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An infinite-dimensional metapopulation SIS model

Jean-François Delmas [a], Dylan Dronnier [a,*], Pierre-André Zitt [b]

[a] CERMICS, École des Ponts, 6-8 avenue Blaise Pascal, 77420 Champs-sur-Marne, France
[b] LAMA, Université Gustave Eiffel, 5 boulevard Descartes, 77420 Champs-sur-Marne, France

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Abstract

In this article, we introduce an infinite-dimensional deterministic metapopulation SIS model which takes into account the heterogeneity of the infections and the social network among a large population. We study the long-time behavior of the dynamic. We identify the basic reproduction number R_0 which determines whether there exists a stable endemic steady state (super-critical case: $R_0 > 1$) or if the only equilibrium is disease-free (critical and sub-critical case: $R_0 \leq 1$). As an application of this general study, we prove that the so-called "leaky" and "all-or-nothing" vaccination mechanism have the same effect on R_0. This framework is also very natural and intuitive to model lockdown policies and study their impact.

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Keywords: SIS model; Infinite-dimensional ODE; Graphon; Banach lattice; Vaccination; Lockdown

1. Introduction

1.1. Motivation

Our goal in this paper is to provide a generalization of the classical SIS model to non-homogeneous population and study its properties, under very weak assumptions. For pedagogical

* Corresponding author.
E-mail addresses: jean-francois.delmas@cermics.enpc.fr (J.-F. Delmas), dylan.dronnier@enpc.fr (D. Dronnier), pierre-andre.zitt@univ-eiffel.fr (P.-A. Zitt).

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purposes, we start by discussing in detail how one can arrive at this generalization, starting from the one-dimensional model, and building our way through the finite dimensional model of Lajmanovich and Yorke, before introducing the general framework. The huge literature concerning these models, how our framework is situated in this picture, and comparison with related works, are discussed below in Section 1.6.

1.1.1. The SIS model

Some infections do not confer any long-lasting immunity. With such infections, individuals become susceptible again once they have recovered from the disease. The simplest deterministic way to model this kind of epidemics in a constant size population is the following system of ordinary differential equations, introduced by Kermack and McKendrick in [1] and known as the SIS (susceptible/infected/susceptible) model:

$$\begin{cases} \dot{S} = -\frac{K}{N} I S + \gamma I, \\ \dot{I} = \frac{K}{N} I S - \gamma I, \end{cases}$$

where $S = S(t)$ and $I = I(t)$ are the number of susceptible and infected individuals, the total size $N = S(t) + I(t)$ of the population is constant in time, and K and γ are two positive numbers which represent the infectiousness and the recovery rate of the disease. The proportion $U(t) = I(t)/N$ of infected individuals in the population evolves autonomously, according to:

$$\dot{U} = (1 - U) K U - \gamma U. \tag{1}$$

Looking at a time change of U given by $V(t) = U(t/\gamma)$ and setting $R_0 = K/\gamma$, one gets that $\dot{V} = (1 - V) R_0 V - V$. The parameter R_0 can be interpreted as the number of infected individuals one infected individual generates on average over the course of its infectious period, in an otherwise uninfected population. This basic reproduction number was first introduced by Macdonald [2], and appears in a large class of models in epidemiology; see the monograph [3] from Brauer and Castillo-Chavez. The ordinary differential equation in V is well-posed and admits an explicit solution. If $V(0) = 0$, then $V(t) = 0$ for all t: as V represents the proportion of infected individuals, this constant solution is called the *disease-free equilibrium*. Now assume $V(0) = V_0 \in (0, 1]$. If $R_0 \neq 1$, the proportion of infected individuals in the population for $t \geq 0$ is given by:

$$U(\gamma t) = V(t) = \frac{R_0 - 1}{R_0 + ((1 - R_0)/V_0 - R_0)\mathrm{e}^{(1-R_0)t}}.$$

If $R_0 = 1$, then the proportion of infected individuals in the population is given by:

$$U(\gamma t) = V(t) = \frac{1}{(1/V_0) + t}.$$

Hence, one can identify three possible longtime behaviors for the dynamical system:

Sub-critical regime If $R_0 < 1$, $U(t)$ converges exponentially fast to 0, and the only equilibrium is the disease-free solution $U(t) = 0$.

Critical regime If $R_0 = 1$, $U(t)$ still converges to 0 but not exponentially. The disease-free equilibrium is still the only one.

Super-critical regime If $R_0 > 1$, the constant solution 0 becomes unstable and another equilibrium appears, $G^* = 1 - R_0^{-1}$. This equilibrium is called *endemic*, and is globally stable in the sense that $U(t)$ converges towards G^* for all initial positive conditions.

1.1.2. The multidimensional Lajmanovich Yorke extension

In a pioneering paper [4], Lajmanovich and Yorke introduced an extension of the SIS model for the propagation of gonorrhea, which takes into account the fact that the propagation of the virus is highly non homogeneous among the population — we refer to the survey [5, Section V.A.2] from Pastor-Satorras, Castellano, van Mieghem and Vespignani, and more precisely Section 2 therein, for broader context and more details.

In this model the population is divided into n groups and the transmission rates of the disease between these groups are not equal, leading to a system of coupled ODEs:

$$\dot{U}_i = (1 - U_i) \sum_{j=1}^{n} K_{i,j}\, U_j - \gamma_i U_i, \qquad \forall i \in \{1, 2, \ldots, n\} \tag{2}$$

where U_i is the proportion of infected individuals in group i with $U_i(0) \in [0, 1]$ for all $1 \le i \le n$, $K = (K_{i,j})_{1 \le i,j \le n}$ is a non-negative matrix that represents the transmission rates of the infection between the different groups, and the non-negative number $\gamma_i > 0$ is the recovery rate of group i. Since the matrix $K/\gamma = (K_{i,j}/\gamma_j)_{1 \le i,j \le n}$ has non-negative entries, we recall it has a Perron eigenvalue, that is, an eigenvalue $R_0 \in \mathbb{R}_+$ such that all other complex eigenvalues λ of K/γ satisfy $|\lambda| \le R_0$. The following result is given in [4].

1. There exists a unique solution $(U_i(t) :\ t \ge 0)_{1 \le i \le n}$ of Equation (2) and $U_i(t) \in [0, 1]$ for all $t \in \mathbb{R}_+$.
2. If $R_0 \le 1$, $U_i(t)$ converges to 0 for all $1 \le i \le n$, so that the disease-free equilibrium $(0, 0, \ldots, 0)$ is globally stable.
3. If K is irreducible and $R_0 > 1$, then there exists an endemic equilibrium $G^* = (G_i^*)_{1 \le i \le n}$ such that for $i = 1 \ldots n$:

$$\lim_{t \to \infty} U_i(t) = G_i^* \in (0, 1),$$

provided that $U(0) \neq (0, 0, \ldots, 0)$.

Thus, under the assumption that people are connected enough, the epidemic has two possible behaviors exactly like in the one-dimensional model:

> [4, Biotheorem 1] Either the epidemic will die out naturally for every possible initial stage of the epidemic, or when it is not true and the initial number of infectives of at least one group is nonzero, the disease will remain endemic for all the future time. Moreover, the number of infectives and susceptibles of each group will approach nonzero constant levels which are independent of the initial levels.

In 1957, before Lajmanovich and Yorke's seminal work, a similar type of behavior was formally derived by Kendall [6] for a heterogeneous SIR epidemic model but with strong assumptions on the transmission rates; see Equation (4) therein.

1.1.3. Towards a generalization

The epidemiological models discussed so far assume a large population, possibly made of a few groups with different behaviors, so that the epidemics are deterministic. At the opposite side of the modeling spectrum, some probabilistic models of interacting particles may be seen as modeling epidemics.

In 1974, Harris [7] introduced the so-called contact process on $\mathbb{Z}^d$. The contact process is a continuous-time Markov process often used as a model for the spread of an infection. Nodes of the graph represent the individuals of a population. They can either be infected or healthy. Infected individuals become healthy after an exponential time, independently of the configuration. Healthy individuals become infected at a rate which is proportional to the number of infected neighbors. The contact process shares numerous properties with the multigroup SIS equations: the existence of an upper invariant measure, a disease-free invariant measure and a monotone coupling [8,9]. This proximity is not surprising since Equation (2) can be obtained from a mean-field approximation of the contact process [5, Section V.A]. Notice that Equation (2) can also be obtained as a limit of individual based models; see [10].

We refer to [5], and the numerous references therein, for a survey on epidemic processes in complex networks. Since social networks are very large graphs, it is natural to consider epidemic processes on limits of large graphs using theories developed during the last two decades. The first type of limiting objects is called *graphings*, and is used to deal with very sparse graphs, namely those with bounded degree; see [11–13]. At the other end of the spectrum, *graphons* are flexible objects that define a limit for dense graphs where the mean degree is of the same order as the number of vertices; see, for example, [13,14]. We refer to [15–17] for further attempts at defining a limit theory for all kind of graphs.

The SIS equation that we propose in the present paper has to be thought as the limit of the mean-field approximations of the contact processes defined on a convergent sequence of large graphs. Thus, the solutions take values in an abstract space Ω (the set of vertices), which can be interpreted as the set of features of the individuals. The transmission of the disease is given by a kernel κ and the recovery rate by a function γ; see the infinite-dimensional evolution Equation (3) below.

The two main goals of this article are the following:

- introduce an infinite-dimensional SIS model, generalizing the model developed by Lajmanovitch and Yorke (see Equation (3) below), and prove a result similar to [4, Biotheorem 1] in that general setting;
- argue that this general setting is flexible enough to take into account not only the topology of the social network, or the disparities between different subgroups of the population, but also the effect of vaccination policies (see Section 5), or the effect of lockdown (see Section 6), in the spirit of the policies used to slow down the propagation of Covid-19 in 2020.

1.2. The model

It is natural to extend the Lajmanovich and Yorke model (2) to a population with an infinite number of groups. We choose to present this extension in an abstract setting, as this allows us to include general vaccination and lockdown policies. We denote by Ω the set of the features of the individuals in a given population. Since Ω might not be countable, we shall consider a σ-field $\mathscr{F}$ on Ω so that $(\Omega, \mathscr{F})$ is a measurable space. We represent the transmission rate from an infinitesimal part of the population $\mathrm{d}y$ to x by a non-negative kernel $\kappa(x, \mathrm{d}y)$: κ is a function from $\Omega \times \mathscr{F}$ to $\mathbb{R}_+$ such that, for all $A \in \mathscr{F}$, the mapping $x \mapsto \kappa(x, A)$ is measurable and, for all $x \in \Omega$, the mapping $A \mapsto \kappa(x, A)$ is a non-negative measure defined on $(\Omega, \mathscr{F})$. We model the recovery rate of individuals with feature x by $\gamma(x)$, where γ is a non-negative measurable function defined on $(\Omega, \mathscr{F})$. The number $1/\gamma(x)$ can be thought as the typical time of recovery for individuals with feature x. For $x \in \Omega$ and $t \geq 0$, we denote by $u(t, x)$ the probability for an individual (or the proportion of individuals) with feature x to be infected at time t. So the intensity of infection attempts on x coming from infected individuals in $\mathrm{d}y$ is given by $u(t, y)\kappa(x, \mathrm{d}y)$. Recall that in a SIS model, the probability for an infection attempt to succeed is proportional to the number of susceptible individuals, *i.e.*, those who are not already infected; this explains the term $(1 - u(t, x))$ in front of the integral in the next equation. The evolution equation of the function u for the SIS model of the probability for being infected is given by the following differential equation (in infinite dimension):

$$\begin{cases} \partial_t u(t, x) = (1 - u(t, x)) \displaystyle\int_\Omega u(t, y)\,\kappa(x, \mathrm{d}y) - \gamma(x)u(t, x), \quad x \in \Omega, t \in [0, \tau), \\ u(0, x) = u_0(x), \quad x \in \Omega, \end{cases} \tag{3}$$

where the measurable function $u_0 : \ \Omega \to [0, 1]$ is the so-called initial condition and the solution u is defined up to time $\tau \in (0, \infty]$. We shall prove that Equation (3) is well defined up to $\tau = +\infty$, and we will mainly focus our study on the long-time behavior of the solutions to this equation and on the study of existence of equilibria. Once again, we refer to Section 1.6 for a discussion on related work, and in particular the work by Thieme [18] on a spatial SIR model and by Busenberg, Iannelli and Thieme [19] on long-time behavior of an age-structured SIS infection.

Example 1.1 *(Lajmanovich and Yorke model).* Consider a finite set of features, $\Omega = \{1, 2, \dots, n\}$ (with the σ-field $\mathscr{F} = \mathscr{P}(\Omega)$ of all sub-sets of Ω), a finite kernel κ and a positive recovery rate γ. We set for all $i, j \in \Omega$ and $t \geq 0$:

$$K_{i,j} = \kappa(i, \{j\}), \quad \gamma_i = \gamma(i) \quad \text{and} \quad U_i(t) = u(t, i),$$

where u is the solution to Equation (3). The functions U_i, for $1 \leq i \leq n$, clearly solve the finite-dimensional model (2).

There are two natural extensions of Example 1.1 to large bounded degree graphs and large dense graphs, which is a first approach to model large complex social networks.

Example 1.2 *(Graph model).* Consider a representation of the social interaction of a population by a simple graph G, with set of vertices $V(G) = \Omega$ which is at most countable, and set of edges $E(G) \subset \Omega \times \Omega$. For $x \in \Omega$, let $\mathscr{N}(x) = \{y \in G : \ (x, y) \in E(G)\}$ stand for the neighborhood of x in G and $\deg_G(x) = \mathrm{Card}\,(\mathscr{N}(x))$ for its degree. If the degree of the vertices of G is finite, we may consider a kernel with the following form:

$$\kappa(x, \mathrm{d}y) = \beta(x) \sum_{z \in \mathscr{N}(x)} \theta(y)\delta_z(\mathrm{d}y), \tag{4}$$

where β and θ are non-negative functions, which represent the susceptibility and the infectiousness of the individuals respectively, and δ_z is the Dirac mass at z. Then Equation (3) represents the evolution equation for a SIS model on a graph. The strength of the formalism of (3) is that one can consider limit of large bounded degree undirected graphs called graphings; see Section 18 in [13] for the definition of a graphings.

Example 1.3 *(Graphon form).* One of the initial motivations of this work, was to consider a SIS model on *graphons*, which are limit of large dense graphs; see the monograph [13] from Lovàsz. In a recent paper [20], Vizuete, Frasca and Garin studied the stability of deterministic SIS epidemics over a large network generated by a Lipschitz graphon.

Recall the set of features of the individuals in the population is given by a set Ω. In this approach, the typical form of the transmission kernel κ we may consider is:

$$\kappa(x, \mathrm{d}y) = \beta(x)W(x, y)\theta(y)\,\mu(\mathrm{d}y), \tag{5}$$

where β represents the susceptibility and θ the infectiousness of the individuals; W models the graph of the contacts within the population and the quantity $W(x, y) \in [0, 1]$ is interpreted as the probability that x and y are connected, or as the density of contacts between individuals with features x and y; μ is a probability measure on $(\Omega, \mathscr{F})$ and $\mu(\mathrm{d}y)$ represents the infinitesimal proportion of the population with feature y. Formally, β and θ are non-negative measurable functions, and the function $W : \Omega \times \Omega \to [0, 1]$ is symmetric measurable. The quadruple $(\Omega, \mathscr{F}, \mu, W)$ is called a graphon. The degree $\deg_W(x)$ of $x \in \Omega$ (*i.e.*, the average number of his contacts) and the mean degree d_W for a graphon W are defined by:

$$\deg_W(x) = \int_\Omega W(x, y)\,\mu(\mathrm{d}y) \text{ and } \mathrm{d}_W = \int_\Omega \deg_W(x)\,\mu(\mathrm{d}x) = \int_{\Omega^2} W(x, y)\,\mu(\mathrm{d}y)\,\mu(\mathrm{d}x). \tag{6}$$

Let us give a bit more detail in three particular cases.

(i) **Constant graphon.** One elementary example, is the constant graphon, $W = p \in [0, 1]$. In this case, the degree function is constant, equal to the mean degree and thus equal to the parameter p. We recall this constant graphon appears as the limit, as n goes to infinity, of Erdös-Rényi random graphs with n vertices and parameter p (that is: independently, for each pair of vertices, there is an edge between those two vertices with probability p). If furthermore the functions β, θ and γ from (3) are constant on Ω, then we recover the SIS model (1) with $K = p\beta\theta$ and $U(t) = \int_\Omega u(t, x)\mu(\mathrm{d}x)$.

(ii) **Stochastic block model.** The stochastic block model of communities, introduced by [21] (and referred to as step graphons in [13]; see also [22] for further references), corresponds to the case where W is constant by block, *i.e.* there exists a finite partition $(\Omega_i : 1 \le i \le n)$ of Ω such that W is constant on the blocks $\Omega_i \times \Omega_j$ for all i, j, and equal say to $W_{i,j}$. If furthermore, the functions β, θ and γ from (3) are also constant on the partition, then we recover the Lajmanovich and Yorke model (2) with: $K_{i,j} = \beta_i\, W_{i,j}\, \theta_j\, \mu(\Omega_j)$; β_i, θ_i and γ_i are the constant values of β, θ and γ on Ω_i; and $U_i(t) = \int_{\Omega_i} u(t, x)\mu(\mathrm{d}x)/\mu(\Omega_i)$.

(iii) **Geometric graphon.** In this case, which is a natural generalization of the Random Geometric Graph (see [23] for a survey and [24] and references therein for related models), the probability of contact between x and y depends on their relative distance. For example, consider the population uniformly spread on the unit circle: $\Omega = [0, 2\pi]$ and $\mu(\mathrm{d}x) = \mathrm{d}x/2\pi$. Let f be a measurable non-negative function defined on $\mathbb{R}$ which is bounded by 1 and

2π-periodic. Define the corresponding geometric graphon W_f by $W_f(x, y) = f(x - y)$ for $x, y \in \Omega$. In this case, the degree of $x \in [0, 1]$ is constant with:

$$\deg_{W_f}(x) = \mathrm{d}_{W_f} = \frac{1}{2\pi} \int_{[0,2\pi]} f(y)\, \mathrm{d}y.$$

1.3. Main assumptions and definition of the reproduction rate

In order for Equation (3) to make sense, we will need the following assumption. It will always be in force throughout this paper without supplementary specification.

Assumption 0. *The function γ is positive, bounded and the non-negative kernel κ is uniformly bounded:*

$$\sup_{x \in \Omega} \kappa(x, \Omega) < \infty. \tag{7}$$

Assuming the recovery rate γ to be bounded is equivalent to require the time of recovery $1/\gamma$ to be bounded from below by a positive constant. The function $1/\gamma$ is also finite for all individuals because γ is supposed to be positive. It is possible with Assumption 0 to have individuals with arbitrary large time of recovery, though. Finally, Equation (7) limits the maximal force of infection that can be put upon a susceptible individual.

In Examples 1.1, 1.2 and 1.3, we observe that the kernel has a density with respect to a reference measure (the counting measure in the first two examples and the probability measure μ in the third one). From an epidemiological point of view, the reference measure μ can be seen as a way to quantify the size of the population and its sub-groups (defined by a given feature such as sex, spatial coordinates, social condition, health background, ...). If the measure μ is finite, then for every measurable set A, the number $\mu(A)/\mu(\Omega)$ is the proportion of individuals in the population whose features belong to A. We shall consider the case where the density k of κ with respect to the reference measure μ satisfies some mild integrability condition. We emphasize that the space Ω is not equipped with a topology and, as a consequence, we do not assume any smoothness condition on the density k. By a slight abuse of language, we will also call the density k a kernel.

Assumption 1. *There exists a finite positive measure μ on $(\Omega, \mathscr{F})$, a non-negative measurable function $k : \Omega \times \Omega \to \mathbb{R}_+$ such that for all $x \in \Omega$, $\kappa(x, \mathrm{d}y) = k(x, y)\mu(\mathrm{d}y)$. Besides, there exists $q > 1$ such that:*

$$\sup_{x \in \Omega} \int_{\Omega} \frac{k(x, y)^q}{\gamma(y)^q}\, \mu(\mathrm{d}y) < \infty. \tag{8}$$

Note that the kernel $k(x, y)/\gamma(y)$ that appears in Assumption 1 is the analogue of the ratio K/γ in the multi-dimensional Lajmanovich–Yorke model.

Since we assume that γ is bounded, Equation (8) implies the following integrability condition for the kernel k:

$$\sup_{x \in \Omega} \int_{\Omega} k(x, y)^q \, \mu(\mathrm{d}y) < \infty. \tag{9}$$

We shall study in Section 4.5 an example which does not satisfy the integrability condition (8) nor (9).

Finally, some results in the supercritical regime will only hold under the following connectivity assumption.

Assumption 2 *(Connectivity). The kernel k is* connected*, that is,*

$$\int_{A \times A^c} k(x, y) \, \mu(\mathrm{d}x)\mu(\mathrm{d}y) > 0 \tag{10}$$

for any measurable set A such that $\mu(A) > 0$ and $\mu(A^c) > 0$.

The sociological interpretation of the connectivity assumption is that we cannot separate the population into two groups of individuals with no interaction. Contrary to Assumption 0 which is assumed throughout the text, we will specify each time whether Assumptions 1 or 2 are needed.

Remark 1.4 *(The finite dimensional case).* Assumption 1 is automatically satisfied in the finite-dimensional model of Example 1.1, where we supposed Assumption 0. We can indeed take μ to be the counting measure and Equation (8) is true because k is bounded from above and γ is bounded from below by a positive constant as it is positive. Notice Assumption 2 is equivalent to the matrix of transmission rates $K = (K_{i,j})_{1 \leq i,j \leq n}$ being irreducible.

The basic reproduction number of an infection, denoted by R_0, has originally been defined as the number of cases one typical individual generates on average over the course of its infectious period, in an otherwise uninfected population. This number plays a fundamental role in epidemiology as it provides a scale to measure how difficult to control an infectious disease is. More importantly, in many models, the particular value $R_0 = 1$ turns out to be a threshold: the disease will die out if $R_0 < 1$, and invade the population if $R_0 > 1$.

In mathematical epidemiology, Diekmann, Heesterbeek and Metz [25] define rigorously the basic reproduction number for a class of models with heterogeneity in the population. They propose to consider the next-generation operator which gives the distribution of secondary cases arising from an infected individual picked randomly according to a certain distribution — the population being assumed uninfected otherwise. In our model, under Assumption 1, following [25, Equation (4.2)], we define the next generation operator, denoted by $T_{k/\gamma}$, as the integral operator:

$$T_{k/\gamma}(g)(x) = \int_{\Omega} \frac{k(x, y)}{\gamma(y)} g(y) \, \mu(\mathrm{d}y) \quad \text{for all } x \in \Omega, \tag{11}$$

which is, thanks to (8), a bounded positive operator on the space $\mathscr{L}^\infty(\Omega)$ of bounded real-valued measurable functions defined on Ω. Following [25, Definition of R_0 in Section 2], the basic reproduction number is then defined by:

$$R_0 = r(T_{k/\gamma}), \tag{12}$$

where r is the spectral radius, whose exact definition in our general setting will be recalled below (Equation (34)). These definitions of the next-generation operator and the basic reproduction number are consistent with the finite dimensional SIS model given in [26].

1.4. Long time behavior of solutions to the evolution equation (3)

We now state our main result concerning solutions of the evolution equation (3). Recall the initial condition of (3), u_0, takes values in $[0, 1]$.

Theorem 1.5. *We have the following properties.*

(i) ***(Equation** (3) **is well defined and** $\tau = +\infty$.)* *Under Assumption 0, there exists a unique solution u to Equation (3). This solution is such that, for all $(x, t) \in \Omega \times \mathbb{R}_+$, $u(t, x) \in [0, 1]$.*

(ii) ***(Disease free equilibrium in the critical and sub-critical case.)*** *Assume that Assumptions 0 and 1 are in force. Let R_0 be defined by (12). If $R_0 \leq 1$, then the disease dies out: for all $x \in \Omega$,*

$$\lim_{t\to\infty} u(t, x) = 0.$$

(iii) ***(Stable endemic equilibrium in the super-critical case.)*** *Assume that Assumptions 0, 1 and 2 are in force. If $R_0 > 1$, then there exists a unique equilibrium $g^* : \Omega \to [0, 1]$ different from 0 and it is positive μ-a.e. For all initial condition u_0 such that its integral is positive:*

$$\int_\Omega u_0(x)\, \mu(\mathrm{d}x) > 0,$$

the solution u to (3) converges pointwise to g^*, i.e., for all $x \in \Omega$:

$$\lim_{t\to\infty} u(t, x) = g^*(x).$$

If $u_0 = 0$ μ-a.e. then the solution u to (3) converges pointwise to 0.

For property (i), see Proposition 2.7; property (ii) is a consequence of Theorems 4.6 and 4.7; and property (iii) follows from Corollary 4.9 and Theorem 4.13.

Remark 1.6 *(Uniform convergence).* The convergence of $u(t, \cdot)$ towards 0 in (ii) or towards g^* in (iii) in Theorem 1.5 is uniform on any measurable subset $A \subset \Omega$ such that $\inf_A \gamma > 0$; see Theorem 4.17. In particular these convergences hold in uniform norm if the recovery rate γ is bounded from below.

1.5. Modeling vaccination policies, vaccination mechanisms and lockdown

1.5.1. Vaccination

In Section 5, we propose extensions of Equation (3) which take into account the effect of a vaccination policy. Vaccination confers a direct protection on the targeted individuals but also acts indirectly on the rest of the population through herd immunity. However, all vaccinated individuals will not be totally immune to the disease. In [27], Smith, Rodrigues and Fine propose two possible models to explain vaccine efficacy. In the first model, the vaccine offers complete protection to a portion of the vaccinated individuals but does not take in the remainder of vaccinated individuals. The second model supposes that the vaccination confers a partial protection to every vaccinated individual. In [28], Halloran, Lugini and Struchiner called the former mechanism the *all-or-nothing* vaccination and the latter one the *leaky* vaccination. We define below one infinite-dimensional SIS model for each of these two mechanisms.

In order to write down the vaccination model, we first adapt the one-group SIR models proposed by Shim and Galvani in [28] to the one-group SIS model. Let us denote by η_{v} the proportion of vaccinated individuals in the total population, and let $\eta_{\mathrm{u}} = 1 - \eta_{\mathrm{v}}$. Let U_{v} and U_{u} be the proportion of infected individuals in the vaccinated and unvaccinated population respectively, so that $\eta_{\mathrm{v}} U_{\mathrm{v}} + \eta_{\mathrm{u}} U_{\mathrm{u}}$ is the proportion of infected individuals in the total population. For both models, we assume that vaccinated individuals who are nevertheless infected by the disease become less contagious (see [29,30] for instance). We will denote the vaccine efficacy for infectiousness, that is, the relative reduction of infectiousness for vaccinated individuals by a parameter $\delta \in [0, 1]$. In what follows, K and γ represent the transmission rate and the recovery rate of the disease as in the model (1) or (2) and are assumed to be the same for the vaccinated and unvaccinated population. We now introduce two models for the so-called vaccine efficacy e, see [31,32,28,27] for discussion on this parameter.

In the leaky vaccination, we define the efficacy $e \in [0, 1]$ as the relative reduction of susceptibility for vaccinated individual. Following [28, Equations (1)-(8)], with the parameters δ and e corresponding to σ and α in [28], the evolution equations for the leaky vaccination are then given by:

$$\begin{cases} \dot{U_{\mathrm{v}}} = (1 - U_{\mathrm{v}})(1 - e)K((1 - \delta)\eta_{\mathrm{v}} U_{\mathrm{v}} + \eta_{\mathrm{u}} U_{\mathrm{u}}) - \gamma U_{\mathrm{v}}, \\ \dot{U_{\mathrm{u}}} = (1 - U_{\mathrm{u}})K((1 - \delta)\eta_{\mathrm{v}} U_{\mathrm{v}} + \eta_{\mathrm{u}} U_{\mathrm{u}}) - \gamma U_{\mathrm{u}}. \end{cases} \tag{13}$$

In the all-or-nothing vaccination, we denote the proportion of vaccinated individuals immunized to the disease (people who can neither contract not transmit the disease) by the parameter $1 - e \in [0, 1]$. Following [28, Equations (13)-(20)], the evolution equations for the all-or-nothing vaccination in the SIS setting are given by:

$$\begin{cases} \dot{U_{\mathrm{v}}} = (1 - e - U_{\mathrm{v}})K((1 - \delta)\eta_{\mathrm{v}} U_{\mathrm{v}} + \eta_{\mathrm{u}} U_{\mathrm{u}}) - \gamma U_{\mathrm{v}}, \\ \dot{U_{\mathrm{u}}} = (1 - U_{\mathrm{u}})K((1 - \delta)\eta_{\mathrm{v}} U_{\mathrm{v}} + \eta_{\mathrm{u}} U_{\mathrm{u}}) - \gamma U_{\mathrm{u}}. \end{cases} \tag{14}$$

Since vaccinated individuals that are immunized cannot get the disease, we have $U_{\mathrm{v}}(t) \leq 1 - e$ for all $t \in \mathbb{R}_+$.

Remark 1.7. Notice that, in both models, the unvaccinated population can be viewed as a population inoculated with a vaccine of efficacy equal to 0.

In Section 5, we derive in Equations (66) and (69) the analogue of (13) and (14) in the infinite-dimensional setting. Those two equations can be seen as a particular case of Equation (3). We also prove that, as far as the basic reproduction number is concerned the two different vaccination mechanisms, the all-or-nothing and leaky mechanisms, have the same effect in the infinite dimensional model; see Proposition 5.2. This result was already observed in a one-group model by Shim and Galvani [28]. In the case of a perfect vaccine, where vaccinated people cannot be infected nor infect others, the evolution equation of the proportion of infected among the non vaccinated population is also given by Equation (3) with the kernel $\kappa(x, \mathrm{d}y)$ replaced by $\eta^0(y)\kappa(x, \mathrm{d}y)$ where $\eta^0(y)$ is the proportion of individuals with feature $x \in \Omega$ which are not vaccinated; see Equation (73). We shall study in a future work the optimal vaccination in this setting with the basic reproduction number as an objective function to minimize.

1.5.2. Effect of lockdown policies

Eventually, we model the effect of lockdown in Section 6 for graphon models presented in Example 1.3, in the spirit of the policies used to slow down the propagation of Covid-19 in 2020; see for example the study [33]. In particular, we prove that a lockdown which bounds the number of contacts of the individuals (this roughly corresponds to reduce significantly the number of contacts for highly connected groups) is enough to reduce the basic reproduction number; see Proposition 6.3. Recall the definition of the degree $\deg_W(x)$ of x and the mean degree d_W for a graphon W defined in (6). Following Remark 6.4, we get that the heterogeneity in the degree for the graphon model implies larger value of the basic reproduction number. In this direction, see also [34, Section 1.1] on the SIS model from Pastor-Satorras and Vespignani, where the basic reproduction number increases with the variance of the degrees of the nodes in a finite graph.

Corollary 1.8. *Consider the SIS model* (3) *with transmission kernel given in a graphon form* (5) *(so that $k(x, y) = \beta(x)W(x, y)\theta(y)$). Assume that the susceptibility β, the infectiousness θ and the recovery rate γ are constant and positive. The weakest value of the basic reproduction number R_0 defined by* (12) *among all graphons W with mean degree $\mathrm{d}_W \geq p$ for some threshold $p \in [0, 1]$ is obtained for graphons with constant degree equal to p (i.e. graphon W such that $\deg_W(x) = p$ for all $x \in \Omega$).*

We recall from Example 1.3 (i) and (iii), that the constant graphon and the geometric graphons have constant degree. Considering a geometric graphon with (mean) degree p, we get that $R_0 = \gamma^{-1}\beta\theta p$, and for $R_0 > 1$, we deduce (directly or from Proposition 2.17), that the equilibrium g^* is constant equal to $1 - R_0^{-1}$ (compare with model (1) with $K = \beta\theta p$). Furthermore, the example of the geometric graphon with a given mean degree, indicates that, if the parameters β, θ and γ are constant, then the contamination distance (or support of the function f; see the end of Remark 6.4) from an infected individual is not relevant for the value of the basic reproduction number nor for the equilibria.

1.6. Discussion and related results

The dichotomy of possible dynamics described in [4, Biotheorem 1] has been established for many other compartmental models and possibly their multigroup version by using Lyapunov function techniques; see, for instance, [35,36]. For a survey, we refer to Fall, Iggidr, Sallet and Tewa [37]. In [38, Section 6] and [39], Hirsch and Smith proved the long-time behavior of Equation (2) thanks to their theory of order preserving systems, thereby giving a completely new perspective to the study of mathematical epidemic models. Their work greatly inspired Li and Muldowney [40] in their important proof of the global stability of the endemic equilibrium of the SEIR model (susceptible-exposed-infected-recovered) which was a long-standing conjecture at that time.

Continuous models involving transmission rates that depend on the localization of the individuals [6,41,42] or their age [43] have long been studied. Both of these can be thought of as multigroup models with a continuous set of groups and therefore lead to differential equations in infinite-dimensional space. In this setting, results about global stability of the endemic or the disease-free equilibrium have also been obtained. We outline some of them below and highlight how they differ from our framework.

In [19], Busenberg, Iannelli and Thieme established the long-time behavior of an age-structured SIS infection. They proved, thanks to semi-group theory and positive operators methods, that the system converges to a unique endemic equilibrium if it exists. Otherwise, it converges to the disease-free equilibrium. In this work the transmission kernel is assumed to be bounded from above and below by product kernels (see Equation (2.9) therein). This represents a restriction (see the discussion at the end of [19]) as it is not possible to forbid contacts between some but not all groups. By contrast, in the setting of Example 1.3, it is easy and natural to model the absence of contact between individuals with feature x and y by imposing that $W(x, y) = 0$, without imposing conditions on the probability of contact between x and other features than y.

In [44] Feng, Huang and Castillo-Chavez considered a similar dynamic for a multigroup age-structured SIS model, but where the endemic equilibrium exists but is not globally stable. They assume that the system has a quasi-irreducibility property (see Definition 3.1 therein) which is a weaker assumption than Assumption 2, but impose bounds on the transmission kernel.

In [45], Thieme also used an operator approach to study a SIR model with variable susceptibility (see Section 4 therein). In particular, he studied the close relation between the spectral bound of the operator $T_k - \gamma$ and of the basic reproduction number R_0 which is the spectral radius of the operator $T_{k/\gamma}$. In [18], Thieme analyzed a space-structured SIR model with birth. In this model, the incidence term, *i.e.* the equivalent of $(1 - u(t, x))u(t, y)\kappa(x, \mathrm{d}y)$ in Equation (3), is replaced by a non bilinear term $f(x, y, 1 - u(t, x), u(t, y))\,\mathrm{d}y$, where the function f is continuous, locally Lipschitz continuous and increasing in its third and fourth argument. Imposing also that the recovery rate γ is bounded away from 0, he proved an analogue of [4, Biotheorem 1] using Lyapunov functions; see Theorems 7.1, 8.2, 9.1 and 12.1 therein. Part of those results would not hold in general if $\inf \gamma = 0$. In contrast to these works, we consider very few regularity assumptions on the parameters, and in particular allow that $\inf \gamma = 0$.

Other works introduce movement of populations, either between discrete patches – see for example [46] and the biological examples therein – or in a continuous space. In [47], Ruan and Xiao obtained the global stability of the steady states for a general spatial SIS model with delay, a diffusion term for the infected population and a non-local kernel governing the transmission of the disease. It is assumed that the transmission kernel is smooth and satisfies a constant-degree assumption, so that the endemic equilibrium is unique and constant. Here, we do not consider the constant degree assumption as this condition intrinsically does not hold when considering vaccination strategies; see Section 5. In the recent [48], Almeida, Bliman, Nadin, Perthame and Vauchelet studied a spatial SEIR model with or without diffusion. The case without diffusion is formally very close to our model, and makes no smoothness assumptions on the infection kernel; however, the infection kernel is supposed to be bounded and positive everywhere, and the various rates bounded above below. Once again, the positiveness assumption of the infection kernel breaks down when taking into account vaccination strategies or lockdown policies. Let us mention also that removing the boundedness assumption on the infection kernel could lead to the existence of an infinite number of positive equilibria; see Section 4.5. For a deeper discussion on spatial epidemic models and a detailed review, we refer the reader to [49].

The principal tools we use to prove Theorem 1.5 can be summarized as follows.

Cooperative systems The function $g \mapsto F(g) = (1-g)T_\kappa(g) - \gamma g$ is *cooperative* (see Definition 2.1 and Remarks 2.2 and 2.3), which implies that the solution of (3) is well defined and the corresponding dynamical system is order preserving. For approaches based on cooperation (or quasi-monotonicity) and monotone dynamical systems on various models, see [10,38,39,50–52].

Positive operators Under Assumption 1 and Equation (8), the integral operator $T_{k/\gamma}$ can be seen as an Hille-Tamarkin operator on $L^p(\mu)$ with the corresponding compactness property see [53, Theorem 41.6]. Then the positivity of the operator $T_{k/\gamma}$ allows to use Krein-Rutman theorem to get that its spectral radius is an eigenvalue with a non-negative eigenfunction. This argument has been widely used; for example in [19] (where the operator is of rank one, and thus is compact) and also in [18,45].

Connectivity Under Assumption 2 on the connectivity of kernel k (which in finite dimension corresponds to the irreducibility of non-negative matrices and is related to the Perron-Frobenius theorem), we can consider the unique corresponding eigenvector, thanks to the Perron-Jentzsch theorem; see [54, Theorem V.6.6] or [55, Theorem 5.2]. This eigenvector is an essential tool to study the long-time behavior of the solution to Equation (3) in the super-critical regime. In finite dimension, we refer the reader to [4], where the matrix K from (2) is assumed to be irreducible, or [10] for a more general finite-dimensional model. In infinite dimension, see [44] for a weaker quasi-irreducibility condition.

Finally let us remark that we do not use the standard tool of Lyapunov functions, in contrast with many previous works; see for example [18,35,36].

1.7. Structure of the paper

In Section 2 we construct the semi-flow associated to the infinite dimensional SIS model (3), and prove its main regularity and monotonicity properties. We introduce in Section 3 some important tools of spectral analysis in Banach lattices. This allows us to define in Section 4.1 the basic reproduction number R_0. The convergence of the system towards an equilibrium is established in Section 4. In Section 5, we take into account the effect of vaccination policies on the propagation of the disease. Eventually, in Section 6, we model the impact of lockdown policies on the propagation of the disease when κ takes the graphon form of Example 1.3.

2. Model analysis

2.1. Preamble

In this paragraph, we recall some definitions of functional analysis. Most of them can be found in [56]. Let $(X, \|\cdot\|)$ be a Banach space. The topological dual $X^\star$ of X is the space of all bounded linear forms and we use the notation $\langle x^\star, x\rangle$ for the value of an element $x^\star \in X^\star$ at $x \in X$. We consider K a proper cone on X, *i.e.* a closed convex subset of X such that $\lambda K \subset K$ for all $\lambda \geq 0$

and $K \cap (-K) = \{0\}$. The proper cone K defines a partial ordering $\leq$ on X: $x \leq y$ if $y - x \in K$. It is said to be reproducing if $K - K = X$ (any element $x \in X$ can be expressed as a difference of elements of K). The dual cone of K is the set $K^\star \subset X^\star$ consisting of all $x^\star$ such that $\langle x^\star, x\rangle \geq 0$ for all $x \in K$. If the proper cone K is reproducing then the set $K^\star$ is a proper cone (see beginning of [56, Section 19.2]).

We denote by $\mathcal{L}(X)$ the space of bounded linear operators from X to X. The operator norm of a bounded operator $A \in \mathcal{L}(X)$ is given by:

$$\|A\| = \sup\{\, \|Ax\| : \ x \in X, \ \|x\| \leq 1\,\}.$$

The topology associated to $\|\cdot\|$ in $\mathcal{L}(X)$ is called the uniform operator topology. A linear bounded operator $A \in \mathcal{L}(X)$ is said to be positive (with respect to the proper cone K) if $AK \subset K$.

Let F be a function defined on an open domain $D \subset X$ and taking values in X. The function F is said to be Fréchet differentiable at $x \in D$, if there exists a bounded linear operator $\mathcal{D}F[x]$ such that:

$$\lim_{y \to 0} \|F(x+y) - F(x) - \mathcal{D}F[x](y)\| / \|y\| = 0.$$

The operator $\mathcal{D}F[x]$ is called the Fréchet derivative of F at point x.

We define the cooperativeness property which is related to the definition of quasimonotony firstly introduced by Volkmann [57] for abstract operators.

Definition 2.1 *(Cooperative function). Let $D_1, D_2 \subset X$. A function $F : X \to X$ is said to be cooperative on $D_1 \times D_2$ (with respect to K) if, for all $(x, y) \in D_1 \times D_2$ such that $x \leq y$ and for all $z^\star \in K^\star$, we have the following property:*

$$\langle z^\star, x - y\rangle = 0 \implies \langle z^\star, F(x) - F(y)\rangle \leq 0. \tag{15}$$

We shall mainly consider the cases $D_1 = X$ or $D_2 = X$.

Remark 2.2. For a better understanding of the cooperativeness property, let us examine the finite dimensional case. Let $d \geq 2$, $X = \mathbb{R}^d$ and $K = \mathbb{R}^d_+$. Then, for a smooth function $F = (F_1, F_2, \ldots, F_d)$, it is easy to see that F is cooperative on $X \times X$ with respect to K if and only if:

$$\frac{\partial F_j}{\partial x_i}(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^d \text{ and all } i \neq j. \tag{16}$$

We recover the definition of cooperativeness introduced by Hirsch [38]. Suppose the vector x represents the utilities of a group of agents $\{1, 2, \ldots d\}$ and F is the dynamics of the system, that is, $\dot{x} = F(x)$. Then, the higher the utilities of agents $j \neq i$ are, the more beneficial the situation is for agent i, as it increases the value of the time derivative of x_i. For this reason, the function F satisfying (16) is called cooperative.

We extend the differential version of cooperativeness of Remark 2.2 to infinite dimension in the next remark.

Remark 2.3. Let X be a Banach space, D an open domain and $F : X \to X$ be a Fréchet differentiable function. Assume that F is cooperative on $D \times X$. Let $(x, z) \in D \times K$ and let $z^\star \in K^\star$ such that $\langle z^\star, z\rangle = 0$. Since F is cooperative on $D \times X$, we have:

$$\langle z^\star, (F(x + \lambda z) - F(x))/\lambda\rangle \geq 0,$$

for all $\lambda > 0$. Letting λ go to 0, we obtain the following inequality:

$$\langle z^\star, \mathcal{D}F[x](z)\rangle \geq 0. \tag{17}$$

Using path integrals in Banach space, we can prove the reverse implication in the case $D = X$. Indeed, for all $x, y \in X$ and all $z^\star \in X^\star$, we have:

$$\langle z^\star, F(x) - F(y)\rangle = -\int_0^1 \langle z^\star, \mathcal{D}F[(1-\lambda)x + \lambda y](y - x)\rangle \,\mathrm{d}\lambda. \tag{18}$$

Assume (17) holds for $z^\star \in K^\star$ and $z \in K$. Then, if $x \leq y$, $z^\star \in K^\star$ and $\langle z^\star, y - x\rangle = 0$, we get that $\langle z^\star, F(x) - F(y)\rangle$ is non-positive thanks to Equation (17). Thus the function F is cooperative.

Ordinary differential equations (ODEs) driven by cooperative vector fields enjoy a number of nice properties that we now review. Let us first recall a few definitions and classical properties of ODEs. Let $F : X \to X$ be a locally Lipschitz function. The Picard–Lindelöf theorem ensures the existence of $0 < \tau \leq \infty$ and of a continuously differentiable function y from $J = [0, \tau)$ to X which is the unique solution of the Cauchy problem:

$$\begin{cases} y'(t) = F(y(t)) \quad t \in J, \\ y(0) = y_0, \end{cases} \tag{19}$$

where $y_0 \in X$ is the so-called initial condition (see [58, Section 1.1]). A solution y defined on an interval $[0, \tau)$ is said to be maximal if there is no solution of Equation (19) defined on $[0, \tau')$ with $\tau' > \tau$. A solution is said to be global if it is defined on $[0, \infty)$.

All comparison properties will be derived from the following key result.

Theorem 2.4 *(Comparison theorem). Let K be a proper cone of X with non-empty interior. Denote by $\leq$ the corresponding partial order. Let $F : X \to X$ be locally Lipschitz, $D_1, D_2 \subset X$, $\tau > 0$, and let $a : [0, \tau) \to D_1$ and $b : [0, \tau) \to D_2$ be $\mathcal{C}^1$ paths. Suppose that F is cooperative on $D_1 \times X$ or on $X \times D_2$, and that:*

$$a'(t) - F(a(t)) \leq b'(t) - F(b(t)) \quad \forall t \in [0, \tau). \tag{20}$$

If $a(0) \leq b(0)$, then $a(t) \leq b(t)$ for all $t \in [0, \tau)$.

This result, in the spirit of [58, Theorem 5.2], is a generalization to infinite dimensional systems of classical comparison theorems for ODEs. Note in particular that (20) holds if a and b

solve the ODE $u' = F(u)$, yielding the monotony of the flow of cooperative vector fields as a corollary; see Proposition 2.8 below. For the sake of completeness, a proof of Theorem 2.4 is given in Appendix A.

2.2. Notations

In this section, we will work in the Banach space $\mathscr{L}^\infty(\Omega)$ of measurable bounded real-valued functions defined on Ω equipped with the supremum norm $\|\cdot\|$. We shall write $\mathscr{L}^\infty$ when there is no ambiguity on the underlying space. The set:

$$\mathscr{L}^\infty_+ = \{ f \in \mathscr{L}^\infty : \ f(x) \geq 0 \quad \forall x \in \Omega \}, \tag{21}$$

is a proper cone in $\mathscr{L}^\infty$ with non-empty interior. The order defined by this proper cone is the usual order: $g \leq h$ if $g(x) \leq h(x)$ for all $x \in \Omega$.

We denote by $\mathscr{L}^{\infty,\star}$, the topological dual of $\mathscr{L}^\infty$. It can be identified as the space of bounded and finitely additive signed measures on Ω equipped with the total variation norm (see [59, Section 2]). Since $\mathscr{L}^\infty_+$ is reproducing, the dual cone $\mathscr{L}^{\infty,\star}_+$ is a proper cone. It consists of the continuous linear positive forms on $\mathscr{L}^\infty$.

Let κ be a non-negative kernel on $\mathscr{L}^\infty$ (endowed with its Borel σ-field) satisfying Assumption 0. We denote by T_κ the operator:

$$\begin{aligned} T_\kappa \quad : \quad & \mathscr{L}^\infty \to \mathscr{L}^\infty \\ & g \mapsto \left(x \mapsto \int_\Omega g(y)\, \kappa(x, \mathrm{d}y) \right). \end{aligned} \tag{22}$$

According to Assumption 0, the operator T_κ is a bounded linear operator with:

$$\|T_\kappa\| = \sup_{x\in\Omega} \kappa(x, \Omega) < \infty. \tag{23}$$

Since, for all $x \in \Omega$, $\kappa(x, \mathrm{d}y)$ is a positive measure, the operator T_κ is moreover positive. Defining now a function F from $\mathscr{L}^\infty$ to $\mathscr{L}^\infty$ by:

$$F(g) = (1 - g)T_\kappa(g) - \gamma g, \tag{24}$$

we may rewrite Equation (3) as an ODE in the Banach space $(\mathscr{L}^\infty, \|\cdot\|)$:

$$\begin{cases} \partial_t u = F(u), & t \in [0, \tau) \\ u(0, \cdot) = u_0, \end{cases} \tag{25}$$

where $u_0 \in \mathscr{L}^\infty$ and $\tau \in (0, \infty]$. Let Δ be the set of non-negative functions bounded by 1:

$$\Delta = \{ f \in \mathscr{L}^\infty : \ 0 \leq f \leq 1 \}. \tag{26}$$

Since the solution $u(t, x)$ of Equation (3) defines the proportion of x-type individuals being infected at time t, it should remain below 1 and above 0. Hence, for (3) to make a biological sense, the initial condition should belong to Δ and the solution, if it exists, should remain in Δ. This will be checked in Proposition 2.7.

2.3. *Properties of the vector field*

Recall that Assumption 0 is in force. The main results of this section are gathered in the following proposition.

Proposition 2.5 *(Properties of F). The function F defined in (24) has the following properties.*

(i) *F is of class $\mathcal{C}^\infty$ on $\mathscr{L}^\infty$ in the Fréchet sense.*
(ii) *F and its repeated derivatives are bounded on bounded sets.*
(iii) *F is continuous on Δ with respect to the topology of pointwise convergence.*
(iv) *F is cooperative both on $(1-\mathscr{L}_+^\infty)\times\mathscr{L}^\infty$ and on $\mathscr{L}^\infty\times(1-\mathscr{L}_+^\infty)$, where:*

$$1-\mathscr{L}_+^\infty=\{g\in\mathscr{L}^\infty\,:\, g\leq 1\}. \tag{27}$$

Proof. The bilinear map $(g,h)\mapsto gh$ and the linear maps $g\mapsto \gamma g$ and $g\mapsto T_\kappa(g)$ are bounded on $\mathscr{L}^\infty$ (hence, smooth as they are linear). Since the function F is a sum of compositions of the previous maps, properties (i) and (ii) are proved.

In order to prove property (iii), consider $(g_n,\ n\in\mathbb{N})$, a sequence of functions in Δ converging pointwise to $g\in\Delta$. Let $x\in\Omega$. The functions g_n are dominated by the function equal to 1 everywhere. The latter is integrable with respect to the measure $\kappa(x,\mathrm{d}y)$ since $\kappa(x,\Omega)<\infty$ according to (7). Therefore, we can apply the dominated convergence theorem and obtain:

$$\lim_{n\to\infty}\int_\Omega g_n(y)\kappa(x,\mathrm{d}y)=\int_\Omega g(y)\kappa(x,\mathrm{d}y).$$

Thus, the operator T_κ is continuous on Δ with respect to the pointwise convergence topology. The maps $(h_1,h_2)\mapsto h_1h_2$ and $(h_1,h_2)\mapsto h_1+h_2$ are also continuous with respect to the pointwise convergence topology. Hence, property (iii) is proved since F is a composition of these functions.

Finally, let us prove property (iv). Let $g,h\in\mathscr{L}^\infty$ such that $g\leq 1$ and $g\leq h$, and let $\nu\in\mathscr{L}_+^{\infty,\star}$ such that $\langle \nu, g-h\rangle=0$. We have:

$$\begin{aligned}\langle \nu, F(g)-F(h)\rangle &= \langle \nu, (1-g)T_\kappa(g-h)+(h-g)(T_\kappa(h)+\gamma)\rangle\\ &= \langle \nu, (1-g)T_\kappa(g-h)\rangle\,,\end{aligned}$$

where we used Lemma 2.6 below (with g replaced by $h-g$ and h by $T_\kappa(h)+\gamma$) in order to get that $\langle \nu, (h-g)(T_\kappa(h)+\gamma)\rangle$ is equal to 0. Since T_κ is a positive operator and $g\leq 1$, the function $(1-g)T_\kappa(g-h)$ is non-positive. The number $\langle \nu, (1-g)T_\kappa(g-h)\rangle$ is also non-positive because $\nu\in\mathscr{L}_+^{\infty,\star}$. Hence, we get that $\langle \nu, F(g)-F(h)\rangle\leq 0$. This proves that F is cooperative on $(1-\mathscr{L}_+^\infty)\times\mathscr{L}^\infty$, thanks to Definition 2.1. If $(g,h)\in\mathscr{L}^\infty\times(1-\mathscr{L}_+^\infty)$ satisfy $g\leq h$, then g is itself bounded above by 1, and the exact same proof applies, showing that F is also cooperative on $\mathscr{L}^\infty\times(1-\mathscr{L}_+^\infty)$. $\square$

The proof of Property (iv) of Proposition 2.5 uses the following lemma.

Lemma 2.6. *Let $g \in \mathscr{L}_+^\infty$ and $\nu \in \mathscr{L}_+^{\infty,\star}$ such that $\langle \nu, g\rangle = 0$. Then, for all $h \in \mathscr{L}^\infty$, we have $\langle \nu, hg\rangle = 0$.*

Proof. Let $g \in \mathscr{L}_+^\infty$ and $\nu \in \mathscr{L}_+^{\infty,\star}$ such that $\langle \nu, g\rangle = 0$. Since g is everywhere non-negative, we have:

$$-\|h\|\, g \le hg \le \|h\|\, g.$$

Since $\nu \in \mathscr{L}_+^{\infty,\star}$, the previous inequalities give:

$$-\|h\|\, \langle \nu, g\rangle \le \langle \nu, hg\rangle \le \|h\|\, \langle \nu, g\rangle\,.$$

By assumption, $\langle \nu, g\rangle$ is equal to 0. Hence, the lemma is proved. □

2.4. Properties of the ODE semi-flow

The aim of this subsection is to define a semi-flow associated to Equation (3) and to study its main properties. Proposition 2.5 (ii) enables to apply the Picard-Lindelöf theorem and show the existence of local solutions of in $\mathscr{L}^\infty$ of Equation (3). We can actually prove a stronger result. Recall that $\Delta = \{ f \in \mathscr{L}^\infty :\ 0 \le f \le 1 \}$.

Proposition 2.7. *Let F defined by* (24).

(i) *The domain Δ is* forward invariant*: if $u_0 \in \Delta$ and u solves* (25) *on $[0, \tau)$, then $u(t) \in \Delta$ for all $0 \le t < \tau$.*
(ii) *Maximal solutions of Equation* (25) *such that $u_0 \in \Delta$ are global, i.e., they are defined on $\mathbb{R}_+$.*

Proof. We first prove property (i). Let $u_0 \in \Delta$, and suppose that u solves (25) on $[0, \tau)$. Let $a(t) \in (1 - \mathscr{L}_+^\infty)$ be equal to the constant function 0, for all t; let $b(t) = u(t)$. Since $F(0) = 0$, and $b(t) = u(t)$ solves the ODE, we have for all $t < \tau$:

$$a'(t) - F(a(t)) = 0 = b'(t) - F(b(t)).$$

Since $0 = a(0) \le u(0) = u_0$, we may apply the comparison principle from Theorem 2.4, noting that F is locally Lipschitz and cooperative on $(1 - \mathscr{L}_+^\infty) \times \mathscr{L}^\infty$ by Lemma 2.5. We deduce that $0 \le u(t)$ for all $t < \tau$.

Similarly, letting now $a(t) = u(t) \in \mathscr{L}^\infty$ and $b(t) \in (1 - \mathscr{L}_+^\infty)$ be the constant function 1 for all t, and remarking that $F(b(t)) = -\gamma \le 0$, we may apply Theorem 2.4 again, using this time the cooperativeness on $\mathscr{L}^\infty \times (1 - \mathscr{L}_+^\infty)$, to get $u(t) \le 1$ for all $t < \tau$.

Now we prove property (ii). Let $(y, [0, \tau))$ be a solution of Equation (3) with $y(0) \in \Delta$. Assume that τ is a positive finite number. Property (i) asserts that $y(t) \in \Delta$, for all $0 \le t < \tau$. Since F is bounded on Δ (see Proposition 2.5 (ii)), $s \mapsto F(y(s))$ is integrable and:

$$\lim_{t\to\tau^-} y(t) = y(0) + \lim_{t\to\tau^-} \int_0^t F(y(s))\, \mathrm{d}s = y(0) + \int_0^\tau F(y(s))\, \mathrm{d}s.$$

The solution y can be extended up to its right boundary, *i.e.*, on $[0, \tau]$. By the Picard–Lindelöf theorem, it may thus be extended to $[0, \tau')$ for a $\tau' > \tau$. This shows that y is not maximal. We deduce that the maximal solution is defined on $\mathbb{R}_+$. $\square$

Thanks to Proposition 2.7, it is possible to define the semi-flow associated to the autonomous differential equation (3) on Δ, *i.e.*, the unique function $\phi : \mathbb{R}_+ \times \Delta \to \Delta$ solution of:

$$\begin{cases} \partial_t \phi(t, g) = F(\phi(t, g)), \\ \phi(0, g) = g. \end{cases} \tag{28}$$

It satisfies the semi-group property, that is, for all $g \in \Delta$ and for all $t, s \in \mathbb{R}_+$, we have:

$$\phi(t + s, g) = \phi(t, \phi(s, g)).$$

The result below is a fundamental property about the semi-flow of the SIS model. It expresses the intuitive idea that if epidemics are worse everywhere compared to a reference state, it will remain worse compared to the evolution of this reference state in the future.

Proposition 2.8 *(Order-preserving flow). If $0 \leq g \leq h \leq 1$, then we have $\phi(t, g) \leq \phi(t, h)$ for all $t \in \mathbb{R}_+$.*

Proof. Since $\partial_t \phi(t, g) - F(\phi(t, g)) = 0$ and $\partial_t \phi(t, h) - F(\phi(t, h)) = 0$, the inequality (20) is satisfied on $\mathbb{R}_+$ for the paths $a : t \mapsto \phi(t, g)$ and $b : t \mapsto \phi(t, h)$. By assumption, we have also that $g = \phi(0, g) \leq \phi(0, h) = h$. Furthermore F is locally Lipschitz (see Proposition 2.5 (ii)) and cooperative on $(1 - \mathscr{L}^\infty_+) \times \mathscr{L}^\infty$ (see Proposition 2.5 (iv)), and $a(t) = \phi(t, g) \in (1 - \mathscr{L}^\infty_+)$ by Proposition 2.7. Hence, we can apply Theorem 2.4 to obtain that $\phi(t, g) \leq \phi(t, h)$ for all $t \in \mathbb{R}_+$. $\square$

As a consequence of the previous proposition, we have the following result.

Corollary 2.9 *(Local monotony implies global monotony). Let $g \in \Delta$. Suppose that there exist $0 \leq a < b$ such that, for all $t \in [a, b)$, the inequality $\phi(a, g) \leq \phi(t, g)$ (resp. $\phi(a, g) \geq \phi(t, g)$) holds. Then, $t \mapsto \phi(t, g)$ is non-decreasing (resp. non-increasing) on $[a, \infty)$.*

Proof. It is sufficient to show that $t \mapsto \phi(t, g)$ is non-decreasing on all subintervals of $[a, \infty)$ whose lengths are bounded from above by $b - a$. Let $t > s \geq a$ such that $t - s < b - a$. By assumption, we have: $\phi(a, g) \leq \phi(a + t - s, g)$. Thus, Proposition 2.8 gives:

$$\phi(s - a, \phi(a, g)) \leq \phi(s - a, \phi(a + t - s, g)).$$

By the semi-group property of the semi-flow, this implies that $\phi(s, g) \leq \phi(t, g)$. $\square$

Proposition 2.10. *Let $g \in \Delta$. The path $t \mapsto \phi(t, g)$ is non-decreasing (resp. non-increasing) if and only if $F(g) \geq 0$ (resp. $F(g) \leq 0$).*

Proof. Let $g \in \Delta$, and suppose $F(g) \geq 0$. Let $a(t) = g$ for all t, and let $b(t) = \phi(t, g)$. Since $(a(t), b(t)) \in (1 - \mathscr{L}^\infty_+) \times \mathscr{L}^\infty$ for all t, and

$$a'(t) - F(a(t)) = -F(g) \leq 0 = b'(t) - F(b(t)),$$

we may apply the comparison Theorem 2.4: for all $t \geq 0$,

$$g = a(t) \leq b(t) = \phi(t, g).$$

We may now apply Corollary 2.9, proving that $t \mapsto \phi(t, g)$ is non-decreasing.

Now, suppose that $t \mapsto \phi(t, g)$ is non-decreasing. For all $t > 0$, the function $(\phi(t, g) - g)/t$ belongs to $\mathscr{L}^\infty_+$. Since $\mathscr{L}^\infty_+$ is closed, it follows that $F(g) = \lim_{t \to 0+}(\phi(t, g) - g)/t$ also belongs to $\mathscr{L}^\infty_+$.

The equivalence between $F(g) \leq 0$ and the fact that $t \mapsto \phi(t, g)$ is non-increasing is proved the same way. $\square$

Now we give some results about the regularity of the semi-flow.

Proposition 2.11 *(Flow regularity). Let $\phi : \mathbb{R}_+ \times \Delta \to \Delta$ be the semi-flow defined by Equation* (28).

(i) *For all $g \in \Delta$, $t \mapsto \phi(t, g)$ is $\mathcal{C}^\infty$ and its repeated derivatives are bounded.*
(ii) *For all $t \in \mathbb{R}_+$, $g \mapsto \phi(t, g)$ is Lipschitz with respect to $\|\cdot\|$.*
(iii) *For all $t \in \mathbb{R}_+$, $g \mapsto \phi(t, g)$ is continuous with respect to the pointwise convergence topology.*

Remark 2.12. Stronger regularity property than (ii) could be proved as in finite dimension. Since we use only the Lipschitz continuity property, we didn't go further in this direction.

Proof. We begin with property (i). The smoothness of the semi-flow with respect to the time variable can be shown by recurrence in a classical way. We have indeed:

$$\partial_t \phi(t, g) = F(\phi(t, g)), \qquad \partial_t^2 \phi(t, g) = \mathcal{D}F[\phi(t, g)](\partial_t \phi(t, g)), \qquad \ldots$$

Since F is of class $\mathcal{C}^\infty$ and its repeated derivatives are bounded on Δ (see (i) and (ii) in Proposition 2.5), the function $t \mapsto \phi(t, g)$ is of class $\mathcal{C}^\infty$ and its repeated derivatives are bounded for all $g \in \Delta$.

We prove (ii). Recall that, since ϕ is the semi-flow associated to Equation (25), the following equality holds for all $g \in \Delta$ and $t \in \mathbb{R}_+$:

$$\phi(t, g) = g + \int_0^t F(\phi(s, g))\, \mathrm{d}s. \tag{29}$$

Let $g, h \in \Delta$. We have the following control:

$$\|\phi(t,g)-\phi(t,h)\| \le \|g-h\| + \int_0^t \|F(\phi(s,g)) - F(\phi(s,h))\|\,\mathrm{d}s$$
$$\le \|g-h\| + C\int_0^t \|\phi(s,g)-\phi(s,h)\|\,\mathrm{d}s,$$

where C is the Lipschitz coefficient of F on Δ (see Proposition 2.5 (ii)). We conclude by applying Grönwall's inequality.

We prove property (iii). Let $(g_n,\ n\in\mathbb{N})$ be a sequence of functions in Δ converging pointwise toward $g\in\Delta$. We define for $n\in\mathbb{N}$:

$$\overline{g}_n = \sup_{j\ge n} g_j \quad \text{and} \quad \underline{g}_n = \inf_{j\ge n} g_j.$$

The sequence $(\overline{g}_n,\ n\in\mathbb{N})$ is non-increasing while $(\underline{g}_n,\ n\in\mathbb{N})$ is non-decreasing. We also have $\underline{g}_n \le g_n \le \overline{g}_n$ for all natural number n. Since the semi-flow is order-preserving by Proposition 2.8, the following inequalities hold for all $(t,x)\in\mathbb{R}_+\times\Omega$ and all $n\in\mathbb{N}^*$:

$$\phi(t,\underline{g}_{n-1})(x) \le \phi(t,\underline{g}_n)(x) \le \phi(t,g_n)(x) \le \phi(t,\overline{g}_n)(x) \le \phi(t,\overline{g}_{n-1})(x). \tag{30}$$

Thus, we can define two measurable functions $v,w:\mathbb{R}_+\times\Omega\to[0,1]$ by:

$$v(t,x)=\lim_{n\to\infty}\phi(t,\underline{g}_n)(x), \qquad w(t,x)=\lim_{n\to\infty}\phi(t,\overline{g}_n)(x),$$

for all $(t,x)\in\mathbb{R}_+\times\Omega$. Notice that $v(t,x)\le w(t,x)$ by construction.

Fix $x\in\Omega$ and $t\ge 0$. We have:

$$\phi(t,\overline{g}_n)(x) = \overline{g}_n(x) + \int_0^t F(\phi(s,\overline{g}_n))(x)\,\mathrm{d}s.$$

The sequence of functions $(\overline{g}_n(x),\ n\in\mathbb{N})$ converges to $g(x)$ while the sequence of functions $(\phi(s,\overline{g}_n),\ n\in\mathbb{N})$ converges pointwise to $w(s,\cdot)\in\Delta$ for all $s\ge 0$. By continuity (see Proposition 2.5 (iii)), $F(\phi(s,\overline{g}_n))(x)$ converges to $F(w(s,\cdot))(x)$. Furthermore, the functions $s\mapsto F(\phi(s,\overline{g}_n))(x)$ are uniformly bounded since F is bounded on Δ (see Proposition 2.5 (ii)). Hence, we deduce from the dominated convergence theorem that:

$$w(t,x) = g(x) + \int_0^t F(w(s,\cdot))(x)\,\mathrm{d}s.$$

The previous equality is true for all $x\in\Omega$ and $t\ge 0$. Since $t\mapsto\phi(t,g)$ is the only solution of (3) having g as initial condition, we have necessarily $w(t,\cdot)=\phi(t,g)$. We prove that $v(t,\cdot)=\phi(t,g)$ the same way. Letting n go to infinity in (30) proves that $\phi(t,g_n)$ converges pointwise to $\phi(t,g)$, for all $t\ge 0$. □

2.5. Equilibria

A function $g \in \Delta$ is an equilibrium of the dynamical system (Δ, ϕ) (also called a stationary point) if for all $t \in \mathbb{R}_+$, $\phi(t, g) = g$. The latter assertion is equivalent to $F(g) = 0$. The function equal to 0 everywhere is a trivial stationary point. In mathematical epidemiology, the other equilibria, if they exist, are called *endemic states* because they model a situation where the infection is constantly maintained at a baseline level in the population.

The following result gives an easy way to identify those special states in the system. It is a well-known fact in dynamical system theory, we give a short proof for completeness.

Proposition 2.13 *(Limit points are equilibria). Let $g \in \Delta$. If $t \mapsto \phi(t, g)$ converges pointwise to a limit $h^* \in \Delta$ when t goes to ∞, then the function h^* is an equilibrium.*

Proof. For all $x \in \Omega$ and $s \geq 0$, we have

$$\phi(s, h^*)(x) = \lim_{t\to\infty} \phi(s, \phi(t, g))(x) = \lim_{t\to\infty} \phi(s + t, g)(x) = h^*(x),$$

where the first inequality follows from the continuity of ϕ with respect to the pointwise convergence topology given in Proposition 2.11 (iii). Thus, h^* is an equilibrium. □

In the next remark, we check that any equilibrium is continuous with respect to an intrinsic distance on Ω based on κ and γ.

Remark 2.14 *(Continuity of the equlibria).* We consider for all $x, y \in \Omega$:

$$r(x, y) = \|\kappa(x, \cdot) - \kappa(y, \cdot)\|_{\mathrm{TV}} + |\gamma(x) - \gamma(y)|,$$

where $\|\cdot\|_{\mathrm{TV}}$ is the total variation norm. The function r defines a pseudo-metric on the space Ω. This pseudo-metric can be thought as an extension of the neighborhood distance on graphons (see [13, Section 13.3]). Notice that the Borel σ-field associated to the topology defined by r is included in $\mathscr{F}$ since γ is measurable and κ is a kernel.

We have that if h^* is an equilibrium of the dynamical system (Δ, ϕ), then it is continuous with respect to r. Indeed, we have for all $x \in \Omega$:

$$h^*(x) = \frac{\lambda(x)}{\lambda(x) + \gamma(x)} \quad \text{with} \quad \lambda(x) = \int_\Omega h^*(z)\, \kappa(x, \mathrm{d}z).$$

Both λ and γ are continuous with respect to r and the function $(a, b) \mapsto a/(a + b)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+^*$. This implies that h^* is continuous.

2.6. The maximal equilibrium

As a consequence of Proposition 2.7 and Corollary 2.9, the path $t \mapsto \phi(t, 1)$ is non-increasing and bounded below by 0. Thus, the path $t \mapsto \phi(t, 1)$ converges pointwise to a limit say g^* when t goes to infinity:

$$g^*(x) = \lim_{t\to+\infty} \phi(t, 1)(x), \qquad \forall x \in \Omega. \tag{31}$$

Proposition 2.15. *Let g^* be defined by (31). We have the following properties.*

(i) *The function g^* is the maximal equilibrium of the dynamical system (Δ, ϕ), i.e., if h^* is an equilibrium, then $h^* \leq g^*$.*
(ii) *For all $g^* \leq g \leq 1$, $\phi(t, g)$ converges pointwise to g^* as t goes to infinity.*

Proof. We first prove property (i). The function g^* is an equilibrium according to Proposition 2.13. Let h^* be another equilibrium in Δ. By Proposition 2.8, $h^* = \phi(t, h^*) \leq \phi(t, 1)$ for all t; sending t to infinity yields $h^* \leq g^*$. The function g^* is thus the maximal equilibrium.

To prove property (ii), we consider $g^* \leq g \leq 1$. By Proposition 2.8, we know that:

$$g^* = \phi(t, g^*) \leq \phi(t, g) \leq \phi(t, 1).$$

Since the rightmost term converges to g^* by (31), this implies that $\phi(t, g)$ converges to g^* as t tends to infinity for the pointwise convergence. □

Remark 2.16. Since $\gamma(x) > 0$ for all $x \in \Omega$ according to Assumption 0 and $F(g^*) = 0$, we have that $g^*(x) < 1$ for all $x \in \Omega$.

There is no closed-form formula for g^* in the general case, even in a finite dimensional model. However, if the function $x \mapsto \kappa(x, \Omega)/\gamma(x)$ is constant, then the formula used for the one-group model can be extended.

Proposition 2.17. *Suppose that there exists $C \in \mathbb{R}_+$ such that $\kappa(x, \Omega)/\gamma(x) = C$ for all $x \in \Omega$. Then, g^* is a constant function equal to* $\max(0, 1 - 1/C)$.

Proof. It is straightforward to check that the function $x \mapsto \max(0, 1 - 1/C)$ is an equilibrium. Now, we prove that it is maximal. Let $h^* \in \Delta$ be an equilibrium. From $F(h^*)/\gamma = 0$, we obtain the inequality:

$$h^* \leq C(1 - h^*)\|h^*\|.$$

Taking a sequence $(x_n, n \in \mathbb{N})$ such that $h^*(x_n)$ converges to $\|h^*\|$, we obtain at the limit that $\|h^*\| \leq C(1 - \|h^*\|)\|h^*\|$. It follows that $\|h^*\| \leq \max(0, 1 - 1/C)$. □

Since we cannot determine g^* in the general case, the important question that naturally arises is to find out whether the epidemic can survive in the population or if it will die out whatever the initial condition is, *i.e.*, we have to determine if $g^*(x) = 0$ for all $x \in \Omega$. In the following, we answer this question with Assumption 1 which imposes further conditions on the transmission kernel κ and the recovery rate γ.

3. Tools from operator theory

In this section, we introduce some tools that we will use in Section 4.

3.1. Compactness, weak compactness and the Dunford Pettis property

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. Recall that T is *compact* if the image of the unit ball in X is relatively compact for the strong topology on Y. Similarly it is *weakly compact* if the image of the unit ball in X by T is relatively compact for the weak topology on Y (that is, its weak closure is weakly compact). We recall the following results on weak compactness ([60, Corollary VI.4.3, Theorem VI.4.5]).

Theorem 3.1 *(Weak compactness). If X and Y are Banach spaces, one of which is reflexive, and T is a bounded operator from X to Y, then T is weakly compact.*

The composition of a bounded operator and a weakly compact operator, in any order, is weakly compact.

The following important property is given in [54] (the space $\mathscr{L}^\infty$ is a so-called "abstract-max space" by [54, II §7, example 3 p. 103], so that [54, Theorem II.9.9] applies).

Theorem 3.2 *(Dunford Pettis property). If Y is a Banach space, and if $T : \mathscr{L}^\infty \to Y$ is weakly compact, then T is absolutely continuous, that is, it maps weakly convergent sequences in $\mathscr{L}^\infty$ to strongly convergent sequences in Y.*

Corollary 3.3. *If $T : \mathscr{L}^\infty \to \mathscr{L}^\infty$ is weakly compact then T^2 is compact.*

Proof. Let x_n be a bounded sequence in $\mathscr{L}^\infty$. Since T is weakly compact there exists a subsequence such that (Tx_n) converges weakly. By the Dunford Pettis property, T is absolutely continuous so $T(Tx_n)$ converges strongly along the subsequence. Therefore T^2 is compact. □

3.2. Banach lattices

Let us first recall standard definitions on Banach lattices; we refer the reader to the standard texts [53] and [54] for a more detailed introduction to the subject. Banach lattices provide a convenient framework to study positive operators and generalizations of the Perron–Frobenius theorem; the two examples we have in mind are the space $\mathscr{L}^\infty$ of bounded functions and the space $L^p(\mu)$.

Let $(X, \leq)$ be a set equipped with a partial order. The set X is a *lattice* if for any x and y in X, there exist two elements i and s in X such that for all z,

$$(z \leq x \text{ and } z \leq y) \implies z \leq i; \qquad (x \leq z \text{ and } y \leq z) \implies s \leq z.$$

The infimum i and supremum s are customarily denoted $x \wedge y$ and $x \vee y$ respectively. A *Riesz space* is a vector space X endowed with a lattice structure (denoted $\leq$), such that the two following compatibility conditions are satisfied:

Translation invariance For all x, y and z in X, if $x \leq y$ then $x + z \leq y + z$.

Positive homogeneity For all x, y in X, if $x \leq y$, then $\lambda x \leq \lambda y$, for all non negative scalar $\lambda \geq 0$.

The *absolute value* $|x|$ of an element x of a Riesz space is defined by $|x| = x \vee (-x)$. We proceed with some further definitions.

Definition 3.4. *A Banach lattice $(X, \leq, \|\cdot\|)$ is a Riesz space $(X, \leq)$ equipped with a complete norm $\|\cdot\|$ and such that, for any $x, y \in X$, we have:*

$$|x| \leq |y| \implies \|x\| \leq \|y\|. \tag{32}$$

In the Banach lattice X, the positive cone:

$$X_+ = \{ x \in E : \ x \geq 0 \},$$

is a proper cone, as it is a closed (see Theorem 15.1 (ii) in [53]) convex set such that $\lambda X_+ \subset X_+$ for all $\lambda \in \mathbb{R}_+$, and $X_+ \cap (-X_+) = \{0\}$. It is also a reproducing cone ($X = X_+ - X_+$) as every element x in X can be decomposed as $x = (x \vee 0) - ((-x) \vee 0)$ and $y \vee 0 \in X_+$ for all $y \in X$.

3.3. Spectral analysis in Banach lattices

The main result of this section concerns the spectrum of operators on Banach lattices. Let us first recall a few classical definitions of spectral theory in Banach spaces. Let $(X, \|\cdot\|)$ be a Banach space. The *spectrum* $\sigma(A)$ of a bounded operator A on X is the set of all complex numbers λ such that $A - \lambda\mathrm{Id}$ does not have a bounded inverse operator. It is well known that the spectrum of a bounded operator is a compact set in $\mathbb{C}$. The *essential spectrum* of an operator is the part that cannot be removed by a compact perturbation:

$$\sigma_{\mathrm{ess}}(A) = \bigcap_{P \text{ compact operator}} \sigma(A + P).$$

Note that there are several conflicting definitions of the essential spectrum; see [61, Section 1.4]. By [61, Theorem 9.1.4, p 422], our definition corresponds to $\sigma_{e4}(A)$ defined p 37 in [61]. Let us remark that other definitions would lead to the same essential spectral radius in the definition below; see [61, Corollary 1.4.11].

For a bounded operator A on X, the spectral bound, the spectral radius and the essential spectral radius are defined as:

$$s(A) = \sup\{ \mathrm{Re}(\lambda) : \ \lambda \in \sigma(A) \}, \tag{33}$$

$$r(A) = \sup\{ |\lambda| : \ \lambda \in \sigma(A) \} = \lim_{n \to +\infty} \|A^n\|^{1/n} = \inf_{n \in \mathbb{N}^*} \|A^n\|^{1/n}, \tag{34}$$

$$r_{\mathrm{ess}}(A) = \sup\{ |\lambda| : \ \lambda \in \sigma_{\mathrm{ess}}(A) \}, \tag{35}$$

respectively, with the convention that $\sup \emptyset = 0$. Since $\sigma_{\mathrm{ess}}(A) \subset \sigma(A)$, we get:

$$r_{\mathrm{ess}}(A) \leq r(A) \leq \|A\|. \tag{36}$$

The spectral theory of positive bounded operator on Banach lattice extends the Perron-Frobenius theory in infinite dimension. Let A be a positive operatoron a Banach lattice $(X, \leq,$

$\|\cdot\|$), that is, $AX_+ \subset X_+$, such that its spectral radius $r(A)$ is positive. Recall $X_+^\star$ is the dual cone of X_+. A vector $x \in X_+ \setminus \{0\}$ (resp. $x^\star \in X_+^\star \setminus \{0\}$) such that $Ax = r(A)x$ (resp. $A^\star x^\star = r(A)x^\star$) is called a right (resp. left) Perron eigenvector. We have the following important result.

Theorem 3.5. *Let $(X, \leq, \|\cdot\|)$ be a Banach lattice. Let A, B be positive bounded operators on X. We have the following properties.*

(i) *If $B - A$ is a positive operator, then $r(A) \leq r(B)$.*
(ii) *The spectral radius $r(A)$ belongs to $\sigma(A)$ and thus $r(A) = s(A)$.*
(iii) *If $r_{\text{ess}}(A) < r(A)$, then, there exists $x \in X_+ \setminus \{0\}$ such that: $Ax = r(A)x$.*

Proof. Property (i) is proved in [62, Theorem 4.2]. Property (ii) is proved in [63] (notice that (32) implies that X_+ is normal in the setting of [63]); see also [53, Lemma 41.1.(ii)]. Property (iii) was shown by Nussbaum in [64, Corollary 2.2] (notice that a reproducing cone is total), where the essential spectrum in [64] is defined in [65] and corresponds to $\sigma_{e5}(A)$ in [61, p. 37]. However, the essential spectral radius of $\sigma_{e5}(A)$ is equal to $r_{\text{ess}}(A)$ the essential spectral radius of $\sigma_{e4}(A)$, according to [61, Theorem I.4.10]. □

It A is assumed to be a compact operator, then Theorem 3.5 (iii) is the so-called Krein-Rutman theorem; see [53, Theorem 41.2]. We will also need the following result proved in [66, Propositions 2.1-2.2].

Proposition 3.6 *(Collatz-Wielandt inequality). Let $(X, \leq, \|\cdot\|)$ be a Banach lattice and A be a positive bounded operator on X. We have:*

$$\sup\{\lambda \in \mathbb{R} : \exists x \in X_+ \setminus \{0\},\ Ax \geq \lambda x\} \leq r(A).$$

3.4. The Banach lattice of bounded measurable functions

The Banach space $(\mathscr{L}^\infty, \|\cdot\|)$ equipped with the partial order $\leq$ defined by the proper cone $\mathscr{L}^\infty_+$ from (21) is a Banach lattice.

Let ν be a finite signed measure on $(\Omega, \mathscr{F})$. For $g \in \mathscr{L}^\infty$, we write $\langle \nu, g\rangle = \int_\Omega g(x)\,\nu(\mathrm{d}x)$ and thus identify ν as an element of $\mathscr{L}^{\infty,\star}$, the dual space of $\mathscr{L}^\infty$ (recall that $\mathscr{L}^{\infty,\star}$ can be identified as the space of bounded and finitely additive signed measures on $(\Omega, \mathscr{F})$).

Let μ be a given finite positive measure on $(\Omega, \mathscr{F})$. For $q \in (1, +\infty)$, denote by $(L^q(\mu), \|\cdot\|_q)$ the usual Banach space of real-valued measurable functions f defined on $(\Omega, \mathscr{F})$ such that $\|f\|_q = \left(\int_\Omega |f(x)|^q\,\mu(\mathrm{d}x)\right)^{1/q}$ is finite and where we have identified functions which agree μ-almost everywhere.

Let ι be the natural linear application ι from $\mathscr{L}^\infty$ to $L^p(\mu)$, with $p = q/(q-1)$ the conjugate of q, and $\iota^\star$ its dual. For $f \in L^q(\mu)$, we can see $\iota^\star(f)$ as the bounded σ-finite signed measure $f(x)\mu(\mathrm{d}x)$ elements of $\mathscr{L}^{\infty,\star}$. By convention, for $f \in L^q(\mu)$ and g in $\mathscr{L}^\infty$, we write:

$$\langle f, g\rangle = \langle \iota^\star(f), g\rangle = \int_\Omega f(x)g(x)\,\mu(\mathrm{d}x).$$

Let k be a non-negative measurable function defined on $(\Omega \times \Omega, \mathscr{F} \otimes \mathscr{F})$ such that $\sup_{x\in\Omega} \int \mathsf{k}(x, y)\,\mu(\mathrm{d}y) < \infty$. We define the integral operator T_k as the operator T_κ defined by (22) with kernel $\kappa(x, \mathrm{d}y) = \mathsf{k}(x, y)\, \mu(\mathrm{d}y)$. Let $q \in (1, +\infty)$. We assume the following condition holds:

$$\sup_{x \in \Omega} \int \mathsf{k}(x, y)^q \, \mu(\mathrm{d}y) < \infty. \tag{37}$$

Then, we can also define the bounded operator:

$$\begin{aligned} \tilde{T}_{\mathsf{k}} \quad : \quad & L^p(\mu) \to \mathscr{L}^\infty \\ & g \mapsto \left(x \mapsto \int_\Omega g(y)\, \mathsf{k}(x, y)\, \mu(\mathrm{d}y) \right). \end{aligned}$$

With this notation, $T_{\mathsf{k}} = \tilde{T}_{\mathsf{k}} \iota$. We also define a bounded operator $\hat{T}_{\mathsf{k}}$ from $L^p(\mu)$ to $L^p(\mu)$:

$$\hat{T}_{\mathsf{k}} = \iota \tilde{T}_{\mathsf{k}}. \tag{38}$$

To sum up we have the following commutative diagram:

$$\begin{tikzcd} \mathscr{L}^\infty \arrow[r, "\iota"] \arrow[d, "T_{\mathsf{k}}"] & L^p(\mu) \arrow[d, "\hat{T}_{\mathsf{k}}"] \arrow[dl, "\tilde{T}_{\mathsf{k}}"] \\ \mathscr{L}^\infty \arrow[r, "\iota"] & L^p(\mu) \end{tikzcd}$$

The following lemma has a fundamental importance for the development of Section 4. The last property on connected integral operator is part of the Perron-Jentzsch theorem; see [54, Theorem V.6.6 and Example V.6.5.b].

Lemma 3.7. *Let k be a non-negative measurable function defined on $(\Omega \times \Omega, \mathscr{F} \otimes \mathscr{F})$ such (37) holds for some $q \in (1, +\infty)$. Then, the positive bounded operators $T_{\mathsf{k}} : \mathscr{L}^\infty \to \mathscr{L}^\infty$ and $\hat{T}_{\mathsf{k}} : L^p(\mu) \to L^p(\mu)$, with $p = q/(q-1)$, satisfy:*

(i) *If $g = 0$ μ-a.e. then we have $T_{\mathsf{k}} g = 0$.*
(ii) *The operator T_{k} is weakly compact.*
(iii) *The operators T_{k}^2 and $\hat{T}_{\mathsf{k}}$ are compact.*
(iv) *The operators T_{k} and $\hat{T}_{\mathsf{k}}$ have the same spectrum, and thus $r(T_{\mathsf{k}}) = r(\hat{T}_{\mathsf{k}})$.*
(v) *If $r(T_{\mathsf{k}}) > 0$, then the operator T_{k} has a right Perron eigenvector in $\mathscr{L}^\infty_+ \backslash \{0\}$ and a left Perron eigenvector in $L^q_+(\mu) \backslash \{0\} \subset \mathscr{L}^{\infty,\star}_+ \backslash \{0\}$.*
(vi) *If k is connected in the sense of Assumption 2, then $r(T_{\mathsf{k}}) > 0$ and the right and left Perron eigenvector are unique (up to a multiplicative constant) and are μ-a.e. positive, with the left Perron eigenvector seen as an element of $L^q_+(\mu) \backslash \{0\}$.*

Proof. Property (i) is straightforward.

To prove property (ii), one may write T_k as the composition $T_\mathsf{k} = \tilde{T}_\mathsf{k} \circ \iota$, where $\tilde{T}_\mathsf{k}$ is bounded, and ι is weakly compact by the first part of Theorem 3.1 since $L^p(\mu)$ is reflexive ([60, Corollary IV.8.2]). By the second part of Theorem 3.1, T_k is weakly compact.

The first part of Property (iii), that is, the compactness of T_k^2, follows directly from Corollary 3.3. Consider now the operator $\hat{T}_\mathsf{k} = \iota \circ \tilde{T}_\mathsf{k}$. By Theorem 3.1, both $\tilde{T}_\mathsf{k}$ and ι are weakly compact. From any bounded sequence x_n in $L^p(\mu)$, we may extract a subsequence such that $\tilde{T}_\mathsf{k} x_n$ converges weakly in $\mathscr{L}^\infty$. By the Dunford Pettis property (Theorem 3.2), the weakly compact operator ι maps this weakly convergent subsequence to a strongly convergent subsequence in $L^p(\mu)$. Therefore $\hat{T}_\mathsf{k}$ is compact.

Let us prove property (iv). If Ω is finite then the operators T_k and $\hat{T}_\mathsf{k}$ coincide and there is nothing to prove. So, we assume that Ω is infinite. In this case, $\sigma_{\text{ess}}(T_\mathsf{k})$ and $\sigma_{\text{ess}}(\hat{T}_\mathsf{k})$ are non empty according to [67, Footnote 2, p. 243]. As $\hat{T}_\mathsf{k}$ and T_k^2 are compact, we deduce from [60, Theorems VII.4.5 and VII.4.6] respectively, that the essential spectra of $\hat{T}_\mathsf{k}$ and T_k are reduced to $\{0\}$, and that the non-null elements of their spectrum are eigenvalues. Then, use that $\iota T_\mathsf{k} = \hat{T}_\mathsf{k} \iota$ and property (i), to deduce that if $f \in \mathscr{L}^\infty \setminus \{0\}$ is an eigenvector of T_k, then $\iota(f)$ belongs to $L^p(\mu)\setminus\{0\}$ thanks to property (i) and that $\iota(f)$ is thus an eigenvector of $\hat{T}_\mathsf{k}$ corresponding to the same eigenvalue. If $v \in L^p(\mu)\setminus\{0\}$ is an eigenvector of $\hat{T}_\mathsf{k}$ corresponding to the eigenvalue λ, then $f = \tilde{T}_\mathsf{k}(v)$ belongs to $\mathscr{L}^\infty$ and $f \neq 0$ (as $\iota(f) = \hat{T}_\mathsf{k}(v) = \lambda v$). We have $T_\mathsf{k}(f) = \tilde{T}_\mathsf{k} \iota \tilde{T}_\mathsf{k}(v) = \tilde{T}_\mathsf{k}(\hat{T}_\mathsf{k}(v)) = \lambda \tilde{T}_\mathsf{k}(v) = \lambda f$, thus λ is also an eigenvalue of T_k. We deduce that $\sigma(T_\mathsf{k}) = \sigma(\hat{T}_\mathsf{k})$.

Let us now prove property (v). We have seen that $\sigma_{\text{ess}}(T_\mathsf{k}) \subset \{0\}$ and thus $r_{\text{ess}}(T_\mathsf{k}) = 0$. According to Theorem 3.5 (iii) (or the Krein-Rutman theorem) there exists a right Perron eigenvector for T_k. Since $\hat{T}_\mathsf{k}^\star$ is a compact operator, thanks to Schauder Theorem [60, Theorem VI.5.2], with the same spectrum as $\hat{T}_\mathsf{k}$, thanks to [60, Lemma VII.3.7], and which is clearly positive, we deduce from Theorem 3.5 (iii) that there exists a right Perron eigenvector, $v^\star \in L^q_+(\mu)\setminus\{0\}$, for $\hat{T}_\mathsf{k}^\star$. Since $T_\mathsf{k}^\star \iota^\star = \iota^\star \hat{T}_\mathsf{k}^\star$, we deduce that $\iota^\star(v^\star)$, and thus $v^\star$ by convention, is also a left Perron eigenvector for T_k. This gives property (v).

Finally let us prove property (vi). Set $\lambda = r(T_\mathsf{k}) = r(\hat{T}_\mathsf{k})$ thanks to property (iv). According to the Perron-Jentzsch theorem [54, Theorem V.6.6 and Example V.6.5.b], since k is connected in the sense of Assumption 2, we have $\lambda > 0$ and there exists a unique (up to a multiplicative constant) eigenvector v of $\hat{T}_\mathsf{k}$ associated to the eigenvalue λ, and it can be chosen such that μ-a.e. $v > 0$. According to the proof of property (iv), we get that $f = \tilde{T}_\mathsf{k}(v)$ is an eigenvector of T_k associated to λ. Notice that $f \geq 0$ as $\mathsf{k} \geq 0$. Since $\iota(f) = \hat{T}_\mathsf{k}(v) = \lambda v$, we deduce that μ-a.e. $f > 0$. Assume that $g \in \mathscr{L}^\infty \setminus \{0\}$ is a right Perron eigenvector of T_k, then $\iota(g)$ is a right Perron eigenvector of $\hat{T}_\mathsf{k}$ and thus (up to a multiplicative constant chosen to be equal to λ), we have μ-a.e. $\iota(g) = \lambda v = \iota(f)$. We deduce that μ-a.e. $g - f = 0$ and thanks to property (i), we deduce that $\lambda(f - g) = T_\mathsf{k}(f - g) = 0$. So the right Perron eigenvector of T_k is unique and μ-a.e. positive.

Let $f^\star$ be a left Perron eigenvector of T_k. Then $v^\star = \tilde{T}_\mathsf{k}^\star(f^\star)$ is an eigenvector of $\hat{T}_\mathsf{k}^\star$ associated to λ and $v^\star \in L^q_+(\mu)\setminus\{0\}$ as $\mathsf{k} \geq 0$. By the Perron-Jentzsch theorem, we get that $v^\star$ is unique (up to a multiplicative constant) and that μ-a.e. $v^\star > 0$. Since $\iota^\star(v^\star) = \lambda f^\star$, we deduce that μ-a.e. $f^\star > 0$ and that $f^\star$ is unique (up to a multiplicative constant). □

Remark 3.8. As a consequence of Lemma 3.7 (i), under Assumption 1, if h^* is an equilibrium which is μ-a.e. equal to 0, then it is equal to 0 everywhere.

4. Infinite-dimensional SIS model when the kernel has a density

The objective of this section is to study the long time behavior of the solutions of (3) under Assumption 0 and Assumption 1 (but for Section 4.2 where the latter is not assumed). Recall the definition of the spectral bound given in (33). We will consider the spectral bound $s(T_k - \gamma)$ of the bounded operator $T_k - \gamma$ on $\mathscr{L}^\infty$ to characterize three different regimes: sub-critical, critical and super-critical, corresponding to the cases $s(T_k - \gamma) <, =, > 0$ respectively. In the first part of the section, we establish a link between $s(T_k - \gamma)$ and the basic reproduction number $R_0 = r(T_{k/\gamma})$ associated to (3).

4.1. Basic reproduction number and spectral bound

Recall that Assumption 0 is in force. If we assume $\inf \gamma > 0$, then the operator $T_{\kappa/\gamma}$, where κ/γ is the kernel defined by $(\kappa/\gamma)(x, \mathrm{d}y) = \kappa(x, \mathrm{d}y)/\gamma(y)$ is bounded. The following result is a direct consequence of a theorem of Thieme [45, Theorem 3.5].

Proposition 4.1 *(Equivalent conditions for criticality). If $\inf \gamma > 0$, then $r(T_{\kappa/\gamma}) - 1$ has the same sign as $s(T_\kappa - \gamma)$ (i.e. these two numbers are simultaneously negative, zero, or positive).*

Proof. Consider the operators $A = T_\kappa - \gamma$ and $B = -\gamma$, where $-\gamma$ is the operator corresponding to the multiplication by $-\gamma$. It is clear from [45, Definition 3.1] that the operator B is a resolvent-positive operator, as the operator $\lambda - B = \lambda + \gamma$ is invertible and its inverse is positive for $\lambda > 0$. We also get that $s(B) = s(-\gamma) = -\inf \gamma < 0$. Let $Q = A + \|\gamma\|$. The operator Q is positive. For $\lambda > r(Q)$, the resolvent operator $(\lambda - Q)^{-1}$ is also positive since, thanks to the Neumann series expansion, we have:

$$(\lambda - Q)^{-1} = \sum_{i=0}^{\infty} \frac{1}{\lambda^{i+1}} Q^i \geq 0.$$

We deduce that $(\lambda - A)$ is invertible and its inverse is positive for $\lambda > r(Q) - \|\gamma\|$, thus A is resolvent-positive. Applying [45, Theorem 3.5] (notice that it is required that $\mathscr{L}^\infty_+$ is normal, which is the case – see [56, Proposition 19.1] – as the norm $\|\cdot\|$ is monotonic: $0 \leq f \leq g$ implies $\|f\| \leq \|g\|$), we deduce that $s(A)$ has the same sign as $r(-(A - B)B^{-1}) - 1 = r(T_{\kappa/\gamma}) - 1$. □

Notice that under Assumption 1, we have by definition $R_0 = r(T_{\kappa/\gamma}) = r(T_{k/\gamma})$, as k is the density of κ with respect to μ. In what follows, we also write T_k for T_κ. According to Assumption 1 (see (8)), the operator $T_{k/\gamma}$ defined by (11) is a bounded operator on $\mathscr{L}^\infty$ which satisfies the integrability condition of Lemma 3.7 with $\mathsf{k}(x, y) = k(x, y)/\gamma(y)$.

We wish to prove a result similar to Proposition 4.1 without assuming that $\inf \gamma > 0$. In this case however, $R_0 - 1$ and $s(T_k - \gamma)$ may have different signs: For instance, if one takes $T_k = 0$ and $\inf \gamma = 0$, then we clearly have $s(T_k - \gamma) = s(-\gamma) = 0$ and $R_0 = 0 < 1$. In order to get a result similar to Proposition 4.1, we must therefore settle for a weaker conclusion, which will however be sufficient for our purposes.

Proposition 4.2 *(Equivalent conditions for the supercritical regime). Suppose Assumption 1 is in force. Then, the following assertions are equivalent:*

(i) $s(T_k - \gamma) > 0$.
(ii) $R_0 > 1$.
(iii) *There exists* $\lambda > 0$ *and* $w \in \mathscr{L}_+^\infty \setminus \{0\}$ *such that:*

$$T_k(w) - \gamma w = \lambda w. \tag{39}$$

Proof. It is immediate that property (iii) implies property (i).

We assume property (i) and prove (ii). Let $a \in (0, s(T_k - \gamma))$, so that $s(T_k - (\gamma + a)) = s(T_k - \gamma) - a > 0$. Using Proposition 4.1 (with γ replaced by $\gamma + a$), we get that $r\left(T_{k/(\gamma+a)}\right) > 1$. Since $r(T_{k/\gamma}) \geq r\left(T_{k/(\gamma+a)}\right)$ according to Theorem 3.5 (i), property (ii) is shown.

We assume property (ii) and prove property (iii). For any non-negative real number $a \geq 0$, let $\psi(a) = r(T_{k/(\gamma+a)})$. Property (ii) exactly means that:

$$\psi(0) > 1. \tag{40}$$

Moreover, it follows from the inequality $r(T_{k/(\gamma+a)}) \leq \|T_{k/(\gamma+a)}\| \leq \|T_k\|/a$ (use (36) for the first inequality), that:

$$\lim_{a\to\infty} \psi(a) = 0. \tag{41}$$

Equation (8) of Assumption 1 enables to apply Lemma 3.7 (iii) and we obtain that all the operators $T_{k/(\gamma+a)}$, for $a \in \mathbb{R}_+$, are power compact (as $T_{k/(\gamma+a)}^2$ is compact). According to [68, Theorem p. 21], their spectra are totally disconnected. Moreover, the function $a \mapsto T_{k/(\gamma+a)}$ mapping $\mathbb{R}_+$ to $\mathcal{L}(\mathscr{L}^\infty)$ is continuous as (8) holds. Indeed, for all $0 \leq a_1 \leq a_2$, thanks to Hölder's inequality, we have:

$$\|T_{k/(\gamma+a_1)} - T_{k/(\gamma+a_2)}\| \leq \left\| \frac{a_2 - a_1}{\gamma + a_2} \right\|_p \sup_{x\in\Omega} \left(\int_\Omega k(x,y)^q / \gamma(y)^q \, \mu(\mathrm{d}y) \right)^{1/q},$$

and by dominated convergence $\left\| \frac{a_2-a_1}{\gamma+a_2} \right\|_p$ converges to 0 as $|a_2 - a_1|$ converges to 0. Thanks to [69, Theorem 11], we get that the application $a \mapsto \sigma(T_{k/(\gamma+a)})$ mapping $\mathbb{R}_+$ to the set $\mathcal{K}(\mathbb{C})$ of non-empty compact subsets endowed with the Hausdorff distance (see Appendix B for the definition of the Hausdorff distance) is continuous. Hence, the function ψ is continuous according to Lemma B.1. From the continuity of ψ and Equations (40) and (41), we conclude that there exists $\lambda > 0$ such that $\psi(\lambda) = 1$. According to Lemma 3.7 (v), there exists a function $v \in \mathscr{L}_+^\infty \setminus \{0\}$ such that:

$$T_k\left(\frac{v}{\gamma + \lambda}\right) = v.$$

Then, Equation (39) holds with $w = v/(\gamma + \lambda)$ which proves property (iii). □

Remark 4.3. Using Lemma 3.7 (i), it is easy to show that w in Proposition 4.2 (iii) should satisfy $\int_\Omega w(x)\, \mu(\mathrm{d}x) > 0$.

The next result is stronger than the implication (i) $\implies$ (ii) in Proposition 4.2.

Lemma 4.4. *Under Assumption 1, the following inequality holds:*

$$s(T_k - \gamma) \leq \max(\|\gamma\|(R_0 - 1), 0). \tag{42}$$

Proof. If $s(T_k - \gamma) \leq 0$, the result is obviously true. Suppose $s(T_k - \gamma) > 0$. Since $T_k - \gamma + \|\gamma\|$ is a positive operator, Theorem 3.5 (ii) implies that $r(T_k - \gamma + \|\gamma\|) = s(T_k - \gamma + \|\gamma\|)$. Since $s(T_k - \gamma + \|\gamma\|) = s(T_k - \gamma) + \|\gamma\| > \|\gamma\|$, we obtain that:

$$r(T_k - \gamma + \|\gamma\|) > \|\gamma\|.$$

Besides, we have $\|\gamma\| \geq r(\|\gamma\| - \gamma)$ according to Theorem 3.5 (i) and $r(\|\gamma\| - \gamma) \geq r_{\text{ess}}(\|\gamma\| - \gamma)$ according to Equation (36). We deduce that:

$$r(T_k - \gamma + \|\gamma\| > r_{\text{ess}}(\|\gamma\| - \gamma).$$

The operator T_k is weakly compact thanks to Lemma 3.7 (ii) since k satisfies (37); see Assumption 1 and more precisely (9). Since $\mathscr{L}^\infty$ has the Dunford-Pettis property according to Theorem 3.2, we deduce from [70, Theorem 3.1] (where $\sigma_{\text{ess}}(A)$ in our setting corresponds to $\sigma_{e5}(A)$ in [70]) that $r_{\text{ess}}(\|\gamma\| - \gamma) = r_{\text{ess}}(T_k - \gamma + \|\gamma\|)$. Therefore, we get that:

$$r(T_k - \gamma + \|\gamma\|) > r_{\text{ess}}(T_k - \gamma + \|\gamma\|).$$

Hence, we can apply Theorem 3.5 (iii) with the positive operator $T_k - \gamma + \|\gamma\|$, to get the existence of a function $w \in \mathscr{L}^\infty_+ \setminus \{0\}$ such that:

$$T_k(w) - \gamma w = s(T_k - \gamma) w, \tag{43}$$

where we used $r(T_k - \gamma + \|\gamma\|) = s(T_k - \gamma + \|\gamma\|) = s(T_k - \gamma) + \|\gamma\|$ for the equality. We have shown that one can actually take $\lambda = s(T_k - \gamma)$ in Equation (39). Thus, we obtain:

$$T_{k/\gamma}(\gamma w) = T_k(w) = (\gamma + s(T_k - \gamma))w \geq \left(1 + \frac{s(T_k - \gamma)}{\|\gamma\|}\right) \gamma w.$$

According to Proposition 3.6, we conclude that:

$$R_0 = r(T_{k/\gamma}) \geq 1 + \frac{s(T_k - \gamma)}{\|\gamma\|}. \tag{44}$$

We deduce that Equation (42) holds. $\square$

The next result gives information about the behavior of R_0 when we modify the susceptibility of individuals, and will be needed below in the proof of Theorem 4.13. For $g \in \Delta$, we define $R_0(g) = r(g T_{k/\gamma})$.

Proposition 4.5 *(A continuity property of R_0). Suppose Assumption 1 holds. The function $g \mapsto R_0(g)$ defined on Δ is non-decreasing and continuous with respect to the $L^1(\mu)$ topology.*

Proof. The fact that $g \mapsto R_0(g)$ is non-decreasing is a direct consequence of Theorem 3.5 (i).

For $g \in \Delta$, the bounded operator $A_g = \hat{T}_{\mathsf{k}}$ on $L^p(\mu)$ defined in Equation (38) with the kernel $\mathsf{k}(x, y) = g(x)k(x, y)/\gamma(y)$ is compact according to Lemma 3.7 (iii). According to Lemma 3.7 (iv), we have that for all $g \in \Delta$:

$$R_0(g) = r(A_g). \tag{45}$$

Besides, the function $g \mapsto A_g$ mapping Δ to $\mathcal{L}(L^p(\mu))$ is continuous with respect to the $L^p(\mu)$ norm. We deduce from [69, Theorem 11], that the function $g \mapsto \sigma(A_g)$ from $(\Delta, \|\cdot\|_p)$ to $(\mathcal{K}(\mathbb{C}), d_{\mathrm{H}})$ is continuous, where $\mathcal{K}(\mathbb{C})$ is the set of non-empty compact subsets and d_{H} is the Hausdorff distance (see Appendix B for the definition of the Hausdorff distance). Using Lemma B.1 and then Equation (45), we get that $g \mapsto R_0(g)$ defined on $(\Delta, \|\cdot\|_p)$ is continuous. In order to conclude, we notice that the topologies induced by $L^p(\mu)$ and $L^1(\mu)$ are equal on Δ because Δ is a bounded subset of $L^\infty(\mu)$. This proves that $g \mapsto R_0(g)$ defined on $(\Delta, \|\cdot\|_1)$ is continuous. $\square$

4.2. The subcritical regime: $s(T_\kappa - \gamma) < 0$

We show here that in the subcritical regime, the solutions of Equation (25) converge exponentially fast to 0 in norm.

Theorem 4.6 *(Uniform exponential extinction). Suppose that $s(T_\kappa - \gamma) < 0$. Then, for all $c \in (0, -s(T_\kappa - \gamma))$, there exists a finite constant $\theta = \theta(c)$ such that, for all $g \in \Delta$, we have:*

$$\|\phi(t, g)\| \le \theta \|g\| \,\mathrm{e}^{-ct}. \tag{46}$$

In particular, the maximal equilibrium g^* is equal to 0 everywhere.

Proof. Recall $T_\kappa - \gamma$ is a bounded operator. For all $t \in \mathbb{R}_+$, define:

$$v(t) = \mathrm{e}^{t(T_\kappa - \gamma)} 1 = \sum_{n \in \mathbb{N}} \frac{t^n}{n!} (T_\kappa - \gamma)^n \, 1. \tag{47}$$

Since multiplication by the constant $\|\gamma\|$ commutes with $T_\kappa - \gamma$, we also have:

$$\mathrm{e}^{\|\gamma\| t} v(t) = \mathrm{e}^{t(T_\kappa - \gamma + \|\gamma\|)} 1 = \sum_{n \in \mathbb{N}} \frac{t^n}{n!} (T_\kappa - \gamma + \|\gamma\|)^n \, 1.$$

As $T_\kappa - \gamma + \|\gamma\|$ is positive, we deduce that $v(t) \ge 0$. As T_κ is positive, we deduce that:

$$v'(t) - F(v(t)) = (T_\kappa - \gamma)(v(t)) - F(v(t)) = v(t) T_\kappa(v(t)) \ge 0.$$

Thus, the following inequality holds for all $g \in \Delta$ and all $t \ge 0$:

$$0 = \partial_t\phi(t,g) - F(\phi(t,g)) \le v'(t) - F(v(t)).$$

As F is cooperative on $(1-\mathscr{L}^\infty_+)\times\mathscr{L}^\infty$ according to Proposition 2.5 (iv), we can apply Theorem 2.4 with the positive cone $K = \mathscr{L}^\infty_+$, $D_1 = 1-\mathscr{L}^\infty_+$, $D_2 = \mathscr{L}^\infty$, $a(t) = \phi(t,g)$ and $b(t) = v(t)$, to obtain that:

$$\phi(t,g) \le v(t) \quad \text{for all } t \in \mathbb{R}_+. \tag{48}$$

Besides, since $T_k - \gamma$ is a bounded operator, its growth bound (*i.e.*, the left member of the equality below) is equal to its spectral bound according to [71, Theorem I.4.1]:

$$\inf\left\{\eta\in\mathbb{R}:\ \sup_{t\in\mathbb{R}_+} \mathrm{e}^{-\eta t}\|\exp(t(T_\kappa-\gamma))\| < \infty\right\} = s(T_\kappa - \gamma). \tag{49}$$

We deduce from Equations (47), (48) and (49), that for all $c \in (0, -s(T_\kappa-\gamma))$, there exists a finite constant θ such that Equation (46) is true. In particular, $t \mapsto \phi(t,1)$ converges uniformly to 0. It then follows from Equation (31) that g^* is equal to 0 everywhere. □

4.3. *Critical regime:* $s(T_k-\gamma)=0$

We suppose here that Assumption 1 holds, so that the kernel κ has a density k, and we write T_k for T_κ. We give the main result of this section.

Theorem 4.7 *(Extinction at criticality). Suppose Assumption 1 is in force and* $s(T_k-\gamma)=0$. *Then the maximal equilibrium* g^* *is equal to* 0 *everywhere. In other words, for all* $g\in\Delta$ *and all* $x\in\Omega$, *we have that:*

$$\lim_{t\to\infty}\phi(t,g)(x) = 0.$$

Proof. Suppose, to derive a contradiction, that g^* is not equal to 0 μ-almost everywhere. We know according to Remark 2.16 that $1-g^*$ is positive everywhere. Hence, we get:

$$T_{k/\gamma}(\gamma g^*) = T_k(g^*) = \left(1+\frac{g^*}{1-g^*}\right)\gamma g^* \ge \gamma g^*. \tag{50}$$

According to Proposition 3.6, R_0 is then greater than or equal to 1.

We now prove that the inequality is strict: $R_0 > 1$. Consider the support of g^*: $A = \{x\in\Omega:\ g^*(x)>0\}$. Equation (50) remains true by replacing k by $k' = \mathbb{1}_A\, k\, \mathbb{1}_A$ (*i.e.* $k'(x,y) = \mathbb{1}_A(x)k(x,y)\mathbb{1}_A(y)$):

$$T_{k'/\gamma}(\gamma g^*) = \left(1+\frac{g^*}{1-g^*}\right)\gamma g^* \ge \gamma g^*. \tag{51}$$

Using Proposition 3.6, we get that $r(T_{k'/\gamma}) \ge 1$. Since Assumption 1 is in force, $T_{k'/\gamma}$ has a left Perron eigenvector h in $L^q_+(\mu)\backslash\{0\}$ (see Lemma 3.7 (v)). By multiplying both members of Equation (51) by h and integrating with respect to μ, we obtain:

$$(r(T_{k'/\gamma}) - 1)\,\langle h, \gamma g^*\rangle = \langle h, (g^*)^2\gamma/(1-g^*)\rangle \tag{52}$$

It is clear that $h\mathbb{1}_{A^c} = 0$. Since $h \in L^q_+(\mu)\setminus\{0\}$, we have necessarily:

$$\int_A h(x)\,\mu(\mathrm{d}x) > 0.$$

Hence, both brackets in Equation (52) are positive. Thus, we get that $r(T_{k'/\gamma}) > 1$. Using Theorem 3.5 (i) and that the operator $T_{k/\gamma} - T_{k'/\gamma}$ is positive, we deduce that $R_0 \geq r(T_{k'/\gamma}) > 1$. This is in contradiction with Proposition 4.2 which asserts that $R_0 \leq 1$ as $s(T_k - \gamma) = 0$. Thus, we obtain that μ-a.e. $g^* = 0$. We conclude using Remark 3.8. $\square$

4.4. *Supercritical regime: $s(T_k - \gamma) > 0$*

Assumption 1 is in force in this section, where we consider the case $s(T_k - \gamma) > 0$. We begin by proving that g^* is different from 0, then we show the convergence of the system to g^*.

Informally, the main idea in the following is to write the linear approximation of the dynamics (25) near the zero equilibrium: $\partial_t u = (T_k - \gamma)u$, and apply the results from Section 3 (Theorem 3.5 (iii) and Lemma 3.7) in order to identify a dominant component of the evolution near 0 as a Perron eigenvector of $T_k - \gamma$. A particularly nice feature is that this procedure yields a monotonous trajectory.

In order to control the non-linearity, we give ourselves a little room by choosing ε small enough so that $(1-\varepsilon)R_0 > 1$. Since $r((1-\varepsilon)T_{k/\gamma}) = (1-\varepsilon)r(T_{k/\gamma}) = (1-\varepsilon)R_0 > 1$, Proposition 4.2 ensures the existence of a vector $w_\varepsilon \in \mathscr{L}^\infty_+\setminus\{0\}$ and a positive real number $\lambda(\varepsilon) > 0$, such that:

$$(1-\varepsilon)T_k(w_\varepsilon) = (\gamma + \lambda(\varepsilon))w_\varepsilon. \tag{53}$$

We can take w_ε such that $\|w_\varepsilon\| < \varepsilon$. Moreover, according to Remark 4.3:

$$\int_\Omega w_\varepsilon(x)\,\mathrm{d}x > 0. \tag{54}$$

Then, we get the following proposition.

Proposition 4.8 *(Increasing trajectory). Suppose Assumption 1 is in force and that $s(T_k - \gamma) > 0$, and let w_ε be defined by (53). The trajectory starting from w_ε is monotonous: the map $t \mapsto \phi(t, w_\varepsilon)$ is non-decreasing.*

Proof. Using Equation (53) and the fact that $\|w_\varepsilon\| < \varepsilon$, we have:

$$0 \leq \lambda(\varepsilon)w_\varepsilon = (1-\varepsilon)T_k(w_\varepsilon) - \gamma w_\varepsilon \leq (1-w_\varepsilon)T_k(w_\varepsilon) - \gamma w_\varepsilon = F(w_\varepsilon).$$

This implies the stated monotony by Proposition 2.10. $\square$

Proposition 4.8 shows that the equilibrium 0 is not asymptotically stable, in the sense that we can find initial conditions arbitrarily close to 0 in norm such that $\phi(t, g)$ does not converge to 0 pointwise (note that $\|w_\varepsilon\|$ may be chosen arbitrarily small). Since $w_\epsilon \leq 1$, we get by monotony and the comparison Theorem that $w_\varepsilon \leq \phi(t, w_\varepsilon) \leq \phi(t, 1)$, which implies that $w_\varepsilon \leq g^*$. In particular we get the following corollary.

Corollary 4.9. *If Assumption 1 is in force and $s(T_k - \gamma) > 0$, then we have:*

$$\int_\Omega g^*(x)\, \mu(\mathrm{d}x) > 0.$$

We deduce from Proposition 4.8 that $t \mapsto \phi(t, w_\varepsilon)$ converges pointwise as t tends to infinity since $\phi(t, w_\varepsilon) \leq 1$ for all t. According to Proposition 2.13, the limit is an equilibrium. It is not 0 but it might be different from g^*. We will use Assumption 2 to ensure that 0 and g^* are the only equilibria. In order to prove this result, we need the following lemma.

Lemma 4.10 *(Instantaneous propagation of the infection). Suppose Assumptions 1 and 2 are in force. If $g \in \Delta$ satisfies $\int_\Omega g(x)\, \mu(\mathrm{d}x) > 0$, then, for all $t > 0$, $\phi(t, g)$ is μ-a.e. positive.*

Proof. Since the flow is order-preserving (see Proposition 2.8), it is sufficient to show the proposition for g such that $\|g\| < 1/2$. It follows from Equation (29) that:

$$\phi(t, g) \leq \|g\| + t\|T_k\|.$$

Thus, for all $t \in [0, c)$, with $c = (1 - 2\|g\|)/2\|T_k\|$ (and $c = +\infty$ if $\|T\| = 0$), we have that $\phi(t, g) < 1/2$. Now, we define the function:

$$u(t) = \mathrm{e}^{-\|\gamma\| t}\, \mathrm{e}^{t T_k/2} g.$$

Taking $c > 0$ smaller if necessary, we get $u(t) \leq \|g\|(1 + t\|T_k\|) < 1/2$ for $t \in [0, c)$. Then we get for $t \in [0, c)$:

$$\begin{aligned} u'(t) - F(u(t)) &= (T_k/2 - \|\gamma\|)u(t) - (1 - u(t))T_k u(t) + \gamma u(t) \\ &= (u(t) - 1/2)T_k u(t) - (\|\gamma\| - \gamma)u(t) \\ &\leq 0 = b'(t) - F(b(t)), \end{aligned}$$

where $b(t) = \phi(t, g)$. Using the comparison Theorem 2.4, we deduce

$$\phi(t, g) \geq u(t). \tag{55}$$

Now, we fix $t \in [0, c)$. We denote by $A = \{x \in \Omega :\ u(t)(x) > 0\}$ the support of $u(t)$. We have:

$$0 = \langle \mathbb{1}_{A^c}, u(t) \rangle = \mathrm{e}^{-\|\gamma\| t} \sum_{n \in \mathbb{N}} \frac{1}{n!} \langle \mathbb{1}_{A^c}, (tT_k/2)^n(g) \rangle .$$

This implies that $\langle \mathbb{1}_{A^c}, (tT_k/2)^n(g)\rangle = 0$ for all n, and thus that $\langle \mathbb{1}_{A^c}, T_k u(t)\rangle = 0$. We deduce that:

$$\int_{A^c \times A} k(x, y)\, \mu(\mathrm{d}x)\mu(\mathrm{d}y) = 0.$$

Since the set A contains the support of g, we get $\mu(A) > 0$. It follows from Assumption 2 that $\mu(A^c) = 0$. This means that $u(t)$ is μ-a.e. positive. Hence, from Equation (55), we get that, for $t \in [0, c)$, $\phi(t, g)$ is μ-a.e. positive. Using the semi-group property of the semi-flow this results propagate on the whole positive half-line and the result is proved. □

Remark 4.11. One can check from its proof, that Lemma 4.10 does not require the integrability condition (8) in Assumption 1 to be true.

Now we can show the following important result.

Proposition 4.12 *(Uniqueness of the endemic state). Under Assumptions 1 and 2, the maximal equilibrium g^*:*

(i) *is positive μ-a.e.,*
(ii) *is the unique equilibrium different from* 0.

Proof. From Lemma 4.10 together with Remark 3.8, we deduce that every equilibrium different from 0 is positive μ-a.e. This proves Point (i) as $\int g^* \mathrm{d}\mu > 0$ in the supercritical regime according to Corollary 4.9.

We now prove Point (ii). Let h^* be another equilibrium different from 0. Since g^* is the maximal equilibrium, we have $h^* \leq g^*$. We shall prove that h^* is equal to g^* almost everywhere. Let us define the non-negative kernel k by:

$$\mathsf{k}(x, y) = (1 - g^*(x))\, \frac{k(x, y)}{\gamma(y)} \quad \text{for } x, y \in \Omega.$$

Notice that k satisfies (37). Since $T_{\mathsf{k}}(\gamma g^*) = \gamma g^*$, we deduce from Proposition 3.6 that $r(T_{\mathsf{k}}) \geq 1$. Let $v \in L^q(\mu)_+ \backslash \{0\}$ be a left Perron vector of the operator T_{k} (given by Lemma 3.7 (v)). The kernel k satisfies Assumption 2 as k does and $1 - g^*$ is positive everywhere (see Remark 2.16). Hence, v can be chosen positive μ-a.e. according to Lemma 3.7 (vi). The following computation:

$$\langle v, \gamma g^* \rangle = \langle v, T_{\mathsf{k}}(\gamma g^*) \rangle = r(T_{\mathsf{k}})\, \langle v, \gamma g^* \rangle\,,$$

shows that $r(T_{\mathsf{k}})$ is actually equal to 1 since $\langle v, \gamma g^* \rangle > 0$. Now we compute:

$$\begin{aligned} 0 &= \langle v, F(h^*) \rangle \\ &= \langle v, T_{\mathsf{k}}(\gamma h^*) - \gamma h^* \rangle + \langle v, (g^* - h^*) T_{k/\gamma}(\gamma h^*) \rangle \\ &= \langle v, (g^* - h^*) T_k(h^*) \rangle\,, \end{aligned}$$

where we used that $\langle v, T_{\mathsf{k}} f - f\rangle = 0$ as $r(T_{\mathsf{k}}) = 1$ and v is a left Perron eigenvector. According to the first part of the proof, h^* is μ-a.e. positive. Since we have $T_k(h^*) = \gamma h^*/(1-h^*)$, the function $T_k(h^*)$ is also μ-a.e. positive. Hence g^* and h^* are equal μ-a.e. since v is μ-a.e. positive, see Lemma 3.7 (vi). This implies in particular that $T_k(h^*) = T_k(g^*)$ by Lemma 3.7 (i). We deduce that, for all $x \in \Omega$:

$$h^*(x) = T_k(h^*)(x)/(\gamma(x) + T_k(h^*)(x)) = T_k(g^*)(x)/(\gamma(x) + T_k(g^*)(x)) = g^*(x).$$

Therefore g^* is then unique equilibrium different from 0. □

Now we can prove the main result of this section on the pointwise convergence of $\phi(t, g)$. If g is μ-a.e. equal to 0, then clearly, as γ is positive, we get that $\lim_{t\to\infty} \phi(t, g) = 0$ pointwise, so we only need to consider the case where g is not μ-a.e. equal to 0.

Theorem 4.13 *(Convergence towards the endemic equilibrium). Suppose that Assumptions 1 and 2 are in force. Let $g \in \Delta$ such that $\int_\Omega g(x)\, \mu(\mathrm{d}x) > 0$. Then, we have that for all $x \in \Omega$:*

$$\lim_{t\to\infty} \phi(t, g)(x) = g^*(x).$$

Proof. By Lemma 4.10, it is enough to show the result for g μ-a.e. positive. The idea is similar to the proof of Proposition 4.8, that is, to try and find a monotonous trajectory; the difference here is that we look for a trajectory that is below $\phi(t, g)$, and we have to adapt the proof accordingly. For such a g, the functions $(1-\varepsilon) g \mathbb{1}_{g\geq\varepsilon}$ converge in $L^1(\mu)$ to g when ε goes to zero. Besides, R_0 is greater than 1 by Proposition 4.2. Hence, according to Proposition 4.5, for ε small enough, we get

$$R_0\left((1-\varepsilon)\, \mathbb{1}_{g\geq\varepsilon} T_{k/\gamma}\right) > 1.$$

By Proposition 4.2 (iii), applied to the kernel $(1-\varepsilon)\mathbb{1}_{g(x)\geq\varepsilon} k(x, y)$, there exists $w_\varepsilon \in \mathscr{L}^\infty_+ \setminus \{0\}$ and $\lambda(\varepsilon) > 0$ such that:

$$(1-\varepsilon)\, \mathbb{1}_{g\geq\varepsilon} T_k(w_\varepsilon) = (\gamma + \lambda(\varepsilon)) w_\varepsilon. \tag{56}$$

We may and will assume additionally that $\|w_\varepsilon\| \leq \varepsilon$. Since (56) implies that $w_\varepsilon(x) = 0$ when $g(x) < \varepsilon$, we know that $w_\varepsilon \leq g$. The monotony of the semi-flow (see Proposition 2.8) then implies that, for all $t \in \mathbb{R}_+$:

$$\phi(t, w_\varepsilon) \leq \phi(t, g) \leq \phi(t, 1). \tag{57}$$

Besides, we have:

$$\begin{aligned} 0 \leq \lambda(\varepsilon) w_\varepsilon &= (1-\varepsilon)\mathbb{1}_{g\geq\varepsilon} T_k(w_\varepsilon) - \gamma w_\varepsilon \\ &\leq (1-\varepsilon) T_k(w_\varepsilon) - \gamma w_\varepsilon \\ &\leq (1-w_\varepsilon) T_k(w_\varepsilon) - \gamma w_\varepsilon \\ &= F(w_\varepsilon), \end{aligned}$$

where the last inequality follows from the fact that $\|w_\varepsilon\| \le \varepsilon$. Thus, the path $t \mapsto \phi(t, w_\varepsilon)$ is non-decreasing according to Proposition 2.10. Hence, it converges pointwise to a limit $h^* \neq 0$ since $w_\varepsilon \in \mathscr{L}^\infty_+ \setminus \{0\}$. This limit has to be an equilibrium by Proposition 2.13. Since 0 and g^* are the only equilibria by Proposition 4.12, we have necessarily $h^* = g^*$. We conclude thanks to Equation (57). □

4.5. Endemic states in the critical regime

Here we show by a counter-example that the integral condition (8) is necessary to obtain the convergence towards the disease-free equilibrium in the critical regime. In the following example, the transmission kernel has a bounded density with respect to a finite measure μ and we have $\inf \gamma > 0$ and $R_0 = 1$. However, there exists a continuum of distinct equilibria.

Consider the set $\mathbb{N}^*$ equipped with some finite measure μ such that $\mu_n = \mu(\{n\}) > 0$ for all $n \in \mathbb{N}^*$. We choose γ constant equal to 1 and the kernel κ defined for $i, j \in \mathbb{N}^*$ by:

$$\kappa(i, \{j\}) = \begin{cases} \frac{2i+2}{2i-1} & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases} \tag{58}$$

Clearly Assumption 0 is satisfied. Moreover, the kernel κ admits with respect to μ the density k defined by $k(i, j) = \kappa(i, \{j\})/\mu(\{j\})$, for $i, j \in \mathbb{N}^*$. However condition (9), and thus (8) from Assumption 1, is not satisfied. Indeed, for all $q > 1$ we have:

$$\sup_{n \in \mathbb{N}^*} \int_{\mathbb{N}^*} k(x, y)^q \, \mu(\mathrm{d}y) = \sup_{n \in \mathbb{N}^*} k(n, n+1)^q \, \mu_{n+1} = \lim_{n \to \infty} \frac{(2n+1)^q}{(2n-1)^q} \, \mu_{n+1}^{1-q} = +\infty,$$

where divergence of the sequence follows from the convergence of μ_{n+1} to 0 (because μ is a finite measure). The following proposition asserts that we are in the critical regime.

Proposition 4.14. *Let κ be defined by (58), k be its density and $\gamma = 1$. We have for the reproduction number: $R_0 = r(T_{k/\gamma}) = 1$, and for the spectral bound: $s(T_k - \gamma) = 0$.*

Proof. Since γ is the function constant equal to 1, we have $s(T_k - \gamma) = R_0 - 1$ and $R_0 = r(T_k)$. We compute the spectral radius of T_k using Gelfand's formula:

$$r(T_k) = \lim_{n \to \infty} \|T_k^n\|^{1/n} = \lim_{n \to \infty} \left(\prod_{i=1}^n \frac{2i+2}{2i-1} \right)^{1/n} = 1.$$

The limit is found by applying the logarithm to the sequence and using Cesàro lemma. □

The following result shows that even if we are in the critical regime, the maximal equilibrium g^* is not equal to 0 everywhere, and there exists infinitely many distinct equilibria. For $\alpha \in [0, 1]$, we define the function g^*_α on $\mathbb{N}^*$ by $g^*_\alpha(1) = \alpha$ and for $n \in \mathbb{N}^*$:

$$g^*_\alpha(n+1) = \begin{cases} \frac{2n-1}{2n+2} \, \frac{g^*_\alpha(n)}{1 - g^*_\alpha(n)} & \text{if } g^*_\alpha(n) < 1, \\ 0 & \text{if } g^*_\alpha(n) \ge 1. \end{cases}$$

Proposition 4.15. *Let κ be defined by (58), k be its density and $\gamma = 1$.*

(i) *The equilibria of Equation (3) are $\{ g^*_\alpha : \alpha \in [0, 1/2] \}$.*
(ii) *The function $\alpha \mapsto g^*_\alpha$ defined on $[0, 1/2]$ and taking values in $\Delta \subset \mathscr{L}^\infty$ is increasing and continuous (with respect to $\|\cdot\|$). In particular, the set of equilibria is totally ordered, compact and connected.*
(iii) *The equilibrium $g^*_{1/2}$ is the maximal equilibrium and is given by $g^*_{1/2}(n) = 1/(2n)$ for $n \in \mathbb{N}^*$.*
(iv) *For every $\alpha \in (0, 1/2)$, there exists a constant c_α such that $g^*_\alpha(n) \sim c_\alpha n^{-3/2}$. Moreover, the map $\alpha \to c_\alpha$ is strictly increasing and continuous on $[0, 1/2)$, with the convention $c_0 = 0$.*

Proof. We start by remarking that for $\alpha = 1/2$, the induction may be solved explicitly, so that $g^*_{1/2}(n) = 1/(2n)$. Similarly $g^*_0(n) = 0$.

We prove property (ii) first. Let Γ denote the function $\alpha \mapsto g^*_\alpha$ defined on $[0, 1/2]$ and taking values in $\mathscr{L}^\infty$. Since the function $x \mapsto \lambda x/(1 - x)$ is increasing on $[0, 1)$ for all $\lambda > 0$, we deduce by induction that $0 \le g^*_\alpha(n) < g^*_\beta(n) \le g^*_{1/2}(n)$ for all $0 \le \alpha < \beta \le 1/2$ and $n \in \mathbb{N}^*$. As g^*_0 and $g^*_{1/2}$ both belong to Δ, we deduce that Γ takes values in Δ by monotonicity. It is also immediate to check that the function Γ is continuous for the pointwise convergence in Δ. Since $\lim_{n\to\infty} \sup_{\alpha\in[0,1/2]} g^*_\alpha(n) = \lim_{n\to\infty} g^*_{1/2}(n) = 0$, this continuity also holds with respect to the uniform convergence in Δ. This proves property (ii).

We now prove property (i). It is clear that if h^* is an equilibrium, then $h^*(n) < 1$ for all $n \in \mathbb{N}^*$ thanks to Remark 2.16 and by the definition of the kernel κ we have that:

$$h^*(n+1) = \frac{2n-1}{2n+2}\,\frac{h^*(n)}{1-h^*(n)} \quad \text{for all } n \in \mathbb{N}^*. \tag{59}$$

This readily implies that $h^\star = g^*_\alpha$ where $\alpha = h^*(0)$, so that the only possible equilibria are the g^*_α for $\alpha \in [0, 1]$.

If $\alpha \in [0, 1/2]$, g^*_α is indeed an equilibrium as $g^*_\alpha(n) \le g^*_{1/2}(n) = 1/(2n)$ and g^*_α solves (59). On the contrary, since $g^*_1(1) = 1$, the function g^*_1 is not an equilibrium.

Let $\alpha \in (1/2, 1)$. We shall now prove by contradiction that there exists $n \in \mathbb{N}^*$ such that $g^*_\alpha(n) \ge 1$. Let us assume that $g^*_\alpha(n) < 1$ for all $n \in \mathbb{N}^*$. Arguing as in the first part of the proof, we get $g^*_\alpha(n) > g^*_{1/2}(n)$ for all $n \in \mathbb{N}^*$. Thus the sequence $v = (v_n : n \in \mathbb{N}^*)$ with $v_n = 2ng^*_\alpha(n)$ satisfies the following recurrence for $n \in \mathbb{N}^*$:

$$v_{n+1} = v_n \frac{2n-1}{2n-v_n} \quad \text{and} \quad 1 < v_n < 2n.$$

We deduce that the sequence v is increasing, and thus $v_{n+1} \ge v_n \frac{2n-1}{2n-2\alpha}$, as $v_1 = 2\alpha$. We deduce that $v_n \ge c\, n^{\alpha-1/2}$ for some positive constant c. This in turn implies that $v_{n+1} \ge v_n \frac{2n-1}{2n-c\,n^{\alpha-1/2}}$ and thus $v_n \ge c' \exp(c'' n^{\alpha-1/2})$ for some positive constants c' and c''. This contradicts the fact that $v_n < 2n$ for $n \in \mathbb{N}^*$. As a conclusion, there exists $n \in \mathbb{N}^*$ such that $g^*_\alpha(n) \ge 1$. This implies that g^*_α can not be an equilibrium. This ends the proof of property (i).

We end the proof of property (iii). We have already computed $g^*_{1/2}$. We deduce from properties (i) and (ii) that $g^*_{1/2}$ is the maximal equilibrium.

Finally, let us prove the asymptotic result (iv). The asymptotics of $g^*_\alpha(n)$ is obtained by starting from an easy bound on its decay, and then refining it by plugging it back into the induction relation (59).

Linear decay. Since $\alpha \le 1/2$, we already know that $g^*_\alpha(n) \le g^*_{1/2}(n) = 1/(2n)$. We can prove a little bit better. The sequence $w = (w_n : n \in \mathbb{N}^*)$ defined by $w_n = ng^*_\alpha(n)/\alpha$ satisfies

$$w_{n+1} = \frac{2n-1}{2n - 2\alpha w_n} w_n.$$

Since $2\alpha w_n = 2ng^*_\alpha(n) \le 1$, the sequence w is non-increasing. In particular $w_n \le w_1 = 1$, so we deduce that:

$$g^*_\alpha(n) \le \frac{\alpha}{n}\cdot \tag{60}$$

Sublinear decay. Let q_n be the quotient $q_n = g^*_\alpha(n+1)/g^*_\alpha(n)$. By the recurrence relation and the linear bound (60), we get successively

$$q_n \le \frac{1 - 1/(2n)}{1 + 1/n} \frac{1}{1 - \alpha/n} \le 1 - \left(\frac{3}{2} - \alpha\right)\frac{1}{n} + O(n^{-2}) \quad \text{and} \quad \log(q_n) \le -(\frac{3}{2} - \alpha)\frac{1}{n} + r_n,$$

where $r_n = O(n^{-2})$. Summing the terms from 1 to $n-1$, we get

$$\log(g^*_\alpha(n)) - \log(g^*_\alpha(1)) \le -(3/2 - \alpha)\log(n) + O(1).$$

Therefore, there exists a constant C (that may depend on α) such that

$$g^*_\alpha(n) \le \frac{C}{n^{3/2-\alpha}}\cdot \tag{61}$$

Optimal decay. We come back to the recurrence relation, and use the sublinear bound (61) on the term $1 - g^*_\alpha(n)$ that appears in the denominator. This yields successively

$$q_n = \left(1 - \frac{3}{2n} + O(n^{-2})\right)\left(1 + O(n^{-(3/2-\alpha)})\right) = 1 - \frac{3}{2n} + O(n^{-(3/2-\alpha)}),$$

which gives $\log(q_n) = -\frac{3}{2n} + r_n$ where $r_n = O(n^{-(3/2-\alpha)})$. As $\sum_m r_m$ is finite, summing from 1 up to $n-1$ and taking the exponential yields

$$g^*_\alpha(n) = \alpha \exp\left(-(3/2)\ln(n) + O(1)\right).$$

In other words we obtain, as claimed, the existence of a (non-explicit) constant c_α such that $g^*_\alpha(n) \sim c_\alpha n^{-3/2}$.

Properties of c_α. Notice that for $\alpha \in [0, 1/2)$ the limit $d_\alpha = \lim_{n\to\infty} \prod_{k=1}^n (1 - g^*_\alpha(k))^{-1}$ exists and is positive. Let $0 \le \alpha \le \beta < 1/2$ and set $D_n = g^*_\beta(n) - g^*_\alpha(n)$ for $n \ge 1$. We have $D_{n+1} = \frac{2n-1}{2n+2}(1 - g^*_\beta(n))^{-1}(1 - g^*_\alpha(n))^{-1} D_n$. So we deduce that:

$$D_n = (\beta - \alpha) \left(\prod_{k=1}^{n-1} \frac{2k-1}{2k+2} \right) \left(\prod_{k=1}^{n-1} (1 - g_\alpha^*(k))^{-1} \right) \left(\prod_{k=1}^{n-1} (1 - g_\beta^*(k))^{-1} \right).$$

Using that $\prod_{k=0}^{n-1} \frac{2k-1}{2k+2} \sim C\, n^{-3/2}$ for some finite constant $C > 0$, we deduce that

$$c_\beta - c_\alpha = C(\beta - \alpha) d_\alpha d_\beta.$$

This gives the strict monotonicity of the map $\alpha \mapsto c_\alpha$. Then, use that $d_\alpha \le d_\beta \le d_{\beta'} < +\infty$ for some $\beta' \in (\beta, 1/2)$ to get the continuity. □

We are not able to describe entirely the basins of attraction of each equilibrium. However, the asymptotic behavior in n of the starting point g tells us quite a lot.

Proposition 4.16. *For all $g \in \Delta$, and for all $\alpha \in (0, 1/2)$, we have:*

$$\limsup_n n^{3/2} g(n) \le c_\alpha \implies \limsup_{t \to \infty} \phi(t, g) \le g_*^\star,$$
$$\liminf_n n^{3/2} g(n) \ge c_\alpha \implies \liminf_{t \to \infty} \phi(t, g) \ge g_*^\star.$$

In particular, we have:

$$\limsup_n n^{3/2} g(n) = 0 \implies \phi(t, g) \to 0,$$
$$\liminf_n n^{3/2} g(n) = \infty \implies \phi(t, g) \to g_{1/2}^*.$$

Proof. Since k is upper-triangular, the long-time behavior of the dynamic does not depend on the first terms of the initial condition. Indeed, for $n \ge 2$, consider the subspace $E_n = \{ g \in \mathscr{L}^\infty : \ g(p) = 0 \text{ for } 1 \le p < n \}$ of functions whose first $n-1$ terms are 0. Denote by P_n the canonical projection from $\mathscr{L}^\infty$ on E_n. For $n \ge 2$ and $g \in \Delta$, we have:

$$P_n \phi(t, g) = P_n \left(\phi(t, P_n(g)) \right). \tag{62}$$

Let us denote by $\preceq$ the partial order defined by $g \preceq h$ if there exists $n \ge 2$ such that $P_n(g) \le P_n(h)$.

Suppose that $\limsup n^{3/2} g(n) \le c_\alpha$. Since $\alpha \to c_\alpha$ is strictly increasing, for any $\alpha < \beta < 1/2$, the asymptotics of g and g_β^* imply that $g \preceq g_\beta^*$. Since the flow is order-preserving, this entails $\limsup \phi(t, g) \le g_\beta^*$. This inequality holds for all $\beta > \alpha$: we get the conclusion by continuity of the map $\Gamma : \alpha \to g_\alpha^*$. The proof of the other implication is similar. □

4.6. Uniform convergence

In the subcritical case, Theorem 4.6 shows an exponentially fast convergence, in the uniform norm. By contrast, the convergence results in the critical and supercritical case from Sections 4.3 and 4.4 only hold pointwise.

In the next result, we show how to recover a form of uniformity; in particular we recover uniform convergence in the particular case where $\inf \gamma > 0$.

Theorem 4.17. *Suppose that Assumption 1 and 2 are in force and let $A \in \mathscr{F}$. If γ is bounded away from 0 on A, that is:*

$$\inf_{x \in A} \gamma(x) > 0,$$

then, for $g \in \Delta$, with positive integral if $g^* \neq 0$, we have:

$$\lim_{t \to \infty} \sup_{x \in A} |\phi(t, g)(x) - g^*(x)| = 0.$$

Proof. Set $m = \inf_{x \in A} \gamma(x)$. Let us first study the convergence of the trajectory starting from 1. For $s \in \mathbb{R}_+$, we have:

$$\begin{aligned}
\partial_t(\phi(s,1) - g^*) &= F(\phi(s,1)) - F(g^*) \\
&\leq (1 - \phi(s,1)) T_k(\phi(s,1) - g^*) - \gamma(\phi(s,1) - g^*) \\
&\leq T_k(\phi(s,1) - g^*) - \gamma(\phi(s,1) - g^*) \\
&\leq M \, \|\phi(s,1) - g^*\|_p - \gamma(\phi(s,1) - g^*),
\end{aligned}$$

where we used that T_k is positive for the second inequality and Hölder inequality for the last with $M = \sup_{x \in \Omega} \left(\int_\Omega k(x,y)^q \, \mu(\mathrm{d}y)\right)^{1/q} < \infty$. For $s \in \mathbb{R}_+$, set $v_s = \mathrm{e}^{ms}(\phi(s,1) - g^*)$. Notice that $v_s \geq 0$ and that $\partial_t v_s(x) \leq M \, \|v_s\|_p$ for $x \in A$. Integrating for $s \in [0,t]$, we deduce that for $x \in A$:

$$\begin{aligned}
0 \leq (\phi(t,1) - g^*)(x) &\leq \mathrm{e}^{-mt}(1 - g^*) + M \int_0^t \mathrm{e}^{-m(t-s)} \|\phi(s,1) - g^*\|_p \, \mathrm{d}s \\
&\leq \mathrm{e}^{-mt} + M \int_0^t \mathrm{e}^{-ms} \|\phi(t-s,1) - g^*\|_p \, \mathrm{d}s.
\end{aligned}$$

Note the right hand-side does not depend on x. As $\phi(s,1)$ converges pointwise to g^* (see Equation (31)) and is bounded by 1, using the dominated convergence theorem, we deduce that the right hand-side goes to 0 as t goes to infinity. So, we obtain that:

$$\lim_{t \to +\infty} \sup_{x \in A} |\phi(t,1)(x) - g^*(x)| = 0. \tag{63}$$

If $g^* = 0$, use that $0 \leq \phi(t,g) \leq \phi(t,1)$ for all $g \in \Delta$ and $t \in \mathbb{R}_+$ to conclude.

If g^* is non zero (which corresponds to the super-critical case), consider a function $f \leq g$ with positive integral such that $f \leq g^*$. By monotonicity of the flow, this implies that $0 \leq g^* - \phi(s,f)$ for all $s \in \mathbb{R}_+$. Arguing similarly as above, we get for $s \in \mathbb{R}_+$:

$$\partial_t(g^* - \phi(s,f)) \leq M \, \|g^* - \phi(s,f)\|_p - \gamma(g^* - \phi(s,f)).$$

Using that $\phi(s,f)$ converges pointwise to g^* (see Theorem 4.13), we similarly get that

$$\lim_{t\to+\infty} \sup_{x\in A} |\phi(t, f)(x) - g^*(x)| = 0. \tag{64}$$

Then, use the monotonicity of the flow which implies that $\phi(t, f) \leq \phi(t, g) \leq \phi(t, 1)$ for $f \leq g \leq 1$ as well as (63) and (64) to conclude. □

5. Vaccination model

5.1. Infinite-dimensional models

We write an infinite-dimensional model with two goals in mind: take into account the heterogeneity in the transmission of the infectious disease, in the spirit of (3), and model the effect of vaccination by generalizing Equations (13) and (14), allowing even for different types of vaccine. Recall that the measurable space $(\Omega, \mathscr{F})$ represents the features of the individuals in a given population, the finite measure μ describes the size of the population and its sub-groups, and the number $\gamma(x)$ is the recovery rate of individuals with feature $x \in \Omega$. The transmission kernel κ describes the way the disease is spread among the population without vaccination.

Suppose that we have different vaccines or treatments available that we can give to individuals in order to fight the disease upstream. The set of vaccines is represented by a set Σ which is finite in practice. We endow Σ with a σ-field $\mathscr{G}$. We are also given two measurable functions $e, \delta : \Omega \times \Sigma \to [0, 1]$. For both models, the number $\delta(x, \xi)$ is the relative reduction of infectiousness for people with feature x vaccinated by the vaccine ξ. The coefficient $e(x, \xi)$ is the efficacy of vaccine ξ given on individuals with feature x. We encode the absence of vaccination by a particular type of vaccine $\xi_0 \in \Sigma$. This vaccination has no efficacy upon the individuals: $e(x, \xi_0) = 0$ and $\delta(x, \xi_0) = 0$ for all $x \in \Omega$. We define a vaccination policy as a non-negative kernel $\eta : \Omega \times \mathscr{G} \to [0, 1]$. The probability for an individual with feature type x to be vaccinated by a vaccine in the measurable set $A \in \mathscr{G}$ under the policy η is equal to $\eta(x, A)$. The recovery rate can be affected by the vaccine, and in this case γ is then a non-negative measurable function defined on $\Omega \times \Sigma$, with $\gamma(x, \xi)$ the recovery rate of individuals with feature x and vaccine ξ. The number $u(t, x, \xi)$ is the probability for an individual with feature x which has been inoculated by the vaccine ξ to be infected at time t. The total number of infected individuals at time t is therefore given by:

$$\int_{\Omega\times\Sigma} u(t, x, \xi)\, \eta(x, \mathrm{d}\xi) \mu(\mathrm{d}x). \tag{65}$$

5.1.1. The leaky vaccination mechanism

In this setting, $e(x, \xi)$ denotes the leaky vaccine efficacy of $\xi \in \Sigma$ on an individual with feature x, *i.e.*, the relative reduction in the transmission rate. We generalize Equation (13) to get the following infinite dimensional evolution equation:

$$\begin{aligned}\partial_t u(t, x, \xi) &= -\gamma(x, \xi)\, u(t, x, \xi) \\ &\quad + (1 - u(t, x, \xi))(1 - e(x, \xi)) \int_{\Omega\times\Sigma} (1 - \delta(y, \zeta)) u(t, y, \zeta) \kappa(x, \mathrm{d}y) \eta(y, \mathrm{d}\zeta).\end{aligned} \tag{66}$$

The evolution Equation (66) can be seen as the SIS evolution Equation (3) on an extended feature space:

- the feature $\boldsymbol{x} = (x, \xi)$ lives in $\boldsymbol{\Omega} = \Omega \times \Sigma$ endowed with the σ-field $\mathscr{F} \otimes \mathscr{G}$,
- the recovery rate is given by $\boldsymbol{\gamma}(\boldsymbol{x}) = \gamma(x, \xi)$,
- the extended transmission kernel is given by:

$$\boldsymbol{\kappa}^a(\boldsymbol{x}, \mathrm{d}\boldsymbol{y}) = (1 - e(x, \xi))(1 - \delta(y, \zeta))\kappa(x, \mathrm{d}y)\eta(y, \mathrm{d}\zeta). \tag{67}$$

Remark 5.1. In the leaky mechanism, we suppose that the vaccine acts directly on the susceptibility and the infectiousness of the individuals. Protective gears (like respirators or safety glasses) which are designed to protect the wearer from absorbing airborne microbes or transmitting them have a similar effect. Hence, Equation (66) is not limited to vaccination and can also be used as a model for distribution of equipment in the population.

5.1.2. The all-or-nothing mechanism

In this setting, $e(x, \xi)$, is defined as the probability to immunize completely the individual with feature x to the disease with vaccine ξ. We generalize Equation (14) to get the following infinite dimensional evolution equation:

$$\begin{aligned}\partial_t u(t, x, \xi) &= -\gamma(x, \xi)\, u(t, x, \xi) \\ &\quad + (1 - e(x, \xi) - u(t, x, \xi)) \int_{\Omega\times\Sigma} (1 - \delta(y, \zeta))u(t, y, \zeta)\kappa(x, \mathrm{d}y)\eta(y, \mathrm{d}\zeta).\end{aligned} \tag{68}$$

The probability $v(t, x, \xi) = u(t, x, \xi)/(1 - e(x, \xi))$ for an individual with feature x which has not been vaccinated by the inoculation of vaccine ξ to be infected at time t satisfies the following equation:

$$\begin{aligned}\partial_t v(t, x, \xi) &= -\gamma(x)v(t, x, \xi) \\ &\quad + (1 - v(t, x, \xi)) \int_{\Omega\times\Sigma} (1 - \delta(y, \zeta))v(t, y, \zeta)(1 - e(y, \zeta))\kappa(x, \mathrm{d}y)\eta(y, \mathrm{d}\zeta).\end{aligned} \tag{69}$$

As before, the evolution Equation (69) can be seen as the SIS evolution Equation (3) on the same extended feature space $\boldsymbol{\Omega} = \Omega \times \Sigma$, still endowed with the σ-field $\mathscr{F} \otimes \mathscr{G}$, with the same recovery rate $\boldsymbol{\gamma}(\boldsymbol{x}) = \gamma(x, \xi)$, but the transmission kernel now reads:

$$\boldsymbol{\kappa}^\ell(\boldsymbol{x}, \mathrm{d}\boldsymbol{y}) = (1 - e(y, \zeta))(1 - \delta(y, \zeta))\kappa(x, \mathrm{d}y)\eta(y, \mathrm{d}\zeta). \tag{70}$$

Notice the difference between the evolution Equation (66) for leaky mechanism and the evolution Equation (69) for the all-or-nothing mechanism is that $e(y, \zeta)$ in (69) (or in the kernel $\boldsymbol{\kappa}^a$ from (70)) is replaced by $e(x, \xi)$ in (66) (or in the kernel $\boldsymbol{\kappa}^\ell$ from (67)).

5.2. Discussion on the basic reproduction number

Suppose that Assumption 1 is in force. Then, we can define a new basic reproduction number for the vaccination models. We consider the following bounded operators on $\mathscr{L}^\infty(\Omega \times \Sigma)$:

$$T(g)(x,\xi) = \int_{\Omega\times\Sigma} (1-\delta(y,\zeta))g(y,\zeta)\,\frac{\kappa(x,\mathrm{d}y)}{\gamma(y,\zeta)}\,\eta(y,\mathrm{d}\zeta),$$

$$M(g)(x,\xi) = (1-e(x,\xi))g(x,\xi).$$

Following Section 4.1, the all-or-nothing vaccination reproduction number $R_0^a(\eta)$ associated to Equation (69) and vaccine policy η is:

$$R_0^a(\eta) = r(TM), \tag{71}$$

where we recall that r stands for the spectral radius. For the leaky vaccination, the basic reproduction number associated to Equation (66) and vaccine policy η is:

$$R_0^\ell(\eta) = r(MT). \tag{72}$$

In [28], the authors already remarked that the two vaccination mechanisms actually lead to the same basic reproduction number for the one-group models. This result also holds in the infinite-dimension SIS model. Notice that Assumption 1 insures that those two basic reproduction numbers are well defined.

Proposition 5.2. *We assume Assumption 1 holds. Let η be a vaccination policy. The basic reproduction number for the leaky vaccination and the for the all-or-nothing vaccination are the same:*

$$R_0^\ell(\eta) = R_0^a(\eta).$$

Proof. Thanks to the definition of the spectral radius (34) and the basic reproduction numbers defined in (71) and (72), the result is a direct consequence of the following equality on the spectra:

$$\sigma(MT)\cup\{0\} = \sigma(TM)\cup\{0\}.$$

We prove this later equality by following [72, Appendix A1]. Let $\lambda \in \mathbb{C}\backslash(\sigma(MT)\cup\{0\})$. By definition, there exists a bounded operator A on $\mathscr{L}^\infty(\Omega\times\Sigma)$ such that:

$$A(\lambda\mathrm{Id} - MT) = (\lambda\mathrm{Id} - MT)A = \mathrm{Id},$$

where Id is the identity operator. Then, one can check easily that $\lambda^{-1}(\mathrm{Id} + TAM)$ is the inverse of $\lambda\mathrm{Id} - TM$, whence $\lambda \in \mathbb{C}\backslash(\sigma(TM)\cup\{0\})$. This gives that $\sigma(TM)\cup\{0\}) \subset \sigma(MT)\cup\{0\})$. The other inclusion is proved similarly. $\square$

5.3. The perfect vaccine

The simplest case is a situation where there is only one vaccine with complete efficacy on every individual: $\Sigma = \{\xi_0, \xi_1\}$ with $e(x, \xi_1) = 1$ and $\delta(x, \xi_1) = 1$ for all $x \in \Omega$. Recall that ξ_0 corresponds to the absence of vaccine. For simplicity, we denote by $\eta^0(x) = \eta(x, \{\xi_0\})$ the probability for (or the proportion of) individuals of type $x \in \Omega$ which are not vaccinated. We assume for simplicity that initially no vaccinated individuals are infected, that is $u(0, x, \xi_1) = 0$. Since individuals that have been vaccinated are fully immunized, we have $u(t, x, \xi_1) = 0$ for all x and t. The only equation that matters is the one on $u^0(t, x) = u(t, x, \xi_0)$ which represents the proportion of unvaccinated individuals that are infected. For both mechanisms (all-or-nothing and leaky vaccination), the evolution equation of u^0 writes:

$$\partial_t u^0(t, x) = (1 - u^0(t, x)) \int_\Omega u^0(t, y)\eta^0(y)\kappa(x, \mathrm{d}y) - \gamma(x)u^0(t, x). \tag{73}$$

We shall use this formulation in a future work to find optimal vaccination policies for a given cost.

6. Limiting contacts within the population

Motivated by the recent lockdown policies taken by many countries all around the world to slow down the propagation of Covid-19 in 2020, we propose to investigate the possible impact on our SIS model of the limitations of contacts within the population. We consider the case where κ takes the form of Example 1.3:

$$\kappa_W(x, \mathrm{d}y) = \beta(x)W(x, y)\theta(y)\,\mu(\mathrm{d}y),$$

where β is the susceptibility function, θ is the infectiousness function, μ is a probability measure on the space Ω of features of the individuals and the graphon W represents the initial graph of the contacts between individuals of the population (recall that $W(x, y) = W(y, x) \in [0, 1]$ is the probability that x and y are connected and can be also seen as the density of contact between the individuals with features x and y). In order to stress the dependence in W, we write $R_0(W) = r(T_{\kappa_W/\gamma})$ the corresponding basic reproduction number and ϕ_W the semi-flow (28) associated to $F = F_W$ in (24) given by $F_W(g) = (1 - g)T_{\kappa_W}(g) - \gamma g$. We model the impact of a policy which reduces the contacts between the individuals, by a new graph of contact given by a new graphon W'. We say that W' is a perfect lockdown with respect to W if:

$$W'(x, y) \le W(x, y), \qquad \forall x, y \in \Omega. \tag{74}$$

Intuitively x and y have a lesser probability to be connected in the graphon W' than in the graphon W. We get the following intuitive result as a direct application of Theorem 3.5 (i) and Corollary 2.4.

Proposition 6.1 *(Perfect lockdown). Assume that β and θ are bounded and γ is bounded away from 0. If W' is a perfect lockdown with respect to W then $R_0(W') \le R_0(W)$ and $\phi_{W'}(t, g) \le \phi_W(t, g)$ for all initial condition $g \in \Delta$.*

However, assuming that all the contacts within the population are reduced might be unrealistic (*e.g.* people can have stronger contacts with their family in lockdown). Instead, we can suppose as a weaker condition, that each individual reduces the average number of contacts he has. Recall (6) for the definition of the degree $\deg_W(x)$ of an individual $x \in \Omega$ (*i.e.* the average number of his contacts) and the mean degree d_W over the population for a graphon W as:

$$\deg_W(x) = \int_\Omega W(x, y)\, \mu(\mathrm{d}y) \quad \text{and} \quad \mathrm{d}_W = \int_\Omega \deg_W(x)\, \mu(\mathrm{d}x) = \int_{\Omega^2} W(x, y)\, \mu(\mathrm{d}y)\, \mu(\mathrm{d}x).$$

Recall that $\|\cdot\|_1$ is the usual $L^1(\mu)$ norm. The following lemma bounds the basic reproduction number with the supremum and the mean degree of the graphon.

Lemma 6.2. *Let W be a graphon. Assume that β and θ/γ are bounded. We have that:*

$$\frac{1}{\|\gamma/\beta\theta\|_1} \mathrm{d}_W \le R_0(W) \le \|\beta\theta/\gamma\| \sup_{x\in\Omega} \deg_W(x). \tag{75}$$

Proof. Recall T_k is the operator defined by (22) with $\kappa(x, \mathrm{d}y) = \mathsf{k}(x, y)\, \mu(\mathrm{d}y)$. Let $M(v)$ be the operator corresponding to the multiplication by the function v. We have:

$$\begin{aligned}
R_0(W) &= r(M(\beta)\, T_W\, M(\theta/\gamma)) \\
&= r(M(\beta\theta/\gamma)\, T_W) \\
&\le \|M(\beta\theta/\gamma)\, T_W\| \\
&= \sup_{x\in\Omega} \frac{\beta(x)\theta(x)}{\gamma(x)} \int_\Omega W(x, y)\, \mu(\mathrm{d}y) \\
&\le \|\beta\theta/\gamma\| \sup_{x\in\Omega} \deg_W(x),
\end{aligned}$$

where we used the definition of the basic reproduction number (12) for the first equality, arguments similar as in the proof of Proposition 5.2 for the second, and the (third) definition of the spectral radius (34) for the first inequality.

Using similar arguments, we have:

$$R_0(W) = r(M(\beta)\, T_W\, M(\theta/\gamma)) = r\left(M(v)\, T_W\, M(v)\right),$$

with $v = \sqrt{\beta\theta/\gamma}$. Recall notations from Lemma 3.7, and notice that $M(v)\, T_W\, M(v) = T_\mathsf{k}$ is a bounded integral operator on $\mathscr{L}^\infty$ associated to the symmetric kernel $\mathsf{k}(x, y) = v(x)W(x, y)v(y)$. According to Lemma 3.7 (iv) with $q = p = 1/2$ and $\hat{T}_\mathsf{k}$ the integral operator on $L^2(\mu)$ with the same kernel k, defined in (38), we get $R_0(W) = r(\hat{T}_\mathsf{k})$. The operator $\hat{T}_\mathsf{k}$ is self-adjoint, as k is symmetric, and compact according to (iii). Thanks to the Courant-Fischer-Weyl min-max principle, we obtain:

$$R_0(W) = r(\hat{T}_\mathsf{k}) = \sup_{g\in L^2(\mu)\setminus\{0\}} \frac{\langle M(v)\, g,\, T_W\, M(v)\, g\rangle}{\langle g, g\rangle}.$$

Taking $g = 1/v$, we get $M(v)g = 1$ and thus:

$$R_0(W) \geq \frac{\langle 1, T_W 1\rangle}{\|\gamma/\beta\theta\|_1} = \frac{d_W}{\|\gamma/\beta\theta\|_1}.$$

This ends the proof of Lemma 6.2. □

We deduce from Lemma 6.2 the following result for a lockdown policy W' for which the degree of each individual is less than the average degree of the initial graphon W.

Corollary 6.3 *(Partial lockdown). Assume that β and θ/γ are bounded. If W' is a partial lockdown of W, that is:*

$$\sup_{x\in\Omega} \deg_{W'}(x) \leq C \mathrm{d}_W \quad \textit{with} \quad C = \frac{1}{\|\beta\theta/\gamma\| \, \|\gamma/\beta\theta\|_1}, \tag{76}$$

then we have $R_0(W') \leq R_0(W)$.

In the general case, we have $C \leq 1$. But, if the functions β, θ and γ are constants (or simply if $\beta\theta/\gamma$ constant), then we have $C = 1$ since μ is a probability measure.

Remark 6.4. Suppose that β, θ and γ are constants (or that $\beta\theta/\gamma$ is constant). Inequality (75) shows that the graphon W which corresponds to a minimal basic reproduction number $R_0(W)$, when the mean degree d_W is fixed, say equal to p, is any graphon with constant degree equal to p, that is $\deg_W(x) = p$ for all $x \in \Omega$. We then deduce from Lemma 6.2 that $R_0(W) = p\beta\theta/\gamma$.

This is in particular the case for the constant graphon $W = p \in [0, 1]$. According to Example 1.2(i), this corresponds to the one dimensional SIS model (1).

This is also the case for the geometric graphon, where the probability of edges between x and y depends only on the distance between x and y. Keeping notations from Example 1.2(iii), we consider the population uniformly spread on the unit circle: $\Omega = [0, 2\pi]$ and $\mu(\mathrm{d}x) = \mathrm{d}x/2\pi$, and the graphon W_f defined by $W_f(x, y) = f(x - y)$ for $x, y \in \Omega$, where f is a measurable non-negative function defined on $\mathbb{R}$ which is bounded by 1 and 2π-periodic. Let $p = (2\pi)^{-1} \int_{[0,2\pi]} f(y)\,\mathrm{d}y$. We have: $\deg_W(x) = \mathrm{d}_W = p$; the basic reproduction number $R_0(W_f) = p\beta\theta/\gamma$ and the maximal equilibrium $g^* = \max(0, 1 - R_0^{-1})$. Furthermore, the graphon W_f minimizes the basic reproduction number among all graphons with mean degree p. It is interesting to notice that $R_0(W_f)$ does not depend on the support of f or even on $\sup\{|r| : r \in [-\pi, \pi] \text{ and } f(r) > 0\}$, which can be seen as the maximal contamination distance from an infected individual.

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Appendix A. Proof of Theorem 2.4

We use notation from Section 2.1 and let X be a Banach space. Let us first recall a few definitions and classical properties of ODEs. Let $a > 0$. We consider a function $G: [0, a) \times X \to X$. We suppose that G is locally Lipschitz in the second variable, that is: for all $(t, x) \in [0, a) \times X$, there exist $\eta = \eta(t, x) > 0$, $L = L(t, x) > 0$ and a neighborhood U_x of x such that $\|G(s, y) - G(s, z)\| \le L\|y - z\|$ for all $s \in [0, a) \cap [t, t + \eta]$ and $y, z \in U_x$. With this assumption over G, the Picard–Lindelöf theorem ensures the existence of $0 < b \le a$ and a continuously differentiable function y from $J = [0, b)$ to X which is the unique solution of the Cauchy problem:

$$\begin{cases} y'(t) = G(t, y(t)) & t \in J, \\ y(0) = y_0, \end{cases} \tag{A.1}$$

where $y_0 \in X$ is the so-called initial condition (see [58, Section 1.1]). A solution y defined on an interval $[0, b)$ is said to be maximal if there is no solution of Equation (A.1) defined on $[0, c)$ with $c > b$. A solution is said to be global if it is defined on $[0, a)$.

Global existence, existence and theorems on differential inequalities are intimately connected with the flow invariance of certain subsets in the domain of G, *i.e.*, the question whether every solution starting in D remains in D as long as it exists. We recall the definition of flow invariance given in [58, Section 5].

Definition A.1 *(Forward invariance). A set $D \subset X$ is said to be forward invariant with respect to G if the maximal solution (y, J) of the Cauchy problem (A.1) takes values in D for $t \in J$ provided that $y_0 \in D$.*

In most applications, the set D owns a structure which make the forward invariance easier to show. For instance, when D is the translation of a cone, the forward invariance is implied by the following condition:

Theorem A.2. *Let $G: [0, a) \times X \to X$ be locally Lipschitz in the second variable. Let K be a proper cone of X with non-empty interior and $y \in X$. If for all $(x, t) \in \partial K \times [0, a)$ and for all $x^\star \in K^\star$ such that $\langle x^\star, x\rangle = 0$, we have: $\langle x^\star, G(t, y + x)\rangle \ge 0$, then $y + K$ is forward invariant with respect to G.*

Before proving this result let us state two lemmas. Let X be a Banach space. For $x \in X$ and $D \subset X$, we denote by $\rho(x, D)$ the distance between x and the set D:

$$\rho(x, D) = \inf_{y \in D} \|x - y\|. \tag{A.2}$$

Let $a > 0$ and $G: [0, a) \times X \to X$ be a locally Lipschitz function with respect to the second variable. Recall Definition A.1 of a forward invariant set with respect to G. The following result appears in [58, Theorem 5.2].

Lemma A.3. *Let D be a closed convex set with non-empty interior. Suppose that G satisfies:*

$$\lim_{\lambda \to 0^+} \frac{1}{\lambda} \rho(x + \lambda G(t, x), D) = 0, \qquad \forall (t, x) \in (0, a) \times \partial D. \tag{A.3}$$

Then D is forward invariant with respect to G.

If the set D is a proper cone, the following equivalence enables to establish (A.3) more easily. It is a consequence of [58, Lemma 4.1] and [58, Example 4.1.ii]

Lemma A.4. *Let K be a proper cone and let $x \in \partial K$ and $z \in X$. The following conditions are equivalent:*

(i) $\lim_{\lambda \to 0^+} \lambda^{-1} \rho(x + \lambda z, K) = 0$.
(ii) *For all $x^\star \in K^\star$ such that $\langle x^\star, x\rangle = 0$, we have $\langle x^\star, z\rangle \geq 0$.*

We have now all the tools to prove Theorem A.2.

Proof of Theorem A.2. Let $y \in X$. We assume that, for all $(x, t) \in \partial K \times [0, a)$ and for all $x^\star \in K^\star$ such that $\langle x^\star, x\rangle = 0$, we have: $\langle x^\star, G(t, y + x)\rangle \geq 0$. According to Lemma A.4, we obtain:

$$\lim_{\lambda \to 0^+} \lambda^{-1} \rho(x + \lambda G(t, y + x), K) = 0,$$

for all $(x, t) \in \partial K \times [0, a)$. Since $\rho(y + x + \lambda G(t, y + x), y + K) = \rho(x + \lambda G(t, y + x), K)$ by Equation (A.2), we can conclude the proof using Lemma A.3 with $D = y + K$. $\square$

Finally, the main comparison result used in the text, Theorem 2.4, may be proved as a corollary.

Proof of Theorem 2.4. We suppose that F is cooperative on $D_1 \times X$ and the inequality (20) holds. Let $w = v - u$. The function w is solution of the ODE $w' = G(t, w)$ where:

$$G(t, x) = F(u(t) + x) - F(u(t)) + d(t) \qquad \text{and} \qquad d(t) = v'(t) - F(v(t)) - u'(t) + F(u(t)).$$

First we show that G is locally Lipschitz with respect to the second variable. Let $(t, x) \in [0, a) \times X$. Let U be a neighborhood of $u(t) + x$ such that F is Lipschitz on U with a Lipschitz constant L. By continuity of u, there exist a neighborhood V_x of x and a positive constant η, such that $u(s) + y \in U$, for all $s \in [t, t + \eta] \cap [0, a)$ and $y \in V_x$. Thus, for all $s \in [t, t + \eta] \cap [0, a)$ and $y, z \in V_x$, we have $\|G(s, y) - G(s, z)\| \leq L\|y - z\|$.

Let $t \in [0, a)$, $x \in \partial K$ and let $x^\star \in K^\star$ such that $\langle x^\star, x\rangle = 0$. Let us prove that $\langle x^\star, G(t, x)\rangle \geq 0$. By (20), we know that $d(t)$ belongs to K. Furthermore, the inequality $\langle x^\star, F(u(t) + x) - F(u(t))\rangle \geq 0$ holds because the function F is cooperative on $D_1 \times X$. Thus, $\langle x^\star, G(t, x)\rangle$ is non-negative. Hence, we can apply Theorem A.2 with $y = 0$ and obtain that K is forward invariant with respect to G. Since $w(0) \in K$, this shows that $w(t) \in K$ for all $t \in [0, a)$, *i.e.*, $u(t) \leq v(t)$ for all $t \in [0, a)$.

When F is cooperative on $X \times D_2$, the proof is similar. $\square$

Appendix B. The Hausdorff distance on the compact sets of $\mathbb{C}$

Let $\mathcal{K}(\mathbb{C})$ be the set of non-empty compact subsets of $\mathbb{C}$. The Hausdorff distance between A and B in $\mathcal{K}(\mathbb{C})$ is defined as:

$$d_{\mathrm{H}}(A, B) = \max \left\{ \sup_{z_1 \in A} \inf_{z_2 \in B} |z_1 - z_2|, \quad \sup_{z_2 \in B} \inf_{z_1 \in A} |z_2 - z_1| \right\}. \tag{B.1}$$

We recall that the space $(\mathcal{K}(\mathbb{C}), d_{\mathrm{H}})$ is a metric space; see [73, Section 7.3.1]. Since:

$$\sup\{\,|z| : z \in A\,\} = d_{\mathrm{H}}(A, \{0\}),$$

for all $A \in \mathcal{K}(\mathbb{C})$, we deduce the following result.

Lemma B.1. *The map $A \mapsto \sup\{\,|z| : z \in A\,\}$ from $(\mathcal{K}(\mathbb{C}), d_{\mathrm{H}})$ to $\mathbb{R}$ endowed with the usual Euclidean distance is continuous.*

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