(i) ***The case*** $b > 0$. *A strategy is Pareto optimal if and only if it belongs to* $\mathcal{S}^{\perp \mathrm{id}}$. *In particular, for any* $c \in [0, 1]$, *the strategy* $(1 - c)\mathbb{1}$ *costs* c *and is Pareto optimal. The anti-Pareto optimal strategies are* $\mathbb{1}_{B(x,t)}$ *for* $x \in \mathbb{S}^{d-1}$ *and* $t \in [-1, 1]$. *In other words:*

$$\mathcal{P} = \mathcal{S}^{\perp \mathrm{id}} \quad \text{and} \quad \mathcal{P}^{\mathrm{Anti}} = \mathcal{S}^{\mathrm{balls}}.$$

(ii) ***The case*** $b < 0$. *A strategy is anti-Pareto optimal if and only if it belongs to* $\mathcal{S}^{\perp \mathrm{id}}$. *In particular, for any* $c \in [0, 1]$, *the strategy* $(1 - c)\mathbb{1}$ *costs* c *and is anti-Pareto optimal. The Pareto optimal strategies are* $\mathbb{1}_{B(x,t)}$ *for* $x \in \mathbb{S}^{d-1}$ *and* $t \in [-1, 1]$. *In other words:*

$$\boxed{\overset{\mathcal{P}}{\cancel{\mathcal{P}^{\mathrm{Anti}}}} = \mathcal{S}^{\mathrm{balls}}} \quad \text{and} \quad \boxed{\overset{\mathcal{P}^{\mathrm{Anti}}}{\cancel{\mathcal{S}^{\mathrm{uni}}}} = \mathcal{S}^{\perp \mathrm{id}}.}$$

In both cases, we have $c_\star = 1$ *and* $c^\star = 0$.

Example 7.10 We consider the kernel $\mathrm{k} = 1 + b\langle \cdot, \cdot \rangle$ on the sphere $\mathbb{S}^{d-1}$, with $d = 2$. This model has the same Pareto and anti-Pareto frontiers as the equivalent model given by $\Omega = [0, 1)$ endowed with the Lebesgue measure and the kernel $(x, y) \mapsto 1 + b \cos(\pi(x - y))$, where the equivalence holds in the sense of (Delmasetal. 2021b, Section 7), with an obvious deterministic coupling $\theta \mapsto \exp(2i\pi\theta)$. We provide the Pareto and anti-Pareto frontiers in Fig. 12 with $b = 1$ (top) and with $b = -1$ (bottom).

Proof The proof of Proposition 7.9 is decomposed in four steps. *Step 1:* $R_e(\eta)$ is the eigenvalue of a 2×2 matrix $M(\eta)$. Without loss of generality, we shall assume that $R_0 = a = 1$. Since k is positive a.s., we deduce that $c_\star = 1$ and $c^\star = 0$ thanks to Lemma 3.1; and the strategy $\mathbb{1}$ (resp. $\mathbb{0}$) is the only Pareto optimal as well as the only anti-Pareto optimal strategy with cost 0 (resp. 1). So we shall only consider strategies $\eta \in \Delta$ such that $C(\eta) \in (0, 1)$.

Let $z_0 \in \mathbb{S}^{d-1}$. Write $b = \varepsilon\lambda^2$ with $\varepsilon \in \{-1, +1\}$ and $\lambda \in (0, 1]$, and define the function α on $\mathbb{S}^{d-1}$ by:

$$\alpha = \lambda \langle \cdot, z_0 \rangle.$$

Let $\eta \in \Delta$ with cost $c \in (0, 1)$. As $c_\star = 1 > C(\eta)$, we get that $R_e(\eta) > 0$. We deduce from the special form of the kernel k that the eigenfunctions of $T_{\mathrm{k}\eta}$ are of the form $\zeta + \beta\lambda\langle \cdot, y \rangle$ with $\zeta, \beta \in \mathbb{R}$ and $y \in \mathbb{S}^{d-1}$. Since $R_e(\eta) > 0$, the right Perron eigenfunction, say h_η, being non-negative, can be chosen such that $h_\eta = 1 + \beta_\eta\lambda\langle \cdot, y_\eta \rangle$ with $\beta_\eta \geq 0$ and $\beta_\eta\lambda \leq 1$. Up to a rotation on the vaccination strategy, we shall take $y_\eta = z_0$, that is:

$$h_\eta = 1 + \beta_\eta \, \alpha.$$

From the equality $R_e(\eta) h_\eta = T_{\mathrm{k}\eta} h_\eta$, we deduce that:

$$R_e(\eta) = \int_{\mathbb{S}^{d-1}} \eta(y) \, \mu(\mathrm{d}y) + \beta_\eta \, \lambda \int_{\mathbb{S}^{d-1}} \eta(y) \, \langle y, z_0 \rangle \, \mu(\mathrm{d}y), \tag{76}$$