Proposition 4.8 shows that the equilibrium 0 is not asymptotically stable, in the sense that we can find initial conditions arbitrarily close to 0 in norm such that $\phi(t, g)$ does not converge to 0 pointwise (note that $\|w_\varepsilon\|$ may be chosen arbitrarily small). Since $w_\epsilon \leq 1$, we get by monotony and the comparison Theorem that $w_\varepsilon \leq \phi(t, w_\varepsilon) \leq \phi(t, 1)$, which implies that $w_\varepsilon \leq g^*$. In particular we get the following corollary.

(or equivalently $R_0 > 1$)

Corollary 4.9. *If Assumption 1 is in force and $s(T_k - \gamma) > 0$, then we have:*

$$\int_\Omega g^*(x)\, \mu(\mathrm{d}x) > 0.$$

We deduce from Proposition 4.8 that $t \mapsto \phi(t, w_\varepsilon)$ converges pointwise as t tends to infinity since $\phi(t, w_\varepsilon) \leq 1$ for all t. According to Proposition 2.13, the limit is an equilibrium. It is not 0 but it might be different from g^*. We will use Assumption 2 to ensure that 0 and g^* are the only equilibria. In order to prove this result, we need the following lemma.

Lemma 4.10 *(Instantaneous propagation of the infection). Suppose Assumptions 1 and 2 are in force. If $g \in \Delta$ satisfies $\int_\Omega g(x)\, \mu(\mathrm{d}x) > 0$, then, for all $t > 0$, $\phi(t, g)$ is μ-a.e. positive.*

Proof. Since the flow is order-preserving (see Proposition 2.8), it is sufficient to show the proposition for g such that $\|g\| < 1/2$. It follows from Equation (29) that:

$$\phi(t, g) \leq \|g\| + t\|T_k\|.$$

Thus, for all $t \in [0, c)$, with $c = (1 - 2\|g\|)/2\|T_k\|$ (and $c = +\infty$ if $\|T\| = 0$), we have that $\phi(t, g) < 1/2$. Now, we define the function:

$$u(t) = \mathrm{e}^{-\|\gamma\| t}\, \mathrm{e}^{tT_k/2} g.$$

Taking $c > 0$ smaller if necessary, we get $u(t) \leq \|g\|(1 + t\|T_k\|) < 1/2$ for $t \in [0, c)$. Then we get for $t \in [0, c)$:

$$\begin{aligned} u'(t) - F(u(t)) &= (T_k/2 - \|\gamma\|)u(t) - (1 - u(t))T_k u(t) + \gamma u(t) \\ &= (u(t) - 1/2)T_k u(t) - (\|\gamma\| - \gamma)u(t) \\ &\leq 0 = b'(t) - F(b(t)), \end{aligned}$$

where $b(t) = \phi(t, g)$. Using the comparison Theorem 2.4, we deduce

$$\phi(t, g) \geq u(t). \tag{55}$$

Now, we fix $t \in [0, c)$. We denote by $A = \{x \in \Omega : u(t)(x) > 0\}$ the support of $u(t)$. We have:

$$0 = \langle \mathbb{1}_{A^c}, u(t)\rangle = \mathrm{e}^{-\|\gamma\| t} \sum_{n \in \mathbb{N}} \frac{1}{n!} \langle \mathbb{1}_{A^c}, (tT_k/2)^n(g)\rangle\,.$$

This implies that $\langle \mathbb{1}_{A^c}, (tT_k/2)^n(g)\rangle = 0$ for all n, and thus that $\langle \mathbb{1}_{A^c}, T_k u(t)\rangle = 0$. We deduce that:

$$\int_{A^c\times A} k(x,y)\,\mu(\mathrm{d}x)\mu(\mathrm{d}y) = 0.$$

Since the set A contains the support of g, we get $\mu(A) > 0$. It follows from Assumption 2 that $\mu(A^c) = 0$. This means that $u(t)$ is μ-a.e. positive. Hence, from Equation (55), we get that, for $t \in [0, c)$, $\phi(t, g)$ is μ-a.e. positive. Using the semi-group property of the semi-flow this results propagate on the whole positive half-line and the result is proved. □

Remark 4.11. One can check from its proof, that Lemma 4.10 does not require the integrability condition (8) in Assumption 1 to be true.

Now we can show the following important result.

If $R_0 > 1$ then

Proposition 4.12 *(Uniqueness of the endemic state). Under Assumptions 1 and 2, the maximal equilibrium g^*:*

(i) *is positive μ-a.e.,*
(ii) *is the unique equilibrium different from* 0.

Proof. From Lemma 4.10 together with Remark 3.8, we deduce that every equilibrium different from 0 is positive μ-a.e. This proves Point (i) as $\int g^* \mathrm{d}\mu > 0$ in the supercritical regime according to Corollary 4.9.

We now prove Point (ii). Let h^* be another equilibrium different from 0. Since g^* is the maximal equilibrium, we have $h^* \leq g^*$. We shall prove that h^* is equal to g^* almost everywhere. Let us define the non-negative kernel k by:

$$\mathsf{k}(x,y) = (1 - g^*(x))\,\frac{k(x,y)}{\gamma(y)} \quad \text{for } x, y \in \Omega.$$

Notice that k satisfies (37). Since $T_{\mathsf{k}}(\gamma g^*) = \gamma g^*$, we deduce from Proposition 3.6 that $r(T_{\mathsf{k}}) \geq 1$. Let $v \in L^q(\mu)_+ \setminus \{0\}$ be a left Perron vector of the operator T_{k} (given by Lemma 3.7 (v)). The kernel k satisfies Assumption 2 as k does and $1 - g^*$ is positive everywhere (see Remark 2.16). Hence, v can be chosen positive μ-a.e. according to Lemma 3.7 (vi). The following computation:

$$\langle v, \gamma g^*\rangle = \langle v, T_{\mathsf{k}}(\gamma g^*)\rangle = r(T_{\mathsf{k}})\,\langle v, \gamma g^*\rangle\,,$$

shows that $r(T_{\mathsf{k}})$ is actually equal to 1 since $\langle v, \gamma g^*\rangle > 0$. Now we compute:

$$\begin{aligned}
0 &= \langle v, F(h^*)\rangle \\
&= \langle v, T_{\mathsf{k}}(\gamma h^*) - \gamma h^*\rangle + \langle v, (g^* - h^*) T_{k/\gamma}(\gamma h^*)\rangle \\
&= \langle v, (g^* - h^*) T_k(h^*)\rangle\,,
\end{aligned}$$

where we used that $\langle v, T_k f - f\rangle = 0$ as $r(T_k) = 1$ and v is a left Perron eigenvector. According to the first part of the proof, h^* is μ-a.e. positive. Since we have $T_k(h^*) = \gamma h^*/(1-h^*)$, the function $T_k(h^*)$ is also μ-a.e. positive. Hence g^* and h^* are equal μ-a.e. since v is μ-a.e. positive, see Lemma 3.7 (vi). This implies in particular that $T_k(h^*) = T_k(g^*)$ by Lemma 3.7 (i). We deduce that, for all $x \in \Omega$:

$$h^*(x) = T_k(h^*)(x)/(\gamma(x) + T_k(h^*)(x)) = T_k(g^*)(x)/(\gamma(x) + T_k(g^*)(x)) = g^*(x).$$

Therefore g^* is then unique equilibrium different from 0. □

Now we can prove the main result of this section on the pointwise convergence of $\phi(t, g)$. If g is μ-a.e. equal to 0, then clearly, as γ is positive, we get that $\lim_{t\to\infty} \phi(t, g) = 0$ pointwise, so we only need to consider the case where g is not μ-a.e. equal to 0.

Theorem 4.13 *(Convergence towards the endemic equilibrium). Suppose that Assumptions 1 and 2 are in force. Let $g \in \Delta$ such that $\int_\Omega g(x)\,\mu(\mathrm{d}x) > 0$. Then, we have that for all $x \in \Omega$:*

and $R_0 > 1$

$$\lim_{t\to\infty} \phi(t, g)(x) = g^*(x).$$

Proof. By Lemma 4.10, it is enough to show the result for g μ-a.e. positive. The idea is similar to the proof of Proposition 4.8, that is, to try and find a monotonous trajectory; the difference here is that we look for a trajectory that is below $\phi(t, g)$, and we have to adapt the proof accordingly. For such a g, the functions $(1-\varepsilon)g\mathbb{1}_{g\geq\varepsilon}$ converge in $L^1(\mu)$ to g when ε goes to zero. Besides, R_0 is greater than 1 by Proposition 4.2. Hence, according to Proposition 4.5, for ε small enough, we get

$$R_0\left((1-\varepsilon)\,\mathbb{1}_{g\geq\varepsilon} T_{k/\gamma}\right) > 1.$$

By Proposition 4.2 (iii), applied to the kernel $(1-\varepsilon)\mathbb{1}_{g(x)\geq\varepsilon} k(x, y)$, there exists $w_\varepsilon \in \mathscr{L}_+^\infty \setminus \{0\}$ and $\lambda(\varepsilon) > 0$ such that:

$$(1-\varepsilon)\,\mathbb{1}_{g\geq\varepsilon} T_k(w_\varepsilon) = (\gamma + \lambda(\varepsilon))w_\varepsilon. \tag{56}$$

We may and will assume additionally that $\|w_\varepsilon\| \leq \varepsilon$. Since (56) implies that $w_\varepsilon(x) = 0$ when $g(x) < \varepsilon$, we know that $w_\varepsilon \leq g$. The monotony of the semi-flow (see Proposition 2.8) then implies that, for all $t \in \mathbb{R}_+$:

$$\phi(t, w_\varepsilon) \leq \phi(t, g) \leq \phi(t, 1). \tag{57}$$

Besides, we have:

$$\begin{aligned} 0 \leq \lambda(\varepsilon) w_\varepsilon &= (1-\varepsilon)\mathbb{1}_{g\geq\varepsilon} T_k(w_\varepsilon) - \gamma w_\varepsilon \\ &\leq (1-\varepsilon) T_k(w_\varepsilon) - \gamma w_\varepsilon \\ &\leq (1-w_\varepsilon) T_k(w_\varepsilon) - \gamma w_\varepsilon \\ &= F(w_\varepsilon), \end{aligned}$$

$$D_n = (\beta - \alpha)\left(\prod_{k=1}^{n-1} \frac{2k-1}{2k+2}\right)\left(\prod_{k=1}^{n-1}(1 - g_\alpha^*(k))^{-1}\right)\left(\prod_{k=1}^{n-1}(1 - g_\beta^*(k))^{-1}\right).$$

Using that $\prod_{k=0}^{n-1} \frac{2k-1}{2k+2} \sim C\, n^{-3/2}$ for some finite constant $C > 0$, we deduce that

$$c_\beta - c_\alpha = C(\beta - \alpha) d_\alpha d_\beta.$$

This gives the strict monotonicity of the map $\alpha \mapsto c_\alpha$. Then, use that $d_\alpha \le d_\beta \le d_{\beta'} < +\infty$ for some $\beta' \in (\beta, 1/2)$ to get the continuity. $\square$

We are not able to describe entirely the basins of attraction of each equilibrium. However, the asymptotic behavior in n of the starting point g tells us quite a lot.

Proposition 4.16. *For all $g \in \Delta$, and for all $\alpha \in (0, 1/2)$, we have:*

$$\limsup_n n^{3/2} g(n) \le c_\alpha \implies \limsup_{t\to\infty} \phi(t, g) \boxed{\le \cancel{g^*_*}, \quad g^*_\alpha}$$
$$\liminf_n n^{3/2} g(n) \ge c_\alpha \implies \liminf_{t\to\infty} \phi(t, g) \ge \boxed{\cancel{g^*_*}. \quad g^*_\alpha}$$

In particular, we have:

$$\limsup_n n^{3/2} g(n) = 0 \implies \phi(t, g) \to 0,$$
$$\liminf_n n^{3/2} g(n) = \infty \implies \phi(t, g) \to g^*_{1/2}.$$

Proof. Since k is upper-triangular, the long-time behavior of the dynamic does not depend on the first terms of the initial condition. Indeed, for $n \ge 2$, consider the subspace $E_n = \{g \in \mathscr{L}^\infty : g(p) = 0 \text{ for } 1 \le p < n\}$ of functions whose first $n-1$ terms are 0. Denote by P_n the canonical projection from $\mathscr{L}^\infty$ on E_n. For $n \ge 2$ and $g \in \Delta$, we have:

$$P_n \phi(t, g) = P_n\left(\phi(t, P_n(g))\right). \tag{62}$$

Let us denote by $\preceq$ the partial order defined by $g \preceq h$ if there exists $n \ge 2$ such that $P_n(g) \le P_n(h)$.

Suppose that $\limsup n^{3/2} g(n) \le c_\alpha$. Since $\alpha \to c_\alpha$ is strictly increasing, for any $\alpha < \beta < 1/2$, the asymptotics of g and g^*_β imply that $g \preceq g^*_\beta$. Since the flow is order-preserving, this entails $\limsup \phi(t, g) \le g^*_\beta$. This inequality holds for all $\beta > \alpha$: we get the conclusion by continuity of the map $\Gamma : \alpha \to g^*_\alpha$. The proof of the other implication is similar. $\square$

4.6. Uniform convergence

In the subcritical case, Theorem 4.6 shows an exponentially fast convergence, in the uniform norm. By contrast, the convergence results in the critical and supercritical case from Sections 4.3 and 4.4 only hold pointwise.

In the next result, we show how to recover a form of uniformity; in particular we recover uniform convergence in the particular case where $\inf \gamma > 0$.

The evolution Equation (66) can be seen as the SIS evolution Equation (3) on an extended feature space:

- the feature $\boldsymbol{x} = (x, \xi)$ lives in $\boldsymbol{\Omega} = \Omega \times \Sigma$ endowed with the σ-field $\mathscr{F} \otimes \mathscr{G}$,
- the recovery rate is given by $\boldsymbol{\gamma}(\boldsymbol{x}) = \gamma(x, \xi)$,
- the extended transmission kernel is given by:

κ^{ℓ}

$$\boxed{\boldsymbol{\kappa}^{\ell}(\boldsymbol{x}, \mathrm{d}\boldsymbol{y})} = (1 - e(x, \xi))(1 - \delta(y, \zeta))\kappa(x, \mathrm{d}y)\eta(y, \mathrm{d}\zeta). \tag{67}$$

Remark 5.1. In the leaky mechanism, we suppose that the vaccine acts directly on the susceptibility and the infectiousness of the individuals. Protective gears (like respirators or safety glasses) which are designed to protect the wearer from absorbing airborne microbes or transmitting them have a similar effect. Hence, Equation (66) is not limited to vaccination and can also be used as a model for distribution of equipment in the population.

5.1.2. *The all-or-nothing mechanism*

In this setting, $e(x, \xi)$, is defined as the probability to immunize completely the individual with feature x to the disease with vaccine ξ. We generalize Equation (14) to get the following infinite dimensional evolution equation:

$$\begin{aligned}\partial_t u(t, x, \xi) &= -\gamma(x, \xi)\, u(t, x, \xi) \\ &\quad + (1 - e(x, \xi) - u(t, x, \xi)) \int_{\Omega \times \Sigma} (1 - \delta(y, \zeta)) u(t, y, \zeta) \kappa(x, \mathrm{d}y) \eta(y, \mathrm{d}\zeta).\end{aligned} \tag{68}$$

The probability $v(t, x, \xi) = u(t, x, \xi)/(1 - e(x, \xi))$ for an individual with feature x which has not been vaccinated by the inoculation of vaccine ξ to be infected at time t satisfies the following equation:

$$\begin{aligned}\partial_t v(t, x, \xi) &= -\gamma(x) v(t, x, \xi) \\ &\quad + (1 - v(t, x, \xi)) \int_{\Omega \times \Sigma} (1 - \delta(y, \zeta)) v(t, y, \zeta)(1 - e(y, \zeta)) \kappa(x, \mathrm{d}y) \eta(y, \mathrm{d}\zeta).\end{aligned} \tag{69}$$

As before, the evolution Equation (69) can be seen as the SIS evolution Equation (3) on the same extended feature space $\boldsymbol{\Omega} = \Omega \times \Sigma$, still endowed with the σ-field $\mathscr{F} \otimes \mathscr{G}$, with the same recovery rate $\boldsymbol{\gamma}(\boldsymbol{x}) = \gamma(x, \xi)$, but the transmission kernel now reads:

κ^{a}

$$\boxed{\boldsymbol{\kappa}^{a}(\boldsymbol{x}, \mathrm{d}\boldsymbol{y})} = (1 - e(y, \zeta))(1 - \delta(y, \zeta))\kappa(x, \mathrm{d}y)\eta(y, \mathrm{d}\zeta). \tag{70}$$

Notice the difference between the evolution Equation (66) for leaky mechanism and the evolution Equation (69) for the all-or-nothing mechanism is that $e(y, \zeta)$ in (69) (or in the kernel $\boldsymbol{\kappa}^a$ from (70)) is replaced by $e(x, \xi)$ in (66) (or in the kernel $\boldsymbol{\kappa}^{\ell}$ from (67)).