Adv. Appl. Prob. **53,** 537–574 (2021)
*doi:*10.1017/apr.2020.70

EXACT SIMULATION OF THE GENEALOGICAL TREE FOR A STATIONARY BRANCHING POPULATION AND APPLICATION TO THE ASYMPTOTICS OF ITS TOTAL LENGTH

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Abstract

We consider a model of a stationary population with random size given by a continuous-state branching process with immigration with a quadratic branching mechanism. We give an exact elementary simulation procedure for the genealogical tree of n individuals randomly chosen among the extant population at a given time. Then we prove the convergence of the renormalized total length of this genealogical tree as n goes to infinity; see also Pfaffelhuber, Wakolbinger and Weisshaupt (2011) in the context of a constant-size population. The limit appears already in Bi and Delmas (2016) but with a different approximation of the full genealogical tree. The proof is based on the ancestral process of the extant population at a fixed time, which was defined by Aldous and Popovic (2005) in the critical case.

Keywords: Stationary branching processes; real trees; genealogical trees; ancestral process; simulation

2010 Mathematics Subject Classification: Primary 60J80
Secondary 60J85

1. Introduction

Continuous-state branching (CB) processes are stochastic processes that can be obtained as the scaling limits of sequences of Galton–Watson processes when the initial number of individuals tends to infinity. They can thus be seen as a model for a large branching population. The genealogical structure of a CB process can be described by a continuum random tree (CRT), introduced first by Aldous [5] for the quadratic critical case; see also Le Gall and Le Jan [24] and Duquesne and Le Gall [18] for the general critical and sub-critical cases. We shall only consider the quadratic case; it is characterized by a branching mechanism ψ_θ:

$$\psi_\theta(\lambda) = \beta\lambda^2 + 2\beta\theta\lambda, \qquad \lambda \in [0, +\infty), \tag{1.1}$$

where $\beta > 0$ and $\theta \in \mathbb{R}$. The sub-critical (resp. critical) case corresponds to $\theta > 0$ (resp. $\theta = 0$). The parameter β can be seen as a time-scaling parameter, and θ as a population size parameter.

Received 18 April 2018; revision received 8 October 2020.

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In this model the population dies out almost surely (a.s.) in the critical and sub-critical cases. In order to model a branching population with stationary size distribution, which corresponds to what is observed at an ecological equilibrium, one can simply condition a sub-critical or a critical CB to not die out. This gives a Q-process (see Roelly-Coppoleta and Rouault [27], Lambert [23] and Abraham and Delmas [1]), which can also be viewed as a CB with a specific immigration. The genealogical structure of the Q-process in the stationary regime is a tree with an infinite spine. This infinite spine has to be removed if one adopts the immigration point of view; in this case the genealogical structure can be seen as a forest of trees. For $\theta > 0$, let $Z = (Z_t, t \in \mathbb{R})$ be this Q-process in the stationary regime, so that Z_t is the size of the population at time $t \in \mathbb{R}$. The process Z is a Feller diffusion (see for example Section 7 in [14]), solution of the stochastic differential equation

$$dZ_t = \sqrt{2\beta Z_t}\, dB_t + 2\beta(1 - \theta Z_t)dt, \tag{1.2}$$

where $(B_t, t \geq 0)$ is a standard Brownian motion. See Chen and Delmas [14] for studies on this model in a more general framework. See Section 3.2.4 for other contour processes associated with the process Z. Let A_t be the time to the most recent common ancestor of the population living at time t; see (5.3) for a precise definition. According to [14], we have $\mathbb{E}[Z_t] = 1/\theta$ and $\mathbb{E}[A_t] = 3/4\beta\theta$, so that θ is indeed a population size parameter and β is a time parameter.

Aldous and Popovic [6] (see also Popovic [26]) give a description of the genealogical tree of the extant population at a fixed time using the so-called ancestral process, which is a point process representation of the height of the branching points of a planar tree in a setting very close to $\theta = 0$ in the present model. We extend the presentation of [6] to the case $\theta \geq 0$, which can be summarized as follows. Set $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. The ancestral process (see Definition 3.1) is a point measure

$$\mathcal{A}(du, d\zeta) = \sum_{i \in \mathcal{I}} \delta_{u_i, \zeta_i}(du, d\zeta)$$

on $\mathbb{R}^* \times (0, +\infty)$ with countable many atoms, where u_i represents the (position of the) individual i in the extant population and ζ_i its 'age'. (The position 0 will correspond to the position of the immortal individual. The order on $\mathbb{R}$ provides a natural order on the individuals through their positions, which means that we are dealing with an ordered or planar genealogical tree.) From this ancestral process, we informally construct a genealogical tree $\mathfrak{T}(\mathcal{A})$ as follows. We view this process as a sequence of vertical segments in $\mathbb{R}^2$, the tops of the segments being the u_i and their lengths being the ζ_i. We add the half-line $\{0\} \times (-\infty, 0]$ to this collection of segments. We then attach the bottom of each segment such that $u_i > 0$ (resp. $u_i < 0$) to the first longer segment to the left (resp. right) of it. See Figure 1 for an example. This provides the planar tree $\mathfrak{T}(\mathcal{A})$ associated with the ancestral process $\mathcal{A}$ (see Proposition 3.1 for the definition and properties of this locally compact real tree with a unique semi-infinite branch). To state our result, we decompose the extant population at time t into two sub-populations so that its size Z_t is distributed as $E_\mathrm{g} + E_\mathrm{d}$, where E_g (resp. E_d) is the size of the population grafted on the left (resp. on the right) of the infinite spine. Below we state the main result of Section 3; see Propositions 3.2. and 3.3.

Theorem 1.1. *Let $\theta \geq 0$. Let $E_\mathrm{g}, E_\mathrm{d}$ be independent exponential random variables with mean $1/2\theta$, and with the convention that $E_\mathrm{d} = E_\mathrm{g} = +\infty$ if $\theta = 0$. Conditionally given $(E_\mathrm{g}, E_\mathrm{d})$, the ancestral process $\mathcal{A}(du, d\zeta)$ is a Poisson point measure with intensity*

$$\mathbf{1}_{(-E_\mathrm{g}, E_\mathrm{d})}(u)\, du\, |c'_\theta(\zeta)| d\zeta,$$

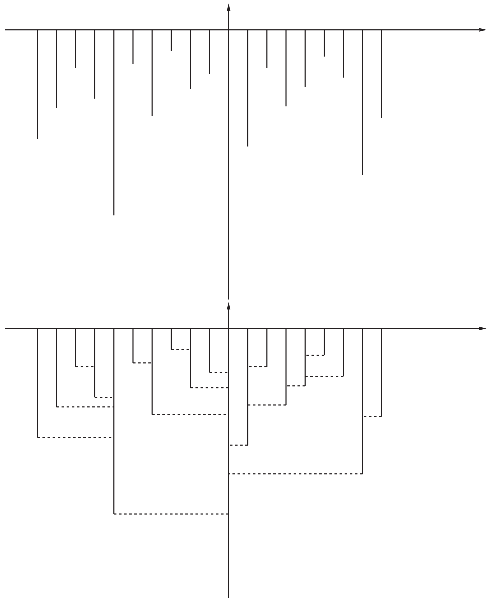

FIGURE 1: An example of an ancestral process and the corresponding genealogical tree.

where $c_\theta(h)$ is the probability (under the excursion measure) that the CB process with branching mechanism (1.1) will survive up to time h, and is given by

$$\forall h>0, \quad c_\theta(h)=\begin{cases} \frac{2\theta}{\mathrm{e}^{2\beta\theta h}-1} & \text{if } \theta>0 \ (\textit{sub-critical case}), \\ (\beta h)^{-1} & \text{if } \theta=0 \ (\textit{critical case}). \end{cases} \tag{1.3}$$

Furthermore, the tree $\mathfrak{T}(\mathcal{A})$ is distributed as the genealogical tree of the extant population at a fixed time $t\in\mathbb{R}$.

The ancestral process description allows us to give elementary exact simulations of the genealogical tree of n individuals randomly chosen in the extant population at time 0 (or at some time $t\in\mathbb{R}$, as the population has a stationary distribution). We present here the static simulation for fixed $n\geq 2$ given in Subsection 4.1; see also Lemma 4.1. See Figure 7 for an illustration for $n=5$.

Theorem 1.2. *Let $\theta>0$. Let $n\in\mathbb{N}^*$.*

(i) ***Size of the extant population****.*
Let $E_\mathrm{g}, E_\mathrm{d}$ be independent exponential random variables with mean $1/2\theta$. ($E_\mathrm{g}+E_\mathrm{d}$ corresponds to the size of the extant population.)

(ii) ***Picking n individuals in the extant population****.*
Let $(X_k, k\in\{1, \ldots, n\})$ be, conditionally on $(E_\mathrm{g}, E_\mathrm{d})$, independent uniform random variables on $[-E_\mathrm{g}, E_\mathrm{d}]$, and set $X_0=0$. (The individual 0 corresponds to the infinite spine.)

(iii) ***The 'age' of the individuals***.
For $k \in \{1, \dots, n\}$, set Δ_k as the length of the intermediate interval to the next X_j on the right if $X_k < 0$ or on the left if $X_k > 0$:

$$\Delta_k = \begin{cases} X_k - \max\{X_j,\ X_j < X_k \text{ and } 0 \le j \le n\} & \text{if } X_k > 0, \\ -X_k + \min\{X_j,\ X_j > X_k \text{ and } 0 \le j \le n\} & \text{if } X_k > 0. \end{cases}$$

Conditionally on $(E_{\mathrm{g}}, E_{\mathrm{d}}, X_1, \dots, X_n)$, let $(\zeta_k^{\mathrm{S}}, 1 \le k \le n)$ be independent random variables such that ζ_k^{S} is distributed as follows, where U is uniform on $[0, 1]$:

$$\frac{1}{2\theta\beta} \log\left(1 - \frac{2\theta\Delta_k}{\log(U)}\right).$$

(iv) ***The tree***.
Let $\mathfrak{T}_n^{\mathrm{S}}$ be the tree associated with the ancestral process

$$\sum_{k=1}^{n} \delta_{\left(X_k, \zeta_k^{\mathrm{S}}\right)}.$$

Then the tree $\mathfrak{T}_n^{\mathrm{S}}$ is distributed as the genealogical tree of n individuals picked uniformly at random among the extant population.

The notion of genealogical tree is appropriate for certain abstractions of genetic relations (e.g. mitochondrial DNA that is maternally inherited when ignoring paternal leakage or hetero-mitochondrial inheritance) in diploid organisms. It is however unclear how to extend our exact simulation algorithm to pedigree-conditioned genealogies as formalized in Sainudiin, Thatte, and Véber [29].

In the spirit of Theorem 1.2, we also provide two dynamic simulations in Subsections 4.2 and 4.3, where the individuals are taken one by one and the genealogical tree is then updated. Our framework allows us also to simulate the genealogical tree of n extant individuals conditionally given the time A_0 to the most recent common ancestor of the extant population; see Subsection 4.4. Let us stress that the existence of an elementary simulation method is new in the setting of branching processes (in particular because this method avoids the size-biased effect on the population which usually comes from picking individuals at random), and the question goes back to Lambert [22] and to [14, Theorem 4.7].

The ancestral process description also allows us to compute the limit distribution of the total length of the genealogical tree of the extant population at time $t \in \mathbb{R}$. More precisely, let $\Lambda_{t,n}$ be the total length of the tree of n individuals randomly chosen in the extant population at time t; see (5.5) for a precise definition. Below we state the main result of Section 5; see Theorem 5.1.

Theorem 1.3. *Let $\theta > 0$. The sequence $\left(\Lambda_{t,n} - \mathbb{E}[\Lambda_{t,n}|Z_t], n \in \mathbb{N}^*\right)$ converges a.s. and in L^2 towards a limit, say $\mathcal{L}_t$, as n tends to $+\infty$. And we have*

$$\mathbb{E}[\Lambda_{t,n}|Z_t] = \frac{Z_t}{\beta} \log\left(\frac{n}{2\theta Z_t}\right) + O(n^{-1}\log(n)).$$

In the Kingman coalescent model, which corresponds to a model where the population has constant size, the first coalescent time of n individuals randomly chosen in the extant population is distributed according to an exponential random variable with mean $2/n(n-1)$. We deduce that the total length of the genealogical tree of n individuals randomly chosen in the extant population at time t, say $L^K_{t,n}$, is distributed as

$$\sum_{k=2}^{n} k \times 2k^{-1}(k-1)^{-1}E_{k-1} = 2\sum_{k=1}^{n-1} k^{-1}E_k,$$

where $(E_k, k \in \mathbb{N}^*)$ are independent exponential random variables with mean 1. Elementary computation on independent exponential random variables gives that $L^K_{t,n}$ is also distributed as $2\max_{1\le k\le n-1} E_k$. This implies that $L^K_{t,n} - 2\log(n)$ converges in distribution towards a Gumbel distribution as n goes to infinity. (See Pfaffelhuber, Wakolbinger and Weisshaupt [25] for more results on this model.) The fact that the same shift in $\log(n)$ appears in the Kingman coalescent model and in Theorem 1.3 comes from the fact that the speed of coming down from infinity (or the birth rate of new branches near the top of the tree in forward time) is of the same order for the Kingman coalescent (see [8]) and for this model (see Corollary 6.5 and Remark 6.6 in [14]).

As part of Theorem 5.1, we also get that $\mathcal{L}_t$ coincides with the limit of the shifted total length L_ε of the genealogical tree up to $t-\varepsilon$ of the individuals alive at time t obtained in [11]: the sequence $(L_\varepsilon - \mathbb{E}[L_\varepsilon|Z_t], \varepsilon > 0)$ converges a.s. towards $\mathcal{L}_t$ as ε goes down to zero. Intuitively, taking ε_n (random) such $t-\varepsilon_n$ is the first time backward where the number of ancestors of the extant population at time t is n, we get that L_{ε_n} is distributed as $\Lambda_{t-\varepsilon_n,n}$. However, one expects that at the coalescent time $t-\varepsilon_n$ the size of the population $Z_{t-\varepsilon_n}$ is stochastically smaller than Z_t; see [14, Proposition 4.5]. Thus, $\Lambda_{t-\varepsilon_n,n}$ does not have the same distribution. But, as $\lim_{n\to\infty} \varepsilon_n = 0$, and thus $\lim_{n\to\infty} Z_{t-\varepsilon_n} = Z_t$, one expects that this difference might disappear in the limit. The proof of Theorem 1.3 is in fact based on technical L^2 computations.

We refer to [11] for properties of the stationary process $(\mathcal{L}_t, t \in \mathbb{R})$. We only recall that the Laplace transform of $\mathcal{L}_t$ is given, for $\lambda > 0$, by

$$\mathbb{E}\left[e^{-\lambda\mathcal{L}_t}|Z_t\right] = e^{-2\theta Z_t\,\varphi(\lambda/(2\beta\theta))} \quad \text{with} \quad \varphi(\lambda) = -\lambda\int_0^1 \frac{1-v^\lambda}{1-v}\,dv, \tag{1.4}$$

and that φ is the Laplace exponent of a Lévy process as

$$\varphi(\lambda) = \int_0^\infty (1-e^{-\lambda t} - \lambda t)\,\frac{e^t}{(e^t-1)^2}\,dt.$$

The paper is organized as follows. We first introduce in Section 2 the framework of real trees and we define the Brownian CRT that describes the genealogy of the CB in the quadratic case. Section 3 is devoted to the description via a Poisson point measure of the ancestral process of the extant population at time 0 and Section 4 gives the different simulations of the genealogical tree of n individuals randomly chosen in this population. Then, Section 5 concerns the asymptotic length of the genealogical tree for those n sampled individuals.

2. Notation

We set $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$, $\mathbb{N}^* = \{1, 2, \ldots, \}$, and $\mathbb{N} = \mathbb{N}^*\cup\{0\}$. Usually I will denote a generic index set, which might be finite, countable or uncountable.

2.1. Excursion measure for Brownian motion with drift

In this section we state some well-known results on excursion measures of the Brownian motion with drift. Let $B = (B_t, t \geq 0)$ be a standard Brownian motion and let $\beta > 0$ be fixed. Let $\theta \in \mathbb{R}$. We consider $B^{(\theta)} = (B_t^{(\theta)}, t \geq 0)$ a Brownian motion with drift -2θ and scale $\sqrt{2/\beta}$:

$$B_t^{(\theta)} = \sqrt{\frac{2}{\beta}}\, B_t - 2\theta t, \qquad t \geq 0. \tag{2.1}$$

Consider the minimum process $I^{(\theta)} = (I_t^{(\theta)}, t \geq 0)$ of $B^{(\theta)}$, defined by

$$I_t^{(\theta)} = \min_{u \in [0,t]} B_u^{(\theta)}.$$

Let $n^{(\theta)}(de)$ be the excursion measure of the process $B^{(\theta)} - I^{(\theta)}$ above 0 associated with its local time at 0 given by $-\beta I^{(\theta)}$. This normalization agrees with the one in [18] given for $\theta \geq 0$; see the remark below. Let $\sigma = \sigma(e) = \inf\{s > 0,\ e(s) = 0\}$ and $\zeta = \zeta(e) = \max_{s \in [0,\sigma]}(e_s)$ be the length and the maximum of the excursion e.

Remark 2.1. In this remark, we assume that $\theta > 0$ (i.e. the Brownian motion has a negative drift). In the framework of [18] (see Section 1.2 therein), $B^{(\theta)}$ is the height process which codes the Brownian continuum random tree (CRT) with branching mechanism ψ_θ defined by (1.1). It is obtained from the underlying Lévy process $X = (X_t, t \geq 0)$, which in the case of quadratic branching mechanism is the Brownian motion with drift: $X_t = \beta B_t^{(\theta)} = \sqrt{2\beta}\, B_t - 2\beta\theta t$ (see the formula (1.7) in [18]). According to [18, Section 1.1.2], considering the minimum process $I = (I_t, t \geq 0)$, with $I_t = \min_{u \in [0,t]} X_u$, the authors choose the normalization in such a way that $-I$ is the local time at 0 of $X - I$. The choice of the normalization of the local time at 0 of $B^{(\theta)} - I^{(\theta)}$ is justified by the fact that $I = \beta I^{(\theta)}$. Recall the definition of c_θ in (1.3). Then from Section 3.2.2 and Corollary 1.4.2 in [18], we have that for $\theta \geq 0$,

$$n^{(\theta)}\left[1 - \mathrm{e}^{-\lambda\sigma}\right] = \psi_\theta^{-1}(\lambda), \qquad \lambda > 0, \tag{2.2}$$

and

$$n^{(\theta)}(\zeta \geq h) = n^{(\theta)}(\zeta > h) = c_\theta(h), \qquad h > 0. \tag{2.3}$$

For $\theta \in \mathbb{R}$, let $\mathbb{P}_\theta^{\uparrow}(de)$ be the law of $B^{(\theta)} - 2I^{(\theta)}$. According to Proposition 14 and Theorem 20 in Section VII of [10], $\mathbb{P}_\theta^{\uparrow}(de)$ is the law of $B^{(\theta)}$ conditionally on being positive. For $\theta \in \mathbb{R}$, let n_θ be the excursion measure of $B^{(\theta)}$ outside 0 associated with the local time $L^0 = L^0(B^{(\theta)})$. For completeness, we give at the end of this section a proof of the following known result. Let $\mathcal{C}([0, +\infty))$ be the set of real-valued continuous functions defined on $[0, +\infty)$. Recall that, according to Definition (2.1) of $B^{(\theta)}$, the case $\theta < 0$ corresponds to a positive drift for the Brownian motion.

Lemma 2.1. *We have for $\theta \in \mathbb{R}$ and $A \in \mathcal{C}([0, +\infty))$ a measurable subset*

$$n_\theta(e \in A) = \frac{\beta}{2}\left[n^{(|\theta|)}(e \in A) + n^{(|\theta|)}(-e \in A) + 2|\theta|\mathbb{P}_\theta^{\uparrow}(-\mathrm{sgn}(\theta)e \in A)\right]. \tag{2.4}$$

We also have that

$$\mathbb{P}_\theta^{\uparrow}(de) = \mathbb{P}_{-\theta}^{\uparrow}(de) \quad \text{and} \quad n^{(\theta)}(de)\mathbf{1}_{\{\sigma < +\infty\}} = n^{(|\theta|)}(de), \tag{2.5}$$

and for $\theta < 0$

$$n^{(\theta)}(\sigma = +\infty) = 2|\theta| \quad \text{and} \quad n^{(\theta)}(de)\mathbf{1}_{\{\sigma=+\infty\}} = 2|\theta|\mathbb{P}_\theta^{\uparrow}(de). \tag{2.6}$$

Furthermore, if $\theta < 0$, then $-\beta I_\infty^{(\theta)}$ is exponentially distributed with parameter $2|\theta|$.

Remark 2.2. The excursion measure $n^{(\theta)}$ corresponds also to the excursion measure $\bar{n}^{(\theta)}$ introduced in [2] of the height process in the super-critical case, i.e. for $\theta < 0$. Indeed Corollary 4.4 in [2] gives that $\bar{n}^{(\theta)}(de)\mathbf{1}_{\{\sigma<+\infty\}} = n^{(|\theta|)}(de)$, and Lemma 4.6 in [2] gives that $\bar{n}^{(\theta)}(\sigma = +\infty) = 2|\theta|$.

Let $\theta \geq 0$ and let $\mathcal{N}(dh, d\varepsilon, de) = \sum_{i\in\mathcal{I}} \delta(h_i, e_i)(dh, de)$ be a Poisson point measure on $\mathbb{R}_+ \times \mathcal{C}([0, +\infty))$ with intensity $\beta\mathbf{1}_{\{h\geq 0\}}\, dh\, n^{(\theta)}(de)$. For every $i \in \mathcal{I}$, we set

$$a_i = \sum_{j\in\mathcal{I}} \mathbf{1}_{\{h_j<h_i\}}\sigma(e_j) \quad \text{and} \quad b_i = a_i + \sigma(e_i),$$

where $\sigma(e_i)$ is the length of excursion e_i. For every $t \geq 0$, we set i_t to be the only index $i \in \mathcal{I}$ such that $a_i \leq t < b_i$. Notice that i_t is a.s. well-defined except on a Lebesgue-null set of values of t. We define the process $(Y, J) = ((Y_t, J_t), t \geq 0)$ by

$$Y_t = e_{i_t}(t - a_{i_t}) \quad \text{and} \quad J_t = h_{i_t} \quad \text{for } t \geq 0,$$

with the convention $Y_t = 0$ and $J_t = \sup\{J_s, s < t\}$ for t such that i_t is not well-defined. Since $n^{(\theta)}$ is the excursion measure of $B^{(\theta)} - I^{(\theta)}$ above 0 associated to its local time at 0 given by $-\beta I^{(\theta)}$, we deduce the following corollary from excursion theory.

Corollary 2.1. *Let $\theta \geq 0$. We have that (Y, J) is distributed as $(B^{(\theta)} - I^{(\theta)}, -I^{(\theta)})$, and thus $Y - J$ and $Y + J$ are respectively distributed as $B^{(\theta)}$ and $B^{(\theta)} - 2I^{(\theta)}$.*

According to [28, Theorem 1], and taking into account the scale $\sqrt{2/\beta}$, the process $B^{(\theta)} - 2I^{(\theta)}$ is a diffusion on $[0, +\infty)$ with infinitesimal generator

$$\beta^{-1}\,\partial_x^2 + 2|\theta|\coth(\beta|\theta|x)\,\partial_x, \quad x \in [0, +\infty). \tag{2.7}$$

Proof of Lemma 2.1. Since $B^{(\theta)} - 2I^{(\theta)}$ and $B^{(-\theta)} - 2I^{(-\theta)}$ have the same distribution (see [28, Theorem 1]), we deduce that $\mathbb{P}_\theta^{\uparrow}(de) = \mathbb{P}_{-\theta}^{\uparrow}(de)$, which gives the first part of (2.5).

For $\theta, \lambda \in \mathbb{R}$, we set $\varphi_\theta(\lambda) = \psi_\theta(\lambda/\beta) = \beta^{-1}\lambda^2 - 2\theta\lambda$ so that $\mathbb{E}[\exp(\lambda B_t^{(\theta)})] = \exp(t\varphi_\theta(\lambda))$. Elementary computations give

$$\int_0^\infty e^{-\lambda x} c_\theta(x)^{-1}\, dx = \frac{1}{\varphi_\theta(\lambda)} \quad \text{for all } \lambda > 2\beta\max(\theta, 0).$$

This implies that $1/c_\theta$ is the scale function of $B^{(\theta)}$; see Theorem 8 in Section VII of [10]. Thanks to Theorem 8 and Proposition 15 in Section VII of [10], there exists a positive constant k_θ such that for all $t > 0$ and A in the σ-field $\mathcal{E}_t$ generated by $(e(s), s \leq t)$,

$$n^{(\theta)}(A, \sigma > t) = k_\theta \mathbb{E}_\theta^{\uparrow}\left[c_\theta(e(t))\mathbf{1}_A\right], \tag{2.8}$$

and $n^{(\theta)}(\zeta > h) = k_\theta c_\theta(h)$ for all $h > 0$. We deduce from (2.3) and the latter equality that $k_\theta = 1$ for $\theta \geq 0$.

We now prove that $k_\theta = 1$ for $\theta < 0$ as well. Assume that $\theta < 0$. Letting t go to infinity in (2.8) (with A fixed) and using that $\mathbb{P}^{\uparrow}_\theta(de)$-a.s., $\lim_{t\to+\infty} e(t) = +\infty$, we deduce that

$$n^{(\theta)}(A, \sigma = +\infty) = 2|\theta|\, k_\theta \mathbb{P}^{\uparrow}_\theta(A) \quad \text{for all } A \in \mathcal{E}_t \text{ and all } t \geq 0, \tag{2.9}$$

and taking for A the whole state space, we get that

$$n^{(\theta)}(\sigma = +\infty) = 2|\theta| k_\theta. \tag{2.10}$$

By excursion theory and the chosen normalization, we get that $-\beta I^{(\theta)}_\infty$ is exponential with parameter $n^{(\theta)}(\sigma = +\infty)$. Since by scaling $-I^{(\theta)}_\infty$ is also distributed as $\inf\{B_t - \beta\theta t,\ t \geq 0\}$, we deduce from [12, IV-5-32, p. 70] that $-I^{(\theta)}_\infty$ is exponential with parameter $2\beta|\theta|$ and thus that $-\beta I^{(\theta)}_\infty$ is exponential with parameter $2|\theta|$. This implies that $n^{(\theta)}(\sigma = +\infty) = 2|\theta|$, which gives the first part of (2.6); thus, thanks to (2.10), we get $k_\theta = 1$. We then use (2.9) with $k_\theta = 1$ to get the second part of (2.6).

Let $\theta < 0$. Let $t > 0$ and $A \in \mathcal{E}_t$. We have

$$\begin{aligned} n^{(\theta)}(A, \sigma > t) &= \mathbb{E}^{\uparrow}_\theta\left[c_\theta(e(t))\mathbf{1}_A\right] \\ &= 2|\theta|\mathbb{P}^{\uparrow}_\theta(A) + \mathbb{E}^{\uparrow}_{|\theta|}\left[c_{|\theta|}(h)(e(t))\mathbf{1}_A\right] \\ &= n^{(\theta)}(A, \sigma = +\infty) + n^{(|\theta|)}(A, \sigma > t), \end{aligned}$$

where, for the first equality, we used (2.8) and $k_\theta = 1$; for the second, we used the fact that $c_\theta(h) = 2|\theta| + c_{|\theta|}(h)$ for all $h > 0$, by (1.3), and that $\mathbb{P}^{\uparrow}_\theta(de) = \mathbb{P}^{\uparrow}_{-\theta}(de)$; and for the last, we used (2.8) with $|\theta|$ instead of θ and $k_{|\theta|} = 1$, as well as (2.9). This implies the last part of (2.5) for $\theta < 0$. We also deduce from (2.2) that $n^{(\theta)}(\sigma = +\infty) = 0$ for $\theta \geq 0$. Thus the second part of (2.5) holds also for $\theta \geq 0$ and thus for $\theta \in \mathbb{R}$.

We shall now prove (2.4). Recall that n_θ is the excursion measure of $B^{(\theta)}$ outside 0 associated with the local time $L^0 = L^0(B^{(\theta)})$, $n^{(\theta)}(de)$ is the excursion measure of $B^{(\theta)} - I^{(\theta)}$ above 0, and $n^{(-\theta)}(de)$ is the excursion measure of $B^{(-\theta)} - I^{(-\theta)}$ above 0. Notice that $B^{(-\theta)} - I^{(-\theta)}$ is distributed as $-(B^{(\theta)} - M^{(\theta)})$, where $M^{(\theta)} = (M^{(\theta)}_t, t \geq 0)$, defined by $M^{(\theta)}_t = \sup_{u\in[0,t]} B^{(\theta)}_u$, is the maximum process, and thus $n^{(-\theta)}(d(-e))$ is the excursion measure of $B^{(\theta)} - M^{(\theta)}$ below 0. According to [9, p. 334] (where the statement is for $\beta = 2$, but can clearly be adapted to $\beta > 0$ using a scaling in time), we get that

(i) $n^{(\theta)}(de)$, the excursion measure of $B^{(\theta)} - I^{(\theta)}$ above 0, is equal, up to a multiplicative constant due to the choice of the normalization of the local times, to $\mathbf{1}_{\{e>0\}} n_\theta(de)$;

(ii) $n^{(-\theta)}(d(-e))$, the excursion measure of $B^{(\theta)} - M^{(\theta)}$ below 0, is equal, up to a multiplicative constant due to the choice of the normalization of the local times, to $\mathbf{1}_{\{e>0\}} n_\theta(de)$.

Thus, we have, for some positive constants a_θ and b_θ, that

$$n_\theta(de) = a_\theta n^{(\theta)}(de) + b_\theta n^{(-\theta)}(d(-e)).$$

Thanks to (2.5) and (2.6), we find that (2.4) will be proved once we prove that $a_\theta = b_\theta = \beta/2$.

Let us assume for simplicity that $\theta \geq 0$ (the argument is similar for $\theta \leq 0$). By excursion theory, L^0_∞ is exponential with parameter $n_\theta(\sigma = +\infty) = b_\theta n^{(-\theta)}(\sigma = +\infty) = 2\theta b_\theta$. According to [12, V-3-11, p. 90], L^0_∞ is exponential with parameter $\beta\theta$ (use the fact that $(L^0_t, t \geq 0)$

is distributed as $(L^0_{2t/\beta}(W), t \geq 0)$, the local time at 0 of the Brownian motion $W = (W_t = B_t - \beta\theta t, t \geq 0))$. This gives $b_\theta = \beta/2$.

We now prove that $a_\theta = \beta/2$. Let $T = \sup\{B_t^{(\theta)}, t \geq 0\}$. We have

$$\begin{aligned}\mathbb{P}(T < a) &= \mathbb{E}\left[\mathrm{e}^{-n_\theta(\zeta \geq a, e>0)\, L^0_\infty}\right] = \mathbb{E}\left[\mathrm{e}^{-a_\theta\, n^{(\theta)}(\zeta \geq a)\, L^0_\infty}\right] \\ &= \mathbb{E}\left[\mathrm{e}^{-a_\theta c_\theta(a)\, L^0_\infty}\right] = \frac{\beta\theta}{\beta\theta + a_\theta c_\theta(a)},\end{aligned}$$

where we used that L^0_∞ is exponential with parameter $\beta\theta$ for the last equality. Since by scaling T is also distributed as $\sup\{B_t - \beta\theta t,\ t \geq 0\}$, we deduce from [12, IV-5-32, p. 70] that T is exponential with parameter $2\beta\theta$. This gives $\mathbb{P}(T < a) = 1 - \mathrm{e}^{-2\beta\theta a}$. Using (1.3), we deduce that $a_\theta = \frac{\beta}{2}$. This ends the proof of the lemma. □

2.2. Real trees

The study of real trees was originally motivated by algebraic and geometric problems; see in particular the survey [15]. Real trees were first used to study CRTs in [21]; see also [20].

Definition 2.1. (*Real tree*.) A real tree is a metric space $(\mathbf{t}, d_\mathbf{t})$ with the following properties:

(i) For every $x, y \in \mathbf{t}$, there is a unique isometric map $f_{x,y}$ from $[0, d_\mathbf{t}(x, y)]$ to $\mathbf{t}$ such that $f_{x,y}(0) = x$ and $f_{x,y}(d_\mathbf{t}(x, y)) = y$.

(ii) For every $x, y \in \mathbf{t}$, if ϕ is a continuous injective map from $[0, 1]$ to $\mathbf{t}$ such that $\phi(0) = x$ and $\phi(1) = y$, then $\phi([0, 1]) = f_{x,y}([0, d_\mathbf{t}(x, y)])$.

Notice that a real tree is a length space as defined in [13]. A *rooted* real tree is given by $(\mathbf{t}, d_\mathbf{t}, \partial_\mathbf{t})$, where $\partial = \partial_\mathbf{t}$ is a distinguished vertex of $\mathbf{t}$, which will be called the root. We remark that the set $\{\partial\}$ is a rooted tree that contains only the root.

Let $\mathbf{t}$ be a compact rooted real tree and let $x, y \in \mathbf{t}$. We denote by $[\![x, y]\!]$ the range of the map $f_{x,y}$ described in Definition 2.1. We also set $[\![x, y[\![= [\![x, y]\!] \setminus \{y\}$. We define the out-degree of x, denoted by $k_\mathbf{t}(x)$, as the number of connected components of $\mathbf{t} \setminus \{x\}$ that do not contain the root. If $k_\mathbf{t}(x) = 0$ (resp. $k_\mathbf{t}(x) > 1$), then x is called a leaf (resp. a branching point). A tree is said to be binary if the out-degree of its vertices belongs to $\{0, 1, 2\}$. The skeleton of the tree $\mathbf{t}$ is the set $\mathrm{sk}(\mathbf{t})$ of points of $\mathbf{t}$ that are not leaves. Notice that $\mathrm{cl}\,(\mathrm{sk}(\mathbf{t})) = \mathbf{t}$, where $\mathrm{cl}\,(A)$ denote the closure of A.

We denote by $\mathbf{t}_x$ the sub-tree of $\mathbf{t}$ above x, i.e.,

$$\mathbf{t}_x = \{y \in \mathbf{t},\ x \in [\![\partial, y]\!]\},$$

rooted at x. We say that x is an ancestor of y, and write $x \preccurlyeq y$, if $y \in \mathbf{t}_x$. We write $x \prec y$ if furthermore $x \neq y$. Notice that $\preccurlyeq$ is a partial order on $\mathbf{t}$. We denote by $x \wedge y$ the most recent common ancestor (MRCA) of x and y in $\mathbf{t}$, i.e. the unique vertex of $\mathbf{t}$ such that $[\![\partial, x]\!] \cap [\![\partial, y]\!] = [\![\partial, x \wedge y]\!]$.

We denote by $h_\mathbf{t}(x) = d_\mathbf{t}(\partial, x)$ the height of the vertex x in the tree $\mathbf{t}$ and by $H(\mathbf{t})$ the height of the tree $\mathbf{t}$:

$$H(\mathbf{t}) = \max\{h_\mathbf{t}(x),\ x \in \mathbf{t}\}.$$

Recall that $\mathbf{t}$ is a compact rooted real tree; let $(\mathbf{t}_i, i \in I)$ be a family of rooted trees, and $(x_i, i \in I)$ a family of vertices of $\mathbf{t}$. Let $\mathbf{t}_i^\circ = \mathbf{t}_i \setminus \{\partial_{\mathbf{t}_i}\}$. We define the tree $\mathbf{t} \circledast_{i\in I} (\mathbf{t}_i, x_i)$ obtained by grafting the trees $\mathbf{t}_i$ onto the tree $\mathbf{t}$ at points x_i by

$$\mathbf{t} \circledast_{i\in I} (\mathbf{t}_i, x_i) = \mathbf{t} \sqcup \left(\bigsqcup_{i\in I} \mathbf{t}_i^\circ \right),$$

$$d_{\mathbf{t}\circledast_{i\in I}(\mathbf{t}_i,x_i)}(y, y') = \begin{cases} d_{\mathbf{t}}(y, y') & \text{if } y, y' \in \mathbf{t}, \\ d_{\mathbf{t}_i}(y, y') & \text{if } y, y' \in \mathbf{t}_i^\circ, \\ d_{\mathbf{t}}(y, x_i) + d_{\mathbf{t}_i}(\partial_{\mathbf{t}_i}, y') & \text{if } y \in \mathbf{t} \text{ and } y' \in \mathbf{t}_i^\circ, \\ d_{\mathbf{t}_i}(y, \partial_{\mathbf{t}_i}) + d_{\mathbf{t}}(x_i, x_j) & \\ \qquad + d_{\mathbf{t}_j}(\partial_{\mathbf{t}_j}, y') & \text{if } y \in \mathbf{t}_i^\circ \text{ and } y' \in \mathbf{t}_j^\circ \text{ with } i \neq j, \end{cases}$$

$$\partial_{\mathbf{t}\circledast_{i\in I}(\mathbf{t}_i,x_i)} = \partial_{\mathbf{t}},$$

where $A \sqcup B$ denotes the disjoint union of the sets A and B. Notice that $\mathbf{t} \circledast_{i\in I} (\mathbf{t}_i, x_i)$ might not be compact.

We say that two rooted real trees $\mathbf{t}$ and $\mathbf{t}'$ are equivalent (and we write $\mathbf{t} \sim \mathbf{t}'$) if there exists a root-preserving isometry that maps $\mathbf{t}$ onto $\mathbf{t}'$. We denote by $\mathbb{T}$ the set of equivalence classes of compact rooted real trees. The metric space $(\mathbb{T}, d_{GH})$, with the so-called Gromov–Hausdorff distance d_{GH}, is Polish; see [21]. This allows us to define random real trees.

2.3. Coding a compact real tree by a function and the Brownian CRT

Let $\mathcal{E}$ be the set of continuous functions $g:[0, +\infty) \longrightarrow [0, +\infty)$ with compact support and such that $g(0) = 0$. For $g \in \mathcal{E}$, we set $\sigma(g) = \sup\{x,\ g(x) > 0\}$. Let $g \in \mathcal{E}$, and assume that $\sigma(g) > 0$, i.e. g is not identically zero. For every $s, t \geq 0$, we set

$$m_g(s, t) = \inf_{r\in[s\wedge t, s\vee t]} g(r)$$

and

$$d_g(s, t) = g(s) + g(t) - 2m_g(s, t). \tag{2.11}$$

It is easy to check that d_g is a pseudo-metric on $[0, +\infty)$. We then say that s and t are equivalent if and only if $d_g(s, t) = 0$, and we denote by T_g the associated quotient space. We keep the notation d_g for the induced distance on T_g. Then the metric space (T_g, d_g) is a compact real tree; see [19]. We denote by p_g the canonical projection from $[0, +\infty)$ to T_g. We will view (T_g, d_g) as a rooted real tree with root $\partial = p_g(0)$. We will call (T_g, d_g) the real tree coded by g, and conversely we will say that g is a contour function of the tree T_g. We denote by F the application that associates with a function $g \in \mathcal{E}$ the equivalence class of the tree T_g.

Conversely, every rooted compact real tree (T, d) can be coded by a continuous function g (up to a root-preserving isometry); see [16].

We define the Brownian CRT, $\tau = F(e)$, as the (equivalence class of the) tree coded by the positive excursion e under $n^{(\theta)}$; see Section 2.1. And we define the measure $\mathbb{N}^{(\theta)}$ on $\mathbb{T}$ as the 'distribution' of τ, i.e. the pushforward of the measure $n^{(\theta)}$ by the application F. Notice that $H(\tau) = \zeta(e)$.

Let e have 'distribution' $n^{(\theta)}(de)$, and let $(\Lambda_s^a, s \geq 0, a \geq 0)$ be the local time of e at time s and level a. Then we define the local time measure of τ at level $a \geq 0$, denoted by $\ell_a(dx)$, as

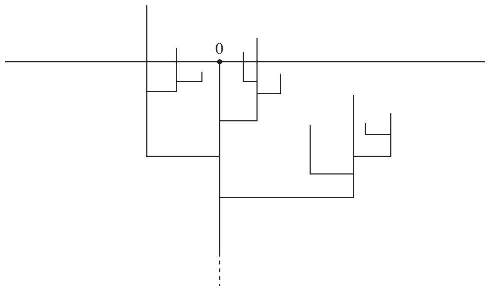

FIGURE 2: An example of a tree with a semi-infinite branch.

the pushforward of the measure $d\Lambda_s^a$ by the map F; see [19, Theorem 4.2]. We shall define ℓ_a for $a \in \mathbb{R}$ by setting $\ell_a = 0$ for $a \in \mathbb{R} \setminus [0, H(\tau)]$.

2.4. Trees with one semi-infinite branch

The goal of this section is to describe the genealogical tree of a stationary CB with immigration (restricted to the population that appeared before time 0). For this purpose, we add an immortal individual living from $-\infty$ to 0, which will be the spine of the genealogical tree (i.e. the semi-infinite branch) and will be represented by the half straight line $(-\infty, 0]$; see Figure 2. Since we are interested in the genealogical tree, we do not record the population generated by the immortal individual after time 0. The distinguished vertex in the tree will be the point 0; in the terminology of Section 2.2, this would be the root of the tree. In what follows, however, in accordance with the natural intuition, we will speak of it as the distinguished leaf. In the same spirit, we will give another definition for the height of a vertex in such a tree in order to allow negative heights.

2.4.1. *Forests.* A forest $\mathbf{f}$ is a family $((h_i, \mathbf{t}_i),\ i \in I)$ of points of $\mathbb{R} \times \mathbb{T}$. Using an immediate extension of the grafting procedure, for an interval $\mathfrak{I} \subset \mathbb{R}$ we define the real tree

$$\mathbf{f}_{\mathfrak{I}} = \mathfrak{I} \circledast_{i \in I, h_i \in \mathfrak{I}} (\mathbf{t}_i, h_i). \tag{2.12}$$

For $i \in I$, let us denote by d_i the distance on the tree $\mathbf{t}_i$ and by $\mathbf{t}_i^\circ = \mathbf{t}_i \setminus \{\partial_{\mathbf{t}_i}\}$ the tree $\mathbf{t}_i$ without its root. The distance on $\mathbf{f}_{\mathfrak{I}}$ is then defined, for $x, y \in \mathbf{f}_{\mathfrak{I}}$, by

$$d_{\mathbf{f}}(x, y) = \begin{cases} d_i(x, y) & \text{if } x, y \in \mathbf{t}_i^\circ, \\ h_{\mathbf{t}_i}(x) + |h_i - h_j| + h_{\mathbf{t}_j}(y) & \text{if } x \in \mathbf{t}_i^\circ,\ y \in \mathbf{t}_j^\circ \text{ with } i \neq j, \\ |x - h_j| + h_{\mathbf{t}_j}(y) & \text{if } x \in \sqcup_{i \in I} \mathbf{t}_i^\circ,\ y \in \mathbf{t}_j^\circ, \\ |x - y| & \text{if } x, y \in \sqcup_{i \in I} \mathbf{t}_i^\circ. \end{cases}$$

Let us recall the following lemma (see [3]).

Lemma 2.2. *Let $\mathfrak{I} \subset \mathbb{R}$ be a closed interval. If, for every $a, b \in \mathfrak{I}$ such that $a < b$, and every $\varepsilon > 0$, the set $\{i \in I,\ \ h_i \in [a, b],\ \ H(\mathbf{t}_i) > \varepsilon\}$ is finite, then the tree $\mathbf{f}_{\mathfrak{I}}$ is a complete locally compact length space.*

2.4.2. *Trees with one semi-infinite branch.*

Definition 2.2. We define $\mathbb{T}_1$ as the set of forests $\mathbf{f} = ((h_i, \mathbf{t}_i),\ i \in I)$ such that

- for every $i \in I$, $h_i \leq 0$;
- for every $a < b$ and every $\varepsilon > 0$, the set $\{i \in I,\ h_i \in [a, b],\ H(\mathbf{t}_i) > \varepsilon\}$ is finite.

The following corollary, which is an elementary consequence of Lemma 2.2, associates with a forest $\mathbf{f} \in \mathbb{T}_1$ a complete and locally compact real tree.

Corollary 2.2. *Let* $\mathbf{f} = ((h_i, \mathbf{t}_i),\ i \in I) \in \mathbb{T}_1$. *Then the tree* $\mathbf{f}_{(-\infty,0]}$ *defined by* (2.12) *is a complete and locally compact real tree.*

Conversely, let $(\mathbf{t}, d_{\mathbf{t}}, \rho_0)$ be a complete and locally compact rooted real tree. We denote by $\mathcal{S}(\mathbf{t})$ the set of vertices $x \in \mathbf{t}$ such that at least one of the connected components of $\mathbf{t} \setminus \{x\}$ that do not contain ρ_0 is unbounded. If $\mathcal{S}(\mathbf{t})$ is not empty, then it is a tree which contains ρ_0. We say that $\mathbf{t}$ has a unique semi-infinite branch if $\mathcal{S}(\mathbf{t})$ is non-empty and has no branching point. We let $(\mathbf{t}_i^\circ, i \in I)$ denote the connected components of $\mathbf{t} \setminus \mathcal{S}(\mathbf{t})$. For every $i \in I$, we let x_i denote the unique point of $\mathcal{S}(\mathbf{t})$ such that $\inf\{d_{\mathbf{t}}(x_i, y),\ y \in \mathbf{t}_i^\circ\} = 0$, and

$$\mathbf{t}_i = \mathbf{t}_i^\circ \cup \{x_i\}, \qquad h_i = -d(\rho_0, x_i).$$

We shall say that x_i is the root of $\mathbf{t}_i$. Notice first that $(\mathbf{t}_i, d_{\mathbf{t}}, x_i)$ is a bounded rooted tree. It is also compact, since, according to the Hopf–Rinow theorem (see [13, Theorem 2.5.26]), it is a bounded closed subset of a complete locally compact length space. Thus it belongs to $\mathbb{T}$.

The family $\mathbf{f} = ((h_i, \mathbf{t}_i),\ i \in I)$ is therefore a forest with $h_i < 0$. To check that it belongs to $\mathbb{T}_1$, we need to prove that the second condition in Definition 2.2 is satisfied, which is a direct consequence of the fact that the tree $\mathbf{f}_{[a,b]}$ is locally compact.

We can therefore identify the set $\mathbb{T}_1$ with the set of (equivalence classes of) complete locally compact rooted real trees with a unique semi-infinite branch. We can follow [4] to endow $\mathbb{T}_1$ with a Gromov–Hausdorff-type distance for which $\mathbb{T}_1$ is a Polish space.

We extend the partial order defined for trees in $\mathbb{T}$ to forests in $\mathbb{T}_1$, with the idea that the distinguished leaf $\rho_0 = 0$ is at the tip of the semi-infinite branch. Let $\mathbf{f} = (h_i, \mathbf{t}_i)_{i\in I} \in \mathbb{T}_1$ and write $\mathbf{t} = \mathbf{f}_{(-\infty,0]}$, viewed as a real tree rooted at $\rho_0 = 0$ (with a unique semi-infinite branch $\mathcal{S}(\mathbf{t}) = (-\infty, 0]$). For $x, y \in \mathbf{t}$, we set $x \preccurlyeq y$ if either $x, y \in S(\mathbf{t})$ and $d_{\mathbf{f}}(x, \rho_0) \geq d_{\mathbf{f}}(y, \rho_0)$, or $x, y \in \mathbf{t}_i$ for some $i \in I$ and $x \preccurlyeq y$ (with the partial order for the rooted compact real tree $\mathbf{t}_i$), or $x \in \mathcal{S}(\mathbf{t})$ and $y \in \mathbf{t}_i$ for some $i \in I$ and $d_{\mathbf{f}}(x, \rho_0) \geq |h_i|$. We write $x \prec y$ if furthermore $x \neq y$. We define $x \wedge y$, the MRCA of $x, y \in \mathbf{t}$, as x if $x \preccurlyeq y$, as $x \wedge y$ if $x, y \in \mathbf{t}_i$ for some $i \in I$ (with the MRCA for the rooted compact real tree $\mathbf{t}_i$), and as $h_i \wedge h_j$ if $x \in \mathbf{t}_i$ and $y \in \mathbf{t}_j$ for some $i \neq j$. We define the height of a vertex $x \in \mathbf{t}$ as

$$h_{\mathbf{f}}(x) = d_{\mathbf{f}}(x, \rho_0 \wedge x) - d_{\mathbf{f}}(\rho_0, \rho_0 \wedge x).$$

Notice that the definition of the height function $h_{\mathbf{f}}$ for a forest $\mathbf{f} = (h_i, \mathbf{t}_i)_{i\in I} \in \mathbb{T}_1$ is different from the height function of the tree $\mathbf{t} = \mathbf{f}_{(-\infty,0]}$ viewed as a tree in $\mathbb{T}$, as in the former case the root ρ_0 is viewed as a distinguished vertex above the semi-infinite branch (all elements of this semi-infinite branch have negative heights for $h_{\mathbf{f}}$, whereas all the heights are nonnegative for $h_{\mathbf{t}}$).

2.4.3. *Coding a forest by a contour function.* The construction of a tree of the type $\mathbf{f}_{[0,+\infty)}$ via a contour function as in Section 2.3 is already present in [17] and in [7, Section 7.4]. This construction is recalled in Subsection 3.2.4 below. We now present a construction of a tree of the type $\mathbf{f}_{(-\infty,0]}$ via a contour function as in Section 2.3. Let $\mathcal{E}_1$ be the set of continuous functions g defined on $\mathbb{R}$ such that $g(0)=0$ and $\liminf_{x\to-\infty} g(x)=\liminf_{x\to+\infty} g(x)=-\infty$. For such a function, we still consider the pseudo-metric d_g defined by (2.11) (but for $s, t\in\mathbb{R}$) and define the tree T_g^- as the quotient space on $\mathbb{R}$ induced by this pseudo-metric. We set p_g as the canonical projection from $\mathbb{R}$ onto T_g^-.

Lemma 2.3. *Let $g\in\mathcal{E}_1$. The triplet $(T_g^-, d_g, p_g(0))$ is a complete locally compact rooted real tree with a unique semi-infinite branch.*

When there is no possibility of confusion, we write T_g for T_g^-.

Proof. We define the infimum function $\underline{g}(x)$ on $\mathbb{R}$ as the infimum of g between 0 and x: $\underline{g}(x)=\inf_{[x\wedge 0, x\vee 0]} g$. The function $g-\underline{g}$ is nonnegative and continuous. Let $((a_i, b_i), i\in I)$ be the excursion intervals of $g-\underline{g}$ above 0. Because of the hypothesis on g, the intervals (a_i, b_i) are bounded. For $i\in I$, set $h_i=g(a_i)$ and $g_i(x)=g((a_i+x)\wedge b_i)-h_i$, so that $g_i\in\mathcal{E}$. Consider the forest $\mathbf{f}=((h_i, T_{g_i}),\ i\in I)$.

It is elementary to check that $(\mathbf{f}_{(-\infty,g(0)]}, d_{\mathbf{f}}, g(0))$ and $(T_g, d_g, p_g(0))$ are root-preserving and isometric. To conclude, it is enough to check that $\mathbf{f}\in\mathbb{T}_1$. First observe that, by definition, $h_i\le 0$ for every $i\in I$. Let $r>0$, and set $r_{\mathrm{g}}=\inf\{x,\ \underline{g}(x)\ge g(0)-r\}$ and $r_{\mathrm{d}}=\sup\{x,\ \underline{g}(x)\ge g(0)-r\}$. Because of the hypothesis on g, we have that r_{g} and r_{d} are finite. By continuity of $g-\underline{g}$ on $[r_{\mathrm{g}}, r_{\mathrm{d}}]$, we deduce that for any $\varepsilon>0$, the set $\{i\in I;\, (a_i, b_i)\subset[r_{\mathrm{g}}, r_{\mathrm{d}}]$ and $\sup_{(a_i,b_i)}(g-\underline{g})>\varepsilon\}$ is finite. Since this holds for any $r>0$ and since $H(T_{g_i})=\sup_{(a_i,b_i)}(g-\underline{g})$ for all $i\in I$, we deduce that $\mathbf{f}\in\mathbb{T}_1$. This concludes the proof. □

2.4.4. *Genealogical tree of an extant population.* For a tree $\mathbf{t}\in\mathbb{T}$ or $\mathbf{t}\in\mathbb{T}_1$ (recall that we identify a forest $\mathbf{f}\in\mathbb{T}_1$ with the tree $\mathbf{t}=\mathbf{f}_{(-\infty,0]}$, with a different definition for the height function) and $h\ge 0$, we define $\mathcal{Z}_h(\mathbf{t})=\{x\in\mathbf{t},\ h_{\mathbf{t}}(x)=h\}$ to be the set of vertices of $\mathbf{t}$ at level h, also called the extant population at time h, and we define the genealogical tree of the vertices of $\mathbf{t}$ at level h by

$$\mathcal{G}_h(\mathbf{t})=\{x\in\mathbf{t};\, \exists y\in\mathcal{Z}_h(\mathbf{t}) \text{ such that } x\preccurlyeq y\}. \tag{2.13}$$

For $\mathbf{f}\in\mathbb{T}_1$, we write $\mathcal{G}_h(\mathbf{f})$ for $\mathcal{G}_h(\mathbf{f}_{(-\infty,0]})$.

3. Ancestral process

Usually, the ancestral process records the genealogy of n extant individuals at time 0 picked at random from the whole population. Using the ideas of [6], we are able to describe in the case of a Brownian forest the genealogy of all extant individuals at time 0 by a simple Poisson point process on $\mathbb{R}^2$.

3.1. Construction of a tree from a point measure

Definition 3.1. A point process $\mathcal{A}(dx, d\zeta)=\sum_{i\in\mathcal{I}}\delta_{(x_i,\zeta_i)}(dx, d\zeta)$ on $\mathbb{R}^*\times(0,+\infty)$ is said to be an ancestral process if the following properties hold:

(i) For all $i, j\in\mathcal{I}$, $i\ne j\Longrightarrow x_i\ne x_j$.

(ii) For all $a, b\in\mathbb{R}$ and $\varepsilon>0$, $\mathcal{A}([a,b]\times[\varepsilon,+\infty))<+\infty$.

(iii) $\sup\{\zeta_i, x_i>0\}=+\infty$ if $\sup_{i\in\mathcal{I}}x_i=+\infty$; and $\sup\{\zeta_i, x_i<0\}=+\infty$ if $\inf_{i\in\mathcal{I}}x_i=-\infty$.

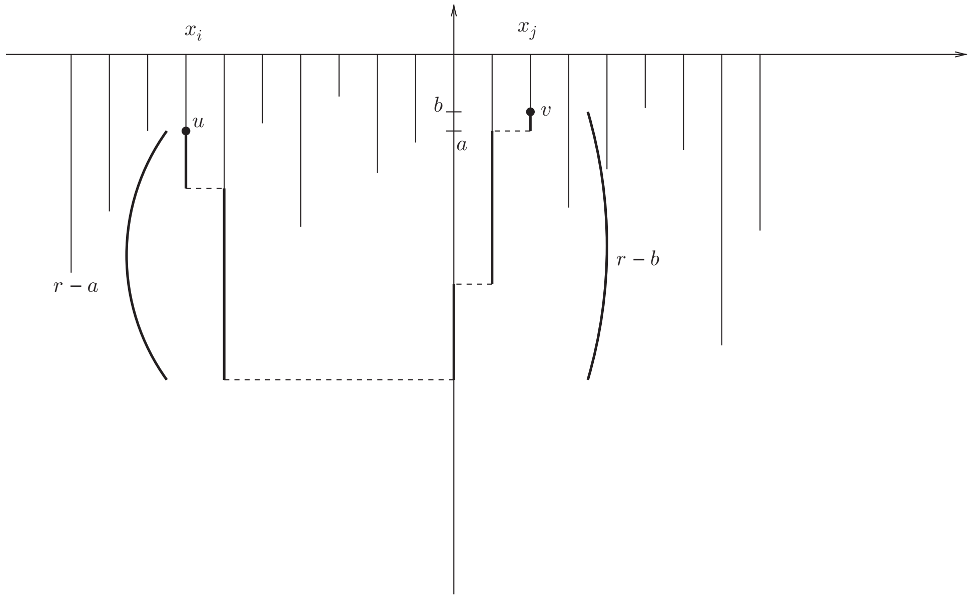

FIGURE 3: An example of the distance $d(u, v)$ defined in (3.2).

Let $\mathcal{A} = \sum_{i\in\mathcal{I}} \delta_{(x_i,\zeta_i)}$ be a point process on $\mathbb{R}^* \times [0, +\infty)$ satisfying (i) and (ii) from Definition 3.1. We shall associate with this ancestral process a genealogical tree. Informally the genealogical tree is constructed as follows. We view this process as a sequence of vertical segments in $\mathbb{R}^2$, the tips of the segments being the x_i and their lengths being the ζ_i. We then attach the bottom of each segment such that $x_i > 0$ (resp. $x_i < 0$) to the first left (resp. first right) longer segment, or to the half-line $\{0\} \times (-\infty, 0]$ if such a segment does not exist. This gives a (non-compact, unrooted) real tree that may not be complete. See Figure 1 for an example.

Let us turn to a more formal definition. Let us define $\mathcal{I}^{\mathrm{d}} = \{i \in \mathcal{I},\ x_i > 0\}$ and $\mathcal{I}^{\mathrm{g}} = \{i \in \mathcal{I},\ x_i < 0\} = \mathcal{I} \setminus \mathcal{I}^{\mathrm{d}}$. We also set $\mathcal{I}_0 = \mathcal{I} \sqcup \{0\}$, $x_0 = 0$, and $\zeta_0 = +\infty$. For every $i \in \mathcal{I}_0$, we denote by $S_i = \{x_i\} \times (-\zeta_i, 0]$ the vertical segment in $\mathbb{R}^2$ that links the points $(x_i, 0)$ and $(x_i, -\zeta_i)$. Notice that we omit the lowest point of the vertical segment. Finally we define

$$\mathfrak{T} = \bigsqcup_{i\in\mathcal{I}_0} S_i. \tag{3.1}$$

We now define a distance on $\mathfrak{T}$. We first define the distance between leaves of $\mathfrak{T}$, i.e. points $(x_i, 0)$ with $i \in \mathcal{I}_0$, then extend it to every point of $\mathfrak{T}$. For $i, j \in \mathcal{I}_0$ such that $x_i < x_j$, we set

$$d((x_i, 0), (x_j, 0)) = 2 \max\{\zeta_k,\ x_k \in J(x_i, x_j)\}, \tag{3.2}$$

where, for $x < y$, $J(x, y) = (x, y]$ (resp. $[x, y)$, resp. $[x, y]\backslash\{0\}$) if $x \geq 0$ (resp. $y \leq 0$, resp. $x < 0$ and $y > 0$), with the convention $\max \emptyset = 0$. For $u = (x_i, a) \in S_i$ and $v = (x_j, b) \in S_j$, we set

$$d(u, v) = |a - b| \mathbf{1}_{\{x_i = x_j\}} + (|a - r| + |b - r|) \mathbf{1}_{\{x_i \neq x_j\}}, \tag{3.3}$$

with $r = \frac{1}{2} d((x_i, 0), (x_j, 0))$. See Figure 3 for an example. It is easy to verify that d is a distance on $\mathfrak{T}$. Notice that $\mathfrak{T}$ is not compact, in particular because of the infinite half-line attached to

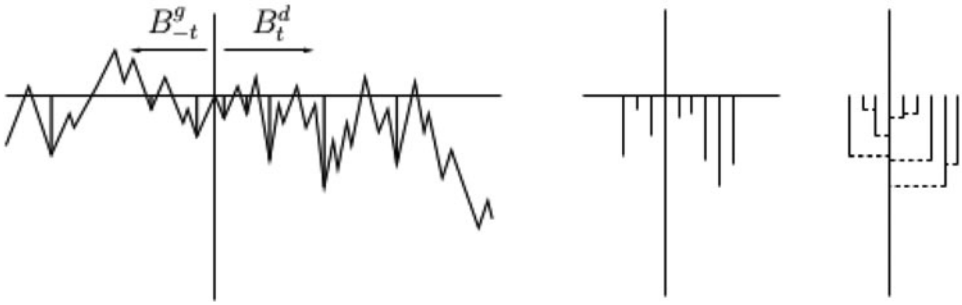

FIGURE 4: The Brownian motions with drift, the ancestral process, and the associated genealogical tree.

(0, 0). In order to stick to the framework of Section 2.4, we will consider the origin (0, 0) to be the distinguished point in $\mathfrak{T}$, located at height $h = 0$.

Finally, we define $\mathfrak{T}(\mathcal{A})$, with the metric d, as the completion of the metric space $(\mathfrak{T}, d)$.

Remark 3.1. For every $i \in \mathcal{I}^{\mathrm{d}}$, we set i_ℓ to be the index in $\mathcal{I}_0$ such that

$$x_{i_\ell} = \max\{x_j,\ 0 \le x_j < x_i \text{ and } \zeta_j > \zeta_i\}.$$

Notice that i_ℓ is well-defined, since there are only a finite number of indices $j \in \mathcal{I}_0$ such that $x_j \in [0, x_i)$ and $\zeta_j > \zeta_i$. Similarly, for $i \in \mathcal{I}^{\mathrm{g}}$, we set i_r to be the index in $\mathcal{I}_0$ such that

$$x_{i_r} = \min\{x_j,\ x_i < x_j \le 0 \text{ and } \zeta_j > \zeta_i\}.$$

The distance d identifies the point $(x_i, -\zeta_i)$ (which does not belong to $\mathfrak{T}$ by definition) with the point $(x_{i_\ell}, -\zeta_i)$ if $x_i > 0$ and with the point $(x_{i_r}, -\zeta_i)$ if $x_i < 0$, as illustrated on the right-hand side of Figure 4.

Proposition 3.1. *Let $\mathcal{A}$ be an ancestral process. The tree $(\mathfrak{T}(\mathcal{A}), d, (0, 0))$ is a complete and locally compact rooted real tree with a unique semi-infinite branch, and the associated forest belongs to $\mathbb{T}_1$.*

We shall call $\mathfrak{T}(\mathcal{A})$ the tree associated with the ancestral process $\mathcal{A}$.

Proof. By construction of $(\mathfrak{T}, d)$ (see (3.1), (3.2), and (3.3)), it is easy to check that $\mathfrak{T}$ is connected and that d satisfies the so-called four-points condition (see Lemma 3.12 in [20]). To conclude, use the fact that those two conditions characterize real trees (see Theorem 3.40 in [20]). This implies that both $(\mathfrak{T}, d)$ and its completion are real trees. By construction of $\mathfrak{T}$, it is easy to check that $\mathfrak{T}(\mathcal{A})$ has a unique semi-infinite branch.

Let us now prove that $\mathfrak{T}(\mathcal{A})$ is locally compact. Let $(y_n, n \in \mathbb{N})$ be a bounded sequence of $\mathfrak{T}$.

On one hand, let us assume that there exists $i \in \mathcal{I}_0$ and a subsequence $(y_{n_k}, k \in \mathbb{N})$ such that y_{n_k} belongs to $S_i = \{x_i\} \times (-\zeta_i, 0]$. Since, for $i \in \mathcal{I}$, there exists a unique $j \in \mathcal{I}_0$ such that $S_i \cup \{(x_j, -\zeta_i)\}$ is compact in $(\mathfrak{T}, d)$ (see Remark 3.1), and since, for $i = 0$, $S_0 = \{0\} \times (-\infty, 0]$, we deduce that the bounded sequence $(y_{n_k}, k \in \mathbb{N})$ has an accumulation point in $S_i \cup \{(x_j, -\zeta_i)\}$ if $i \in \mathcal{I}$ or in $\{0\} \times (-\infty, 0]$ if $i = 0$.

On the other hand, let us assume that for all $i \in \mathcal{I}_0$ the sets $\{n,\ y_n \in S_i\}$ are finite. For $n \in \mathbb{N}$, let i_n be uniquely defined by $y_n \in S_{i_n}$. Since $(y_n, n \in \mathbb{N})$ is bounded, we deduce from

Conditions (ii)–(iii) in Definition 3.1 that the sequence $(x_{i_n}, n \in \mathbb{N})$ is bounded in $\mathbb{R}$. In particular, there is a subsequence $(x_{i_{n_k}}, k \in \mathbb{N})$ that converges to a limit, say a. Without loss of generality, we can assume that the subsequence is non-decreasing. We deduce from Condition (ii) in Definition 3.1 that $\lim_{\varepsilon \downarrow 0} \max\{\zeta_i, a - \varepsilon < x_i < a\} = 0$. This implies, thanks to Definition (3.2), that $(\{x_{i_{n_k}}\} \times \{0\}, k \in \mathbb{N})$ is Cauchy in $\mathfrak{T}$ and, using (ii) again, that $\lim_{k \to +\infty} \zeta_{i_{n_k}} = 0$. Then we use the fact that

$$d(y_{n_k}, y_{n_{k'}}) \leq \zeta_{i_{n_k}} + \zeta_{i_{n_{k'}}} + d((x_{n_k}, 0), (x_{n_{k'}}, 0))$$

to conclude that the sequence $(y_{n_k}, k \in \mathbb{N})$ is Cauchy in $\mathfrak{T}$.

We deduce that every bounded sequence in $\mathfrak{T}$ has a Cauchy subsequence. This proves that $\mathfrak{T}(\mathcal{A})$, the completion of $\mathfrak{T}$, is locally compact. □

Remark 3.2. In the proof of Proposition 3.1, Conditions (i) and (ii) in Definition 3.1 ensure that $\mathfrak{T}(\mathcal{A})$ is a tree and Conditions (ii) and (iii) that $\mathfrak{T}(\mathcal{A})$ is locally compact.

3.2. The ancestral process of the Brownian forest

Let $\theta \geq 0$. Let $\mathcal{N}(dh, d\varepsilon, de) = \sum_{i \in I} \delta_{(h_i, \varepsilon_i, e_i)}(dh, d\varepsilon, de)$ be, under $\mathbb{P}^{(\theta)}$, a Poisson point measure on $\mathbb{R} \times \{-1, 1\} \times \mathcal{E}$ with intensity $\beta dh\, (\delta_{-1}(d\varepsilon) + \delta_1(d\varepsilon))\, n^{(\theta)}(de)$, and let $\mathcal{F}^{(\theta)} = ((h_i, \tau_i),\ i \in I)$ be the associated Brownian forest where $\tau_i = T_{e_i}$ is the tree associated with the excursion e_i; see Section 2.3. As explained in Section 3.2.4, this Brownian forest models the evolution of a stationary population directed by the branching mechanism ψ_θ defined in (1.1).

We want to describe the genealogical tree of the extant population at some fixed time, say 0. The genealogical tree under consideration is then $\mathcal{G}_0(\mathcal{F}^{(\theta)})$ as defined by (2.13). To describe the distribution of this tree, we use an ancestral process as described in the previous subsection. We first construct a contour process $(B_t, t \in \mathbb{R})$ (obtained by the concatenation of two independent Brownian motions distributed as $B^{(\theta)}$) which codes for the tree $\mathcal{F}^{(\theta)}_{(-\infty,0]}$ (see Section 2.4 for the notation). The supplementary variables ε_i are needed at this point to determine whether the tree $\mathbf{t}_i$ is located on the left or on the right of the infinite spine. The atoms of the ancestral process are then the pairs formed by the points of growth of the local time at 0 of B and the depth of the associated excursion of B below 0.

3.2.1. *Construction of the contour process.* Let $\theta \geq 0$. Set $\mathcal{I} = \{i \in I; h_i < 0\}$.

For every $i \in \mathcal{I}$, we set

$$a_i = \sum_{j \in \mathcal{I}} \mathbf{1}_{\{\varepsilon_j = \varepsilon_i\}} \mathbf{1}_{\{h_j < h_i\}} \sigma(e_j) \quad \text{and} \quad b_i = a_i + \sigma(e_i),$$

where we recall that $\sigma(e_i)$ is the length of the excursion e_i. For every $t \geq 0$, we denote by i_t^{d} (resp. i_t^{g}) the only index $i \in \mathcal{I}$ such that $\varepsilon_i = 1$ (resp. $\varepsilon_i = -1$) and $a_i \leq t < b_i$. Notice that i_t^{d} and i_t^{g} are a.s. well-defined except on a Lebesgue-null set of values of t. We set $B^{\mathrm{d}} = (B_t^{\mathrm{d}}, t \geq 0)$ and $B^{\mathrm{g}} = (B_t^{\mathrm{g}}, t \geq 0)$, where, for $t \geq 0$,

$$B_t^{\mathrm{d}} = h_{i_t^{\mathrm{d}}} + e_{i_t^{\mathrm{d}}}(t - a_{i_t^{\mathrm{d}}}) \quad \text{and} \quad B_t^{\mathrm{g}} = h_{i_t^{\mathrm{g}}} + e_{i_t^{\mathrm{g}}}(\sigma(e_{i_t^{\mathrm{g}}}) - (t - a_{i_t^{\mathrm{g}}})).$$

We deduce from Corollary 2.1 that the processes B^{d} and B^{g} are two independent Brownian motions distributed as $B^{(\theta)}$. We define the process $B = (B_t, t \in \mathbb{R})$ by $B_t = B_t^{\mathrm{d}} \mathbf{1}_{\{t>0\}} + B_{-t}^{\mathrm{g}} \mathbf{1}_{\{t<0\}}$. By construction, the process B indeed codes for the tree $\mathcal{F}^{(\theta)}_{(-\infty,0]}$.

3.2.2. *The ancestral process.* Let $(L_t^\ell, t \ge 0)$ be the local time at 0 of the process B^ℓ, where $\ell \in \{\mathrm{g}, \mathrm{d}\}$. We denote by $((\alpha_i, \beta_i),\ i \in \mathcal{I}^\ell)$ the excursion intervals of B^ℓ below 0, omitting the last infinite excursion if any, and for every $i \in \mathcal{I}^\ell$ we set $\zeta_i = -\min\{B_s^\ell,\ s \in (\alpha_i, \beta_i)\}$.

We consider the point measure on $\mathbb{R} \times \mathbb{R}_+$ defined by

$$\mathcal{A}^{\mathcal{N}}(du, d\zeta) = \sum_{i \in \mathcal{I}^{\mathrm{d}}} \delta_{(L^{\mathrm{d}}_{\alpha_i}, \zeta_i)}(du, d\zeta) + \sum_{i \in \mathcal{I}^{\mathrm{g}}} \delta_{(-L^{\mathrm{g}}_{\alpha_i}, \zeta_i)}(du, d\zeta).$$

See Figure 4 for a representation of the contour process B, the ancestral process $\mathcal{A}^{\mathcal{N}}$, and the genealogical tree $\mathcal{G}_0(\mathcal{F}^{(\theta)})$. In this figure, the horizontal axis represents the time for Brownian motion on the left-hand figure, whereas it is in the scale of local time for the ancestral process on the two right-hand figures. This will always be the case in the rest of the paper when we are dealing with ancestral processes.

Let $[-E_{\mathrm{g}}, E_{\mathrm{d}}]$ be the closed support of the measure $\mathcal{A}^{\mathcal{N}}(du, \mathbb{R}_+)$:

$$\begin{aligned} E_{\mathrm{d}} &= \inf\{u \ge 0,\ \mathcal{A}([u, +\infty) \times \mathbb{R}_+) = 0\} \\ \text{and} \quad E_{\mathrm{g}} &= \inf\{u \ge 0,\ \mathcal{A}((-\infty, -u] \times \mathbb{R}_+) = 0\}, \end{aligned}$$

with the convention that $\inf \emptyset = +\infty$. Notice that, for $\ell \in \{\mathrm{g}, \mathrm{d}\}$, we also have $E_\ell = L_\infty^\ell$. We now give the distribution of the ancestral process $\mathcal{A}^{\mathcal{N}}$. Recall that c_θ was defined in (1.3).

Proposition 3.2. *Let $\theta \ge 0$. Under $\mathbb{P}^{(\theta)}$, the random variables $E_{\mathrm{g}}, E_{\mathrm{d}}$ are independent and exponentially distributed with parameter 2θ (and mean $1/2\theta$), with the convention that $E_{\mathrm{d}} = E_{\mathrm{g}} = +\infty$ if $\theta = 0$. Under $\mathbb{P}^{(\theta)}$ and conditionally given $(E_{\mathrm{g}}, E_{\mathrm{d}})$, the ancestral process $\mathcal{A}^{\mathcal{N}}(du, d\zeta)$ is a Poisson point measure with intensity*

$$\mathbf{1}_{(-E_{\mathrm{g}}, E_{\mathrm{d}})}(u)\, du\, |c'_\theta(\zeta)| d\zeta .$$

Notice that the random measure $\mathcal{A}^{\mathcal{N}}$ satisfies Conditions (i)–(iii) from Definition 3.1 and is thus indeed an ancestral process.

This result is very similar to Corollary 2 in [11]. The main additional ingredient here is the order (given by the u variable), which will be very useful in the simulation.

Proof. Since B^{d} and B^{g} are independent with the same distribution, we deduce that E_{g} and E_{d} are independent with the same distribution. Let $\theta > 0$. Since B^{d} is a Brownian motion with drift -2θ, we deduce from Lemma 2.1 that E_{d} is exponential with mean $1/2\theta$. The case $\theta = 0$ is immediate.

The excursions below zero of B^{d} conditionally given E_d are excursions of a Brownian motion $B^{(-\theta)}$ with drift 2θ (by symmetry with respect to 0) conditioned on being finite, i.e. excursions of a Brownian motion $B^{(\theta)}$ with drift -2θ; see Lemma 2.1. Moreover, by (1.3), c_θ is exactly the tail distribution of the maximum of an excursion under $n^{(\theta)}$. Standard theory of Brownian excursions gives then the result. □

3.2.3. *Identification of the trees.* Let $\mathfrak{T}^{\mathcal{N}} = \mathfrak{T}(\mathcal{A}^{\mathcal{N}})$ denote the locally compact tree associated with the ancestral process $\mathcal{A}^{\mathcal{N}}$; see Proposition 3.1. Based on the following proposition, we shall say that the ancestral process $\mathcal{A}^{\mathcal{N}}$ codes for the genealogical tree of the extant population at time 0 for the forest $\mathcal{F}^{(\theta)}$.

Proposition 3.3. *Let $\theta \ge 0$. The trees $\mathcal{G}_0(\mathcal{F}^{(\theta)})$ under $\mathbb{P}^{(\theta)}$ and $\mathfrak{T}^{\mathcal{N}}$ belong to the same equivalence class in $\mathbb{T}_1$.*

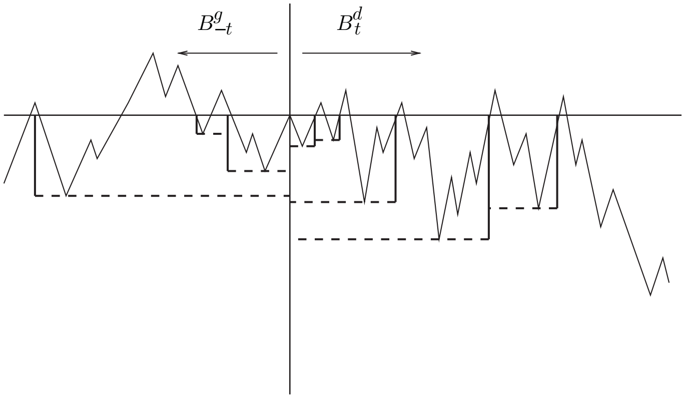

FIGURE 5: The genealogical tree inside the Brownian motions.

Proof. Let us first remark that the genealogical tree $\mathcal{G}_0(\mathcal{F}^{(\theta)})$ can be directly constructed using the process B as described in Figure 5.

More precisely, recall that B is the contour function of the tree $\mathcal{F}^{(\theta)}_{(-\infty,0]}$. Let us denote by p_B the canonical projection from $\mathbb{R}$ to $\mathcal{F}^{(\theta)}_{(-\infty,0]}$ as defined in Section 2.4. Recall that $((\alpha_i, \beta_i),\ i \in \mathcal{I}^\ell)$, with $\ell \in \{\mathrm{g}, \mathrm{d}\}$, are the excursion intervals of B^ℓ below 0. Then $\mathcal{G}_0(\mathcal{F}^{(\theta)})$ is the smallest complete sub-tree of $\mathcal{F}^{(\theta)}_{(-\infty,0]}$ that contains the points $(p_B(\alpha_i), i \in \mathcal{I}_\mathrm{g} \bigcup \mathcal{I}_\mathrm{d})$ and the semi-infinite branch of $\mathcal{F}^{(\theta)}_{(-\infty,0]}$.

Let $i, j \in \mathcal{I}$ with $0 < \alpha_i < \alpha_j$, for instance. By definition of the tree coded by a function, the distance between $p_B(\alpha_i)$ and $p_B(\alpha_j)$ in $\mathcal{G}_0(\mathcal{F}^{(\theta)})$ is given by

$$d(p_B(\alpha_i), p_B(\alpha_j)) = -2 \min_{u \in [\alpha_i, \alpha_j]} B_u.$$

But, by definition of $\mathcal{A}^\mathcal{N}$, we have

$$\begin{aligned} - \min_{u \in [\alpha_i, \alpha_j]} B_u &= \max_{k \in \mathcal{I}\ \alpha_i \le \alpha_k < \alpha_j} \Big(- \min_{u \in [\alpha_k, \beta_k]} B_u \Big) \\ &= \max_{k \in \mathcal{I}\ \alpha_i \le \alpha_k < \alpha_j} \zeta_k. \end{aligned}$$

The other cases $\alpha_j < \alpha_i < 0$ and $\alpha_i < 0 < \alpha_j$ can be handled similarly. We deduce that the distances on a dense subset of leaves of $\mathcal{G}_0(\mathcal{F}^{(\theta)})$ and $\mathfrak{T}^\mathcal{N}$ coincide, which implies the result by completeness of the trees. □

3.2.4. *Local times and other contour processes.* Recall $\theta \ge 0$. The Brownian forest $\mathcal{F}^{(\theta)}$ can be viewed as the genealogical tree of a stationary CB process (associated with the branching mechanism ψ_θ defined in (1.1)); see [14]. To be more precise, for every $i \in I$ let $(\ell^{(i)}_a,\ a \ge 0)$ be the local time measures of the tree τ_i. For every $t \in \mathbb{R}$, we consider the measure $\mathbf{Z}_t$ on $\mathcal{Z}_t(\mathcal{F}^{(\theta)})$ defined by

$$\mathbf{Z}_t(dx) = \sum_{i \in I} \mathbf{1}_{\tau_i}(x)\, \ell^{(i)}_{t-h_i}(dx), \tag{3.4}$$

and we write $Z_t = \mathbf{Z}_t(1)$ for its total mass, which also represents the population size at time t. For $\theta = 0$, we have $Z_t = +\infty$ a.s. for every $t \in \mathbb{R}$. For $\theta > 0$, the process $(Z_t, t \ge 0)$ is a stationary Feller diffusion, solution of the stochastic differential equation (1.2); see Corollary 3.3. in [14], as well as Theorem 1.2 in [17].

In the literature, one also considers the so-called Kesten tree, which is the genealogical tree associated with the Feller diffusion Z^+ solving (1.2) for $t \geq 0$ with initial condition $Z_0^+ = 0$. It corresponds to the genealogy of a sub-critical branching process started from an infinitesimal individual alive at time 0, conditionally on the non-extinction event. In our setting, the genealogical tree corresponds to $\mathcal{F}^{(\theta)}_{[0,+\infty)}$, and the process Z^+ is distributed as $\mathbf{Z}^+(1) = (\mathbf{Z}_t^+(1), t \geq 0)$, with the measure $\mathbf{Z}_t^+$ on $\mathcal{Z}_t(\mathcal{F}^{(\theta)}_{[0,+\infty)})$ defined by

$$\mathbf{Z}_t^+(dx) = \sum_{i \in I} \mathbf{1}_{\{h_i \geq 0\}} \, \mathbf{1}_{\tau_i}(x) \, \ell^{(i)}_{t-h_i}(dx).$$

It can also be described using a contour process obtained by the concatenation at infinity of two independent Brownian motions distributed as $B^{(\theta)}$ conditioned to be positive. We use the description given in [17], which is valid in the general Lévy case; see also [7, Section 7.4], which corresponds to the case $\theta = 0$.

Let $\mathcal{E}_2$ be the set of continuous nonnegative functions g defined on $\mathbb{R}$ such that $g(0) = 0$ and $\lim_{x \to -\infty} g(x) = \lim_{x \to +\infty} g(x) = +\infty$. For such a function, we again consider the pseudo-metric d_g defined by (2.11), but for $s, t \in \mathbb{R}$, and with $m_g(s, t)$ replaced by $m_g(s, t) = \inf_{r \notin [s \wedge t, s \vee t]} g(r)$ if $st < 0$. We define the tree T_g^+ as the quotient space on $\mathbb{R}$ induced by this pseudo-metric. We set p_g to be the canonical projection from $\mathbb{R}$ onto T_g. For $g \in \mathcal{E}_2$, the triplet $(T_g^+, d_g, p_g(0))$ is a complete locally compact rooted real tree with a unique semi-infinite branch. We continue to call g the contour process of T_g^+.

Let $B^+ = (B_t^+, t \in \mathbb{R})$ be such that $(B_t^+, t \geq 0)$ and $(B_{-t}^+, t \geq 0)$ are independent and distributed as $B^{(\theta)} - 2I^{(\theta)}$, which is a diffusion on $\mathbb{R}_+$ with infinitesimal generator given by (2.7). Thanks to Corollary 2.1, we get that the tree $T_{B^+}^+$ with contour process B^+ is distributed as the genealogical tree $\mathcal{F}^{(\theta)}_{[0,+\infty)}$ which is associated to the Feller diffusion $\mathbf{Z}^+(1)$ (solution of (1.2) on $\mathbb{R}^+$ with $Z_0 = 0$).

Let $\theta > 0$. It is also immediate to give the contour process of the genealogical tree conditionally on the extinction being at time 0. Recall the tree defined by its contour process with the concatenation at 0 defined in Lemma 2.3. Set $B^- = -B^+$. It is left to the reader to check that the tree $T_{B^-}^-$ with contour process B^- is distributed as the genealogical tree of the Feller diffusion $\mathbf{Z}^-(1) = (\mathbf{Z}^-(1)_t, t \leq 0)$ conditioned to die at time 0 and started with the stationary distribution at $-\infty$ (solution of (1.2) on $\mathbb{R}_-$ with $Z_0 = 0$), where the measure $\mathbf{Z}_t^-$ on $\mathcal{Z}_t(\mathcal{F}^{(\theta)}_{(-\infty,0]})$ is defined by

$$\mathbf{Z}_t^-(dx) = \sum_{i \in I} \mathbf{1}_{\{\zeta_i + h_i < 0\}} \, \mathbf{1}_{\tau_i}(x) \, \ell^{(i)}_{t-h_i}(dx),$$

where ζ_i is the height of the tree τ_i. This result can also be deduced from the reversal property of the Brownian tree; see [3].

4. Simulation of the genealogical tree ($\theta > 0$)

We use the representation of trees via the ancestral process (see Section 3), which is an atomic measure on $\mathbb{R}^* \times (0, +\infty)$ satisfying conditions of Definition 3.1.

Under $\mathbb{P}^{(\theta)}$, let $\sum_{i \in I} \delta_{(h_i, \varepsilon_i, e_i)}$ be a Poisson point measure on $\mathbb{R} \times \{-1, 1\} \times \mathcal{E}$ with intensity $\beta dh \, (\delta_{-1}(d\varepsilon) + \delta_1(d\varepsilon)) \, n^{(\theta)}(de)$, and let $\mathcal{F}^{(\theta)} = ((h_i, \tau_i), \; i \in I)$ be the associated Brownian forest. We denote by $\ell_a^{(i)}$ the local time measure of the tree τ_i at level a (recall that this local time is zero for $a \notin [0, H(\tau_i)]$), and we denote by ∂_i the root of τ_i. Recall that the extant population at time $h \in \mathbb{R}$ is given by $\mathcal{Z}_h(\mathcal{F}^{(\theta)})$ defined in Section 2.4.4, and the measure $\mathbf{Z}_h$ on $\mathcal{Z}_h(\mathcal{F}^{(\theta)})$ is defined by (3.4).

Let $(\mathfrak{X}_k, k \in \mathbb{N}^*)$ be, conditionally given $\mathcal{F}^{(\theta)}$, independent random variables distributed according to the probability measure $\mathbf{Z}_0/Z_0$. Notice that normalization by Z_0, which is motivated by the sampling approach, is not usual in the branching setting; see for instance Theorem 4.7 in [14], where the sampling is according to $\mathbf{Z}_0$ instead, leading to the bias factor Z_0^n.

For every $k \in \mathbb{N}^*$, we denote by i_k the index in I such that $\mathfrak{X}_k \in \tau_{i_k}$. For every $n \in \mathbb{N}^*$, we set $I_n = \{i_k,\ 1 \le k \le n\}$, and for every $i \in I_n$ we denote by $\tau_i^{(n)}$ the sub-tree of τ_i generated by the $\mathfrak{X}_k$ such that $i_k = i$ and $1 \le k \le n$; i.e.

$$\tau_i^{(n)} = \bigcup_{1\le k\le n,\ i_k=i} [\![\partial_i, \mathfrak{X}_k]\!].$$

We define the genealogical tree T_n of n individuals sampled uniformly at random from the population at time 0 by

$$T_n = (-\infty, 0] \circledast_{i\in I_n} (\tau_i^{(n)}, h_i).$$

Notice that $T_n \subset T_{n+1}$. Since the support of $\mathbf{Z}_h$ is $\mathcal{Z}_h(\mathcal{F}^{(\theta)})$ a.s., we get that a.s. $\mathrm{cl}\left(\bigcup_{n\in\mathbb{N}^*} T_n\right) = \mathcal{G}_0(\mathcal{F}^{(\theta)})$, where $\mathcal{G}_0(\mathcal{F}^{(\theta)})$ (see Definition (2.13)) is the genealogical tree of the forest $\mathcal{F}^{(\theta)}$ at time 0.

Recall c_θ as defined in (1.3). For $\delta > 0$, we will consider in the next sections a positive random variable ζ_δ^* whose distribution is given by, for $h > 0$,

$$\mathbb{P}(\zeta_\delta^* < h) = \mathrm{e}^{-\delta c_\theta(h)}. \tag{4.1}$$

This random variable is easy to simulate, since if U is uniformly distributed on [0, 1], then ζ_δ^* has the same distribution as

$$\frac{1}{2\theta\beta} \log\left(1 - \frac{2\theta\delta}{\log(U)}\right).$$

This random variable appears naturally in the simulation of the ancestral process of $\mathcal{F}^{(\theta)}$, because if $\sum_{i\in I} \delta_{(z_i,\zeta_i)}$ is a Poisson point measure on $\mathbb{R} \times \mathbb{R}_+$ with intensity

$$\mathbf{1}_{[0,\delta]}(z)\, dz\, |c_\theta'(\zeta)| d\zeta$$

(see Proposition 3.2 for the interpretation), then ζ_δ^* is distributed as $\max_{i\in I} \zeta_i$.

We now present many ways to simulate T_n. This will be done by simulating ancestral processes (see Section 3), which code for trees distributed as T_n.

Recall that for an interval I, we write $|I|$ for its length.

4.1. Static simulation

In what follows, S stands for 'static'. Assume $n \in \mathbb{N}^*$ is fixed. We present a way to simulate T_n under $\mathbb{P}^{(\theta)}$ with $\theta > 0$. See Figures 6 and 7 for an illustration for $n = 5$.

(i) The size of the population on the left (resp. right) of the origin is E_g (resp. E_d), where $E_\mathrm{g}, E_\mathrm{d}$ are independent exponential random variables with mean $1/2\theta$. Set $Z_0 = E_\mathrm{g} + E_\mathrm{d}$ for the total size of the population at time 0. Let $(U_k, k \in \mathbb{N}^*)$ be independent random variables uniformly distributed on [0, 1] and independent of $(E_\mathrm{g}, E_\mathrm{d})$. Set $X_0 = 0$ and, for $k \in \mathbb{N}^*$, $X_k = Z_0 U_k - E_\mathrm{g}$ as well as $\mathcal{X}_k = \{-E_\mathrm{g}, E_\mathrm{d}, X_0, \ldots, X_k\}$.

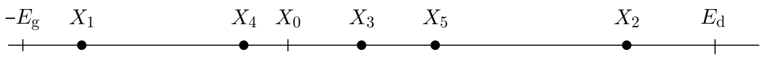

FIGURE 6: One realization of $E_g, E_d, X_1, \ldots, X_5$.

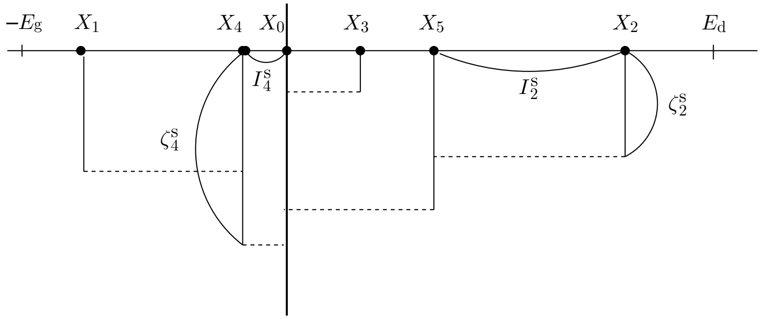

FIGURE 7: One realization of the tree $\mathfrak{T}_5^S$.

(ii) For $1 \le k \le n$, set $X_{k,n}^g = \max\{x \in \mathcal{X}_n,\ x < X_k\}$ and $X_{k,n}^d = \min\{x \in \mathcal{X}_n,\ x > X_k\}$. We also set $I_k^S = [X_{k,n}^g, X_k]$ if $X_k > 0$ and $I_k^S = [X_k, X_{k,n}^d]$ if $X_k < 0$.

(iii) Conditionally on $(E_g, E_d, X_1, \ldots, X_n)$, let $(\zeta_k^S, 1 \le k \le n)$ be independent random variables such that for $1 \le k \le n$, ζ_k^S is distributed as ζ_δ^* (see (4.1)), with $\delta = |I_k^S|$. Consider the tree $\mathfrak{T}_n^S$ corresponding to the ancestral process $\mathcal{A}_n^S = \sum_{k=1}^n \delta_{(X_k, \zeta_k^S)}$.

This gives an exact simulation of the tree T_n, according to the following result.

Lemma 4.1. *Let $\theta > 0$ and $n \in \mathbb{N}^*$. The tree $\mathfrak{T}_n^S$ is distributed as T_n under $\mathbb{P}^{(\theta)}$.*

Proof. Let $B = (B_t, t \in \mathbb{R})$ be the Brownian motion with drift defined in Subsection 3.2.1, and let $(L_t, t \in \mathbb{R})$ be its local time at 0, i.e.

$$L_t = L_t^d \mathbf{1}_{t>0} + L_{-t}^g \mathbf{1}_{t<0}.$$

We set $L_\infty = L_\infty^d + L_\infty^g$, and we consider independent and identically distributed variables $(S_1, \ldots, S_n)$ distributed according to dL_s/L_∞. We denote by $(S_{(1)}, \ldots, S_{(n)})$ the order statistics of $(S_1, \ldots S_n)$, and for every $i \le n$ we set

$$M_i = \begin{cases} -\min_{u \in [S_{(i)}, S_{(i+1)} \wedge 0]} B_u & \text{if } S_{(i)} < 0, \\ -\min_{u \in [S_{(i-1)} \vee 0, S_{(i)}]} B_u & \text{if } S_{(i)} > 0. \end{cases}$$

We set

$$\mathcal{A}_n = \sum_{1 \le i \le n} \delta_{(L_{S_{(i)}}, \zeta_i)},$$

which is an ancestral process (see Definition 3.1), and let $\mathfrak{T}(\mathcal{A}_n)$ be the associated tree. As B is the contour process of the tree $\mathcal{F}_{(-\infty,0]}$, we get that T_n and $\mathfrak{T}(\mathcal{A}_n)$ are equally distributed.

Moreover, by Proposition 3.2, Proposition 3.3, and standard results on Poisson point processes, we get that $\mathfrak{T}(\mathcal{A}_n)$ and $\mathfrak{T}_n^S$ are also equally distributed. $\square$

4.2. Dynamic simulation (I)

We can modify the static simulation of the previous section to provide a natural dynamic construction of the genealogical tree. In what follows, D stands for 'dynamic'. Let $\theta > 0$. We build recursively a family of ancestral processes $(\mathcal{A}_n, n \in \mathbb{N})$, with $\mathcal{A}_0^{\mathrm{D}} = 0$ and

$$\mathcal{A}_n^{\mathrm{D}} = \sum_{k=1}^{n} \delta_{(V_k, \zeta_k^{\mathrm{D}})}$$

for $n \in \mathbb{N}^*$.

(i) Let E_{g}, E_{d}, $(X_n, n \in \mathbb{N})$, and $(\mathcal{X}_n, n \in \mathbb{N}^*)$ be defined as in (i) of Section 4.1. For $n \in \mathbb{N}^*$, set $X_n^{\mathrm{g}} = \max\{x \in \mathcal{X}_n, x < X_n\}$ and $X_n^{\mathrm{d}} = \min\{x \in \mathcal{X}_n, x > X_n\}$. For $n \in \mathbb{N}^*$ and $\ell \in \{\mathrm{g}, \mathrm{d}\}$, define the interval $I_n^\ell = [X_n \wedge X_n^\ell, X_n \vee X_n^\ell]$ and its length $|I_n^\ell| = |X_n - X_n^\ell|$.

We shall consider and check by the induction the following hypothesis: for $n \geq 2$ the random variables $V_1, \dots, V_{n-1}$ are such that

$$X_{(0,n-1)} < V_{(1,n-1)} < X_{(1,n-1)} < \cdots < V_{(n-1,n-1)} < X_{(n-1,n-1)}, \tag{4.2}$$

where $(V_{(1,n-1)}, \dots, V_{(n,n)})$ and $(X_{(0,n-1)}, \dots, X_{(n-1,n-1)})$ respectively are the order statistics of $(V_1, \dots, V_{n-1})$ and of $(X_0, \dots, X_{n-1})$. Notice that (4.2) holds trivially for $n = 1$.

We set $\mathcal{W}_n^{\mathrm{D}} = (E_{\mathrm{g}}, E_{\mathrm{d}}, X_1, \dots, X_n, V_1, \dots, V_{n-1}, \zeta_1^{\mathrm{D}}, \dots, \zeta_{n-1}^{\mathrm{D}})$.

(ii) Assume $n \geq 1$. We work conditionally on $\mathcal{W}_n^{\mathrm{D}}$. On the event $\{X_n^{\mathrm{d}} = E_{\mathrm{d}}\}$ set $I_n = I_n^{\mathrm{g}}$ and on the event $\{X_n^{\mathrm{g}} = -E_{\mathrm{g}}\}$ set $I_n = I_n^{\mathrm{d}}$. On the event $\{X_n^{\mathrm{d}} = E_{\mathrm{d}}\} \bigcup \{X_n^{\mathrm{g}} = -E_{\mathrm{g}}\}$, let V_n be uniform on I_n, and let ζ_n^{D} be distributed as ζ_δ^* (see (4.1)) with $\delta = |I_n|$.

(iii) Assume $n \geq 2$ and that (4.2) holds. We work conditionally on $\mathcal{W}_n^{\mathrm{D}}$. On the event $\{-E_{\mathrm{g}} < X_n^{\mathrm{g}},\ X_n^{\mathrm{d}} < E_{\mathrm{d}}\}$, there exists a unique integer $\kappa_n \in \{1, \dots, n-1\}$ such that $V_{\kappa_n} \in [X_n^{\mathrm{g}},\ X_n^{\mathrm{d}}]$. If $X_n \in [X_n^{\mathrm{g}}, V_{\kappa_n})$, set $I_n = I_n^{\mathrm{g}}$; if $X_n \in [V_{\kappa_n}, X_n^{\mathrm{d}}]$, set $I_n = I_n^{\mathrm{d}}$. Then let V_n be uniform on I_n, and let ζ_n^{D} be distributed as ζ_δ^*, with $\delta = |I_n|$, conditionally on being less than $\zeta_{\kappa_n}^{\mathrm{D}}$.

(iv) Thanks to (ii) and (iii), notice that (4.2) holds with $n-1$ replaced by n, so that the induction is valid. Set

$$\mathcal{A}_n^{\mathrm{D}} = \mathcal{A}_{n-1}^{\mathrm{D}} + \delta_{(V_n, \zeta_n^{\mathrm{D}})}$$

and consider the tree $\mathfrak{T}_n^{\mathrm{D}}$ corresponding to the ancestral process $\mathcal{A}_n^{\mathrm{D}}$.

See Figures 8 and 9 for an example of $\mathfrak{T}_4^{\mathrm{D}}$ and $\mathfrak{T}_5^{\mathrm{D}}$.

Then we have the following result.

Lemma 4.2. *Let $\theta > 0$. The sequences of trees $\left(\mathfrak{T}_n^{\mathrm{D}}, n \in \mathbb{N}^*\right)$ and $(T_n, n \in \mathbb{N}^*)$ under $\mathbb{P}^{(\theta)}$ have the same distribution.*

Proof. We consider $\sum_{i \in \mathcal{I}} \delta_{(u_i, \zeta_i)}$ the ancestral process associated to the Poisson point measure $\sum_{i \in I} \delta_{(h_i, \varepsilon_i, e_i)}$ defined in Subsection 3.2.2. Let $(X_k'', k \in \mathbb{N}^*)$ be independent uniform random variables on $[-E_{\mathrm{g}}, E_{\mathrm{d}}]$. Set $X_0'' = 0$. For $n \geq 1$, let us denote by $(X''_{(k,n)}, 0 \leq k \leq n)$ the order statistic of $(X_0'', \dots, X_n'')$.

For every $n \geq 1$ and every $1 \leq k \leq n$, we set $i_{k,n}$ to be the index in $\mathcal{I}$ such that

$$\zeta_{i_{k,n}} = \max_{X''_{(k-1,n)} \leq u_i < X''_{(k,n)}} \zeta_i.$$

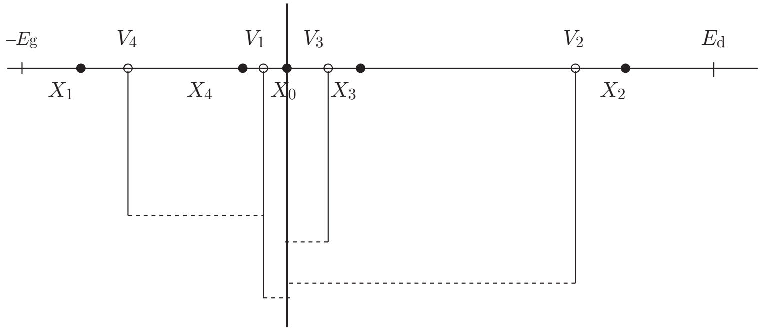

FIGURE 8: An example of the tree $\mathfrak{T}_4^{\mathrm{D}}$.

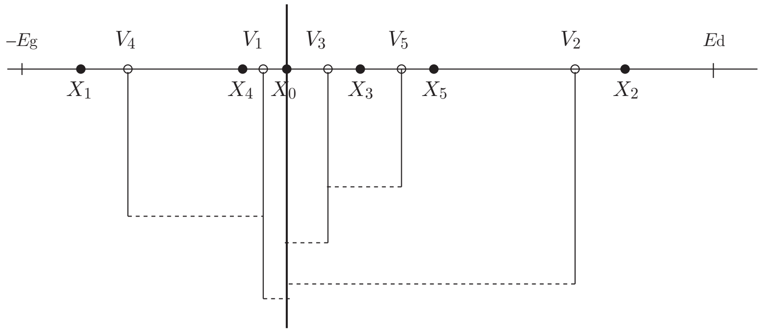

FIGURE 9: An example of the tree $\mathfrak{T}_5^{\mathrm{D}}$. The length of the new branch attached to V_5 is conditioned to be less than that of the previous branch that was in the considered interval attached to V_2.

Note that this index exists, since for every $\varepsilon > 0$, the set $\{i \in \mathcal{I},\ \ \zeta_i > \varepsilon\}$ is a.s. finite. We set $V''_{(k,n)} = u_{i_{k,n}}$ and notice that, by standard Poisson point measure properties, $V''_{(k,n)}$ is, conditionally given $(X''_0, \dots, X''_n)$, uniformly distributed on $[X''_{(k-1,n)}, X''_{(k,n)}]$. We define

$$\mathcal{A}''_n = \sum_{k=1}^{n} \delta_{(V''_{(k,n)}, \zeta_{i_{k,n}})}.$$

By construction, it is easy to check that the order statistics

$$X''_{(0,n)} < V''_{(1,n)} < X''_{(1,n)} < \cdots < V''_{(n,n)} < X''_{(n,n)}$$

are distributed as

$$X_{(0,n)} < V_{(1,n)} < X_{(1,n)} < \cdots < V_{(n,n)} < X_{(n,n)}.$$

For $1 \le k \le n$, let $j_{k,n} \in \{1, \dots, n\}$ be the index such that $V_{(k,n)} = V_{j_{k,n}}$. By construction, we then deduce that $(((V_{(k,n)}, \zeta^{\mathrm{D}}_{j_{k,n}}), 1 \le k \le n), n \in \mathbb{N}^*)$ is distributed as $(((V''_{(k,n)}, \zeta_{i_{k,n}}), 1 \le k \le n), n \in \mathbb{N}^*)$. This implies that the sequences of ancestral processes $(\mathcal{A}''_n, n \in \mathbb{N}^*)$ and $(\mathcal{A}_n, n \in \mathbb{N}^*)$ have the same distribution. Then we use Proposition 3.3 to get that the sequence of trees $(T''_n, n \in \mathbb{N}^*)$, with T''_n associated to $\mathcal{A}''_n$, is distributed as $(T_n, n \in \mathbb{N}^*)$. □

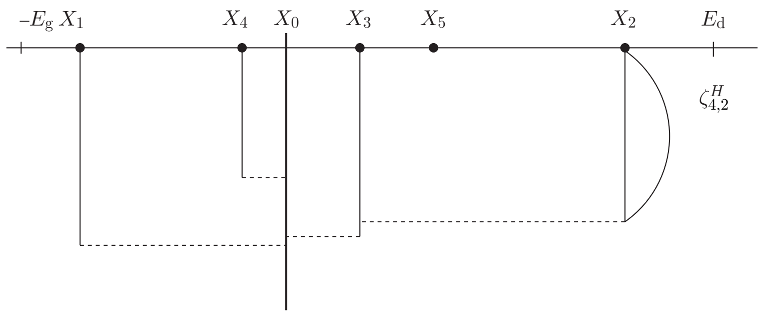

FIGURE 10: An example of the tree $\mathfrak{T}^{\mathrm{H}}_4$ with the new individual X_5.

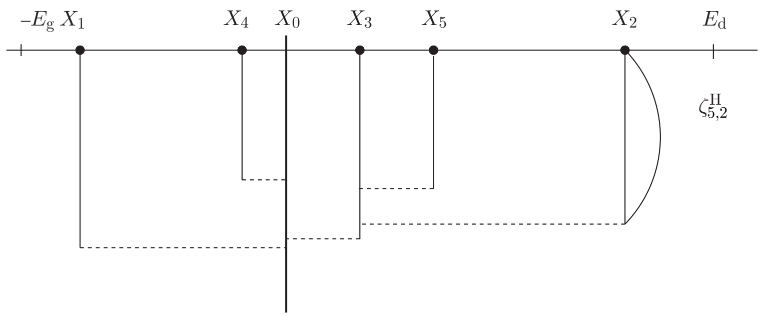

FIGURE 11: An example of the tree $\mathfrak{T}^{\mathrm{H}}_5$ with $\mathfrak{T}^{\mathrm{H}}_4$ given in Figure 10 and the event associated with p_{d} (a new segment is attached to X_5).

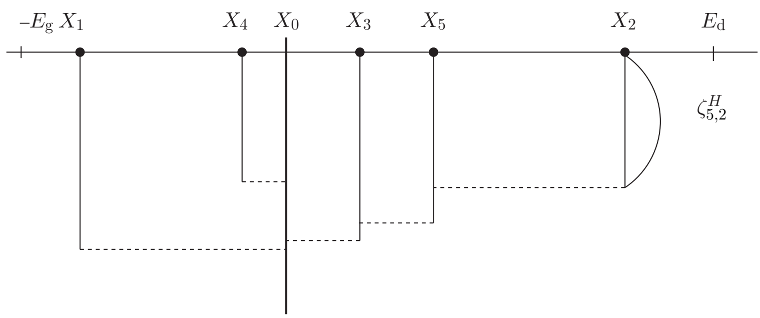

FIGURE 12: An example of the tree $\mathfrak{T}^{\mathrm{H}}_5$ with $\mathfrak{T}^{\mathrm{H}}_4$ given in Figure 10 and the event associated with p_{g} (the segment previously attached to X_2 is now attached to X_5, and a new segment is attached to X_2).

4.3. Dynamic simulation (II)

In a sense, we had to introduce another piece of random information corresponding to the position V_n of the largest spine of the sub-tree containing X_n. The construction in this subsection provides a way to remove this additional information (which is now hidden), but at the expense of possibly exchanging the newly inserted branch with one of its neighbors. In what follows, H stands for 'hidden'. An example is provided for $\mathfrak{T}^{\mathrm{H}}_4$ and $\mathfrak{T}^{\mathrm{H}}_5$ in Figures 10, 11, and 12.

Let $\theta > 0$. We build recursively a family of ancestral processes $(\mathcal{A}_n^{\mathrm{H}}, n \in \mathbb{N})$, with $\mathcal{A}_0^{\mathrm{H}} = 0$ and

$$\mathcal{A}_n^{\mathrm{H}} = \sum_{k=1}^{n} \delta_{\left(X_k, \zeta_{k,n}^{\mathrm{H}}\right)}$$

for $n \in \mathbb{N}^*$.

(i) Let E_{g}, E_{d}, $(X_n, n \in \mathbb{N})$, and $(\mathcal{X}_n, n \in \mathbb{N}^*)$ be defined as in (i) of Subsection 4.1. For $n \in \mathbb{N}^*$, set $X_n^{\mathrm{g}} = \max\{x \in \mathcal{X}_n, x < X_n\}$ and $X_n^{\mathrm{d}} = \min\{x \in \mathcal{X}_n, x > X_n\}$. For $n \in \mathbb{N}^*$ and $\ell \in \{\mathrm{g}, \mathrm{d}\}$, define the interval $I_n^\ell = [X_n \wedge X_n^\ell, X_n \vee X_n^\ell]$ and its length $|I_n^\ell| = |X_n - X_n^\ell|$.

We set $\mathcal{W}_n^{\mathrm{H}} = (E_{\mathrm{g}}, E_{\mathrm{d}}, X_1, \ldots, X_n, \zeta_{1,n-1}^{\mathrm{H}}, \ldots, \zeta_{n-1,n-1}^{\mathrm{H}})$.

(ii) Assume $n \geq 1$. On the event $\{X_n^{\mathrm{d}} = E_{\mathrm{d}}\}$ set $I_n = I_n^{\mathrm{g}}$, and on the event $\{X_n^{\mathrm{g}} = -E_{\mathrm{g}}\}$ set $I_n = I_n^{\mathrm{d}}$. Conditionally on $\mathcal{W}_n^{\mathrm{H}}$, let $\zeta_{n,n}^{\mathrm{H}}$ be distributed as ζ_δ^* (see (4.1)) with $\delta = |I_n|$; and for $1 \leq k \leq n-1$, set $\zeta_{k,n}^{\mathrm{H}} = \zeta_{k,n-1}^{\mathrm{H}}$.

(iii) Assume $n \geq 2$. We work conditionally on $\mathcal{W}_n^{\mathrm{H}}$. We define

$$p_{\mathrm{d}} = \frac{|I_n^{\mathrm{d}}|}{|I_n^{\mathrm{d}}| + |I_n^{\mathrm{g}}|} \quad \text{and} \quad p_{\mathrm{g}} = 1 - p_{\mathrm{d}} = \frac{|I_n^{\mathrm{g}}|}{|I_n^{\mathrm{d}}| + |I_n^{\mathrm{g}}|}.$$

(a) On the event $\{0 \leq X_n^{\mathrm{g}},\ X_n^{\mathrm{d}} < E_{\mathrm{d}}\}$, there exists a unique integer $\kappa_n^{\mathrm{d}} \in \{1, \ldots, n-1\}$ such that $X_{\kappa_n^{\mathrm{d}}} = X_n^{\mathrm{d}}$. For $1 \leq k \leq n-1$ and $k \neq \kappa_n^{\mathrm{d}}$, set $\zeta_{n,k}^{\mathrm{H}} = \zeta_{n-1,k}^{\mathrm{H}}$. Write

$$\zeta_n^{\mathrm{H}} = \zeta_{n-1,\kappa_n^{\mathrm{d}}}^{\mathrm{H}}.$$

With probability p_{d}, set $\zeta_{n,\kappa_n^{\mathrm{d}}}^{\mathrm{H}} = \zeta_n^{\mathrm{H}}$ and let $\zeta_{n,n}^{\mathrm{H}}$ be distributed as ζ_δ^*, with $\delta = |I_n^{\mathrm{g}}|$, conditionally on being less than ζ_n^{H}.

With probability p_{g}, set $\zeta_{n,n}^{\mathrm{H}} = \zeta_n^{\mathrm{H}}$ and let $\zeta_{n,\kappa_n^{\mathrm{d}}}^{\mathrm{H}}$ be distributed as ζ_δ^*, with $\delta = |I_n^{\mathrm{d}}|$, conditionally on being less than ζ_n^{H}.

(b) On the event $\{-E_{\mathrm{g}} < X_n^{\mathrm{g}},\ X_n^{\mathrm{d}} \leq 0\}$, there exists a unique integer $\kappa_n^{\mathrm{g}} \in \{1, \ldots, n-1\}$ such that $X_{\kappa_n^{\mathrm{g}}} = X_n^{\mathrm{g}}$. For $1 \leq k \leq n-1$ and $k \neq \kappa_n^{\mathrm{g}}$, set $\zeta_{n,k}^{\mathrm{H}} = \zeta_{n-1,k}^{\mathrm{H}}$. Write

$$\zeta_n^{\mathrm{H}} = \zeta_{n-1,\kappa_n^{\mathrm{g}}}^{\mathrm{H}}.$$

With probability p_{g}, set $\zeta_{n,\kappa_n^{\mathrm{g}}}^{\mathrm{H}} = \zeta_n^{\mathrm{H}}$ and let $\zeta_{n,n}^{\mathrm{H}}$ be distributed as ζ_δ^*, with $\delta = |I_n^{\mathrm{d}}|$, conditionally on being less than ζ_n^{H}.

With probability p_{d}, set $\zeta_{n,n}^{\mathrm{H}} = \zeta_n^{\mathrm{H}}$ and let $\zeta_{n,\kappa_n^{\mathrm{g}}}^{\mathrm{H}}$ be distributed as ζ_δ^*, with $\delta = |I_n^{\mathrm{g}}|$, conditionally on being less than ζ_n^{H}.

(iv) Let $\mathfrak{T}_n^{\mathrm{H}}$ be the tree corresponding to the ancestral process

$$\mathcal{A}_n^{\mathrm{H}} = \sum_{k=1}^{n} \delta_{(X_k, \zeta_{k,n}^{\mathrm{H}})}.$$

We now have the next result.

Lemma 4.3. *Let $\theta > 0$. The sequences of trees $(\mathfrak{T}_n^{\mathrm{H}}, n \in \mathbb{N}^*)$ and $(T_n, n \in \mathbb{N}^*)$ under $\mathbb{P}^{(\theta)}$ have the same distribution.*

Proof. The proof is left to the reader. It is in the same spirit as the proof of Lemma 4.2, but here we consider the random variables $((V''_{(k,n)}, 1 \le k \le n), n \in \mathbb{N}^*)$ to be unobserved. □

4.4. Simulation of genealogical tree conditional on its maximal height

Let $\mathcal{F}^{(\theta)} = ((\tau_i, h_i),\ i \in I)$ be a Brownian forest under $\mathbb{P}^{(\theta)}$. Recall the definition of A_0, the time to the MRCA of the population living at time 0, given in (5.3). The goal of this section is to simulate the genealogical tree T_n of n individuals uniformly sampled in the population living at time 0, conditionally given that the time to the MRCA of the whole population is h, i.e. given $A_0 = h$.

Let

$$\mathcal{A}(du, d\zeta) = \sum_{j \in \mathcal{I}} \delta_{(u_j, \zeta_j)}(du, d\zeta)$$

be the ancestral process of Definition 3.1. Recall the notation $E_{\mathrm{g}}, E_{\mathrm{d}}$ from Subsection 3.2.2. Let $\zeta_{\max} = \sup\{\zeta_j,\ j \in \mathcal{I}\}$ and define the random index $J_0 \in \mathcal{I}$ so that $\zeta_{\max} = \zeta_{J_0}$. Note that J_0 is well-defined since for every $\varepsilon > 0$, the set $\{j \in \mathcal{I},\ \zeta_j > \varepsilon\}$ is finite. We set $X = u_{J_0} \in (-E_{\mathrm{g}}, E_{\mathrm{d}})$. Note that $\zeta_{\max}$ is distributed as A_0.

For $r \in \mathbb{R}$, let $r_+ = \max(0, r)$ and $r_- = \max(0, -r)$ be respectively the positive and negative part of r. The proof of the next lemma is postponed to the end of this section.

Lemma 4.4. *Let $\theta > 0$. Under $\mathbb{P}^{(\theta)}$, conditionally given $\zeta_{\max} = h$, the random variables $E_g + X_-$, $|X|$, $E_d - X_+$, and $\mathbf{1}_{\{X \ge 0\}}$ are independent; $E_g + X_-$, $|X|$, and $E_d - X_+$ are exponentially distributed with parameter $2\theta + c_\theta(h)$, and $\mathbf{1}_{\{X \ge 0\}}$ is Bernoulli $1/2$.*

Let $h > 0$ be fixed. For $\delta > 0$, let $\zeta_\delta^{*,h}$ be a positive random variable distributed as ζ_δ^* conditionally on $\{\zeta_\delta^* \le h\}$; i.e. for $0 \le u \le h$,

$$\mathbb{P}\big(\zeta_{\delta \le u}^{*,h}\big) = \mathbb{P}\big(\zeta_\delta^{* \le u} m | \zeta_\delta^{* \le h}\big) = \mathrm{e}^{-\delta(c_\theta(u) - c_\theta(h))}.$$

Then the static simulation runs as follows.

(i) Simulate three independent random variables E_1, E_2, E_3 exponentially distributed with parameter $2\theta + c_\theta(h)$, and another independent Bernoulli variable ξ with parameter $1/2$. If $\xi = 0$, set $E_{\mathrm{g}} = E_1,\ X = E_2,\ E_{\mathrm{d}} = E_2 + E_3$, and if $\xi = 1$, set $E_{\mathrm{g}} = E_1 + E_2,\ X = -E_2,\ E_{\mathrm{d}} = E_3$. Let X_k and $\mathcal{X}_k$ be defined as in (i) of Subsection 4.1 for $1 \le k \le n$.

(ii) Let the intervals I_k^{S} be defined as in (ii) of Subsection 4.1 for $1 \le k \le n$.

(iii) Conditionally on $(E_{\mathrm{g}}, E_{\mathrm{d}}, X, X_1, \dots, X_n)$, let $\big(\zeta_k^h, 1 \le k \le n\big)$ be independent random variables such that, for $1 \le k \le n$, ζ_k^h is distributed as $\zeta_\delta^{*,h}$ with $\delta = \big|I_k^{\mathrm{S}}\big|$ if $X \notin I_k^{\mathrm{S}}$, and $\zeta_k^h = h$ if $X \in I_k^{\mathrm{S}}$. Consider the tree $\mathfrak{T}_n^h$ corresponding to the ancestral process

$$\mathcal{A}_n^h = \sum_{k=1} \delta_{\left(X_k, \zeta_k^h\right)}.$$

The proof of the following result, which relies on Lemma 4.4, is similar to that of Lemma 4.1, and is not reproduced here.

Lemma 4.5. *Let $\theta > 0$, $h > 0$, and $n \in \mathbb{N}^*$. The tree $\mathfrak{T}_n^h$ is distributed as T_n under $\mathbb{P}^{(\theta)}$ conditionally given $A_0 = h$.*

Notice that the height of $\mathfrak{T}_n^h$ is less than or equal to h. When it is strictly less than h, this means that no individual of the oldest family has been sampled.

Proof of Lemma 4.4. By Proposition 3.2, the pair $E=(E_\mathrm{g}, E_\mathrm{d})$ under $\mathbb{P}^{(\theta)}$ has density

$$f_E(e_\mathrm{g}, e_\mathrm{d}) = (2\theta)^2 \mathrm{e}^{-2\theta(e_\mathrm{g}+e_\mathrm{d})}\mathbf{1}_{\{e_\mathrm{g}\geq 0, e_\mathrm{d}\geq 0\}}.$$

Moreover, by standard results on Poisson point measures, the conditional density of the pair $(X, \zeta_{\max})$ given $(E_\mathrm{d}, E_\mathrm{g}) = (e_g, e_d)$ exists and is

$$\begin{aligned} f_{X,\zeta_{\max}}^{E=(e_\mathrm{g},e_\mathrm{d})}(x, h) &= \frac{1}{e_\mathrm{g}+e_\mathrm{d}}\mathbf{1}_{[-e_\mathrm{g},e_\mathrm{d}]}(x)\,(e_\mathrm{g}+e_\mathrm{d})\,|c'_\theta(h)|\mathrm{e}^{-c_\theta(h)(e_\mathrm{g}+e_\mathrm{d})}\mathbf{1}_{\{h\geq 0\}} \\ &= \mathbf{1}_{[-e_\mathrm{g},e_\mathrm{d}]}(x)\,|c'_\theta(h)|\mathrm{e}^{-c_\theta(h)(e_\mathrm{g}+e_\mathrm{d})}\mathbf{1}_{\{h\geq 0\}}. \end{aligned}$$

We deduce that the vector $(E_\mathrm{g}, E_\mathrm{d}, X, \zeta_{\max})$ has density

$$f(e_\mathrm{g}, e_\mathrm{d}, x, h) = (2\theta)^2|c'_\theta(h)|\mathrm{e}^{-(2\theta+c_\theta(h))(e_\mathrm{g}+e_\mathrm{d})}\mathbf{1}_{\{e_\mathrm{g}\geq 0,\ e_\mathrm{d}\geq 0,\ -e_\mathrm{g}\leq x\leq e_\mathrm{d},\ h\geq 0\}}$$

and that the random variable $\zeta_{\max}$ has density

$$\begin{aligned} f_{\zeta_{\max}}(h) &= \int (2\theta)^2|c'_\theta(h)|\mathrm{e}^{-(2\theta+c_\theta(h))(e_\mathrm{g}+e_\mathrm{d})}\mathbf{1}_{\{e_\mathrm{g}\geq 0,\ e_\mathrm{d}\geq 0,\ -e_\mathrm{g}\leq x\leq e_\mathrm{d},\ h\geq 0\}}\,de_\mathrm{g}\,de_\mathrm{d}\,dx \\ &= (2\theta)^2|c'_\theta(h)|\frac{2}{(2\theta+c_\theta(h))^3}\,\mathbf{1}_{\{h\geq 0\}}. \end{aligned}$$

Therefore, the conditional density of the vector $(E_\mathrm{g}, E_\mathrm{d}, X)$ given $\zeta_{\max}=h$ is

$$f_{E,X}^{\zeta_{\max}=h}(e_\mathrm{g}, e_\mathrm{d}, x) = \frac{1}{2}(2\theta+c_\theta(h))^3\mathrm{e}^{-(2\theta+c_\theta(h))(e_\mathrm{g}+e_\mathrm{d})}\mathbf{1}_{\{e_\mathrm{g}\geq 0,\ e_\mathrm{d}\geq 0,\ -e_\mathrm{g}\leq x\leq e_\mathrm{d}\}}.$$

For any nonnegative measurable function φ, we have

$$\begin{aligned} &\mathbb{E}^{(\theta)}[\varphi(E_\mathrm{g}+X_-, |X|, E_\mathrm{d}-X_+)\mathbf{1}_{\{X\geq 0\}}m|\zeta_{\max}=h] \\ &\quad= \mathbb{E}^{(\theta)}[\varphi(E_\mathrm{g}, X, E_\mathrm{d}-X)\mathbf{1}_{\{X\geq 0\}}m|\zeta_{\max}=h] \\ &\quad= \int \varphi(e_\mathrm{g}, x, e_\mathrm{d}-x)\frac{1}{2}(2\theta+c_\theta(h))^3\mathrm{e}^{-(2\theta+c_\theta(h))(e_\mathrm{g}+e_\mathrm{d})}\mathbf{1}_{\{e_\mathrm{g}\geq 0,\ e_\mathrm{d}\geq x\geq 0\}}\,de_\mathrm{g}\,de_\mathrm{d}\,dx \\ &\quad= \int \varphi(e_1, e_2, e_3)\frac{1}{2}(2\theta+c_\theta(h))^3\mathrm{e}^{-(2\theta+c_\theta(h))(e_1+e_2+e_3)}\mathbf{1}_{\{e_1\geq 0,\ e_2\geq 0,\ e_3\geq 0\}}\,de_1\,de_2\,de_3, \end{aligned}$$

using an obvious change of variables. Similarly, we get

$$\begin{aligned} &\mathbb{E}^{(\theta)}[\varphi(E_\mathrm{g}+X_-, |X|, E_\mathrm{d}-X_+)\mathbf{1}_{\{X<0\}}m|\zeta_{\max}=h] \\ &\qquad\qquad= \mathbb{E}^{(\theta)}[\varphi(E_\mathrm{g}+X_-, |X|, E_\mathrm{d}-X_+)\mathbf{1}_{\{X\geq 0\}}m|\zeta_{\max}=h]. \end{aligned}$$

This proves the lemma. □

5. Renormalized total length of the genealogical tree

Let $\mathcal{F}^{(\theta)} = ((h_i, \tau_i),\ i\in I)$ be a Brownian forest under $\mathbb{P}^{(\theta)}$ with $\theta>0$. Recall that the tree $\mathcal{F}^{(\theta)}_{(-\infty,0]}$ belongs to $\mathbb{T}_1$. For a forest $\mathbf{f}\in\mathbb{T}_1$, recall that $\mathcal{Z}_h(\mathbf{f})$ denotes the set of vertices of

$\mathcal{F}^{(\theta)}_{(-\infty,0]}$ at level h. We shall also consider $\mathcal{Z}^*_h(\mathbf{f}) = \mathcal{Z}_h(\mathbf{f}) \bigcap \mathcal{S}(\mathbf{f}_{(-\infty,h]})^c$, the extant population at time h except the point on the semi-infinite branch $(-\infty, h]$. For $r \leq h$, we define the set of ancestors at time r in the past of the extant population at time h, forgetting the individual in the infinite spine, as

$$\mathcal{M}^h_r(\mathbf{f}) = \mathcal{G}_h(\mathbf{f}) \bigcap \mathcal{Z}^*_r(\mathbf{f}), \tag{5.1}$$

and denote its cardinality by

$$M^h_r(\mathbf{f}) = \text{Card}\,(\mathcal{M}^h_r(\mathbf{f})). \tag{5.2}$$

We also define the time to the MRCA of $\mathcal{Z}_t(\mathcal{F}^{(\theta)})$ as

$$A_t = t - \sup\left\{r \leq t; M^t_r = 0\right\}. \tag{5.3}$$

We want to define the length of the genealogical tree $\mathcal{G}_t(\mathcal{F}^{(\theta)})$ of all extant individuals at time t (which is a.s. infinite) by approximating this genealogical tree by trees with finite length and taking compensated limits. Without loss of generality we can take $t = 0$ (since the distribution of the Brownian forest is invariant under time translation).

Two approximations may be considered here. For the first, we consider for $\varepsilon > 0$ the genealogical tree of individuals at time $t - \varepsilon$, with descendants at time t, and let ε go down to 0. We define the total length of the genealogical tree of the current population up to $\varepsilon > 0$ in the past as

$$L_\varepsilon = \int_\varepsilon^\infty M^0_{-s}\, ds. \tag{5.4}$$

Set $L = (L_\varepsilon, \varepsilon > 0)$. According to [11], we have that $\mathbb{E}[L_\varepsilon|Z_0] = -Z_0 \log(2\beta\theta\varepsilon)/\beta + O(\varepsilon)$ (see also (5.8), as $\tilde{L}_\varepsilon$ is distributed as L_ε), and that the sequence $(L_\varepsilon - \mathbb{E}[L_\varepsilon|Z_0], \varepsilon > 0)$ converges a.s. as ε goes down to zero towards a limit, say $\mathcal{L}$. We recall (1.4):

$$\mathbb{E}\left[\mathrm{e}^{-\lambda\mathcal{L}}|Z_0\right] = \mathrm{e}^{-2\theta Z_0\, \varphi(\lambda/(2\beta\theta))}, \quad \text{with} \quad \varphi(\lambda) = -\lambda \int_0^1 \frac{1 - v^\lambda}{1 - v}\, dv \quad \text{for all } \lambda > 0.$$

The second approximation consists in looking at the genealogical tree associated with n individuals picked at random from the population at time 0. Recall Definition (3.4) for $\mathbf{Z}_h$. Let $(X_k, k \in \mathbb{N}^*)$ be, conditionally on $\mathcal{F}^{(\theta)}$, independent random variables with distribution $\mathbf{Z}_0(dx)/Z_0$. This models individuals uniformly chosen from the population living at time 0. Define the set of ancestors of $X_1, \ldots, X_n$ at time $s < 0$ as

$$\mathcal{M}^{(n)}_s(\mathcal{F}^{(\theta)}) = \{x \in \mathcal{M}^0_s(\mathcal{F}^{(\theta)}); x \prec X_i \text{ for some } 1 \leq i \leq n\},$$

and let $M^{(n)}_s = \text{Card}\,(\mathcal{M}^{(n)}_s(\mathcal{F}^{(\theta)}))$ denote its cardinality. We define the total length of the genealogical tree of n individuals uniformly chosen in the current population as

$$\Lambda_n = \int_0^\infty M^{(n)}_{-s}\, ds. \tag{5.5}$$

Set $\Lambda = (\Lambda_n, n \in \mathbb{N}^*)$. The next theorem states that the two approximations give the same a.s. limit.

Theorem 5.1. *The sequence* $(\Lambda_n - \mathbb{E}[\Lambda_n|Z_0], n \in \mathbb{N}^*)$ *converges a.s. and in* L^2 *towards* $\mathcal{L}$ *as* n *tends to* $+\infty$. *And we have*

$$\mathbb{E}[\Lambda_n|Z_0] = \frac{Z_0}{\beta} \log\left(\frac{n}{2\theta Z_0}\right) + R_n,$$

with $R_n = O(n^{-1} \log(n))$ *and* $\mathbb{E}[|R_n|] = O(n^{-1} \log(n))$.

The rest of the section is devoted to the proof of this theorem. The idea of this proof is to prove the result for the ancestral tree of Section 3. The static simulation of Subsection 4.1 allows us to give precise asymptotics for the first and second moments (conditionally given the population size) of the quantity Λ_n. Then we prove that $\Lambda_n - L_\varepsilon$ (where L_ε is the length of the genealogical tree up to level $-\varepsilon$ introduced in [11]) converges in L^2 to 0 when ε is of order $1/n$, which easily yields the theorem.

5.1. Preliminary results

Let E_{g} and E_{d} be two independent exponential random variables with parameter 2θ. Let $\mathcal{N}(dz, d\tau)$ be, conditionally given $(E_{\mathrm{g}}, E_{\mathrm{d}})$, distributed as a Poisson point measure with intensity

$$\mathbf{1}_{[-E_{\mathrm{g}},E_{\mathrm{d}}]}(z)\, dz\mathbb{N}^{(\theta)}[d\tau].$$

We denote by $(z_i, \tau_i)_{i\in I}$ the atoms of the measure $\mathcal{N}$, so that

$$\mathcal{N} = \sum_{i\in I} \delta_{z_i,\tau_i}.$$

We define $\tilde{L} = (\tilde{L}_\varepsilon, \varepsilon > 0)$ with

$$\tilde{L}_\varepsilon = \sum_{i\in I} (\zeta_i - \varepsilon)_+,$$

where $\zeta_i = H(\tau_i)$ is the height of τ_i. Notice that only a finite number of trees $(\tau_i)_{i\in I}$ have height greater than ε, so the sum in $\tilde{L}_\varepsilon$ is a.s. finite, and $\tilde{L}_\varepsilon$ is well-defined. Let $(U_k, k \in \mathbb{N}^*)$ be independent random variables uniformly distributed on [0, 1] and independent of $(\mathcal{N}, E_{\mathrm{g}}, E_{\mathrm{d}})$. We set $X_0 = 0$, and $X_k = (E_{\mathrm{g}} + E_{\mathrm{d}})U_k - E_{\mathrm{g}}$ for $k \in \mathbb{N}^*$. Fix $n \in \mathbb{N}^*$. Let $X_{(0,n)} \le \cdots \le X_{(n,n)}$ be the corresponding order statistic of $(X_0, \ldots, X_n)$. We set $X_{(-1,n)} = -E_{\mathrm{g}}$ and $X_{(n+1,n)} = E_{\mathrm{d}}$. We define the interval $I_{k,n} = (X_{(k-1,n)}, X_{(k,n)})$ and its length $\Delta_{k,n} = X_{(k,n)} - X_{(k-1,n)}$ for $0 \le k \le n+1$. We set $\Delta_n = (\Delta_{k,n}, 0 \le k \le n+1)$. For $1 \le k \le n$, we define $\tilde{\Lambda} = (\tilde{\Lambda}_n, n \in \mathbb{N}^*)$ by

$$\tilde{\Lambda}_n = \sum_{k=1}^{n} \zeta^*_{k,n}. \quad \text{with} \quad \zeta^*_{k,n} = \max\{\zeta_i; z_i \in I_{k,n}\}.$$

Recall the definitions of Z_0 in (3.4), $L = (L_\varepsilon, \varepsilon > 0)$ in (5.4), and $\Lambda = (\Lambda_n, n \in \mathbb{N}^*)$ in (5.5). Thanks to Proposition 3.2, we deduce that (Z_0, L, Λ) is distributed as $(E_{\mathrm{g}} + E_{\mathrm{d}}, \tilde{L}, \tilde{\Lambda})$. So to prove Theorem 5.1, it is enough to prove the statement with $\tilde{\Lambda}$ instead of Λ.

For convenience, we set $Z_0 = E_{\mathrm{g}} + E_{\mathrm{d}}$. Elementary computations give the following lemma. Recall that $z_+ = \max(z, 0)$.

Lemma 5.1. *Let $\theta > 0$ and $\varepsilon > 0$. We have*

$$\mathbb{N}^{(\theta)}[(\zeta - \varepsilon)_+] = \int_\varepsilon^\infty c_\theta(h)\, dh = -\frac{1}{\beta}\log(2\beta\theta\varepsilon) + O(\varepsilon), \tag{5.6}$$

$$\mathbb{N}^{(\theta)}[(\zeta - \varepsilon)_+^2] = 2\int_\varepsilon^\infty hc_\theta(h)\, dh - 2\varepsilon\int_\varepsilon^\infty c_\theta(h)\, dh = 2\int_0^\infty hc_\theta(h)\, dh + O(\varepsilon\log(\varepsilon)). \tag{5.7}$$

We deduce that

$$\mathbb{E}[\tilde{L}_\varepsilon | Z_0] = -\frac{Z_0}{\beta}\log(2\beta\theta\varepsilon) + O(\varepsilon), \tag{5.8}$$

$$\mathbb{E}[\tilde{L}_\varepsilon^2 | Z_0] = 2Z_0\int_0^\infty hc_\theta(h)\, dh + \mathbb{E}[\tilde{L}_\varepsilon | Z_0]^2 + O(\varepsilon\log(\varepsilon)), \tag{5.9}$$

where we used that if $\sum_{i\in I}\delta_{x_i}$ is a Poisson point measure with intensity $\mu(dx)$, then

$$\mathbb{E}\left[\left(\sum_{i\in I} f(x_i)\right)^2\right] = \mu(f^2)+\mu(f)^2. \tag{5.10}$$

Eventually, let us notice that with the change of variable $u=c_\theta(h)$ (so that $dh = du/\beta u(u+2\theta)$), we have

$$2\int_0^\infty hc_\theta(h)\,dh = \frac{1}{\beta^2\theta}\int_0^\infty \frac{\log(v+1)}{v(v+1)}\,dv. \tag{5.11}$$

Recall the definition of ζ^*_δ for $\delta>0$ (see (4.1)). Let γ be the Euler constant, and thus

$$\gamma = -\int_0^{+\infty}\log(u)\mathrm{e}^{-u}\,du.$$

We have the following lemma.

Lemma 5.2. *Let $\delta>0$. We have*

$$\mathbb{E}[\zeta^*_\delta] = -\frac{\delta}{\beta}\log(2\theta\delta)+\frac{\delta}{\beta}(1-\gamma)+\frac{\delta}{\beta}g_1(2\theta\delta), \tag{5.12}$$

with $|g_1(x)|\leq x(|\log(x)|+2)$ for $x>0$ and

$$\mathbb{E}[(\zeta^*_\delta)^2] = 2\delta\int_0^\infty hc_\theta(h)\,dh+\frac{\delta}{\beta^2\theta}g_2(2\theta\delta), \tag{5.13}$$

with $|g_2(x)|\leq x(|\log(x)|+2)$ for $x>0$. We also have

$$\mathbb{E}\left[\zeta^*_\delta\sum_{i\in I}(\zeta_i-\varepsilon)_+\right] = 2\delta\int_0^\infty hc_\theta(h)\,dh+g_3(\delta), \tag{5.14}$$

and there exists a finite constant c such that for all $x>0$ and $\varepsilon\in(0,1]$, we have

$$|g_3(x)|\leq cx^2(1+x)(|\log(x)|+1)(|\log(\varepsilon)|+1)+c\varepsilon x(|\log(x)|+1)(1+x)+\varepsilon^2.$$

The end of this section is devoted to the proof of Lemma 5.2.

5.1.1. *Proof of* (5.12). Using (4.1), we get

$$\mathbb{E}[\zeta^*_\delta] = \int_0^\infty \mathbb{P}(\zeta^*_\delta>h)\,dh = \int_0^\infty (1-\mathrm{e}^{-\delta c_\theta(h)})\,dh = \frac{\delta}{\beta}\int_0^\infty (1-\mathrm{e}^{-u})\frac{du}{u(u+2\theta\delta)}, \tag{5.15}$$

where we used the change of variable $u=\delta c_\theta(h)$. It is easy to check that for $a>0$,

$$\log(1+a)\leq|\log(a)|+\log(2). \tag{5.16}$$

Let $a>0$. We have

$$\begin{aligned}\int_0^1(1-\mathrm{e}^{-u})\frac{du}{u(u+a)} &= \int_0^1(1-u-\mathrm{e}^{-u})\frac{du}{u(u+a)}+\log(1+a)-\log(a)\\ &= \int_0^1(1-u-\mathrm{e}^{-u})\frac{du}{u^2}+\log(1+a)-\log(a)+ag_{1,0}(a),\end{aligned}$$

with

$$g_{1,0}(a) = -\int_0^1 (1-u-\mathrm{e}^{-u})\,\frac{du}{u^2(u+a)}$$
$$\leq \int_0^1 \frac{du}{2(u+a)} = \frac{1}{2}(\log(1+a) - \log(a)) \leq |\log(a)| + \frac{1}{2}$$

and $g_{1,0}(a) \geq 0$, where we used that $0 \leq -(1-u-\mathrm{e}^{-u}) \leq u^2/2$ for $u \geq 0$. We also have

$$\int_1^\infty (1-\mathrm{e}^{-u})\,\frac{du}{u(u+a)} = \int_1^\infty (1-\mathrm{e}^{-u})\,\frac{du}{u^2} - a g_{1,1}(a),$$

with

$$g_{1,1}(a) = \int_1^\infty (1-\mathrm{e}^{-u})\,\frac{du}{u^2(u+a)} \leq \int_1^\infty \frac{du}{u^3} \leq \frac{1}{2}\cdot$$

Notice that, by integration by parts, we have

$$\int_0^1 (1-u-\mathrm{e}^{-u})\,\frac{du}{u^2} + \int_1^\infty (1-\mathrm{e}^{-u})\,\frac{du}{u^2}$$
$$= \mathrm{e}^{-1} + \int_0^1 \log(u)\mathrm{e}^{-u} du + 1 - \mathrm{e}^{-1} + \int_1^\infty \log(u)\mathrm{e}^{-u} du = 1-\gamma.$$

We deduce that

$$\int_0^\infty (1-\mathrm{e}^{-u})\,\frac{du}{u(u+a)} = 1-\gamma-\log(a) + g_1(a)$$

with $g_1(a) = \log(1+a) + a g_{1,0}(a) - a g_{1,1}(a)$ and

$$|g_1(a)| = |\log(1+a) + a g_{1,0}(a) - a g_{1,1}(a)| \leq a(|\log(a)| + 2).$$

We then use (5.15) to get (5.12).

5.1.2. *Proof of* (5.13). Using (4.1), we get

$$\mathbb{E}\Big[\big(\zeta_\delta^*\big)^2\Big] = 2\int_0^\infty h\big(1-\mathrm{e}^{-\delta c_\theta(h)}\big)\,dh = 2\frac{\delta}{\beta}\int_0^\infty \frac{1}{2\beta\theta}\log\left(\frac{u+2\theta\delta}{u}\right)(1-\mathrm{e}^{-u})\,\frac{du}{u(u+2\theta\delta)}, \tag{5.17}$$

where we used the change of variable $u = \delta c_\theta(h)$. Let $a > 0$. We set

$$g_{2,1}(a) = \int_1^\infty \log\left(\frac{u+a}{u}\right)(1-\mathrm{e}^{-u})\,\frac{du}{u(u+a)}\cdot$$

Using that $0 \leq \log(1+x) \leq x$ for $x > 0$, we have

$$|g_{2,1}(a)| \leq a\int_1^\infty \frac{du}{u^3} \leq \frac{a}{2}\cdot$$

We also have

$$\int_0^1 \log\left(\frac{u+a}{u}\right)(1-\mathrm{e}^{-u})\,\frac{du}{u(u+a)} = \int_0^1 \log\left(\frac{u+a}{u}\right)\frac{du}{u+a} + g_{2,2}(u)$$
$$= \int_0^\infty \frac{\log(v+1)}{v(v+1)}\,dv - g_{2,3}(a) + g_{2,2}(a),$$

with the change of variable $v = a/u$ as well as

$$g_{2,2}(a) = \int_0^1 \log\left(\frac{u+a}{u}\right)(1-u-\mathrm{e}^{-u})\,\frac{du}{u(u+a)} \quad \text{and} \quad g_{2,3}(a) = \int_0^a \frac{\log(v+1)}{v(v+1)}\,dv.$$

Using $\log(1+v) \leq v$ for $v > 0$ (twice), we have that

$$0 \leq g_{2,3}(a) \leq \int_0^a \frac{dv}{v+1} \leq a.$$

Using the fact that $|1-u-\mathrm{e}^{-u}| \leq u^2/2$ if $u > 0$ for the first inequality and (5.16) for the last, we have that

$$|g_{2,2}(a)| \leq \frac{1}{2}\int_0^1 \log\left(1+\frac{a}{u}\right)\frac{udu}{(u+a)} \leq \frac{a}{2}\int_0^1 \frac{du}{(u+a)} \leq a\Big(|\log(a)| + \frac{1}{2}\Big).$$

We deduce that

$$\int_0^\infty \log\left(\frac{u+a}{u}\right)(1-\mathrm{e}^{-u})\,\frac{du}{u(u+a)} = \int_0^\infty \frac{\log(v+1)}{v(v+1)}\,dv + g_2(a)$$

and

$$|g_2(a)| = |g_{2,1}(a) - g_{2,3}(a) + g_{2,2}(a)| \leq a(|\log(a)| + 2).$$

We then use (5.17) as well as the identity (5.11) to get (5.13).

5.1.3. *Proof of* (5.14). Using properties of Poisson point measures, we get that if $\sum_{j\in J}\delta_{\zeta_j}$ is a Poisson point measure with intensity $\delta\mathbb{N}[d\zeta]$ and $\zeta_\delta^* = \max_{j\in J}\zeta_j$, then for any measurable nonnegative functions f and g, we have

$$\mathbb{E}\left[f(\zeta_\delta^*)\mathrm{e}^{-\sum_{j\in J} g(\zeta_j)}\right] = \mathbb{E}\left[f(\zeta_\delta^*)\mathrm{e}^{-g(\zeta_\delta^*)-G(\zeta_\delta^*)}\right]$$

with

$$G(r) = \delta\mathbb{N}\left[(1-\mathrm{e}^{-g(\zeta)})\mathbf{1}_{\{\zeta<r\}}\right].$$

We deduce that

$$\mathbb{E}\left[\zeta_\delta^*\sum_{i\in I}(\zeta_i-\varepsilon)_+\right] = \mathbb{E}[\zeta_\delta^*(\zeta_\delta^*-\varepsilon)_+] + \delta g_{3,1}(\delta),$$

with

$$g_{3,1}(\delta) = \mathbb{E}\left[\zeta_\delta^*\mathbb{N}\left[(\zeta-\varepsilon_+)\mathbf{1}_{\{\zeta<h\}}\right]_{|h=\zeta_\delta^*}\right].$$

According to (5.12), there exists a finite constant $c > 0$ such that for all $\delta > 0$, we have $\mathbb{E}[\zeta_\delta^*] \leq c\delta(|\log(\delta)|+1)(1+\delta)$. We deduce from (5.6) that there exists a finite constant c independent of $\delta > 0$ and $\varepsilon \in (0, 1]$ such that

$$g_{3,1}(\delta) \leq \mathbb{E}[\zeta_\delta^*]\mathbb{N}[(\zeta-\varepsilon)_+] \leq c\delta(|\log(\delta)|+1)(1+\delta)(|\log(\varepsilon)|+1).$$

We also have

$$\begin{aligned}\mathbb{E}[\zeta_\delta^*(\zeta_\delta^*-\varepsilon)_+] &= \mathbb{E}[(\zeta_\delta^*)^2] - \mathbb{E}[(\zeta_\delta^*)^2\mathbf{1}_{\{\zeta_\delta^*<\varepsilon\}}] - \varepsilon\mathbb{E}[\zeta_\delta^*\mathbf{1}_{\{\zeta_\delta^*>\varepsilon\}}] \\ &= 2\delta\int_0^\infty hc_\theta(h)\,dh + g_{3,2}(\varepsilon, \delta),\end{aligned}$$

with, thanks to (5.12) and (5.13),

$$|g_{3,2}(\varepsilon,\delta)| \le c\delta^2(|\log(\delta)|+1)+\varepsilon^2+c\varepsilon\delta(|\log(\delta)|+1)(1+\delta)$$

for some finite constant c independent of $\delta > 0$ and $\varepsilon > 0$. We deduce that

$$\mathbb{E}\left[\zeta_\delta^* \sum_{i\in I}(\zeta_i-\varepsilon)_+\right] = 2\delta\int_0^\infty h c_\theta(h)\,dh + g_3(\delta),$$

and for some finite constant c independent of $\delta > 0$ and $\varepsilon \in (0, 1]$,

$$|g_3(\delta)| \le c\delta^2(1+\delta)(|\log(\delta)|+1)(|\log(\varepsilon)|+1)+c\varepsilon\delta(|\log(\delta)|+1)(1+\delta)+\varepsilon^2.$$

5.2. A technical lemma

An elementary induction argument shows that for $n \in \mathbb{N}$,

$$\int_0^1 (1-x)^n|\log(x)|\,dx = \frac{H_{n+1}}{n+1} \quad \text{and} \quad \int_0^1 (1-x)^n \log^2(x)\,dx = \frac{2}{n+1}\sum_{k=1}^{n+1}\frac{H_k}{k},$$

where $H_n = \sum_{k=1}^n k^{-1}$ is the harmonic sum. Recall that $H_n = \log(n) + \gamma + (2n)^{-1} + O(n^{-2})$. So we deduce that

$$(n+1)\int_0^1 (1-x)^n|\log(x)|\,dx = \log(n)+\gamma+\frac{3}{2n}+O(n^{-2}). \tag{5.18}$$

It is also easy to deduce that for $a, b \in \{1, 2\}$,

$$\int_0^1 x^a(1-x)^n|\log(x)|^b\,dx = O\left(\frac{\log^b(n)}{n^{a+1}}\right). \tag{5.19}$$

Recall $\tilde{\Lambda}_n$ and Δ_n defined in Subsection 5.1. We give a technical lemma. In this lemma $O(f(n))$ denotes a function, say ϕ, of Z_0 and n, such that $|\phi(Z_0, n)| \le Q(Z_0)f(n)$ for some positive function Q such that $Q(Z_0)$ is integrable. The explicit function Q is unimportant and thus not specified.

Lemma 5.3. *We have*

$$\mathbb{E}\big[\tilde{\Lambda}_n|\Delta_n\big] = \frac{Z_0}{\beta}(1-\gamma) - \sum_{k=1}^n \frac{\Delta_{k,n}}{\beta}\log(2\theta\Delta_{k,n}) + W_n, \tag{5.20}$$

with $\mathbb{E}[|W_n|\,|Z_0] = O(n^{-1}\log(n))$ *and*

$$\mathbb{E}\big[\tilde{\Lambda}_n|Z_0\big] = \frac{Z_0}{\beta}\log\left(\frac{n}{2\theta Z_0}\right) + O(n^{-1}\log(n)). \tag{5.21}$$

We have also

$$\mathbb{E}\big[\tilde{\Lambda}_n^2|Z_0\big] = 2Z_0\int_0^\infty h c_\theta(h)\,dh + \mathbb{E}\big[\tilde{\Lambda}_n\,|\,Z_0\big]^2 + O(n^{-1}\log^2(n)). \tag{5.22}$$

Proof. We first prove (5.20). We have $\mathbb{E}[\tilde{\Lambda}_n|\Delta_n] = \sum_{k=1}^n \mathbb{E}[\zeta_\delta^*]_{|\delta=\Delta_{k,n}}$. We deduce from (5.12) that (5.20) holds with

$$W_n = \frac{\Delta_{0,n} + \Delta_{n+1,n}}{\beta}(\gamma - 1) + \frac{1}{\beta}\sum_{k=1}^n \Delta_{k,n} g_1(2\theta\Delta_{k,n}).$$

Since, conditionally on Z_0, the random variables $\Delta_{k,n}$ are all distributed as $Z_0\tilde{U}_n$, where $\tilde{U}_n$ is independent of Z_0 and has distribution $\beta(1, n+1)$, we deduce using (5.19) that

$$\mathbb{E}[|W_n|\,|Z_0] \le 2\frac{(1-\gamma)Z_0}{\beta}\mathbb{E}[\tilde{U}_n] + n\frac{2\theta Z_0^2}{\beta}\mathbb{E}[\tilde{U}_n^2(|\log(2\theta Z_0\tilde{U}_n)| + 2)|Z_0] = O(n^{-1}\log(n)).$$

We then prove (5.21). Taking the expectation in (5.20) conditionally on Z_0, we get

$$\mathbb{E}[\tilde{\Lambda}_n|Z_0] = \frac{Z_0}{\beta}(1-\gamma) - n\frac{Z_0}{\beta}\mathcal{H}(2\theta Z_0) + \mathbb{E}[W_n|Z_0],$$

where

$$\mathcal{H}(a) = \mathbb{E}[\tilde{U}_n \log(a\tilde{U}_n)]. \tag{5.23}$$

We deduce from (5.18) that

$$n\mathcal{H}(a) = \log(a) - \log(n) + 1 - \gamma + O(n^{-1}\log(n)). \tag{5.24}$$

This gives

$$\mathbb{E}[\tilde{\Lambda}_n|Z_0] = \frac{Z_0}{\beta}\log\left(\frac{n}{2\theta Z_0}\right) + O(n^{-1}\log(n)).$$

We finally prove (5.22). We have

$$\mathbb{E}\left[\tilde{\Lambda}_n^2|\Delta_n\right] = \sum_{k=1}^n \mathbb{E}\left[(\zeta_\delta^*)^2\right]_{|\delta=\Delta_{k,n}} - \sum_{k=1}^n \mathbb{E}\left[\zeta_\delta^*\right]^2_{|\delta=\Delta_{k,n}} + \mathbb{E}\left[\tilde{\Lambda}_n|\Delta_n\right]^2. \tag{5.25}$$

Thanks to (5.13), we have

$$\sum_{k=1}^n \mathbb{E}\left[(\zeta_\delta^*)^2\right]_{|\delta=\Delta_{k,n}} = 2Z_0\int_0^\infty hc_\theta(h)\,dh + W_{1,n},$$

with

$$W_{1,n} = -2(\Delta_{0,n} + \Delta_{n+1,n})\int_0^\infty hc_\theta(h)\,dh + \sum_{k=1}^n \frac{\Delta_{k,n}}{\beta^2\theta} g_2(2\theta\Delta_{k,n}).$$

Using similar computations as the ones used to bound $\mathbb{E}[|W_n|\,|Z_0]$, we get

$$\mathbb{E}[|W_{1,n}|\,|Z_0] = O(n^{-1}\log(n)),$$

so that

$$\mathbb{E}\left[\sum_{k=1}^n \mathbb{E}\left[(\zeta_\delta^*)^2\right]_{|\delta=\Delta_{k,n}}\,|\,Z_0\right] = 2Z_0\int_0^\infty hc_\theta(h)\,dh + O(n^{-1}\log(n)).$$

Thanks to (5.12), we have $\mathbb{E}\left[\zeta_\delta^*\right]^2 \leq c\delta^2(|\log(\delta)|+1)^2(1+\delta)^2$ for some finite constant c which does not depend on δ. We set $\mathcal{H}_2(a)=\mathbb{E}\left[\tilde{U}_n^2\log^2(a\tilde{U}_n)(1+\tilde{U}_n)^2\right]$, and using (5.19), we get

$$\mathcal{H}_2(a)=O(n^{-3}\log^2(n))=O(n^{-2}\log^2(n)). \tag{5.26}$$

We deduce that

$$\mathbb{E}\left[\sum_{k=1}^n \mathbb{E}\left[\zeta_\delta^*\right]^2_{|\delta=\Delta_{k,n}} \mid Z_0\right]=O(n^{-1}\log^2(n)).$$

Then using (5.21), elementary computations give

$$\mathbb{E}\left[\mathbb{E}[\tilde{\Lambda}_n|\Delta_n]^2\,|Z_0\right]=2\frac{Z_0}{\beta}(1-\gamma)\mathbb{E}[\tilde{\Lambda}_n|Z_0]-\frac{Z_0^2}{\beta^2}(1-\gamma)^2+\frac{1}{\beta^2}J_{1,n}+J_{2,n}-\frac{2}{\beta}J_{3,n},$$

with

$$\begin{aligned}
J_{1,n}&=\mathbb{E}\left[\left(\sum_{k=1}^n \Delta_{k,n}\log(2\theta\Delta_{k,n})\right)^2\,\Big|\,Z_0\right],\\
J_{2,n}&=\mathbb{E}[W_n^2|Z_0],\\
J_{3,n}&=\mathbb{E}\left[W_n\left(\sum_{k=1}^n \Delta_{k,n}\log(2\theta\Delta_{k,n})\right)\,\Big|\,Z_0\right].
\end{aligned}$$

By Cauchy–Schwartz, we have $|J_{3,n}|\leq\sqrt{J_{1,n}J_{2,n}}$. Using $\left(\sum_{k=1}^n a_k\right)^2\leq n\sum_{k=1}^n a_k^2$, we also get

$$J_{2,n}\leq\frac{8}{\beta^2}(\gamma-1)^2Z_0^2\mathbb{E}[\tilde{U}_n^2]+\frac{2n}{\beta^2}Z_0^2\mathbb{E}\left[\tilde{U}_n^2g_1^2(2\theta Z_0\tilde{U}_n)\right]=O(n^{-2}).$$

By independence, we obtain

$$J_{1,n}=n(n-1)\mathbb{E}\left[\Delta_{1,n}\log(2\theta\Delta_{1,n})|Z_0\right]^2+n\mathbb{E}\left[\Delta_{1,n}^2\log^2(2\theta\Delta_{1,n})|Z_0\right].$$

Recall the function $\mathcal{H}$ defined in (5.23) and its asymptotic expansion (5.24). We have, using (5.26), that

$$\begin{aligned}
J_{1,n}&=n(n-1)Z_0^2\mathcal{H}(2\theta Z_0)^2+nZ_0^2\mathcal{H}_2(2Z_0)\\
&=Z_0^2\left(-\log\left(\frac{n}{2\theta Z_0}\right)+1-\gamma\right)^2+O(n^{-1}\log^2(n)).
\end{aligned}$$

So we deduce that

$$\begin{aligned}
\frac{1}{\beta^2}J_{1,n}+J_{2,n}-\frac{2}{\beta}J_{3,n}&=\left(-\frac{Z_0}{\beta}\log\left(\frac{n}{2\theta Z_0}\right)+\frac{Z_0}{\beta}(1-\gamma)\right)^2+O(n^{-1}\log^2(n))\\
&=\left(-\mathbb{E}[\tilde{\Lambda}_n|Z_0]+\frac{Z_0}{\beta}(1-\gamma)\right)^2+O(n^{-1}\log^2(n)).
\end{aligned}$$

We deduce that

$$\mathbb{E}\left[\mathbb{E}\left[\tilde{\Lambda}_n|\Delta_n\right]^2\,|Z_0\right]=\mathbb{E}[\tilde{\Lambda}_n\mid Z_0]^2+O(n^{-1}\log^2(n)).$$

So in the end, using (5.25), we get

$$\mathbb{E}\left[\tilde{\Lambda}_n^2 \,|\, Z_0\right] = 2Z_0 \int_0^\infty hc_\theta(h)\,dh + \mathbb{E}[\tilde{\Lambda}_n \,|\, Z_0]^2 + O(n^{-1}\log^2(n)). \qquad \square$$

5.3. Proof of Theorem 5.1

We shall keep the notation from Subsection 5.1. We set $J_n(\varepsilon) = \mathbb{E}\left[\left(\tilde{\Lambda}_n - \tilde{L}_\varepsilon\right)^2 |Z_0\right]$. We have

$$J_n(\varepsilon) = \mathbb{E}[\tilde{\Lambda}_n^2|Z_0] + \mathbb{E}[\tilde{L}_\varepsilon^2|Z_0] - 2\mathbb{E}[\tilde{\Lambda}_n\tilde{L}_\varepsilon|Z_0].$$

By conditioning with respect to Δ_n, and using the independence, we get

$$\begin{aligned}\mathbb{E}[\tilde{\Lambda}_n\tilde{L}_\varepsilon|Z_0] &= \mathbb{E}\left[\mathbb{E}[\tilde{\Lambda}_n\tilde{L}_\varepsilon|\Delta_n]|Z_0\right]\\ &= \Sigma_n + \mathbb{E}\left[\mathbb{E}[\tilde{\Lambda}_n|\Delta_n]\mathbb{E}[\tilde{L}_\varepsilon|\Delta_n]\,\Big|\, Z_0\right] = \Sigma_n + \mathbb{E}[\tilde{\Lambda}_n|Z_0]\mathbb{E}[\tilde{L}_\varepsilon|Z_0],\end{aligned}$$

where we used that $\mathbb{E}[\tilde{L}_\varepsilon|\Delta_n] = \mathbb{E}[\tilde{L}_\varepsilon|Z_0]$ for the last equality, and

$$\begin{aligned}\Sigma_n = \mathbb{E}\Bigg[&\sum_{k=1}^n \mathbb{E}\left[\zeta_{k,n}^* \sum_{z_i\in I_{k,n}} (\zeta_i-\varepsilon)_+ \,\Big|\, \Delta_n\right]\\ &- \sum_{k=1}^n \mathbb{E}[\zeta_{k,n}^*|\Delta_n]\mathbb{E}\left[\sum_{z_i\in I_{k,n}} (\zeta_i-\varepsilon)_+ \,\Big|\, \Delta_n\right] \,\Bigg|\, Z_0\Bigg].\end{aligned}$$

So using (5.9) and (5.22), we get

$$\begin{aligned}J_n(\varepsilon) = {}& 4Z_0\int_0^\infty hc_\theta(h)\,dh - 2\Sigma_n\\ &+ \left(\mathbb{E}[\tilde{\Lambda}_n|Z_0] - \mathbb{E}[\tilde{L}_\varepsilon|Z_0]\right)^2 + O(\varepsilon\log(\varepsilon)) + O(n^{-1}\log^2(n)).\end{aligned}$$

Then, taking $\varepsilon \asymp n^{-1}$, and using (5.8), (5.21) and Lemma 5.4 below, we get

$$J_n(\varepsilon) = \frac{Z_0^2}{\beta^2}\log^2\left(n\varepsilon\frac{\beta}{Z_0}\right) + O(n^{-1}\log^2(n)).$$

We deduce that $\tilde{\Lambda}_n - \tilde{L}_{Z_0/(n\beta)}$ converges in probability to 0 and, by the Borel–Cantelli lemma, almost surely along the subsequence n^3. Recall that the sequence $(\tilde{L}_\varepsilon - \mathbb{E}[\tilde{L}_\varepsilon|Z_0], \varepsilon > 0)$ converges a.s., as ε goes down to 0, towards a limit, say $\tilde{\mathcal{L}}$. Notice that

$$\mathbb{E}[\tilde{L}_{Z_0/n\beta}|Z_0] = \mathbb{E}[\tilde{\Lambda}_n|Z_0] + O(n^{-1}\log(n)),$$

and thus we deduce that $(\tilde{\Lambda}_{n^3} - \mathbb{E}[\tilde{\Lambda}_{n^3}|Z_0], n\in\mathbb{N}^*)$ also converges a.s. towards $\tilde{\mathcal{L}}$. Then we use (5.20) to get that for $k \in [n^3, (n+1)^3)$,

$$\begin{aligned}&\tilde{\Lambda}_{n^3} - \mathbb{E}[\tilde{\Lambda}_{n^3}|Z_0] + O(n^{-1}\log(n))\\ &\quad\le \tilde{\Lambda}_k - \mathbb{E}[\tilde{\Lambda}_k|Z_0] \le \tilde{\Lambda}_{(n+1)^3} - \mathbb{E}[\tilde{\Lambda}_{(n+1)^3}|Z_0] + O(n^{-1}\log(n)).\end{aligned}$$

We conclude that $(\tilde{\Lambda}_n - \mathbb{E}[\tilde{\Lambda}_n|Z_0], n\in\mathbb{N}^*)$ also converges a.s. towards $\mathcal{L}$.

Lemma 5.4. *Let $\varepsilon \asymp n^{-1}$. We have*

$$\Sigma_n = 2Z_0 \int_0^\infty h c_\theta(h)\, dh + O(n^{-1} \log^2(n)).$$

Proof. We have

$$\mathbb{E}\left[\sum_{z_i \in I_{k,n}} (\zeta_i - \varepsilon)_+ \,\middle|\, \Delta_n \right] = \Delta_{k,n} \mathbb{N}[(\zeta - \varepsilon)_+].$$

Thanks to (5.12), (5.18), and (5.19), we get

$$\begin{aligned}\mathbb{E}\left[\sum_{k=1}^n \Delta_{k,n} \mathbb{E}[\zeta^*_{k,n} | \Delta_n] \,\middle|\, Z_0 \right] &= \frac{nZ_0^2}{\beta} \mathbb{E}\left[\tilde{U}_n^2 \left(\log(2\theta Z_0 \tilde{U}_n) + (1-\gamma) + g_1(2\theta Z_0 \tilde{U}_n) \right) |Z_0 \right] \\ &= O(n^{-2} \log(n)).\end{aligned}$$

We deduce from (5.6) with $\varepsilon \asymp n^{-1}$ that

$$\mathbb{E}\left[\sum_{k=1}^n \mathbb{E}[\zeta^*_{k,n} | \Delta_n] \mathbb{E}\left[\sum_{z_i \in I_{k,n}} (\zeta_i - \varepsilon)_+ \,\middle|\, \Delta_n \right] \,\middle|\, Z_0 \right] = O(n^{-1} \log^2(n)).$$

By (5.14), we have

$$\sum_{k=1}^n \mathbb{E}\left[\zeta^*_{k,n} \sum_{z_i \in I_{k,n}} (\zeta_i - \varepsilon)_+ \,\middle|\, \Delta_n \right] = 2Z_0 \int_0^\infty h c_\theta(h)\, dh + W_n''',$$

with

$$W_n''' = -2(\Delta_{0,n} + \Delta_{n+1,n}) \int_0^\infty h c_\theta(h)\, dh + \sum_{k=1}^n g_3(\Delta_{k,n}).$$

Since $\varepsilon \asymp n^{-1}$, we deduce that

$$\mathbb{E}[|W_n'''||Z_0] \le \frac{2Z_0}{n+1} \int_0^\infty h c_\theta(h)\, dh + O(n^{-1} \log^2(n)).$$

This gives the result. □

Acknowledgements

We thank the referees for their valuable comments, which improved the presentation of the paper and broadened the bibliography.

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