

$$v(x) + 2Gv^2(x) - a_D \int_D v(y)\, dy = \frac{1}{2}\, G\varphi\sigma(x). \tag{14}$$

Notice that (14) may have many different nonnegative solutions (see remark 7.3). However, there is a unique nonnegative solution to (14) satisfying the integral condition $4\int_D v(y)^2\, dy = \int_{\partial D} \varphi(y)\, \sigma(dy)$ (the proof of this fact is similar to what follows lemma 7.4).

Proof. From proposition 5.2 (with $\kappa = 0$), we have

$$v_\theta(x) + 2\mathbb{E}_x\left[\int_0^{+\infty} v_\theta(B_s)^2\, \mathrm{e}^{-\theta s}\, ds\right] = \frac{1}{2}\, \mathbb{E}_x\left[\int_0^{+\infty} \varphi(B_s)\, \mathrm{e}^{-\theta s}\, dl_s\right]. \tag{15}$$

From (25), we deduce that

$$\begin{aligned}\mathbb{E}_x&\left[\int_0^{+\infty} \varphi(B_s)\, \mathrm{e}^{-\theta s}\, dl_s\right]\\ &= \int_0^\infty ds\ \mathrm{e}^{-\theta s} \int_D \varphi(y)[p_s(x, y) - a_D]\sigma(dy)\end{aligned}$$

$$
\begin{aligned}
&+a_D \int_0^{\infty} ds\ \mathrm{e}^{-\theta s} \int_D \varphi(y)\sigma(dy) \\
&= G\varphi\sigma(x) + \frac{a_D}{\theta} \int_{\partial D} \varphi(y)\ \sigma(dy) \\
&\quad + \int_0^{\infty} ds\ (\mathrm{e}^{-\theta s} - 1) \int_D \varphi(y)[p_s(x,y) - a_D]\sigma(dy).
\end{aligned}
$$

The third term of the last equality goes to 0 as $\theta \downarrow 0$, thanks to (22) and (23). Therefore, we have

$$
\mathbb{E}_x \left[\int_0^{+\infty} \varphi(B_s)\, \mathrm{e}^{-\theta s}\ dl_s \right] = G\varphi\sigma(x) + \frac{a_D}{\theta} \int_{\partial D} \varphi(y)\ \sigma(dy) + o(1).
$$

By a similar argument, we have

$$
\mathbb{E}_x \left[\int_0^{+\infty} v_\theta(B_s)^2\, \mathrm{e}^{-\theta s}\ ds \right] = G v_\theta^2(x) + \frac{a_D}{\theta} \int_D v_\theta(y)^2\ dy + o(1).
$$

From the second equations of propositions 5.6 and 6.3 we get that

$$
4 \int_D v_\theta(y)^2\ dy + 2\theta \int_D v_\theta(y)\ dy = 4 \int_D v(y)^2\ dy.
$$

Since v_θ increases uniformly to v as θ decreases to θ, we have $v_\theta = v + o(1)$ in $\bar{D}$. We deduce that

$$
\begin{aligned}
&4 \int_D v_\theta(y)^2\ dy = 4 \int_D v(y)^2\ dy - 2\theta \int_D v(y)\ dy + o(1) \quad \text{and} \\
&G v_\theta^2 = G v^2 + o(1) \quad \text{in } \bar{D}.
\end{aligned}
$$

Plugging those results in (15), we get that for $x \in \bar{D}$,

$$
\begin{aligned}
&v(x) + 2Gv^2(x) + 2\frac{a_D}{\theta} \int_D v(y)^2\ dy - a_D \int_D v(y)\ dy \\
&= \frac{1}{2}\, G\varphi\sigma(x) + \frac{1}{2}\, \frac{a_D}{\theta} \int_{\partial D} \varphi(y)\ \sigma(dy) + o(1).
\end{aligned}
$$

Using the second equation of proposition 6.3 we get (14), as θ goes to 0. □

We assume $\varphi \geq 0$ is non zero, that is $\int_{\partial D} \varphi(y)\ \sigma(dy) > 0$. We consider the functions defined on $\bar{D}$ by,

$$
\begin{aligned}
&w_1(x) = w_1 = \frac{1}{2} \left[a_D \int_{\partial D} \varphi(y)\ \sigma(dy) \right]^{1/2} > 0, \\
&w_2(x) = \frac{1}{2}\, G\varphi\sigma(x), \\
&w_n(x) = -2 \sum_{k=1}^{n-1} G(w_k w_{n-k})(x) + c_n, \quad \text{for } n \geq 3,
\end{aligned}
$$

where we set

$$c_n = -\frac{a_D}{2w_1}\int_D \sum_{k=2}^{n-1} w_k(y)w_{n+1-k}(y)\,dy.$$

The functions are well defined, because the function φ is bounded. By symmetry of G, we have for $n \geq 3$,

$$\int_D w_n(y)\,dy = -2\int_D \sum_{k=1}^{n-1} w_k(y)w_{n-k}(y)G\mathbf{1}(y)\,dy + \int_D c_n\,dy = \frac{c_n}{a_D},$$

because $G\mathbf{1} = 0$. In particular we deduce from the definition of c_{n-1}, that for $n \geq 4$,

$$\begin{aligned}\int_D &\sum_{k=1}^{n-1} w_k(y)w_{n-k}(y)\,dy\\ &= 2w_1\int_D w_{n-1}(y)\,dy + \int_D \sum_{k=2}^{n-2} w_k(y)w_{n-k}(y)\,dy = 0.\end{aligned} \tag{16}$$

For f a measurable function defined on $\bar{D}$ (resp. ∂D), we define $\| f \|_\infty = \sup\{|f(x)|; x \in \bar{D}\}$ (resp. $\| f \|_\infty = \sup\{|f(x)|; x \in \partial D\}$).

Proposition 7.2. *(1) There exists $\eta_0 > 0$ (depending on φ), such that the series*

$$v_\eta^+ = \sum_{n\geq 1} \eta^{n/2} w_n \quad \textit{and} \quad v_\eta^- = \sum_{n\geq 1} (-1)^n \eta^{n/2} w_n$$

are absolutely convergent (for the norm $\|\cdot\|_\infty$) for $\eta \in [0, \eta_0)$. The functions v_η^+ and v_η^- are continuous in $\bar{D}$.

(2) For $\eta > 0$, small enough, we have that v_η^+ (resp. v_η^-) is the only nonnegative (resp. non positive) continuous weak solution to the Neumann problem

$$\Delta u = 4u^2 \quad \textit{in } D, \quad \textit{and} \quad \frac{\partial u}{\partial n} = \eta\varphi \quad \textit{on } \partial D. \tag{17}$$

In particular $v_\eta^+(x) = \mathbb{N}_x\left[1 - \mathrm{e}^{-\eta(Z,\varphi)}\right]$, for $\eta > 0$, small enough.

Proof. 1) From (21), (22) and (23), it is clear that there exists a constant c_D, such that for any measurable function f (resp. h) defined on $\bar{D}$ (resp. ∂D),

$$\| Gf \|_\infty \leq c_D \| f \|_\infty \quad \text{and} \quad \| Gh\sigma \|_\infty \leq c_D \| h \|_\infty .$$

We have by recurrence that $\| w_n \|_\infty \leq \beta_n \| \varphi \|_\infty^{n/2}$, where

$$\beta_1 = \frac{1}{2}\sqrt{a_D\int_{\partial D} \sigma(dy)}, \quad \beta_2 = \frac{1}{2}c_D,$$

and for $n \geq 3$,

$$\beta_n = 2c_D \sum_{k=1}^{n-1} \beta_k \beta_{n-k} + \sqrt{a_D}\rho \sum_{k=2}^{n-1} \beta_k \beta_{n+1-k},$$

with $\rho = \left[\|\varphi\|_\infty / \int_{\partial D} \varphi(y)\, \sigma(dy)\right]^{1/2}$. It is easy to check there exists $\tilde{\eta}_0 > 0$ (depending only on β_1, c_D and ρ) such that the series $g(r) = \sum_{k \geq 3} \beta_k r^k$ is convergent for $r \in [0, \tilde{\eta}_0)$, and that $g(r)$ is the smallest solution of

$$g(r) = 2c_D \left[(g(r) + \beta_1 r + \beta_2 r^2)^2 - \beta_1^2 r^2\right] + \frac{1}{r}\sqrt{a_D}\rho \left[g(r) + \beta_2 r^2\right]^2.$$

It is then clear that the series $v_\eta^\delta = v_\eta^+$ or v_η^-, as $\delta = +1$ or $\delta = -1$, are absolutely convergent for $\eta \in [0, \eta_0 = \tilde{\eta}_0 / \|\varphi\|_\infty^{1/2})$.

From the continuity of p, (22), (23), we have that $G\varphi\sigma$ is continuous on $\bar{D}$. By recurrence, we get that w_n is continuous for $n \geq 3$. This implies that v_η^δ is continuous on $\bar{D}$.

2) Furthermore, let us note that, by the product of two series, for $\eta \in [0, \eta_0)$,

$$(v_\eta^\delta)^2 = \sum_{n \geq 2} \delta^n \eta^{n/2} \sum_{k=1}^{n-1} (w_k w_{n-k})$$

and, as $G(w_1^2) = 0$,

$$G\left((v_\eta^\delta)^2\right) = \sum_{n \geq 3} \delta^n \eta^{n/2} \sum_{k=1}^{n-1} G(w_k w_{n-k}).$$

Then, we have that, for $\eta \in [0, \eta_0)$,

$$\begin{aligned} v_\eta^\delta(x) &= \sum_{n \geq 1} \delta^n \eta^{n/2} w_n(x) \\ &= \delta\sqrt{\eta} w_1 + \eta w_2(x) - 2\sum_{n \geq 3} \delta^n \eta^{n/2} \sum_{k=1}^{n-1} G(w_k w_{n-k})(x) + \sum_{n \geq 3} \delta^n \eta^{n/2} c_n \\ &= \delta\sqrt{\eta} w_1 + \frac{\eta}{2} G\varphi\sigma(x) - 2G((v_\eta^\delta)^2)(x) + \sum_{n \geq 3} \delta^n \eta^{n/2} c_n. \end{aligned}$$

From the symmetry of G and the fact that $G\mathbf{1} = 0$, we get

$$\begin{aligned} \int_D v_\eta^\delta(y)\, dy = \frac{\delta\sqrt{\eta}}{a_D} w_1 + \frac{\eta}{2} \int_D G\varphi\sigma(y)\, dy \\ -2\int_D G((v_\eta^\delta)^2)(y)\, dy + \sum_{n \geq 3} \frac{\delta^n \eta^{n/2}}{a_D} c_n \end{aligned}$$

$$
\begin{aligned}
&= \frac{\delta\sqrt{\eta}}{a_D} w_1 + \frac{\eta}{2} \int_{\partial D} \varphi(x) G\mathbf{1}(x) \sigma(dx) \\
&\quad -2 \int_D (v_\eta^\delta(x))^2 G\mathbf{1}(x) dx + \sum_{n\geq 3} \frac{\delta^n \eta^{n/2}}{a_D} c_n \\
&= \frac{\delta\sqrt{\eta}}{a_D} w_1 + \sum_{n\geq 3} \frac{\delta^n \eta^{n/2}}{a_D} c_n.
\end{aligned}
$$

Plugging this in the previous equation, we get that

$$
v_\eta^\delta(x) = \frac{\eta}{2} G\varphi\sigma(x) - 2G((v_\eta^\delta)^2)(x) + a_D \int_D v_\eta^\delta(y)\, dy.
$$

Hence v_η^δ solves (14) with φ replaced by $\eta\varphi$.

Remark 7.3. By considering w'_n defined as w_n but for $c'_n = 0$, it is easy to get that $v'_\eta = \sum_{n\geq 1} \eta^{n/2} w'_n$ is well defined, continuous, nonnegative and solution of (14) for $\eta > 0$ small enough. Since $c_3 \neq 0$, we have $w'_\eta \neq v_\eta^+$. Hence (14) doesn't have a unique nonnegative continuous solution.

We have

$$
\begin{aligned}
\int_D v_\eta^\delta(y)^2\, dy &= \int_D \sum_{n\geq 2} \delta^n \eta^{n/2} \sum_{k=1}^{n-1} w_k(y) w_{n-k}(y)\, dy \\
&= \frac{\delta^2 \eta}{a_D} w_1^2 + \delta^3 \eta^{3/2} w_1 \int_D G\varphi\sigma(y)\, dy \\
&\quad + \sum_{n\geq 4} \eta^{n/2} \delta^n \int_D \sum_{k=1}^{n-1} w_k(y) w_{n-k}(y)\, dy \\
&= \frac{\eta}{4} \int_{\partial D} \varphi(y)\, \sigma(dy),
\end{aligned}
$$

where we used (16) for the last equality as well as the symmetry of G and the fact that $G1 = 0$. Thus we have that v_η^δ solves also

$$
4 \int_D u(y)^2\, dy = \int_{\partial D} \varphi(y)\, \sigma(dy), \tag{18}
$$

with φ replaced by $\eta\varphi$. The next lemma states that the two integral equations (14) and (18) characterize the weak solutions of the Neumann problem (1) (with $\kappa = 0$). Its proof is postponed at the end of this section.

Lemma 7.4. *Any bounded measurable function u satisfying (14) and (18) is a weak solution of the Neumann problem (1) (with $\kappa = 0$). That is, for any test function $\phi \in C_b^2(D) \cap C^1(\bar{D})$, we have:*

$$
\begin{aligned}
&\int_D \Delta\phi(y) u(y)\, dy - 4 \int_D \phi(y) u(y)^2\, dy \\
&= \int_{\partial D} \frac{\partial \phi}{\partial n}(y) u(y)\, \sigma(dy) - \int_{\partial D} \phi(y) \varphi(y)\, \sigma(dy).
\end{aligned} \tag{19}
$$

From this lemma, we get that v_η^+ and v_η^- are continuous weak solution of the Neumann problem (1) (with $\kappa = 0$ and φ replaced by $\eta\varphi$). From propositions 6.2 and 6.3, we have that $v_\eta(x) = \mathbb{N}_x\left[1 - \mathrm{e}^{-\eta(Z,\varphi)}\right]$ is also a continuous weak solution.

To complete the proof of the proposition, we just have to check that $v_\eta^+ = v_\eta$. This will be done once we prove the uniqueness of the continuous weak solutions.

Let $\eta > 0$ and $\varepsilon > 0$ small enough, so that $\left|v_\eta^\delta\right|(x) \geq \varepsilon > 0$ in $\bar{D}$ for $\delta \in \{+1, -1\}$ (this can be done since $w_1 > 0$). Consider u a continuous nonnegative solution of (19), with φ replaced by $\eta\varphi$. Since v_η^+ is a positive solution of (19), by subtraction, we get

$$\int_D w(y)[\Delta\phi(y) + 4q(y)\phi(y)] = \int_{\partial D} \frac{\partial\phi}{\partial n}(y)w(y)\,\sigma(dy),$$

with $w = u - v_\eta^+$ and $q = -4(u + v_\eta^+)$. From theorem 5.5 of [9], we deduce that $w = 0$ (the finiteness of the gauge in [9] is implied by the fact that $q(y) \leq -\varepsilon < 0$ for $y \in \bar{D}$). In particular, v_η^+ is the unique continuous nonnegative solution of (19), for $\eta > 0$ small enough. This implies that for $\eta > 0$, small enough, $v_\eta^+(x) = \mathbb{N}_x\left[1 - \mathrm{e}^{-\eta(Z,\varphi)}\right]$.

Similarly, we get that v_η^- is the unique continuous non positive solution of (19), for $\eta > 0$ small enough. □

Corollary 7.5. *Let $\varphi \geq 0$, such that $\int_{\partial D} \varphi(y)\ \sigma(dy) > 0$. Then $v(x) = \mathbb{N}_x \left[1 - \mathrm{e}^{-(Z,\varphi)}\right]$ is the only nonnegative weak solution of (1) (with $\kappa = 0$).*

Proof. From the last part of the proof of proposition 7.2, concerning uniqueness of weak solution, we see with v_η^+ replaced by v that it is enough to check that $v(x) \geq \varepsilon > 0$ in D. For $\eta \in (0, 1]$ small enough, we have

$$v(x) \geq v_\eta^+(x) = \mathbb{N}_x\left[1 - \mathrm{e}^{-\eta(Z,\varphi)}\right].$$

For $\varepsilon > 0$ and $\eta > 0$ small enough, we get that $v_\eta^+ \geq \varepsilon$ on D, since $w_1 > 0$. □

Proof of lemma 7.4. From the definition of the kernel G and the symmetry of p we get that for any bounded measurable function f:

$$\int_D Gf(y)\ dy = 0 \quad \text{and} \quad \int_{\partial D} Gf\sigma(y)\ dy = 0.$$

From [5], we get that:

- If f is a bounded measurable function defined on D, then Gf is a weak solution of $\frac{\Delta}{2}w = -f + a_D \int_D f(y)\ dy$ with Neumann boundary condition $\frac{\partial w}{\partial n} = 0$. And for any test function $\phi \in C_b^2(D) \cap C^1(\bar{D})$,

$$\int_D \Delta\phi(y)w(y)\ dy + 2\int_D \phi(y)f(y)\ dy - 2a_D \int_D \phi(y)\ dy \int_D f(y)\ dy$$
$$= \int_{\partial D} \frac{\partial\phi}{\partial n}(y)w(y)\ \sigma(dy).$$

- If h is a bounded measurable function defined on ∂D, then $\frac{1}{2}\, Gh\sigma$ is a weak solution of $\frac{\Delta}{2} w = \frac{a_D}{2} \int_{\partial D} h(y)\, \sigma(dy)$ with Neumann boundary condition $\frac{\partial w}{\partial n} = h$. And for any test function $\phi \in C_b^2(D) \cap C^1(\bar{D})$,

$$\begin{aligned}\int_D \Delta\phi(y) w(y)\, dy - a_D \int_D \phi(y)\, dy \int_{\partial D} h(y)\, \sigma(dy)\\ = \int_{\partial D} \frac{\partial \phi}{\partial n}(y) w(y) \sigma(dy) - \int_{\partial D} \phi(y) h(y)\, \sigma(dy).\end{aligned}$$

Let u be a bounded measurable function defined on $\bar{D}$, satisfying (14) and (18). Let $\phi \in C_b^2(D) \cap C^1(\bar{D})$, be a test function. Using the symmetry of G, the above remarks, we then deduce from (14) by multiplying by $\Delta\phi$ and integrating on D, that

$$\begin{aligned}&\int_D \Delta\phi(y) u(y)\, dy - 4 \int_D \phi(y) u(y)^2\, dy + 4a_D \int_D \phi(y)\, dy \int_D u(y)^2\, dy\\ &\quad +2 \int_{\partial D} \frac{\partial \phi}{\partial n}(y) Gu^2(y) \sigma(dy) - a_D \int_D \Delta\phi(y)\, dy \int_D u(y)\, dy\\ &= a_D \int_D \phi(y)\, dy \int_{\partial D} \varphi(y)\, \sigma(dy)\\ &\quad + \int_{\partial D} \frac{\partial \phi}{\partial n}(y)\, \frac{1}{2}\, G\varphi(y) \sigma(dy) - \int_{\partial D} \phi(y) \varphi(y)\, \sigma(dy).\end{aligned}$$

Use (18) for the third term, (14) for the fourth and the Green formula $\int_D \Delta\phi(y)\, dy = \int_{\partial D} \frac{\partial \phi}{\partial n}(y)\, \sigma(dy)$ for the fifth of the left member to get (19). □

8. Properties of *Z*

We can give estimate of the probability of hitting small balls for the measure Z.

Let $x_0 \in \partial D$, and $B_{\partial D}(x_0, \varepsilon)$ be the ball on the boundary of D centered at x, with radius $\varepsilon > 0$: $B_{\partial D}(x_0, \varepsilon) = \{y \in \partial D;\, |x - y| < \varepsilon\}$. We write $Z(B_{\partial D}(x_0, \varepsilon)) = (Z, \mathbf{1}_{B_{\partial D}(x_0,\varepsilon)})$.

Proposition 8.1. *For every compact set $K \subset D$, there exists $1/2 > \varepsilon_0 > 0$ and a constant $c_d > 0$ (which depends on the dimension d) such that for any $x \in K$, $x_0 \in \partial D$, $\varepsilon \in (0, \varepsilon_0)$,*

$$\mathbb{N}_x[Z(B_{\partial D}(x_0, \varepsilon)) > 0] \geq \begin{cases} c_d & \text{if } d = 2 \text{ or } 3,\\ c_d (\log(1/\varepsilon))^{-1} & \text{if } d = 4,\\ c_d \varepsilon^{d-4} & \text{if } d \geq 5.\end{cases}$$

Proof. We fix $\theta > 0$ and notice that $\mathbb{N}_x[Z(B_{\partial D}(x_0, \varepsilon)) > 0] \geq \mathbb{N}_x[Z_\theta(B_{\partial D}(x_0, \varepsilon)) > 0]$. Consequently, it is enough to get a lower bound for Z_θ.

Let us set

$$g_\theta(x, y) = \int_0^{+\infty} p_s(x, y) e^{-\theta s} ds$$

and recall the following estimates: there exists a constant α (which depends on d and θ) such that for all $(x, y) \in \bar{D} \times \bar{D}$,

$$g_\theta(x, y) \le \alpha h(x, y)$$

with

$$h(x, y) = \begin{cases} \left(1 + \ln_+ \frac{1}{\|x-y\|}\right) & \text{if } d = 2 \\ \| x - y \|^{2-d} & \text{if } d \ge 3, \end{cases}$$

where $\ln_+(r) = \max(0, \ln(r))$ (see for instance [3], Corollary 3.3 or [6], Theorem 3.4 (iv)). Now, by Cauchy-Schwartz inequality, we have

$$\mathbb{N}_x[Z_\theta(B_{\partial D}(x_0, \varepsilon)) > 0] \ge \frac{\mathbb{N}_x[Z_\theta(B_{\partial D}(x_0, \varepsilon))]^2}{\mathbb{N}_x[Z_\theta(B_{\partial D}(x_0, \varepsilon))^2]}.$$

The first moment is easy to estimate: we have, by definition of Z_θ

$$\begin{aligned} \mathbb{N}_x[Z_\theta(B_{\partial D}(x_0, \varepsilon))] &= \mathbb{N}_x\left[\int_0^\sigma \mathbf{1}_{\{\hat{W}_s \in B_{\partial D}(x_0,\varepsilon)\}} \mathbf{1}_{\{\hat{R}_s \ge \theta\}} dL_s\right] \\ &= \mathbb{E}_x\left[\int_0^{+\infty} \mathbf{1}_{\{B_s \in B_{\partial D}(x_0,\varepsilon)\}} e^{-\theta s} dl_s\right] \\ &= \int_{B_{\partial D}(x_0,\varepsilon)} g_\theta(x, y)\sigma(dy) \end{aligned}$$

and, as g_θ is bounded below by a constant on $K \times \partial D$, there exists $\varepsilon_0 > 0$ such that for any $x \in K$, $x_0 \in \partial D$, $\varepsilon \in (0, \varepsilon_0]$,

$$\mathbb{N}_x[Z_\theta(B_{\partial D}(x_0, \varepsilon))] \ge c_d \varepsilon^{d-1}.$$

For the second moment, let us first prove the following lemma

Lemma 8.2. *For every nonnegative measurable function φ on ∂D,*

$$\mathbb{N}_x((Z_\theta, \varphi)^2) = 4 \int_D dy\, g_\theta(x, y) \left(\int_{\partial D} \sigma(dz) g_\theta(y, z)\varphi(z)\right)^2.$$

Proof. Using the definition of the measure Z_θ then the Markov property, we have

$$\begin{aligned} \mathbb{N}_x[(Z_\theta, \varphi)^2] &= \mathbb{N}_x\left[\left(\int_0^\sigma dL_u \mathbf{1}_{\{\hat{R}_u \ge \theta\}} \varphi(\hat{W}_u)\right)^2\right] \\ &= 2\mathbb{N}_x\left[\int_0^\sigma dL_u \int_u^\sigma dL_{u'} \mathbf{1}_{\{\hat{R}_u \ge \theta\}} \mathbf{1}_{\{\hat{R}_{u'} \ge \theta\}} \varphi(\hat{W}_u)\varphi(\hat{W}_{u'})\right] \\ &= 2\mathbb{N}_x\left[\int_0^\sigma dL_u \mathbf{1}_{\{\hat{R}_u \ge \theta\}} \varphi(\hat{W}_u) \mathrm{E}^*_{\tilde{W}_u}\left[\int_0^\sigma dL_{u'} \mathbf{1}_{\{\hat{R}_{u'} \ge \theta\}} \varphi(\hat{W}_{u'})\right]\right] \\ &= 4\mathbb{N}_x\left[\int_0^\sigma dL_u \mathbf{1}_{\{\hat{R}_u \ge \theta\}} \varphi(\hat{W}_u) \int_0^{\zeta_u} dt\, \mathbb{N}_{\tilde{W}_u(t)}\big[(Z_\theta, \varphi)\big]\right] \end{aligned}$$

where we used the Poissonian representation, stated in the proof of proposition 5.2, of the Brownian snake under $\mathrm{E}^*_{\tilde{w}}$ (see also proposition 2.5 of [12]). Then, thank to formula (6), we get

$$\mathbb{N}_x[(Z_\theta,\varphi)^2] = 4\mathbb{N}_x\left[\int_0^\sigma dL_u \mathbf{1}_{\{\hat{R}_u\geq\theta\}}\varphi(\hat{W}_u)\int_0^{\zeta_u} dt\, \mathbb{N}_{(W_u(t),+\infty,1,0)}[(Z_\theta,\varphi)]\right].$$

Now, using formula (4) twice, we have

$$\begin{aligned}
\mathbb{N}_x[(Z_\theta,\varphi)^2] &= 4\mathbb{E}_x\left[\int_0^{+\infty} dl_u\, \mathrm{e}^{-\theta u}\,\varphi(B_u)\int_0^u dt\,\mathbb{E}_{B_t}\left[\int_0^{+\infty}\varphi(B_s)\,\mathrm{e}^{-\theta s}\,dl_s\right]\right]\\
&= 4\mathbb{E}_x\left[\int_0^{+\infty} dt\int_t^{+\infty} dl_u\,\mathrm{e}^{-\theta u}\,\varphi(B_u)\mathbb{E}_{B_t}\left[\int_0^{+\infty}\varphi(B_s)\,\mathrm{e}^{-\theta s}\,dl_s\right]\right]\\
&= 4\mathbb{E}_x\left[\int_0^{+\infty} dt\ \mathrm{e}^{-\theta t}\,\mathbb{E}_{B_t}\left[\int_0^{+\infty}\varphi(B_s)\,\mathrm{e}^{-\theta s}\,dl_s\right]^2\right]\\
&= 4\int_D dy\, g_\theta(x,y)\left(\int_{\partial D}\sigma(dz)g_\theta(y,z)\varphi(z)\right)^2.
\end{aligned}$$

□

Applying this result with $\varphi=\mathbf{1}_{B_{\partial D}(x_0,\varepsilon)}$, we have

$$\begin{aligned}
&\mathbb{N}_x\big(Z_\theta(B_{\partial D}(x_0,\varepsilon))^2\big)\\
&\quad= 4\int_D dy\, g_\theta(x,y)\iint_{B_{\partial D}(x_0,\varepsilon)^2}\sigma(dz)\sigma(dz')g_\theta(y,z)g_\theta(y,z')\\
&\quad= 4\iint_{B_{\partial D}(x_0,\varepsilon)^2}\sigma(dz)\sigma(dz')\int_D dy\, g_\theta(x,y)g_\theta(y,z)g_\theta(y,z').
\end{aligned}$$

We set

$$\psi_{x,\theta}(z,z') = \int_D dy\, g_\theta(x,y)g_\theta(y,z)g_\theta(y,z').$$

The upper bounds for the kernel g_θ lead to: for $x\in K$, $z,z'\in\partial D$,

$$\psi_{x,\theta}(z,z')\leq\begin{cases} C & \text{if } d\leq 3\\ C\left(1+\ln_+\frac{1}{|z-z'|}\right) & \text{if } d=4\\ C|z-z'|^{4-d} & \text{if } d\geq 5.\end{cases}$$

We then deduce easily, using the regularity of ∂D, that there exists $\varepsilon_0\in(0,1/2]$, and $c_d>0$, such that for any $x\in K$, $\varepsilon\in(0,\varepsilon_0]$, $x_0\in\partial D$,

$$\mathbb{N}_x\big(Z_\theta(B_{\partial D}(x_0,\varepsilon))^2\big)\leq\begin{cases} c_d\varepsilon^{2(d-1)} & \text{if } d=2 \text{ or } 3\\ c_d\varepsilon^6\left(\ln\frac{1}{\varepsilon}\right) & \text{if } d=4\\ c_d\varepsilon^{d+2} & \text{if } d\geq 5.\end{cases}$$

To finish, it suffices to combine the Cauchy-Schwartz inequality with the estimates for the first and second moment. □

From the upper bound of g_θ, we get the next lemma.

Lemma 8.3. *For $d = 2, 3$, for every $x \in D$ and every $\theta > 0$, the function*

$$\psi_{x,\theta}(y, y') = \int_D dz\, g_\theta(x, z) g_\theta(z, y) g_\theta(z, y')$$

is continuous on $\partial D \times \partial D$.

Proposition 8.4. *If $d = 2$ or 3, the measure Z is absolutely continuous with respect to the surface measure σ, $\mathbb{N}_x$-a.e., for $x \in D$.*

Proof. Mimicking the proof of theorem 5.1 in [2], we get that for $d = 2$ or 3, $x \in D$, $\mathbb{N}_x$-a.e., Z_θ is absolutely continuous with respect to σ for any $\theta > 0$. Let $A \subset \partial D$ be measurable, such that $\int_A \sigma(dy) = 0$. We deduce that $(Z_\theta, \mathbf{1}_A) = 0$, $\mathbb{N}_x$-a.e. Since Z_θ increases to Z as θ decreases to 0, we deduce that $\mathbb{N}_x$-a.e., $(Z, \mathbf{1}_A) = 0$ for any Borel set $A \subset \partial D$ such that $\int_A \sigma(dy) = 0$. Since supp $Z \subset \partial D$, this implies that Z is absolutely continuous with respect to σ. □

If A is a subset of $\mathbb{R}^d$, let $\dim(A)$ denote its Hausdorff dimension. For a measure μ on $\mathbb{R}^d$, let supp μ denote its closed support.

Proposition 8.5. *We have, for every x in D,*

$$\dim(\text{supp } Z) \geq 3 \wedge (d-1) \quad \mathbb{N}_x\text{-a.e. on } \{Z \neq 0\}.$$

Proof. Let $d \geq 4$. We will first prove that

$$\dim(\text{supp } Z_\theta) \geq 3 \wedge (d-1) \quad \mathbb{N}_x\text{-a.e. on } \{Z_\theta \neq 0\}. \tag{20}$$

Notice that since Z_θ increases to Z as θ decreases to 0, we have $\mathbb{N}_x$-a.e.,

$$\text{supp } Z = \bigcup_{k \in \mathbb{N}} \text{supp } Z_{\theta_k}$$

for any sequence $(\theta_k, k \in \mathbb{N})$ that decreases to 0. This implies the proposition.

The proof of (20) is an adaptation of the proof of Theorem 6.1 of [2]. We set, for $\alpha > 0$, $h_\alpha(r) = r^3 |\ln r|^\alpha$. Using lemma 8.2 and a polarization argument, we have for every $\varepsilon > 0$,

$$\mathbb{N}_x \left[\int_D Z_\theta(dy) Z_\theta\big(B_{\partial D}(y, \varepsilon)\big) \right] = 4 \int_{|y-y'|<\varepsilon} \sigma(dy)\sigma(dy')\psi_{x,\theta}(y, y').$$

The upper bounds for $\psi_{x,\theta}$ obtained in the proof of proposition 8.1 yield for ε small enough and $x \in D$, that

$$\mathbb{N}_x \left[\int_D Z_\theta(dy) Z_\theta\big(B_{\partial D}(y, \varepsilon)\big) \right] \leq \begin{cases} C'(x)\varepsilon^3 |\ln \varepsilon| & \text{if } d = 4 \\ C'(x)\varepsilon^3 & \text{if } d \geq 5. \end{cases}$$

In the case $d \geq 5$, we have for $n \in \mathbb{N}$ large enough,

$$\begin{aligned}
&\mathbb{N}_x\left[\int_{\partial D} Z_\theta(dy)\mathbf{1}_{\left\{Z_\theta\left(B_{\partial D}(y,2^{-n})\right)\geq n^\alpha 2^{-3n}\right\}}\right]\\
&\leq n^{-\alpha}2^{3n}\mathbb{N}_x\left[\int_{\partial D} Z_\theta(dy)Z^\theta\left(B_{\partial D}(y,2^{-n})\right)\right]\\
&\leq C'(x)n^{-\alpha}2^{3n}2^{-3n}\\
&= C'(x)n^{-\alpha}.
\end{aligned}$$

If $\alpha > 2$, we deduce that

$$\sum_{n=1}^{+\infty}\mathbf{1}_{\left\{Z_\theta\left(B_{\partial D}(y,2^{-n})\right)\geq n^\alpha 2^{-3n}\right\}} < \infty \qquad Z_\theta(dy) - .\text{a.e.} \qquad \mathbb{N}_x - \text{a.e.}$$

This implies that

$$\limsup_{\varepsilon\to 0}\frac{Z_\theta\left(B_{\partial D}(y,\varepsilon)\right)}{h_\alpha(\varepsilon)} < \infty, \qquad Z_\theta(dy).\text{a.e.} \qquad \mathbb{N}_x - \text{a.e.}$$

and a well-known result gives that the h_α-Hausdorff measure of supp Z_θ is strictly positive $\mathbb{N}_x$-a.e. on $\{Z_\theta \neq 0\}$.

The case $d = 4$ is similar (with $\alpha > 3$). In particular we deduce (20). □

9. Appendix

9.1. Reflected Brownian motion

The properties of the reflected Brownian motion B in $\bar{D}$ are from [5] and [16].

For $t > 0, x, y \in \bar{D}$, let $p_t(x, y)$ be the density of the reflected Brownian motion B_t when $B_0 = x \in \bar{D}$. The density is a continuous function on $(0, \infty) \times \bar{D} \times \bar{D}$. It is also symmetric on $\bar{D} \times \bar{D}$.

For any $\varepsilon_0 > 0$, there exists a constant c such that for any $x \in \bar{D}$, $t \in (0, 1]$, $\varepsilon \in (0, \varepsilon_0]$,

$$\frac{1}{\varepsilon}\int_{D_\varepsilon} p_t(x, y)\, dy \leq c/\sqrt{t}, \tag{21}$$

and

$$\int_{\partial D} p_t(x, y)\, \sigma(dy) \leq c/\sqrt{t}, \tag{22}$$

where $\sigma(dy)$ is the surface measure on ∂D.

There exist two positive constant c and β such that for $t \geq 1$, we have for all $x, y \in \bar{D}$

$$|p_t(x, y) - a_D| \leq c\,\mathrm{e}^{-\beta t}, \tag{23}$$

where $a_D^{-1} = \int_D\, dy$.

We deduce from (22) and (23) that for any $\theta > 0$, there exists a constant c such that, for all $x \in \bar{D}$,

$$\int_0^\infty \mathrm{e}^{-\theta s} \ ds \int_{\partial D} p_s(x, y)\sigma(dy) \le c. \tag{24}$$

The local time of B on ∂D, $l = (l_t, t \ge 0)$, is a continuous additive functional of B with Revuz measure $\sigma(dy)$. In particular we have for any nonnegative function f defined on $\mathbb{R}^+ \times \partial D$

$$\mathbb{E}_x \left[\int_0^\infty f(s, B_s) \ dl_s \right] = \int_0^\infty ds \ \int f(s, y) p_s(x, y) \ \sigma(dy). \tag{25}$$

From this last equation and (22), it is easy to prove by recurrence that for $T > 0$ and $n \ge 1$, there exists a constant K_n such that for all $t \in [0, T]$,

$$\sup_{x \in \bar{D}} \mathbb{E}_x[l_t^n] \le K_n t^{n/2}. \tag{26}$$

The density $p_s(x, y)$ as a function of x belongs to $C^1(\bar{D}) \cap C^2(D)$ for $(s, y) \in (0, \infty) \times \bar{D}$. Furthermore (see [16] p.600) there exists a constant $c > 0$ such that for $(s, y) \in (0, \infty) \times \bar{D}$, and $x = (x_1, \dots, x_d) \in \bar{D}$,

$$\left| \frac{\partial p_s}{\partial x_i}(x, y) \right| \le c \ s^{-(d+1)/2}, \tag{27}$$

and also

$$\int_D \left| \frac{\partial p_s}{\partial x_i}(x, y) \right| dy \le c \ s^{-1/2}. \tag{28}$$

9.2. Convergence lemmas

In this section, we present some convergence results which are used for proving proposition 5.6. They all concern reflected Brownian motion.

Lemma 9.1.

$$\lim_{t \to 0} \int_D \frac{1}{t} \mathbb{E}_x[l_t^2] dx = 0.$$

Proof. Let us recall that we denote by τ_D the exit time of B out of D. For every $x \in D$ $t_0 > 0$, we have for $t \in (0, t_0]$,

$$\begin{aligned} \frac{1}{t}\mathbb{E}_x[l_t^2] &= \frac{1}{t}\mathbb{E}_x[l_t^2 \mathbf{1}_{t > \tau_D}] \\ &\le \frac{1}{t}\mathbb{E}_x[l_t^4]^{1/2} \mathbb{P}_x(t > \tau_D)^{1/2} \\ &\le C\mathbb{P}_x(t > \tau_D)^{1/2}, \end{aligned}$$

thanks to (26). So, for every $x \in D$, we deduce that

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}_x[l_t^2] = 0.$$

Moreover, thanks to (26), $\frac{1}{t}\mathbb{E}_x[l_t^2]$ is bounded and the dominated convergence theorem gives the result. □

The next lemma is lemma 4.13 in [1].

Lemma 9.2. *For every continuous function ϕ on $\bar{D}$ and every continuous function ψ on ∂D,*

$$\lim_{t\to 0} \int_D dx\, \phi(x)\frac{1}{t}\mathbb{E}_x\left[\int_0^t \psi(B_s)dl_s\right] = \int_{\partial D} \sigma(dy)\phi(y)\psi(y).$$

Lemma 9.3. *For every bounded measurable function ϕ on D and every continuous function ψ on D,*

$$\lim_{t\to 0} \int_D dx\, \phi(x)\frac{1}{t}\,\mathrm{e}^{\theta t}\,\mathbb{E}_x\left[\int_0^t \psi(B_s)\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,ds\right] = \int_D \phi(x)\psi(x)dx.$$

Proof. We first write

$$\begin{aligned}
&\int_D dx\, \phi(x)\frac{1}{t}\,\mathrm{e}^{\theta t}\,\mathbb{E}_x\left[\int_0^t \psi(B_s)\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,ds\right]\\
&\quad= \int_D dx\, \phi(x)\frac{1}{t}\,\mathrm{e}^{\theta t}\,\mathbb{E}_x\left[\int_0^t \psi(B_s)ds\right]\\
&\qquad- \int_D dx\, \phi(x)\frac{1}{t}\,\mathrm{e}^{\theta t}\,\mathbb{E}_x\left[\int_0^t \psi(B_s)\left(1-\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\right)ds\right].
\end{aligned}$$

The first term converges to the expected limit by the continuity of B. The second one goes to 0 as t decreases to 0, since for $t \le 1$,

$$\begin{aligned}
&\left|\int_D dx\, \phi(x)\frac{1}{t}\,\mathrm{e}^{\theta t}\,\mathbb{E}_x\left[\int_0^t \psi(B_s)\left(1-\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\right)ds\right]\right|\\
&\quad\le C\int_D \frac{1}{t}\mathbb{E}_x\left[\int_0^t \left(1-\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\right)ds\right]dx\\
&\quad\le C\int_D \frac{1}{t}\mathbb{E}_x\left[\int_0^t \left(\theta s+\frac{1}{2}(\kappa\cdot l)_s\right)ds\right]dx\\
&\quad\le C\int_D \big(t+\mathbb{E}_x[l_t]\big)dx,
\end{aligned}$$

and thanks to (26). □

Lemma 9.4. *For every continuous function ϕ on $\bar{D}$ and every continuous function ψ on ∂D,*

$$\lim_{t\to 0} \int_D \phi(x)\frac{1}{t}\,\mathrm{e}^{\theta t}\,\mathbb{E}_x\left[\int_0^t \psi(B_s)\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,dl_s\right]dx = \int_{\partial D} \phi(y)\psi(y)\sigma(dy).$$

Proof. As for lemma 9.3, we write

$$\begin{aligned}
&\int_D \phi(x)\frac{1}{t}\,\mathrm{e}^{\theta t}\,\mathbb{E}_x\left[\int_0^t \psi(B_s)\,\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\,dl_s\right]dx\\
&\quad=\int_D \phi(x)\frac{1}{t}\,\mathrm{e}^{\theta t}\,\mathbb{E}_x\left[\int_0^t \psi(B_s)dl_s\right]dx\\
&\qquad-\int_D \phi(x)\frac{1}{t}\,\mathrm{e}^{\theta t}\,\mathbb{E}_x\left[\int_0^t \psi(B_s)\left(1-\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\right)dl_s\right]dx.
\end{aligned}$$

The first term converges to the expected expression by lemma 9.2. The second one goes to 0 as t decreases to 0, since for $t \leq 1$,

$$\begin{aligned}
&\left|\int_D \phi(x)\frac{1}{t}\,\mathrm{e}^{\theta t}\,\mathbb{E}_x\left[\int_0^t \psi(B_s)\left(1-\mathrm{e}^{-\theta s}\,\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_s}\right)dl_s\right]dx\right|\\
&\qquad\leq C\int_D \frac{1}{t}\mathbb{E}_x\left[\int_0^t\left(\theta s+\frac{1}{2}(\kappa\cdot l)_s\right)dl_s\right]dx\\
&\qquad\leq C\int_D \mathbb{E}_x[l_t]dx + C\int_D \frac{1}{t}\mathbb{E}_x[l_t^2]dx
\end{aligned}$$

thanks to (26) for the first term and lemma 9.1 for the second. □

Lemma 9.5. *For every bounded measurable function ϕ on $\bar{D}$ and every continuous function ψ on $\bar{D}$,*

$$\lim_{t\to+\infty}\int_D \phi(x)\frac{1}{t}\mathbb{E}_x\left[\psi(B_t)\left(1-\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_t}\right)\right]dx=\frac{1}{2}\int_{\partial D}\phi(y)\psi(y)\kappa(y)\sigma(dy).$$

Proof. We write

$$\begin{aligned}
&\int_D \phi(x)\frac{1}{t}\mathbb{E}_x\left[\psi(B_t)\left(1-\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_t}\right)\right]dx\\
&\quad=\int_D \phi(x)\frac{1}{t}\mathbb{E}_x\left[\psi(B_t)\frac{1}{2}(\kappa\cdot l)_t\right]dx\\
&\qquad+\int_D \phi(x)\frac{1}{t}\mathbb{E}_x\left[\psi(B_t)\left(1-\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_t}-\frac{1}{2}(\kappa\cdot l)_t\right)\right]dx\\
&\quad=\int_D \phi(x)\psi(x)\frac{1}{t}\mathbb{E}_x\left[\frac{1}{2}(\kappa\cdot l)_t\right]dx\\
&\qquad+\int_D \phi(x)\frac{1}{t}\mathbb{E}_x\Big[\big(\psi(B_t)-\psi(x)\big)\frac{1}{2}(\kappa\cdot l)_t\Big]dx\\
&\qquad+\int_D \phi(x)\frac{1}{t}\mathbb{E}_x\left[\psi(B_t)\left(1-\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_t}-\frac{1}{2}(\kappa\cdot l)_t\right)\right]dx.
\end{aligned}$$

The first term converges to the sought-after term by lemma 9.2. The third term is bounded from above by

$$\left|\int_D \phi(x)\frac{1}{t}\mathbb{E}_x\left[\psi(B_t)\left(1-\mathrm{e}^{-\frac{1}{2}(\kappa\cdot l)_t}-\frac{1}{2}(\kappa\cdot l)_t\right)\right]dx\right|\leq C\int_D \frac{1}{t}\mathbb{E}_x[l_t^2]dx$$

and so converges to 0 by lemma 9.1.

For the second term, we fix $\varepsilon > 0$. As ψ is continuous on $\bar{D}$, it is uniformly continuous and there exists a $\eta > 0$ such that

$$\forall x, y \in \bar{D},\ |x - y| < \eta \Longrightarrow \left|\psi(x) - \psi(y)\right| < \varepsilon.$$

Now, we write

$$\begin{aligned}
&\left|\int_D \frac{1}{t}\mathbb{E}_x\left[\big(\psi(B_t) - \psi(x)\big)\frac{1}{2}(\kappa \cdot l)_t\right]dx\right| \\
&\quad \le C \int_D \frac{1}{t}\mathbb{E}_x\Big[\big|\psi(B_t) - \psi(x)\big|\mathbf{1}_{|B_t - x| < \eta} l_t\Big]dx \\
&\qquad + C \int_D \frac{1}{t}\mathbb{E}_x\Big[\big|\psi(B_t) - \psi(x)\big|\mathbf{1}_{|B_t - x| \ge \eta} l_t\Big]dx \\
&\quad \le \varepsilon \int_D \frac{1}{t}\mathbb{E}_x[l_t]dx + C \int_D \frac{1}{t}\mathbb{E}_x[\mathbf{1}_{|B_t - x| \ge \eta} l_t]dx \\
&\quad \le \varepsilon \int_D \frac{1}{t}\mathbb{E}_x[l_t]dx + C \int_D \frac{1}{t}\mathbb{E}_x[l_t^2]^{1/2}\mathbb{P}(|B_t - x| \ge \eta)^{1/2}dx \\
&\quad \le \varepsilon \int_D \frac{1}{t}\mathbb{E}_x[l_t]dx + C \int_D \frac{1}{\sqrt{t}}\mathbb{P}_x(|B_t - x| \ge \eta)^{1/2}dx,
\end{aligned}$$

where we used (26) for the last inequality. Now, by lemma 9.2, the first term is less than some constant times ε for t say less than 1. The second one goes to 0 as t goes to 0 and this complete the proof. $\square$

9.3. Linear boundary problem

Recall D is a bounded domain with C^3 boundary. We first recall some results on the Neumann problem. If $u \in C^1(\bar{D})$, let $\dfrac{\partial u}{\partial n}(x)$ denote the outward normal derivative of u at $x \in \partial D$. Let $\theta \ge 0$ and φ a bounded measurable function defined on ∂D. A function u is a strong solution to the Neumann problem $N(\varphi, \theta)$ if $u \in C^2(D) \cap C^1(\bar{D})$ and

$$\begin{aligned}
\frac{\Delta}{2}u - \theta u &= 0 \quad \text{in } D, \\
\frac{\partial u}{\partial n} &= \varphi \quad \text{on } \partial D.
\end{aligned} \tag{29}$$

A function u is a weak solution to the Neumann problem $N(\varphi, \theta)$ if $u \in C(\bar{D})$ and for any function ϕ such that $\phi \in C_b^2(D) \cap C^1(\bar{D})$ and $\partial\phi/\partial n = 0$ on ∂D, we have

$$\int_D u(x)\frac{\Delta}{2}\phi(x)\, dx - \theta \int_D u(x)\phi(x)\, dx = -\frac{1}{2}\int_{\partial D} \varphi(x)\phi(x)\, \sigma(dx).$$

From the Green formula, it is clear that any strong solution is a weak solution.

Using the local time l, we can represent solution to the Neumann problem in D. We refer to [9] for the next proposition.

Proposition 9.6. *Let $\theta > 0$ and φ be a bounded measurable function defined on ∂D. The function*

$$w_\theta(x) = \frac{1}{2}\,\mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-\theta s}\,\varphi(B_s)\,dl_s\right]$$

is continuous in $\bar{D}$. Furthermore it is also the only weak solution of the Neumann problem $N(\varphi,\theta)$.

If φ is more regular, then we get strong solution to $N(\varphi,\theta)$. For $k \in \mathbb{N}$, $\alpha \in (0,1]$, recall that $C^{k,\alpha}(\Omega)$ is the set of functions defined on Ω which are k times differentiable such that their k^{th} derivative is Hölder with parameter α.

From theorem 2.3 in [16], we have

Proposition 9.7. *Let $\theta > 0$, $f \in C^{0,\alpha}(\bar{D})$ and $\varphi \in C^{0,\alpha}(\partial D)$. Then, the function defined for $x \in \bar{D}$ by*

$$w_\theta(x) = -\mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-\theta s}\,f(B_s)\,ds\right] + \frac{1}{2}\,\mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-\theta s}\,\varphi(B_s)\,dl_s\right]$$

belongs to $C^2(D)\cap C^1(\bar{D})$ and solves

$$\begin{aligned} \frac{\Delta}{2}u - \theta u &= f \quad \textit{in } D,\\ \frac{\partial u}{\partial n} &= \varphi \quad \textit{on } \partial D. \end{aligned} \tag{30}$$

The next proposition is a consequence of (27) and (28).

Proposition 9.8. *Let $\theta > 0$, and f bounded measurable defined on D. The function defined on $\bar{D}$ by*

$$\mathbb{E}_x\left[\int_0^{+\infty} f(B_s)\,\mathrm{e}^{-\theta s}\;ds\right] = \int_0^\infty ds \int_D dy\; p_s(x,y) f(y)\,\mathrm{e}^{-\theta s}$$

belongs to $C^1(\bar{D})$.

From remark 6.3.2.4 in [8], we get the next result.

Proposition 9.9. *Let $\theta > 0$, $f \in C^{0,\alpha}(\bar{D})$ and $\varphi \in C^{1,\alpha}(\partial D)$. There exists a unique strong solution to (30). Furthermore it belongs to $C^{2,\alpha}(\bar{D})$.*

We end this section with well known results for the Dirichlet problem. Let $\theta \geq 0$ and f a measurable function defined on D and g a measurable function defined on ∂D. A function u is a strong solution to the Dirichlet problem if $u \in C^2(D)\cap C^0(\bar{D})$ and

$$\begin{aligned} \frac{\Delta}{2}u - \theta u &= f \quad \text{in } D,\\ u &= g \quad \text{on } \partial D. \end{aligned} \tag{31}$$

The next two results can be found in [7].

Proposition 9.10. *Let $\theta \geq 0$. Let g be a bounded measurable function defined on ∂D. The function defined on $\bar{D}$ by*

$$u_\theta(x) = \mathbb{E}_x \left[g(B_{\tau_D}) \, \mathrm{e}^{-\theta \tau_D} \right]$$

belongs to $C^\infty(D)$ and $\frac{\Delta}{2} u_\theta - \theta u_\theta = 0$ in D. Furthermore, if $g \in C^0(\partial D)$, then $u_\theta \in C^0(\bar{D})$ and $u_\theta = g$ on ∂D.

Proposition 9.11. *Let $\theta \geq 0$. Let f be a bounded measurable function defined on D. The function defined on $\bar{D}$ by*

$$u_\theta(x) = -\mathbb{E}_x \left[\int_0^{\tau_D} f(B_s) \, \mathrm{e}^{-\theta s} \; ds \right]$$

belongs to $C^{0,1}(D) \cap C^0(\bar{D})$ and $u_\theta = 0$ on ∂D. Furthermore, if $f \in C^{0,\alpha}(D)$, then $u_\theta \in C^{2,\alpha}(D)$ and $\frac{\Delta}{2} u_\theta - \theta u_\theta = f$ in D.

From remark 6.3.2.4 in [8], we get the next result.

Proposition 9.12. *Let $\theta \geq 0$, $f \in C^{0,\alpha}(\bar{D})$ and $g \in C^{2,\alpha}(\partial D)$. There exists a unique strong solution to (31). Furthermore it belongs to $C^{2,\alpha}(\bar{D})$.*

9.4. Proof of lemma 3.1

Assume $\alpha > 0$ and let $\varepsilon_0 > 0$ be fixed. For $x \in \bar{D}$, consider

$$u^{\alpha,\varepsilon}(x) = \mathbb{E}_x \left[\int_0^\infty \mathrm{e}^{-\alpha s} \, \frac{1}{\varepsilon} \, \mathbf{1}_{D_\varepsilon}(B_s) \; ds \right] = \int_0^\infty ds \;\, \mathrm{e}^{-\alpha s} \int dy \;\, p_s(x,y) \, \frac{1}{\varepsilon} \, \mathbf{1}_{D_\varepsilon}(y),$$

the α-potential of the continuous additive functional $\int_0^t \frac{1}{\varepsilon} \mathbf{1}_{D_\varepsilon}(B_s) \; ds$ for the reflected Brownian motion in $\bar{D}$. We deduce from (21) and (23) that for $\varepsilon \in (0, \varepsilon_0]$, $x \in \bar{D}$,

$$\left| u^{\alpha,\varepsilon}(x) \right| \leq \int_0^1 c \frac{ds}{\sqrt{s}} + \int_1^\infty \mathrm{e}^{-\alpha s} [c + a_D].$$

Therefore, $u^{\alpha,\varepsilon}$ is uniformly bounded in $\bar{D}$ for $\varepsilon \in (0, \varepsilon_0]$. From the continuity of the density p of B, (21) and (23), it is easy to deduce that $u^{\alpha,\varepsilon}$ converges as ε decreases to 0 to the α-potential of l, the local time on ∂D:

$$u^\alpha(x) = \mathbb{E}_x \left[\int_0^\infty \mathrm{e}^{-\alpha s} \;\, dl_s \right] = \int_0^\infty ds \;\, \mathrm{e}^{-\alpha s} \int \sigma(dy) \;\, p_s(x,y).$$

Furthermore this convergence is uniform in $\bar{D}$. Notice also that the continuity of the density p implies the uniform continuity of u^α and $u^{\alpha,\varepsilon}$ for $\varepsilon \in (0, \varepsilon_0]$ on $\bar{D}$.

Because $L^{\alpha,\varepsilon}$ depends only on the spatial motion W, the three other components of the Brownian snake, that is R, J and K doesn't play any role in what follows. However we shall keep the notation defined in section 2. Let $\tilde{x} \in E$ with first component $x \in \bar{D}$.

Now we compute $I = \mathbb{N}_{\tilde{x}}[L_\sigma^{\alpha,\varepsilon} L_\sigma^{\alpha,\varepsilon'}]$ and show it converges to a limit as ε and ε' decrease to 0. We have

$$I = \mathbb{N}_{\tilde{x}}\left[\int_0^\sigma dL_s^{\alpha,\varepsilon} \int_s^\sigma dL_u^{\alpha,\varepsilon'}\right] + \mathbb{N}_{\tilde{x}}\left[\int_0^\sigma dL_s^{\alpha,\varepsilon} \int_0^s dL_u^{\alpha,\varepsilon'}\right].$$

The time reversal property of the Itô measure and the properties of $\mathbb{N}_{\tilde{x}}$ readily imply that the latter itself enjoys the same invariance property. In particular the two terms of the right member are equal. From the Markov property of the Brownian snake (see [12]), we deduce that

$$I = 2\mathbb{N}_{\tilde{x}}\left[\int_0^\sigma dL_s^{\alpha,\varepsilon}\, \mathrm{E}^*_{\tilde{W}_s}[L_\sigma^{\alpha,\varepsilon'}]\right].$$

We deduce from proposition 2.1 of [13] that for $\tilde{w} \in \mathcal{W}_{\tilde{x}}$,

$$\mathrm{E}^*_{\tilde{w}}[L_\sigma^{\alpha,\varepsilon'}] = 2\int_0^\zeta dt\ \mathbb{N}_{\tilde{w}(t)}[\mathrm{e}^{-\alpha t}\, L_\sigma^{\alpha,\varepsilon'}],$$

where ζ is the lifetime of $\tilde{w}$. Therefore using formula (3) we get that

$$\mathbb{N}_{\tilde{x}}[L_\sigma^{\alpha,\varepsilon'}] = \mathbb{E}_{\tilde{x}}\left[\int_0^\infty \mathrm{e}^{-\alpha s}\, \frac{1}{\varepsilon'}\, \mathbf{1}_{D_{\varepsilon'}}(B_s)\, ds\right] = u^{\alpha,\varepsilon'}(x).$$

Thus, for $\tilde{w} \in \mathcal{W}_{\tilde{x}}$,

$$\mathrm{E}^*_{\tilde{w}}[L_\sigma^{\alpha,\varepsilon'}] = 2\int_0^\zeta dt\ \ \mathrm{e}^{-\alpha t}\, u^{\alpha,\varepsilon'}(W(t)). \tag{32}$$

Using (3) again, we get

$$\begin{aligned} I &= 4\int_0^\infty du\ \mathbb{E}_x\left[\mathrm{e}^{-\alpha u}\, \frac{1}{\varepsilon}\, \mathbf{1}_{D_\varepsilon}(B_u) \int_0^u dt\ \ \mathrm{e}^{-\alpha t}\, u^{\alpha,\varepsilon'}(B_t)\right] \\ &= 4\mathbb{E}_x\left[\int_0^\infty dt\ \ \mathrm{e}^{-\alpha t}\, u^{\alpha,\varepsilon}(B_t) u^{\alpha,\varepsilon'}(B_t)\right]. \end{aligned} \tag{33}$$

Since the function $u^{\alpha,\varepsilon}$ are uniformly bounded and converge as $\varepsilon \downarrow 0$, we deduce form dominated convergence that I converge as ε and ε' decrease to 0. This implies that $L_\sigma^{\alpha,\varepsilon}$ converge in $L^2(\mathbb{N}_{\tilde{x}})$.

Now we use standard techniques to prove the a.e. convergence of $L_s^{\alpha,\varepsilon}$ for $s \geq 0$ (see [13] p. 402). For $s > 0$, we set

$$M_s^\varepsilon = L_s^{\alpha,\varepsilon} + \mathrm{E}^*_{\tilde{W}_s}[L_\sigma^{\alpha,\varepsilon}].$$

The process $M^\varepsilon = (M_s^\varepsilon, s > 0)$ is continuous $\mathbb{N}_{\tilde{x}}$-a.e. thanks to the continuity of $u^{\alpha,\varepsilon}$ and (32). Since $L_\sigma^{\alpha,\varepsilon} \in L^1(\mathbb{N}_{\tilde{x}})$ (recall that $\mathbb{N}_{\tilde{x}}[L_\sigma^{\alpha,\varepsilon}] = u^{\alpha,\varepsilon}(x)$), we deduce from the Markov property of the Brownian snake that M^ε is a continuous martingale under $\mathbb{N}_{\tilde{x}}$. Notice that $M_\infty^\varepsilon = L_\sigma^{\alpha,\varepsilon}$ converges in $L^2(\mathbb{N}_{\tilde{x}})$ as $\varepsilon \downarrow 0$. From the maximal Doob inequality, we get for $\delta > 0$,

$$\mathbb{N}_{\tilde{x}}\left[\sup_{s>0}\left|M_s^\varepsilon - M_s^{\varepsilon'}\right| > \delta\right] \leq \frac{1}{\delta^2}\, \mathbb{N}_{\tilde{x}}[(L_\sigma^{\alpha,\varepsilon} - L_\sigma^{\alpha,\varepsilon'})^2].$$

In particular M^ε converges to a continuous martingale $M = (M_s, s > 0)$ and there exists a sequence $(\varepsilon_k, k \geq 1)$ decreasing to 0, such that $\mathbb{N}_{\tilde{x}}$-a.e. $\lim_{k\to\infty} \sup_{s>0} \left|M_s^{\varepsilon_k} - M_s\right| = 0$. Because of the uniform convergence of $u^{\alpha,\varepsilon}$, we deduce that $\mathbb{N}_{\tilde{x}}$-a.e. for all $s > 0$, $L_s^{\alpha,\varepsilon_k}$ converge to a limit

$$L_s^\alpha = M_s - 2\int_0^{\zeta_s} dt\ \mathrm{e}^{-\alpha t}\, u^\alpha(W_s(t)). \tag{34}$$

Therefore, the process $(L_s^\alpha, s > 0)$ is a continuous additive functional of the Brownian snake.

The measures $dL_s^{\alpha,\varepsilon_k}$ on $\mathbb{R}^+$ converge weakly to dL_s^α. The function $f_{\alpha'}(s) = \mathrm{e}^{(\alpha-\alpha')\zeta_s}$ defined on $\mathbb{R}^+$ is continuous and bounded for any $\alpha' \geq 0$, $\mathbb{N}_{\tilde{x}}$-a.e. We deduce that the measure $dL_s^{\alpha',\varepsilon} = f_{\alpha'}(s) dL_s^{\alpha,\varepsilon}$ converges weakly to $dL_s^{\alpha'} = f_{\alpha'}(s) dL_s^\alpha$. We write L for L^0. The first part of the lemma is proved.

Let F be a nonnegative continuous function defined on $\mathcal{W}_{\tilde{x}}$. Assume F is bounded from above by a. From (3), we have

$$\mathbb{N}_{\tilde{x}}\left[\int_0^\sigma F(W_s) dL_s^{\alpha,\varepsilon_k}\right] = \mathbb{E}_{\tilde{x}}\left[\int_0^\infty F(\Theta^{(u)})\, \mathrm{e}^{-\alpha u}\, \frac{1}{\varepsilon_k} \mathbf{1}_{D_{\varepsilon_k}}(B_u)\, du\right]. \tag{35}$$

From theorem 7.2 of [16], we get that the right member converges, as ε decreases to 0, to

$$\mathbb{E}_{\tilde{x}}\left[\int_0^\infty F(\Theta^{(u)})\, \mathrm{e}^{-\alpha u}\ dl_u\right].$$

To prove the convergence of $\mathbb{N}_{\tilde{x}}\left[\int_0^\sigma F(\tilde{W}_s) dL_s^{\alpha,\varepsilon_k}\right]$ to $\mathbb{N}_{\tilde{x}}\left[\int_0^\sigma F(\tilde{W}_s) dL_s^\alpha\right]$, using Fatou's lemma with F and $a - F$, we see it is enough to check that $\mathbb{N}_{\tilde{x}}\left[L_\sigma^{\alpha,\varepsilon_k}\right]$ converges to $\mathbb{N}_{\tilde{x}}\left[L_\sigma^\alpha\right]$.

We have from the convergence of $u^{\alpha,\varepsilon}$ that

$$\lim_{k\to\infty} \mathbb{N}_{\tilde{x}}\left[L_\sigma^{\alpha,\varepsilon_k}\right] = \lim_{k\to\infty} u^{\alpha,\varepsilon_k}(x) = u^\alpha(x).$$

Thanks to the upper bound of $u^{\alpha,\varepsilon}$, we deduce from (32) by dominated convergence that $\mathbb{N}_{\tilde{x}}[\mathrm{E}^*_{\tilde{W}_s}[L_\sigma^{\alpha,\varepsilon_k}]]$ converge to $\mathbb{N}_{\tilde{x}}[2\int_0^{\zeta_s} dt\ \mathrm{e}^{-\alpha t}\, u^\alpha(W_s(t))]$.

Notice that for $s > 0$, we have from (34)

$$\mathbb{N}_{\tilde{x}}\left[L_\sigma^\alpha\right] = \mathbb{N}_{\tilde{x}}[M_\infty] = \mathbb{N}_{\tilde{x}}[L_s^\alpha] + \mathbb{N}_{\tilde{x}}\left[2\int_0^{\zeta_s} dt\ \mathrm{e}^{-\alpha t}\, u^\alpha(W_s(t))\right]. \tag{36}$$

Using the law of ζ_s under the Itô measure, we have

$$\begin{aligned}\mathbb{N}_{\tilde{x}}\left[2\int_0^{\zeta_s} dt\ \mathrm{e}^{-\alpha t}\, u^\alpha(W_s(t))\right] &= 2\int_0^\infty \mathrm{e}^{-\alpha t}\, \mathbb{E}_x[u^\alpha(B_t)]\, \frac{1}{\sqrt{2\pi s}}\, \mathrm{e}^{-t^2/2s}\ dt\\ &= 2\int_0^\infty \mathrm{e}^{-\alpha\sqrt{s}r}\, \mathbb{E}_x[u^\alpha(B_{\sqrt{s}r})]\, \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-r^2/2}\ dr,\end{aligned}$$

where we set $r = t/\sqrt{s}$. From dominated convergence, using the continuity of u^α and the continuity of the path B, we see that $\lim_{s\to 0} \mathbb{N}_{\tilde{x}}[2\int_0^{\zeta_s} dt \;\mathrm{e}^{-\alpha t}\, u^\alpha(W_s(t))] = u^\alpha(x)$. Using Fatou's lemma we get

$$\begin{aligned}
\mathbb{N}_{\tilde{x}}[L_s^\alpha] &\le \liminf_{k\to\infty} \mathbb{N}_{\tilde{x}}[L_s^{\alpha,\varepsilon_k}] \\
&= \lim_{k\to\infty} \mathbb{N}_{\tilde{x}}[L_\sigma^{\alpha,\varepsilon_k}] - \lim_{k\to\infty} \mathbb{N}_{\tilde{x}}[\mathrm{E}^*_{W_s}[L_\sigma^{\alpha,\varepsilon_k}]] \\
&= u^\alpha(x) - 2\int_0^\infty \mathrm{e}^{-\alpha\sqrt{s}r}\, \mathbb{E}_x[u^\alpha(B_{\sqrt{s}r})]\, \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-r^2/2}\; dr.
\end{aligned}$$

We see that $\lim_{s\to 0} \mathbb{N}_{\tilde{x}}[L_s^\alpha] = 0$. Therefore we deduce from equation (36) that

$$\mathbb{N}_{\tilde{x}}\left[L_\sigma^\alpha\right] = \lim_{s\to 0} \mathbb{N}_{\tilde{x}}\left[2\int_0^{\zeta_s} dt \;\mathrm{e}^{-\alpha t}\, u^\alpha(W_s(t))\right] = u^\alpha(x).$$

As we said, this implies the convergence of $\mathbb{N}_{\tilde{x}}\left[\int_0^\sigma F(\tilde{W}_s) dL_s^{\alpha,\varepsilon_k}\right]$ to $\mathbb{N}_{\tilde{x}}\left[\int_0^\sigma F(\tilde{W}_s) dL_s^\alpha\right]$. From (35), we deduce that

$$\mathbb{N}_{\tilde{x}}\left[\int_0^\sigma F(\tilde{W}_s)\,\mathrm{e}^{-\alpha\zeta_s}\, dL_s\right] = \mathbb{E}_{\tilde{x}}\left[\int_0^\infty F(\Theta^{(u)})\,\mathrm{e}^{-\alpha u}\; dl_u\right].$$

This hold for any bounded continuous function F. By monotone class theorem, this holds also for all nonnegative measurable function F. By monotone convergence, let $\alpha \downarrow 0$ to prove the end of the lemma.

9.5. Proof of lemmas 6.5 and 6.6

Proof of lemma 6.5. Since $g \in C^{2,\alpha}(\partial D)$, we deduce from propositions 9.10 and 9.12 that $\mathbb{E}_x\left[g(B_{\tau_D})\,\mathrm{e}^{-\theta\tau_D}\right]$ belongs to $C^{2,\alpha}(\bar{D})$ and solve (31) with $f = 0$. Since u_θ is bounded, we get from proposition 9.11, that $\mathbb{E}_x\left[\int_0^{\tau_D} u_\theta(B_s)^2\,\mathrm{e}^{-\theta s}\; ds\right]$ belongs to $C^{0,1}(D) \cap C^0(\bar{D})$. From (13), we deduce that u_θ itself belongs to $C^{0,1}(D)$. Using proposition 9.11 and (13) again, we get that u_θ belongs to $C^{2,\alpha}(D)$. We see from (13), we need to check the regularity of $h(x) = \mathbb{E}_x\left[\int_0^{\tau_D} u_\theta(B_s)^2\,\mathrm{e}^{-\theta s}\; ds\right]$ on ∂D to end the proof of this lemma.

Notice that

$$h(x) = H(x) - \mathbb{E}_x\left[H(B_{\tau_D})\,\mathrm{e}^{-\theta\tau_D}\right],$$

where $H(x) = \mathbb{E}_x\left[\int_0^\infty u_\theta(B_s)^2\,\mathrm{e}^{-\theta s}\; ds\right]$. Since u_θ is bounded we have, thanks to proposition 9.8 that H belongs to $C^1(\bar{D})$.

The proof will be complete, once we prove that $\mathbb{E}_x\left[H(B_{\tau_D})\,\mathrm{e}^{-\theta\tau_D}\right]$ belongs to $C^{0,1/2}(\bar{D})$. Indeed, from (13), we then will get that $u_\theta \in C^{0,1/2}(\bar{D})$. This in turn, will imply thanks to proposition 9.12, that $h \in C^{2,\alpha'}(\bar{D})$, with $\alpha' = \min(\alpha, \frac{1}{2})$. From (13), we will deduce that $u_\theta \in C^{2,\alpha'}(\bar{D})$.

To prove that $\mathbb{E}_x\left[H(B_{\tau_D})\,\mathrm{e}^{-\theta\tau_D}\right] \in C^{0,1/2}(\bar{D})$, we will check that if $g \in C^1(\partial D)$, then the function $w(x) = \mathbb{E}_x\left[g(B_{\tau_D})\,\mathrm{e}^{-\theta\tau_D}\right]$ belongs to $C^{0,1/2}(\bar{D})$. Notice B can be replaced by a Brownian motion in $\mathbb{R}^d$, say B', in the definition of w.

Let $D(x) = \{y \in \mathbb{R}^d;\ y + x \in D\}$, and $\tau_{D(x)} = \inf\{t > 0, B'_t \notin D(x)\}$ the exit time of $D(x)$ for B'. Since $g \in C^1(\partial D)$, we have for $x, y \in \bar{D}$,

$$\begin{aligned}
|w(x) - w(y)| &= \left|\mathbb{E}_0\left[g(B'_{\tau_{D(x)}})\,\mathrm{e}^{-\theta\tau_{D(x)}}\right] - \mathbb{E}_0\left[g(B'_{\tau_{D(y)}})\,\mathrm{e}^{-\theta\tau_{D(y)}}\right]\right| \\
&\le c\mathbb{E}_0\left[\left|B'_{\tau_{D(x)}} - B'_{\tau_{D(y)}}\right|\right] + c\mathbb{E}_0\left[\left|\tau_{D(x)} - \tau_{D(y)}\right|\right] \\
&\le c\mathbb{E}_0\left[(B'_{\tau_{D(x)}} - B'_{\tau_{D(y)}})^2\right]^{1/2} + c\mathbb{E}_0\left[\left|\tau_{D(x)} - \tau_{D(y)}\right|\right] \\
&\le c\mathbb{E}_0\left[\left|\tau_{D(x)} - \tau_{D(y)}\right|\right]^{1/2} + c\mathbb{E}_0\left[\left|\tau_{D(x)} - \tau_{D(y)}\right|\right] \\
&\le c\mathbb{E}_0\left[\left|\tau_{D(x)} - \tau_{D(y)}\right|\right]^{1/2},
\end{aligned}$$

since $\sup_{z\in D}\mathbb{E}_0[\tau_{D(z)}] = \sup_{z\in D}\mathbb{E}_z[\tau_D] < \infty$. This last inequality is a consequence of propositions 9.11 and 9.12 with $\theta = 0$, $\varphi = 0$ and $f = 1$, so that the function $F(z) = \mathbb{E}_z[\tau_D]$ belongs to $C^{2,1}(\bar{D})$. Using the strong Markov property of B at time $\tau = \tau_{D(x)} \wedge \tau_{D(y)}$, we get

$$\begin{aligned}
\mathbb{E}_0\left[\left|\tau_{D(x)} - \tau_{D(y)}\right|\right] &= \mathbb{E}_0\left[\mathbb{E}_{B_\tau}\left[\tau_{D(x)} + \tau_{D(y)}\right]\right] \\
&\le \sup_{z\in D;\, d(z,\partial D)\le|x-y|} \mathbb{E}_z[\tau_D] \\
&\le c\,|x-y|,
\end{aligned}$$

because the function $F(z) = \mathbb{E}_z[\tau_D]$ belongs to $C^1(\bar{D})$. In conclusion there exists a constant $c > 0$ such that for $x, y \in \bar{D}$,

$$|w(x) - w(y)| \le c\sqrt{|x-y|}.$$

That is $w \in C^{0,1/2}(\bar{D})$. □

Proof of lemma 6.6. By symmetry we get that w_N is radial. For $y \in B(0, r_0)$, we have $w_N(x_0 - r_0 n_{x_0} + y) = h(|y|)$, and h is defined on $[0, r_0]$ and of class C^2. The function h is the unique nonnegative solution of

$$\begin{aligned}
&h''(r) + \frac{d-1}{r}h'(r) = 4h(r)^2 \quad \text{for } r \in (0, r_0), \\
&h'(0) = 0 \quad \text{and} \quad h(r_0) = N.
\end{aligned} \tag{37}$$

From the maximum principle, we get that for $r \in (0, r_0]$, $y \in B(x_0 - r_0 n_{x_0}, r)$,

$$w_N(y) < \max_{z\in\partial B(x_0 - r_0 n_{x_0}, r)} w_N(z) = h(r).$$

This implies the function h is increasing over $[0, r_0]$. Since, from the maximum principle, $w_N(x_0 - r_0 n_{x_0}) > 0$ we have $h > 0$.

Let $t = \inf\{r \in (0, r_0]; h''(r) \le 0\}$, with the convention $\inf \emptyset = +\infty$. We first assume that $t > 0$. If $t \le r_0$, from the continuity of h'' we deduce that $h''(t) = 0$ and from (37) that $h'(t) > 0$. By differentiating (37), we get that $h'''(t) > 0$. This contradict the fact $h''(t - \varepsilon) > h''(t) = 0$ for any $\varepsilon > 0$ small enough. Hence we have either $t = 0$ or $t = +\infty$.

If $t = 0$, there is a sequence $(t_k > 0, k \ge 1)$ decreasing to 0 such that $h''(t_k) \le 0$. Since $h \in C^2([0, r_0])$, we get $h''(0) \le 0$ by continuity. Since $h' \ge 0$ and $h'(0) = 0$, this implies that $h''(0) = 0$ and $\lim_{r\downarrow 0} h'(r)/r = 0$. Let $r \downarrow 0$ in (37) to get $h(0) = 0$, which is absurd since $w_N > 0$ in $B(x_0 - r_0 n_{x_0}, r_0)$. Therefore, we have $t = +\infty$.

In conclusion, we get that $h''(r) > 0$ on $(0, r_0]$. This implies that

$$rh''(r) + h'(r) \ge h'(r) \quad \text{on } [0, r_0].$$

By integration we deduce that

$$h'(r_0) \ge \frac{N - h(0)}{r_0}.$$

Notice that w_N is bounded from above by the maximal solution $w_{\max}$ of $\Delta u = 4u^2$ in $B(x_0 - r_0 n_{x_0}, r_0)$. This implies

$$h(0) = w_N(x_0 - r_0 n_{x_0}) \le w_{\max}(x_0 - r_0 n_{x_0}) = c_0,$$

where the constant c_0 depends only on r_0 and d. This end the proof of the lemma since $h'(r_0) = \phi_N(x_0)$. □

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