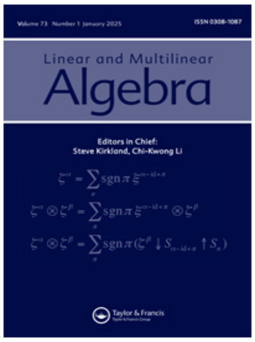

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Transformations preserving the effective spectral radius of a matrix

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ABSTRACT
We discuss transformations on matrices that preserve the effective spectrum and/or the effective spectral radius.

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1. Effective spectral radius of a matrix

Let $n \in \mathbb{N}^* = \{n > 0 : n \in \mathbb{Z}\}$. For K a square matrix of size n, let $\mathrm{spec}(K)$ and $\rho(K) = \max\{|\lambda| : \lambda \in \mathrm{spec}(K)\}$ denote the spectrum and spectral radius of K. By the Perron-Frobenius theorem, if K has nonnegative entries, then $\rho(K)$ is an eigenvalue and thus belongs to $\mathrm{Spec}(K)$. For $\eta \in \mathbb{R}_+^n$, let $\mathrm{Diag}(\eta)$ denote the diagonal matrix with entries given by η. We now define the effective spectrum and effective spectral radius functions associated to K.

Definition 1.1: Let K be a square matrix of size $n \in \mathbb{N}^*$. The effective spectrum $\mathrm{Spec}[K]$ and effective spectral radius $R_e[K]$ functions are defined on $\mathbb{R}_+^n$ by:

$$\mathrm{Spec}[K](\eta) = \mathrm{spec}(K \cdot \mathrm{Diag}(\eta)) \quad \text{and} \quad R_e[K](\eta) = \rho(K \cdot \mathrm{Diag}(\eta)) \quad \text{for} \quad \eta \in \mathbb{R}_+^n.$$

Trivially, two matrices with the same effective spectrum have the same effective spectral radius.

Motivated by the quantitative effect of vaccination strategies on the reproduction number in epidemic models, see [1, Theorem 7.4] or [2] in a more general framework, we give in Ref. [3] examples of transformations on positive compact operators that leave the effective spectral radius and effective spectrum functions invariant (mainly transposition and

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diagonal similarity, see below). The aim of this note is to explore further this invariance property in a finite-dimensional setting, that is, for matrices.

Our main result is a characterization of the equality between effective spectrum, shown in Section 2. Building in particular on results from Refs [4, 5], we then give in Section 3 sufficient conditions for two matrices to have the same effective spectral radius, and show that they are necessary under various additional assumptions.

2. Equivalent conditions for equality

Let us first define some notation. For α and β non-empty subsets of $\{1, \dots, n\}$ we denote by $K[\alpha, \beta]$ the sub-matrix of K obtained by keeping the lines in α and the columns in β, and let $K[\alpha] = K[\alpha, \alpha]$. The determinant of $K[\alpha]$ is called a *principal minor* of K, its *size* is the cardinal of α. It is elementary to check that the characteristic polynomial of K may be written as:

$$\chi_K(t) = \sum_{k=0}^{n} (-1)^k c_{n-k} t^k, \tag{1}$$

where $c_0 = 1$ and, for $j \geq 1$, c_j is the sum of all principal minors of size j of K.

We now give equivalent conditions for the effective spectrum and effective spectral radius for two matrices to be equal.

Theorem 2.1 (Effective spectrum and principal minors): *Let K and $\tilde{K}$ be square matrices of the same size $n \in \mathbb{N}^*$ with nonnegative entries. The following are equivalent.*

- (i) *The functions $R_e[K]$ and $R_e[\tilde{K}]$ coincide on $\mathbb{R}_+^n$.*
- (ii) *The functions $R_e[K]$ and $R_e[\tilde{K}]$ coincide on $\{0, 1\}^n$.*
- (iii) *The functions $\mathrm{Spec}[K]$ and $\mathrm{Spec}[\tilde{K}]$ coincide on $\mathbb{R}_+^n$.*
- (iv) *The functions $\mathrm{Spec}[K]$ and $\mathrm{Spec}[\tilde{K}]$ coincide on $\{0, 1\}^n$.*
- (v) *All principal minors of K and $\tilde{K}$ coincide.*

For simplicity, we write $\mathscr{E}(n) = \{0, 1\}^n$.

Proof: Clearly (iii)⇒(i)⇒(ii), and (iii)⇒(iv)⇒(ii).

Let us check that (v) implies (iii). Assume that all principal minors of K and $\tilde{K}$ coincide. For any vector $\eta \in \mathbb{R}^n$ and any set of indices α, by multi-linearity of the determinant, we get:

$$\det\Big((K \cdot \mathrm{Diag}(\eta))[\alpha]\Big) = \left(\prod_{i \in \alpha} \eta_i\right) \det\left(K[\alpha]\right).$$

Consequently, all principal minors of $K \cdot \mathrm{Diag}(\eta)$ and $\tilde{K} \cdot \mathrm{Diag}(\eta)$ coincide. By (1) this implies that $K \cdot \mathrm{Diag}(\eta)$ and $\tilde{K} \cdot \mathrm{Diag}(\eta)$ have the same spectrum. Thus, Point (iii) holds.

Therefore, it is enough to prove that (ii) implies (v). The proof is an induction on the dimension. The result is clear in dimension 1. Assume that it holds for any square matrix with nonnegative entries of dimension smaller than or equal to n. Let K and $\tilde{K}$ be two square matrices of dimension $n+1$ with nonnegative entries, and assume that $R_e[K]$ and

$R_e[\tilde{K}]$ coincide on $\mathscr{E}(n+1)$. For any non-empty $\alpha \subset \{1,\dots,n+1\}$, let η_α be the column vector $(\mathbb{1}_\alpha(i), 1 \le i \le n+1)$, that is, with 1 in position α and 0 otherwise. Notice that for any matrix K':

$$R_e[K'](\eta_\alpha) = \rho(K' \cdot \mathrm{Diag}(\eta_\alpha)) = \rho(K'[\alpha]).$$

Fix $\alpha \subset \{1,\dots,n+1\}$ non-empty, with $\alpha \neq \{1,\dots,n+1\}$. Let $\beta \subset \alpha$ and set $\tilde{\eta}_\beta = (\mathbb{1}_\beta(i), i \in \alpha)$. We have:

$$\begin{aligned} R_e[K'[\alpha]](\tilde{\eta}_\beta) &= \rho(K'[\alpha] \cdot \mathrm{Diag}(\tilde{\eta}_\beta)) = \rho(K' \cdot \mathrm{Diag}(\eta_\alpha) \cdot \mathrm{Diag}(\eta_\beta)) = \rho(K' \cdot \mathrm{Diag}(\eta_\beta)) \\ &= R_e[K'](\eta_\beta). \end{aligned} \tag{2}$$

Since $\eta_\beta \in \mathscr{E}(n+1)$, we get $R_e[K](\eta_\beta) = R_e[\tilde{K}](\eta_\beta)$ for all $\beta \subset \alpha$. We deduce from (2) that $R_e[K[\alpha]] = R_e[\tilde{K}[\alpha]]$ on $\mathscr{E}(\mathrm{Card}\,\alpha)$. By the induction hypothesis, the principal minors of $K[\alpha]$ and $\tilde{K}[\alpha]$ are equal, that is all principal minors of size less than or equal to n of K and $\tilde{K}$ coincide. It remains to check that the determinants are the same. Since all principal minors of size less than or equal to n coincide, we deduce from (1) that:

$$\chi_K(t) - \det(K) = \chi_{\tilde{K}}(t) - \det(\tilde{K}). \tag{3}$$

Since K and K' have nonnegative entries, by Perron-Frobenius theorem, their spectral radius $\rho(K) = R_e[K](\mathbb{1})$ and $\rho(K') = R_e[K'](\mathbb{1})$ is also an eigenvalue, and thus a root of their characteristic polynomial. As $R_e[K](\mathbb{1}) = R_e[K'](\mathbb{1})$, we deduce from (3) that $\det(K) = \det(\tilde{K})$. This ends the proof of the induction step. ■

According to the proof of Theorem 2.1, we have that (v) implies (iii) and thus (i), (ii) and (iv) without assuming that the entries are nonnegative. We first investigate whether (i) from Theorem 2.1 implies (v) when the entries of the matrices have generals signs. Notice that $R_e[K] = R_e[\tilde{K}]$ automatically implies K and $\tilde{K}$ have the same entries on the diagonal up to their sign (evaluate the effective spectral radii on η with only one non-zero component). So in order for the equality of the effective spectral radii of K and $\tilde{K}$ to imply the equality of all principal minors, it is necessary to assume that the two matrices have the same sign on their diagonal, that is, $\mathrm{sign}(K_{ii}) = \mathrm{sign}(\tilde{K}_{ii})$ for all indices i. It is however not enough, see next example and lemma.

Example 2.1 (Same effective spectral radii do not imply same principal minors in general): Consider the following two matrices:

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{K} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We have $R_e[K] = R_e[\tilde{K}]$ on $\mathbb{R}_+^2$, but, even if all the principal minors of size 1 coincide, the principal minor of size two is different.

The key point is in fact the number of zeroes on the diagonal.

Lemma 2.1: *Let K and $\tilde{K}$ be square matrices of the same size $n \in \mathbb{N}^*$ with the same sign on their diagonal and having at most one zero term in their diagonal. If $R_e[K] = R_e[\tilde{K}]$ (on $\mathbb{R}_+^n$), then all principal minors of K and $\tilde{K}$ coincide.*

Proof: We adapt the proof of (ii) ⇒(v) from Theorem 2.1 which relies on an induction over the dimension. The result of Lemma 2.1 is clear in dimension 1. Assume that it holds for square matrices of dimension smaller than or equal to n. Let K and $\tilde{K}$ be two square matrices of dimension $n+1$ with the same sign on their diagonal and having at most one zero term in their diagonal, and assume that $R_e[K]$ and $R_e[\tilde{K}]$ coincide on $\mathbb{R}_+^{n+1}$. Fix $\alpha \subset \{1, \ldots, n+1\}$ non-empty, with $\alpha \neq \{1, \ldots, n+1\}$. Arguing as in (2), for any square matrix K' of size $n+1$ and $\eta \in \mathbb{R}^{n+1}$ such that $\eta(i) = 0$ if $i \notin \alpha$, we get, with $\eta' = (\eta(i), i \in \alpha)$, that:

$$R_e[K'[\alpha]](\tilde{\eta}) = R_e[K'](\eta).$$

Thus, we deduce that $R_e[K[\alpha]] = R_e[\tilde{K}[\alpha]]$ on $\mathbb{R}_+^{\mathrm{Card}\,\alpha}$. By the induction hypothesis the principal minors of $K[\alpha]$ and $\tilde{K}[\alpha]$ are equal, that is all principal minors of size less than or equal to n of K and $\tilde{K}$ coincide. It remains to check that the determinants are the same.

Since all principal minors of size less than or equal to n coincide, we deduce from (1) that $\chi_K(t) - \det(K) = \chi_{\tilde{K}}(t) - \det(\tilde{K})$. Multiplying all terms in the previous equality by $\prod_{i=1}^{n+1} \eta_i$, where $\eta \in \mathbb{R}_+^{n+1}$, we deduce that:

$$\chi_{K\eta}(t) - \det(K \cdot \mathrm{Diag}(\eta)) = \chi_{\tilde{K}\eta}(t) - \det(\tilde{K} \cdot \mathrm{Diag}(\eta)).$$

As there is at most one zero on the diagonal, without loss of generality (multiplying K and $\tilde{K}$ by -1 and using a permutation of the canonical basis of $\mathbb{R}^{n+1}$ if necessary), we can assume that $K_{11} = \tilde{K}_{11} = a > 0$. Taking $\eta = (1, \varepsilon, \ldots, \varepsilon)$ for $\varepsilon > 0$ small enough, we deduce that the spectral radius of $K \cdot \mathrm{Diag}(\eta)$ (resp. $\tilde{K} \cdot \mathrm{Diag}(\eta)$) is also a simple eigenvalue of $K \cdot \mathrm{Diag}(\eta)$ (resp. $\tilde{K} \cdot \mathrm{Diag}(\eta)$). As $R_e[K](\eta) = R_e[\tilde{K}](\eta)$, we deduce that $\det(K \cdot \mathrm{Diag}(\eta)) = \det(\tilde{K} \cdot \mathrm{Diag}(\eta))$ and thus $\det(K) = \det(\tilde{K})$. Thus, by induction, all the principal minors of K and $\tilde{K}$ coincide. ■

We now check that (ii) from Theorem 2.1 does not imply (i) or (v) when the entries of the matrices have general signs (even with positive entries on the diagonal).

Example 2.2 (Same effective spectral radii on Boolean vectors do not imply same effective spectral radius): Consider the following two matrices:

$$K = \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \quad \text{and} \quad \tilde{K} = \begin{pmatrix} 1 & -\gamma \\ \gamma & 1 \end{pmatrix},$$

where $\gamma > 0$ and $\beta = \sqrt{1+\gamma^2} - 1$. The eigenvalues of K are $\sqrt{1+\gamma^2}$ and $2 - \sqrt{1+\gamma^2}$; the eigenvalues of $\tilde{K}$ are $1 \pm \gamma\, i$. In particular, the two matrices have the same spectral radius $\sqrt{1+\gamma^2}$. The functions $R_e[K]$ and $R_e[\tilde{K}]$ clearly coincide on $\mathscr{E}(2)$. Since $\det(K) \neq \det(\tilde{K})$, we deduce that all the principal minors of K and $\tilde{K}$ do not coincide, and thus $R_e[K] \neq R_e[\tilde{K}]$ on $\mathbb{R}_+^2$ thanks to Lemma 2.1.

3. Matrices with the same effective spectral radius

Let us first recall a few notions. The matrix K is *irreducible* if $K[\alpha, \alpha^c] \neq 0$ for all subsets α such that α and α^c are non-empty. The non-empty subset α is irreducible for K if $K[\alpha]$

is irreducible. Let $\mathscr{A}(K)$ be the family of maximal irreducible sets for the inclusion, and consider the matrix $K^{\mathscr{A}}$ given by:

$$K^{\mathscr{A}}_{ij} = K_{ij} \quad \text{if } i,j \in \alpha \text{ for some } \alpha \in \mathscr{A}(K), \quad \text{and } K^{\mathscr{A}}_{ij} = 0 \text{ otherwise.}$$

The elements of $\mathscr{A}(K)$ correspond to the atoms of K in Ref. [3]. The map $K \mapsto K^{\mathscr{A}}$ is not linear.

The matrix K is *completely reducible* if $K[\alpha, \alpha^c] = 0$ implies $K[\alpha^c, \alpha] = 0$ whenever α and α^c are non-empty, or equivalently if $K = K^{\mathscr{A}}$. We have the following graph interpretation: consider the oriented graph $G = (V, E)$ with $V = \{1, \dots, n\}$ and $ij \in E$, that is ij is an oriented edge of G, if and only if $K_{ij} \neq 0$. Then the matrix K is irreducible if for any choice of vertices $i, j \in V$ there is an oriented path from i to j; the matrix K is completely reducible if for any vertices $i, j \in V$ there is an oriented path from i to j if and only if there is an oriented path from j to i.

Recall the matrix K is *diagonally similar* to a matrix $\tilde{K}$ if there exists a non-singular real diagonal matrix D such that $K = D \cdot \tilde{K} \cdot D^{-1}$. Notice that if K and $\tilde{K}$ have nonnegative entries one can assume without loss of generality that D is also nonnegative. We recall the following well-known result (see [3, Lemma 3.1 and Corollary 5.4] in the infinite-dimensional setting). For $\eta \in \mathbb{R}^n_+$, we denote $\mathbb{1}_{\{\eta>0\}}$ the vector whose i-th component is $\mathbb{1}_{\{\eta_i>0\}}$.

Lemma 3.1 (Sufficient conditions for equality of effective spectrum): *Let K and $\tilde{K}$ be square matrices of the same size $n \in \mathbb{N}^*$ with nonnegative entries. We have:*

(i) $\mathrm{Spec}[K] = \mathrm{Spec}[K^\top] = \mathrm{Spec}[K^{\mathscr{A}}]$.
(ii) *If K and $\tilde{K}$ are diagonally similar, then* $\mathrm{Spec}[K] = \mathrm{Spec}[\tilde{K}]$.
(iii) *For $\eta \in \mathbb{R}^n_+$, we have:*

$$\begin{aligned}\mathrm{Spec}[K \cdot \mathrm{Diag}(\eta)] = \mathrm{Spec}[\mathrm{Diag}(\eta) \cdot K] &= \mathrm{Spec}[\mathrm{Diag}(\mathbb{1}_{\{\eta>0\}}) \cdot K \cdot \mathrm{Diag}(\eta)] \\ &= \mathrm{Spec}[\mathrm{Diag}(\eta) \cdot K \cdot \mathrm{Diag}(\mathbb{1}_{\{\eta>0\}})].\end{aligned}$$

We now try to find necessary conditions for equality of effective spectra. In other words, we would like to see if there are others transformations of matrices that leave the effective spectrum invariant. Following Ref. [6], we introduce the notion of clan.

Definition 3.1 (Clans and clan-free matrix): Let K be a square matrix of size n. A subset α of $\{1, \dots, n\}$ is a *clan* if it satisfies $2 \leq \mathrm{Card}(\alpha) \leq n-2$, and the submatrices $K[\alpha, \alpha^c]$ and $K[\alpha^c, \alpha]$ have rank at most 1. The matrix K is *clan-free* if there exists no clan.

Remark 3.1: A square matrix of size $n \in \{1, 2, 3\}$ is automatically clan-free.

The following proposition gathers known results on necessary conditions for equality of principal minors, and therefore of effective spectrum.

Proposition 3.1: *Let K and $\tilde{K}$ be square matrices of the same size with nonnegative entries, and the same effective spectrum, that is, $R_e[K] = R_e[\tilde{K}]$.*

(i) *If K and $\tilde{K}$ are symmetric, then $\tilde{K} = K$.*
(ii) *If K is symmetric, then $\tilde{K}^{\mathscr{A}}$ is diagonally similar to K.*
(iii) *If K is irreducible and clan-free, then $\tilde{K}$ is diagonally similar to K or to $K^\top$.*

Proof: Applying Theorem 2.1, the principal minors of K and $\tilde{K}$ coincide. The results then follow directly from [7, Theorem 3.5], for the symmetric case, [4, Theorem 3] for the irreducible case when $n \leq 3$ (by Remark 3.1, there can be no clan in this case), and [5, Theorem 1] for the clan-free case when $n \geq 4$. ■

Finally, as a consequence of Theorem 2.1, we show that the clan-free assumption is needed and get an additional sufficient condition for equality. Assume that $\alpha = \{1, \dots, m\}$ is a clan for K (and thus $2 \leq m \leq n-2$). Then, there exists vectors v, w of size m, and b, c of size $n-m$ such that K may be written in block form as:

$$K = \begin{pmatrix} A & vb^\top \\ cw^\top & B \end{pmatrix}. \tag{4}$$

The choice of v, w, b, c is not unique in general. We say that:

$$\tilde{K} = \begin{pmatrix} A^\top & wb^\top \\ cv^\top & B \end{pmatrix} \tag{5}$$

is a *partial transpose* of K (note that the partial transpose is not unique in general).

Remark 3.2: Such transformations have been considered in the special case, where $v = w$ in [5, Lemma 5]; see also Ref. [6], where a similar transformation called *clan reversal* is introduced for skew-symmetric matrices.

Proposition 3.2: *If K is not clan-free, then we have $R_e[K] = R_e[\tilde{K}]$ for any partial transpose $\tilde{K}$ of K.*

Proof: Suppose that K has a clan α, and let $\tilde{K}$ be a partial transpose of K, so that K and $\tilde{K}$ may be given by (4) and (5). For any $\lambda \notin \mathrm{Spec}(B)$, using a classical formula for determinants of block matrices, we get:

$$\begin{aligned}
\det(K - \lambda I) &= \det(A - \lambda I - vb^\top (B - \lambda I)^{-1} cw^\top) \det(B - \lambda I), \\
\det(\tilde{K} - \lambda I) &= \det(A^\top - \lambda I - wb^\top (B - \lambda I)^{-1} cv^\top) \det(B - \lambda I) \\
&= \det(A - \lambda I - vc^\top ((B - \lambda I)^{-1})^\top bw^\top) \det(B - \lambda I).
\end{aligned}$$

Since $b^\top (B - \lambda I)^{-1} c$ is a one-dimensional matrix, it is equal to its transpose, so that $\det(K - \lambda I) = \det(\tilde{K} - \lambda I)$ are equal for all $\lambda \notin \mathrm{Spec}(B)$, and thus for all $\lambda \in \mathbb{C}$ by continuity. Consequently, the matrices K and $\tilde{K}$ have the same spectrum. For any β, it is easily seen that $K[\beta]$ and $\tilde{K}[\beta]$ are partial transposes of each other, so that $K[\beta]$ and $\tilde{K}[\beta]$ also have the same spectrum, and in particular the same spectral radius. Therefore $R_e[K]$ and $R_e[\tilde{K}]$ coincide as (i) and (ii) are equivalent in Theorem 2.1. ■

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