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ScienceDirect
Stochastic Processes and their Applications 125 (2015) 1569–1604
stochastic processes and their applications
www.elsevier.com/locate/spa

Pruning of CRT-sub-trees

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Received 13 December 2012; received in revised form 9 November 2014; accepted 9 November 2014
Available online 15 November 2014

Abstract

We study the pruning process developed by Abraham and Delmas (2012) on the discrete Galton–Watson sub-trees of the Lévy tree which are obtained by considering the minimal sub-tree connecting the root and leaves chosen uniformly at rate λ, see Duquesne and Le Gall (2002). The tree-valued process, as λ increases, has been studied by Duquesne and Winkel (2007). Notice that we have a tree-valued process indexed by two parameters: the pruning parameter θ and the intensity λ. Our main results are: construction and marginals of the pruning process, representation of the pruning process (forward in time that is as θ increases) and description of the growing process (backward in time that is as θ decreases) and distribution of the ascension time (or explosion time of the backward process) as well as the tree at the ascension time. A by-product of our result is that the super-critical Lévy trees independently introduced by Abraham and Delmas (2012) and Duquesne and Winkel (2007) coincide. This work is also related to the pruning of discrete Galton–Watson trees studied by Abraham, Delmas and He (2012).

MSC: 05C05; 60J80; 60J27

Keywords: Pruning; Branching process; Galton–Watson process; Random tree; CRT; Tree-valued process; Girsanov transformation

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http://dx.doi.org/10.1016/j.spa.2014.11.008
0304-4149/

1. Introduction

The study of pruning of Galton–Watson trees has been initiated by Aldous and Pitman [9]. Roughly speaking, it corresponds to the percolation on edges: an edge is uniformly chosen at random in a Galton–Watson tree and is removed, and the connected component containing the root is kept. This procedure is then iterated. This process can be extended backward in time. It corresponds then to a non-decreasing tree-valued process. The ascension time A is then the first time at which this tree-valued process reaches an unbounded tree. In [9], the authors give the joint distribution of A as well as the tree just before the ascension time (in backward time).

The limits of Galton–Watson trees are the so called continuum Lévy trees, see [6,7,20,13]; they are characterized by a branching mechanism ψ which is also a Lévy exponent. The result for the pruning process on Galton–Watson trees was then extended by Abraham and Delmas [1] to a process indexed by time θ whose marginals are continuum Lévy trees. In the setting of the Brownian continuum random tree, which corresponds to a quadratic branching mechanism, the pruning procedure is uniform on the skeleton, see also Aldous and Pitman [8] for a fragmentation point of view in this case. This is the analogue of [9]. However in the general Lévy case, one has to take into account the pruning of nodes with a rate given by their "size" or "mass", which is defined as the asymptotic number of small trees attached to the node. This result in the continuous setting motivated a new pruning procedure on the nodes of Galton–Watson trees, which was developed by Abraham, Delmas and He [2]. In this case, the pruning happens on the nodes with rate depending on the degree of the nodes.

In the present work, we study the pruning process developed in [1] on the discrete Galton–Watson sub-trees of the Lévy tree. The discrete Galton–Watson sub-trees of the Lévy trees are obtained by considering the minimal sub-tree connecting the root and leaves chosen uniformly with rate $\lambda \geq 0$, see Duquesne and Le Gall [13]. The tree-valued process, as λ increases, has been studied by Duquesne and Winkel [14], in particular to construct super-critical Lévy trees. Notice that super-critical Lévy trees have also been defined in [1]. One of the by-product of our results is that the two definitions coincide, see Section 5. Notice that we have a tree-valued process indexed by two parameters θ (as in [9,1]) and λ (as in [14]). The other main results are: construction and marginals of the pruning process in Section 4, representation of the pruning process (forward in time that is as θ increases) and description of the growing process (backward in time that is as θ decreases) in Section 6, some remarks on martingales related to the number of leaves in Section 7, distributions of the ascension time and of the tree at the ascension time in Section 8.

Now, we present more precisely our results. Let ψ be a branching mechanism satisfying the Grey condition (see (6) in Section 2.6). A priori, the branching mechanism ψ is defined on $[0, +\infty)$ but we may extend it on a part of $(-\infty, 0)$ using formula (5). For every θ such that $\psi(\theta)$ exists, we define the branching mechanism ψ_θ by:

$$\psi_\theta(q) = \psi(q+\theta) - \psi(\theta) \quad \text{for all } q \geq 0,$$

and denote by Θ the set of θ for which $\psi(\theta)$ exists. Note that ψ_θ satisfies the Grey condition (6). We consider the tree-valued process $(\mathcal{T}_\theta, \theta \in \Theta)$ introduced in [1], corresponding to a uniform pruning on the skeleton and to a pruning at nodes with rate depending on its size. We recall that $\mathcal{T}_\theta$ is a Lévy tree with branching mechanism ψ_θ. Let $\mathbf{m}^{\mathcal{T}_\theta}$ be its mass measure, which is a uniform measure on the set of leaves. Let $\tau_0(\lambda)$ be the minimal sub-tree of $\mathcal{T}_0$ generated by the root and leaves chosen before time λ according to a Poisson point measure $\mathcal{P}^0$ on $\mathbb{R}_+ \times \mathcal{T}_0$ with intensity $dt\, \mathbf{m}^{\mathcal{T}_0}$. Let M_λ be the number of chosen leaves: $M_\lambda = \mathcal{P}^0([0, \lambda] \times \mathcal{T}_0)$, so that $\tau_0(\lambda)$ is well defined for $M_\lambda \geq 1$. And we set $\tau_\theta(\lambda) = \mathcal{T}_\theta \bigcap \tau_0(\lambda)$ for $\theta \geq 0$. So we get a two-parameter family of sub-trees $(\tau_\theta(\lambda), \lambda \ge 0, \theta \ge 0)$. Let $\mathbb{P}^{\psi,\lambda}$ be the conditional probability given the event $\{M_\lambda \ge 1\}$. We will be interested in the process $\lambda \mapsto \tau_\theta(\lambda)$ which was studied in [14] and in the pruning process $\theta \mapsto \tau_\theta(\lambda)$ where the case of $\lambda = +\infty$ was studied in [1].

Notice that the leaves of $\tau_\theta(\lambda)$ correspond to marked leaves belonging to $\mathcal{T}_\theta$ as well as roots of sub-trees of $\mathcal{T}_0$ with marked leaves which are removed to get $\mathcal{T}_\theta$. If one is interested only in $\hat{\tau}_\theta(\lambda)$, the minimal sub-tree containing the root and the marked leaves belonging to $\mathcal{T}_\theta$, then one would get a process such that $\hat{\tau}_\theta(\lambda)$ has the same distribution as $\tau_\theta(\lambda_\theta)$ with $\lambda_\theta = \psi(\psi_\theta^{-1}(\lambda)))$. This would lead to another natural process indexed by the level-set of the function $(\theta, \lambda) \mapsto \psi_\theta(\psi^{-1}(\lambda))$.

Theorem 3.2.1 in [12] in the sub-critical case and Corollary 4.5 in our paper in the general case give that the sub-tree $\tau_\theta(\lambda)$ is distributed as a Galton–Watson real tree; its reproduction law has generating function $g_{(\psi_\theta,\psi_\theta(\eta))}$, see definition (32), with $\eta = \psi^{-1}(\lambda)$ and exponential individual lifetime with parameter $\psi_\theta'(\eta)$. If we endow $\mathcal{T}_\theta$ with its mass measure and $\tau_\theta(\lambda)$ with a discrete mass measure defined by

$$\mathbf{m}^{\tau_\theta(\lambda)} = \frac{1}{\psi_\theta(\eta)} \sum_{x \text{ a leaf of } \tau_\theta(\lambda)} \delta_x,$$

then we show in Theorem 5.1 the convergence for the Gromov–Hausdorff–Prohorov distance defined in [4] of $\tau_\theta(\lambda)$ to $\mathcal{T}_\theta$ as λ goes to infinity. This result was already in [14] (with the Gromov–Hausdorff distance instead of the Gromov–Hausdorff–Prohorov distance), and this ensures that in the super-critical case the Lévy trees introduced in [14] and in [1] are the same. We give in Theorems 6.1 and 6.6 a precise description of the process $(\tau_\theta(\lambda), \theta \ge 0)$ in forward (decreasing tree-valued process) and backward (increasing tree-valued process) times. By considering the backward process, we see that it is possible to extend the process up to θ_λ backward in time, with θ_λ defined roughly by $\psi_{\theta_\lambda}(\psi^{-1}(\lambda)) = 0$ (see (45) for a precise definition). Usually θ_λ is not the lower bound of Θ. Intuitively, when θ decreases, the tree grows and in order to balance the number of leaves, the intensity for choosing them has to decrease; this can be done up to the lower bound θ_λ.

By considering $L_\theta(\lambda)$, the number of leaves of $\tau_\theta(\lambda)$, we shall show that the process $\theta \mapsto \psi'(\theta) L_\theta(\lambda)/\psi_\theta(\psi^{-1}(\lambda))$ is a backward martingale, see Proposition 7.2. By taking the limit as λ goes to infinity, and since the total mass of $\mathbf{m}^{\tau_\theta(\lambda)}$, that is $L_\theta(\lambda)/\psi_\theta(\psi^{-1}(\lambda))$, converges to the total mass of $\mathbf{m}^{\mathcal{T}_\theta}$, say σ_θ, we get in Proposition 7.1 that $\psi'(\theta)\sigma_\theta$ is also a backward martingale.

Then we consider the process $(\tau_\theta(\lambda), \theta > \theta_\lambda)$ backward in time and consider its ascension time A_λ defined in (52) as the first time θ at which the tree $\tau_\theta(\lambda)$ is unbounded. Of course, this corresponds to the ascension time of $(\mathcal{T}_\theta, \theta \in \Theta)$ when it is larger than θ_λ. We give in Proposition 8.1 the distribution of $(\tau_\theta(\lambda), \theta \ge A_\lambda)$ and identify it in Proposition 8.11 using the pruning of a tree $\tau_0^*(\lambda)$ with an infinite spine defined in Sections 8.2 and 8.3. We also prove the convergence, as λ goes to infinity of the tree $\tau_0^*(\lambda)$ toward the CRT $\mathcal{T}_0^*$ with infinite spine introduced in [1]. The latter can be seen as a sub-tree of Lévy trees with immigration, see [10] for a further work in this direction.

2. Lévy trees and the forest obtained by pruning

2.1. Notations

Let (E, d) be a metric Polish space. We denote by $\mathcal{M}_f(E)$ (resp. $\mathcal{M}_f^{\text{loc}}(E)$) the space of all finite (resp. locally finite) Borel measures on E, where a Borel measure μ is locally finite if for any $x \in E$, $\mu(B(x,r)) < \infty$ for all sufficiently small r. For $x \in E$, let δ_x denote the Dirac measure at point x. For $\mu \in \mathcal{M}_f^{\mathrm{loc}}(E)$ and f a non-negative measurable function, we set $\langle \mu, f \rangle = \int f(x)\, \mu(dx) = \mu(f)$.

2.2. Real trees

We refer to [15] or [19] for a general presentation of random real trees. A metric space (T, d) is a real tree if the following properties are satisfied:

(1) For every $s, t \in T$, there is a unique isometric map $f_{s,t}$ from $[0, d(s,t)]$ to T such that $f_{s,t}(0) = s$ and $f_{s,t}(d(s,t)) = t$.
(2) For every $s, t \in T$, if q is a continuous injective map from $[0, 1]$ to T such that $q(0) = s$ and $q(1) = t$, then $q([0, 1]) = f_{s,t}([0, d(s,t)])$.

If $s, t \in T$, we will note $[\![s, t]\!]$ the range of the isometric map $f_{s,t}$ described above and $[\![s, t[\![$ for $[\![s, t]\!] \setminus \{t\}$.

We say that $(T, d, \emptyset)$ is a rooted real tree with root $\emptyset$ if (T, d) is a real tree and $\emptyset \in T$ is a distinguished vertex.

Let $(T, d, \emptyset)$ be a rooted real tree. The degree $n(x)$ of $x \in T$ is the number of connected components of $T \setminus \{x\}$. The number of children of $x \neq \emptyset$ is $\kappa_x = n(x) - 1$ and the number of children of the root is $\kappa_\emptyset = n(\emptyset)$. We shall consider the set of leaves $\mathrm{Lf}(T) = \{x \in T,\ \kappa_x = 0\}$, the set of branching points $\mathrm{Br}(T) = \{x \in T,\ \kappa_x \geq 2\}$ and the set of infinite branching points $\mathrm{Br}_\infty(T) = \{x \in T,\ \kappa_x = \infty\}$. We say that a tree is discrete if $\{x \in \mathrm{Lf}(T) \cup \mathrm{Br}(T);\ d(\emptyset, x) \leq a\}$ is finite for all a. The skeleton of T is the set of points in the tree that are not leaves: $\mathrm{Sk}(T) = T \setminus \mathrm{Lf}(T)$. The trace of the Borel σ-field of T restricted to $\mathrm{Sk}(T)$ is generated by the sets $[\![s, s']\!]$; $s, s' \in \mathrm{Sk}(T)$. One uniquely defines a σ-finite Borel measure ℓ^T on T, called the length measure of T, such that:

$$\ell^T(\mathrm{Lf}(T)) = 0 \quad \text{and} \quad \ell^T([\![s, s']\!]) = d(s, s').$$

For every $x \in T$, $[\![\emptyset, x]\!]$ is interpreted as the ancestral line of vertex x in the tree. We define a partial order on T by setting $x \preccurlyeq y$ (x is an ancestor of y) if $x \in [\![\emptyset, y]\!]$. If $x, y \in T$, there exists a unique $z \in T$, called the Most Recent Common Ancestor (MRCA) of x and y, such that $[\![\emptyset, x]\!] \cap [\![\emptyset, y]\!] = [\![\emptyset, z]\!]$.

2.3. Measured rooted real trees

A rooted measured metric space $\mathcal{X} = (X, d, \emptyset, \mu)$ is a metric space (X, d) with a distinguished element $\emptyset \in X$ and a locally finite Borel measure $\mu \in \mathcal{M}_f^{\mathrm{loc}}(E)$. Two rooted measured metric spaces $\mathcal{X} = (X, d, \emptyset, \mu)$ and $\mathcal{X}' = (X', d', \emptyset', \mu')$ are called GHP-isometric if there exists an isometric bijection $\Phi : X \to X'$ such that $\Phi(\emptyset) = \emptyset'$ and $\Phi_* \mu = \mu'$, where $\Phi_* \mu$ is the measure μ transported by Φ.

We will denote by $\mathbb{T}$ the set of (GHP-isometry classes of) measured rooted real trees $(T, d, \emptyset, \mathbf{m})$ where $(T, d, \emptyset)$ is a locally compact rooted real tree and $\mathbf{m} \in \mathcal{M}_f^{\mathrm{loc}}(T)$ is a locally finite measure on T. Sometimes, we will write $(T, d^T, \emptyset^T, \mathbf{m}^T)$ for $(T, d, \emptyset, \mathbf{m})$ to stress the dependence in T. Sometimes, when there is no confusion, we will simply write T for $(T, d, \emptyset, \mathbf{m})$ and $\tilde{T}$ for $(T, d, \emptyset)$.

The set $\mathbb{T}$ can be endowed with the so-called Gromov–Hausdorff–Prohorov metric which first appeared in [22] for compact trees endowed with a probability measure (which leads to the same topology as in [16]) and which was extended to the set $\mathbb{T}$ in [3] (see also [17,21] for other distances on the set of measured trees). This metric is defined as follows.

Let (X, d) be a Polish metric space. For $A, B \in \mathcal{B}(X)$, we set:

$$d_{\mathrm{H}}(A, B) = \inf\{\varepsilon > 0,\ A \subset B^{\varepsilon} \text{ and } B \subset A^{\varepsilon}\},$$

the Hausdorff distance between A and B, where $A^{\varepsilon} = \{x \in X, \inf_{y\in A} d(x, y) < \varepsilon\}$ is the ε-halo set of A. If $\mu, \nu \in \mathcal{M}_f(X)$, we set:

$$d_{\mathrm{P}}(\mu, \nu) = \inf\{\varepsilon > 0,\ \mu(A) \le \nu(A^{\varepsilon}) + \varepsilon \text{ and } \nu(A) \le \mu(A^{\varepsilon}) + \varepsilon \ \text{ for all closed set } A\},$$

the Prohorov distance between μ and ν.

Let $\mathcal{X} = (X, d, \emptyset, \mu)$ and $\mathcal{X}' = (X', d', \emptyset', \mu')$ be two compact rooted measured metric spaces endowed with finite measures μ and μ', and define:

$$d^c_{\mathrm{GHP}}(\mathcal{X}, \mathcal{X}') = \inf_{\Phi, \Phi', Z} \left(d^Z_{\mathrm{H}}(\Phi(X), \Phi'(X')) + d^Z(\Phi(\emptyset), \Phi'(\emptyset')) + d^Z_{\mathrm{P}}(\Phi_*\mu, \Phi'_*\mu') \right),$$

where the infimum is taken over all isometric embeddings $\Phi : X \hookrightarrow Z$ and $\Phi' : X' \hookrightarrow Z$ into some common Polish metric space (Z, d^Z).

If $\mathcal{X} = (X, d, \emptyset, \mu)$ is a rooted measured metric space, then for $r \ge 0$ we will consider its restriction to the ball of radius r centered at $\emptyset$, $\mathcal{X}^{(r)} = (X^{(r)}, d^{(r)}, \emptyset, \mu^{(r)})$, where

$$X^{(r)} = \{x \in X; d(\emptyset, x) \le r\},$$

the metric $d^{(r)}$ is the restriction of d to $X^{(r)}$, and the measure $\mu^{(r)}(dx) = \mathbf{1}_{X^{(r)}}(x)\ \mu(dx)$ is the restriction of μ to $X^{(r)}$.

We define the following function on $\mathbb{T}^2$, for $T_1, T_2 \in \mathbb{T}$:

$$d_{\mathrm{GHP}}(T_1, T_2) = \int_0^{\infty} \mathrm{e}^{-r} \left(1 \wedge d^c_{\mathrm{GHP}}\left(T_1^{(r)}, T_2^{(r)}\right)\right) dr.$$

According to Corollary 2.8 in [3], the function d_{GHP} is well defined and $(\mathbb{T}, d_{\mathrm{GHP}})$ is a Polish metric space.

Remark 2.1. In that paper, we always handle elements of $\mathbb{T}$. However, our objects or transformations are a priori only defined on a tree T and not on an equivalence class. To be totally rigorous, we should prove that everything is well defined on $\mathbb{T}$, i.e., that given two representatives of the same equivalence class, the construction leads to two trees that still belong to the same equivalence class. This will be done once for the definition of the grafting procedure in the next section as an example.

2.4. Grafting procedure

We will define in this section a procedure that consists in grafting measured rooted real trees on an existing measured rooted real tree. More precisely, let T be a measured rooted real tree and let $((T_i, x_i), i \in I)$ be a finite or countable family of elements of $\mathbb{T} \times T$. We define the real tree obtained by grafting the trees T_i on T at point x_i. We set $\hat{T} = T \sqcup \left(\bigsqcup_{i\in I} T_i \setminus \{\emptyset^{T_i}\}\right)$ where the symbol $\sqcup$ means that we choose for the sets $(T_i)_{i\in I}$ representatives of GHP-isometry classes in $\mathbb{T}$ which are disjoint subsets of some common set and that we perform the disjoint union of all these sets. We set $\emptyset^{\hat{T}} = \emptyset^T$. The set $\hat{T}$ is endowed with the following metric $d^{\hat{T}}$:

if $s, t \in \hat{T}$,

$$d^{\hat{T}}(s,t) = \begin{cases} d^T(s,t) & \text{if } s,t \in T, \\ d^T(s,x_i) + d^{T_i}(\emptyset^{T_i}, t) & \text{if } s \in T,\ t \in T_i \setminus \{\emptyset^{T_i}\}, \\ d^{T_i}(s,t) & \text{if } s,t \in T_i \setminus \{\emptyset^{T_i}\}, \\ d^T(x_i,x_j) + d^{T_j}(\emptyset^{T_j}, s) + d^{T_i}(\emptyset^{T_i}, t) & \text{if } i \neq j \text{ and } s \in T_j \setminus \{\emptyset^{T_j}\},\ t \in T_i \setminus \{\emptyset^{T_i}\}. \end{cases}$$

We define the mass measure on $\hat{T}$ by:

$$\mathbf{m}^{\hat{T}} = \mathbf{m}^T + \sum_{i \in I} \left(\mathbf{1}_{T_i \setminus \{\emptyset^{T_i}\}} \mathbf{m}^{T_i} + \mathbf{m}^{T_i}(\{\emptyset^{T_i}\}) \delta_{x_i} \right).$$

It is clear that the rooted metric space $(\hat{T}, d^{\hat{T}}, \emptyset^{\hat{T}})$ is still a rooted real tree. (Notice that it is not always true that $\hat{T}$ remains locally compact or that $\mathbf{m}^{\hat{T}}$ defines a locally finite measure on $\hat{T}$.) We will use the following notation for the grafted tree:

$$T \circledast_{i \in I} (T_i, x_i) = (\hat{T}, d^{\hat{T}}, \emptyset^{\hat{T}}, \mathbf{m}^{\hat{T}}), \tag{1}$$

where we make the convention that $T \circledast_{i \in I} (T_i, x_i) = T$ for $I = \emptyset$.

Remark 2.2. We detail here how to justify that this grafting procedure is well defined on $\mathbb{T}$. Let T and T' be two GHP-isometric trees and let φ be an isometry from T onto T'. Let $(T_i)_{i \in I}$ and $(T'_i)_{i \in I}$ be two families of trees such that, for every $i \in I$, T_i and T'_i are GHP-isometric and we denote by φ_i an isometry that maps T_i onto T'_i. Then $T \circledast_{i \in I} (T_i, x_i)$ and $T' \circledast_{i \in I} (T'_i, \varphi(x_i))$ are also isometric. It suffices to define ψ on $T \circledast_{i \in I} (T_i, x_i)$ by

$$\psi_{|T_i} = \varphi_i \quad \text{and} \quad \psi_{|T} = \varphi.$$

In Section 3.2, we shall use the grafting procedure for rooted real trees but without mass measure. Recall that $\tilde{T} = (T, \emptyset^T, d^T)$. We shall use the following notation:

$$\tilde{T} \tilde{\circledast}_{i \in I} (\tilde{T}_i, x_i) = (\hat{T}, d^{\hat{T}}, \emptyset^{\hat{T}}), \tag{2}$$

where we also make the convention that $\tilde{T} \tilde{\circledast}_{i \in I} (\tilde{T}_i, x_i) = \tilde{T}$ for $I = \emptyset$.

2.5. Sub-trees above a given level

For $T \in \mathbb{T}$ we set $H_{\max}(T) = \sup_{x \in T} d^T(\emptyset^T, x)$ the height of T and for $a \geq 0$:

$$T^{(a)} = \{x \in T,\ d(\emptyset, x) \leq a\} \quad \text{and} \quad T(a) = \{x \in T,\ d(\emptyset, x) = a\} \tag{3}$$

the restriction of the tree T under level a and the set of vertices of T at level a respectively. We denote by $(T^{i,\circ}, i \in I)$ the connected components of $T \setminus T^{(a)}$. Let $\emptyset_i$ be the MRCA of all the vertices of $T^{i,\circ}$. We consider the real tree $T^i = T^{i,\circ} \cup \{\emptyset_i\}$ rooted at point $\emptyset_i$ with mass measure $\mathbf{m}^{T^i}$ defined as the restriction of $\mathbf{m}^T$ to $T^{i,\circ}$ (hence $\mathbf{m}^{T^i}(\emptyset_i) = 0$). Notice that $T = T^{(a)} \circledast_{i \in I} (T_i, \emptyset_i)$. We will consider the point measure on $T \times \mathbb{T}$:

$$\mathcal{N}_a^T = \sum_{i \in I} \delta_{(\emptyset_i, T^i)}. \tag{4}$$

2.6. Excursion measure of a Lévy tree

Let $\alpha \in \mathbb{R}$, $\beta \geq 0$ and π be a σ-finite measure on $(0, +\infty)$ such that $\int_{(0,+\infty)} (r \wedge r^2)\pi(dr) < +\infty$. The branching mechanism ψ with characteristic (α, β, π) is defined for every $\lambda \geq 0$ by:

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,+\infty)} \left(\mathrm{e}^{-\lambda r} - 1 + \lambda r\right) \pi(dr). \tag{5}$$

Notice that $\psi'(0) = \alpha$ in this case. The branching mechanism ψ is said to be super-critical, critical, sub-critical if $\alpha < 0, \alpha = 0, \alpha > 0$. We say ψ is (sub)critical if $\alpha \geq 0$. We assume that the Grey condition holds:

$$\int^{+\infty} \frac{d\lambda}{\psi(\lambda)} < +\infty. \tag{6}$$

This assumption is used to ensure that the corresponding Lévy tree is locally compact. The Grey condition also implies that $\beta > 0$ or $\int_{(0,1)} \ell\pi(d\ell) = +\infty$ which is equivalent to the fact that the Lévy process with index ψ is of infinite variation.

Let v be the unique non-negative solution of the equation:

$$\int_{v(a)}^{+\infty} \frac{d\lambda}{\psi(\lambda)} = a.$$

Results from [13] in the (sub)critical case, using the coding of compact real trees by height function, can be extended to the super-critical case, see [3]. They can be stated in the following form. There exists a σ-finite measure $\mathbb{N}^\psi[d\mathcal{T}]$ on $\mathbb{T}$, called the excursion measure or the canonical measure of a Lévy tree, with the following properties.

(i) **Height.** For all $a > 0$, $\mathbb{N}^\psi[H_{\max}(\mathcal{T}) > a] = v(a)$.

(ii) **Mass measure.** The mass measure $\mathbf{m}^\mathcal{T}$ is supported by $\mathrm{Lf}(\mathcal{T})$, $\mathbb{N}^\psi[d\mathcal{T}]$-a.e.

(iii) **Local time.** There exists a $\mathcal{T}$-measure valued process $(\Lambda_a, a \geq 0)$ càdlàg for the weak topology on the set of finite measures on $\mathcal{T}$ such that $\mathbb{N}^\psi[d\mathcal{T}]$-a.e.:

$$\mathbf{m}^\mathcal{T}(dx) = \int_0^\infty \Lambda_a(dx)\, da, \tag{7}$$

$\Lambda_0 = 0$, $\inf\{a > 0; \Lambda_a = 0\} = \sup\{a \geq 0; \Lambda_a \neq 0\} = H_{\max}(\mathcal{T})$ and for every fixed $a \geq 0$, $\mathbb{N}^\psi[d\mathcal{T}]$-a.e.:

- The measure Λ_a is supported on $\mathcal{T}(a)$.
- We have for every bounded continuous function ϕ on $\mathcal{T}$:

$$\begin{aligned}\langle \Lambda_a, \phi \rangle &= \lim_{\varepsilon \downarrow 0} \frac{1}{v(\varepsilon)} \int \phi(x) \mathbf{1}_{\{H_{\max}(\mathcal{T}') \geq \varepsilon\}} \mathcal{N}_a^\mathcal{T}(dx, d\mathcal{T}') \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{v(\varepsilon)} \int \phi(x) \mathbf{1}_{\{H_{\max}(\mathcal{T}') \geq \varepsilon\}} \mathcal{N}_{a-\varepsilon}^\mathcal{T}(dx, d\mathcal{T}'), \quad \text{if } a > 0.\end{aligned}$$

Under $\mathbb{N}^\psi$, the real valued process $(\langle \Lambda_a, 1 \rangle, a \geq 0)$ is distributed as a CSBP with branching mechanism ψ under its canonical measure (which intuitively represents the contributions to the CSBP of the descendants of one single individual).

(iv) **Branching property.** For every $a > 0$, the conditional distribution of the point measure $\mathcal{N}_a^\mathcal{T}(dx, d\mathcal{T}')$ under $\mathbb{N}^\psi[d\mathcal{T} | H_{\max}(\mathcal{T}) > a]$, given $\mathcal{T}^{(a)}$, is that of a Poisson point measure on $\mathcal{T}(a) \times \mathbb{T}$ with intensity $\Lambda_a(dx)\mathbb{N}^\psi[d\mathcal{T}']$.

(v) **Branching points.**

- $\mathbb{N}^{\psi}[d\mathcal{T}]$-a.e., the branching points of $\mathcal{T}$ have 2 children or an infinite number of children.
- The set of binary branching points (i.e. with 2 children) is empty $\mathbb{N}^{\psi}$-a.e if $\beta = 0$ and is a countable dense subset of $\mathcal{T}$ if $\beta > 0$.
- The set $\mathrm{Br}_{\infty}(\mathcal{T})$ of infinite branching points is nonempty with $\mathbb{N}^{\psi}$-positive measure if and only if $\pi \neq 0$. If $\langle \pi, 1\rangle = +\infty$, the set $\mathrm{Br}_{\infty}(\mathcal{T})$ is $\mathbb{N}^{\psi}$-a.e. a countable dense subset of $\mathcal{T}$.

(vi) **Mass of the nodes.** The set $\{d(\emptyset, x),\ x \in \mathrm{Br}_{\infty}(\mathcal{T})\}$ coincides $\mathbb{N}^{\psi}$-a.e. with the set of discontinuity times of the mapping $a \mapsto \Lambda_a$. Moreover, $\mathbb{N}^{\psi}$-a.e., for every such discontinuity time b, there is a unique $x_b \in \mathrm{Br}_{\infty}(\mathcal{T}) \cap \mathcal{T}(b)$ and $\Delta_b > 0$, such that:

$$\Lambda_b = \Lambda_{b-} + \Delta_b \delta_{x_b},$$

where $\Delta_b > 0$ is called the mass of the node x_b. Furthermore Δ_b can be obtained by the approximation:

$$\Delta_b = \lim_{\varepsilon \to 0} \frac{1}{v(\varepsilon)} n(x_b, \varepsilon), \tag{8}$$

where $n(x_b, \varepsilon) = \int \mathbf{1}_{\{x_b\}}(x) \mathbf{1}_{\{H_{\max}(\mathcal{T}') > \varepsilon\}} \mathcal{N}_b^{\mathcal{T}}(dx, d\mathcal{T}')$ is the number of sub-trees with MRCA x_b and height larger than ε.

In order to stress the dependence in $\mathcal{T}$, we may write $\Lambda_{a,\mathcal{T}}$ for Λ_a. We set $\sigma^{\mathcal{T}}$ or simply σ when there is no confusion, for the total mass of the mass measure on $\mathcal{T}$:

$$\sigma = \mathbf{m}^{\mathcal{T}}(\mathcal{T}). \tag{9}$$

Notice that (7) readily implies that $\mathbf{m}^{\mathcal{T}}(\{x\}) = 0$ for all $x \in \mathcal{T}$.

2.7. Related measure on Lévy trees

We define a probability measure on $\mathbb{T}$ as follows. Let $r > 0$ and $\sum_{k \in \mathcal{K}} \delta_{\mathcal{T}^k}$ be a Poisson point measure on $\mathbb{T}$ with intensity $r\mathbb{N}^{\psi}$. Consider $\emptyset$ as the trivial measured rooted real tree reduced to the root with null mass measure. Define $\mathcal{T} = \emptyset \circledast_{k \in \mathcal{K}} (\mathcal{T}^k, \emptyset)$. Using Property (i) as well as (11), one easily gets that $\mathcal{T}$ is a measured locally compact rooted real tree, and thus belongs to $\mathbb{T}$. We denote by $\mathbb{P}_r^{\psi}$ its distribution. Its corresponding local time and mass measure are respectively defined by $\Lambda_a = \sum_{k \in \mathcal{K}} \Lambda_{a, \mathcal{T}^k}$ for $a \geq 0$, and $\mathbf{m}^{\mathcal{T}} = \sum_{k \in \mathcal{K}} \mathbf{m}^{\mathcal{T}^k}$. Furthermore, its total mass is defined by $\sigma = \sum_{k \in \mathcal{K}} \sigma^{\mathcal{T}^k}$. By construction, we have $\mathbb{P}_r^{\psi}(d\mathcal{T})$-a.s. $\emptyset \in \mathrm{Br}_{\infty}(\mathcal{T})$, $\Delta_{\emptyset} = r$ (see Definition (8) with $b = 0$) and $\Lambda_0 = r\delta_{\emptyset}$. Under $\mathbb{P}_r^{\psi}$ or under $\mathbb{N}^{\psi}$, we define the process $\mathcal{Z} = (\mathcal{Z}_a, a \geq 0)$ by:

$$\mathcal{Z}_a = \langle \Lambda_a, 1 \rangle.$$

According to Property (iii), under $\mathbb{P}_r^{\psi}$, the real valued process $\mathcal{Z}$ is distributed as a CSBP with branching mechanism ψ with initial value r. This remark also holds under $\mathbb{N}^{\psi}$, the corresponding measure for CSBPs being the canonical measure. Notice that (under $\mathbb{N}^{\psi}$ or $\mathbb{P}_r^{\psi}$):

$$\sigma = \int_0^{+\infty} \mathcal{Z}_a \, da = \mathbf{m}^{\mathcal{T}}(\mathcal{T}). \tag{10}$$

In particular, as σ is distributed as the total mass of a CSBP under its canonical measure, we have that $\mathbb{N}^\psi$-a.s. $\sigma > 0$ and for $q > 0$ such that $\psi(q) > 0$:

$$\mathbb{N}^\psi \left[1 - \mathrm{e}^{-\psi(q)\sigma}\right] = q \quad \text{and} \quad \mathbb{N}^\psi \left[\sigma \mathrm{e}^{-\psi(q)\sigma}\right] = \frac{1}{\psi'(q)}. \tag{11}$$

The last equation holds for $q = 0$ if $\psi'(0) > 0$, i.e. $\alpha > 0$.

We will consider the following measures on $\mathbb{T}$:

$$\mathcal{N}_\theta^\psi[d\mathcal{T}] = 2\beta\theta\mathbb{N}^\psi[d\mathcal{T}] + \int_{(0,+\infty)} \pi(dr)(1 - \mathrm{e}^{-\theta r})\mathbb{P}_r^\psi(d\mathcal{T}) \tag{12}$$

and

$$\mathbf{N}^\psi[d\mathcal{T}] = \frac{\partial}{\partial\theta} \mathcal{N}_\theta^\psi[d\mathcal{T}]_{|\theta=0} = 2\beta\mathbb{N}^\psi[d\mathcal{T}] + \int_0^{+\infty} r\pi(dr)\, \mathbb{P}_r^\psi(d\mathcal{T}). \tag{13}$$

Elementary computations yield for $q > 0$ such that $\psi(q) > 0$:

$$\begin{aligned} &\mathcal{N}_\theta^\psi \left[1 - \mathrm{e}^{-\psi(q)\sigma}\right] = \psi(\theta + q) - \psi(\theta) - \psi(q) \quad \text{and} \\ &\mathbf{N}^\psi \left[1 - \mathrm{e}^{-\psi(q)\sigma}\right] = \psi'(q) - \psi'(0). \end{aligned} \tag{14}$$

2.8. *Definition of* ψ^{-1}

Let θ^* be the root of ψ' in $[0, +\infty)$ if it exists (as ψ is a convex function, there exists at most one such root). Notice that $\theta^* = 0$ if ψ is critical and that θ^* exists and is positive if ψ is super-critical. If ψ' is everywhere positive, we set $\theta^* = 0$. The function ψ is then a one-to-one mapping from $[\theta^*, +\infty)$ onto $\psi([\theta^*, +\infty))$. We write ψ^{-1} for the inverse of the previous mapping. In particular, if $\psi'(\theta) \geq 0$ then we have $\psi^{-1}(\psi(\theta)) = \theta$; and if $\psi'(\theta) < 0$ then we have $\theta < \theta^* < \psi^{-1}(\psi(\theta))$. We set:

$$q_0 = \psi^{-1}(0). \tag{15}$$

Note that if ψ is super-critical, then $q_0 > 0$ and, thanks to (11), $\mathbb{N}^\psi[\sigma = +\infty] = \psi^{-1}(0) > 0$.

2.9. *Girsanov transformation*

In Eq. (5), the branching mechanism ψ is defined on $[0, +\infty)$. However, the definition may remain valid for negative λ. Therefore we here extend the definition of ψ for such λ.

For θ such that $\psi(\theta)$ is well defined, we consider the branching mechanism

$$\forall \lambda \geq 0, \quad \psi_\theta(\lambda) = \psi(\lambda + \theta) - \psi(\theta), \tag{16}$$

with characteristic

$$\left(\psi'(\theta), \beta, \mathrm{e}^{-\theta r}\pi(dr)\right). \tag{17}$$

We denote by Θ the set of θ such that $\psi(\theta)$ is well defined and such that ψ_θ satisfies the Grey condition (6).

Notice that $\theta^* \in \Theta$ and that ψ_{θ^*} is a (sub)critical branching mechanism.

We recall the Girsanov transformation from [1], which sums up the situation for any branching mechanism ψ. Let ψ be a branching mechanism satisfying (6). Let $\theta \in \Theta$ and $a > 0$. We set:

$$M_a^{\psi,\theta} = \exp\left\{\theta \mathcal{Z}_0 - \theta \mathcal{Z}_a - \psi(\theta)\int_0^a \mathcal{Z}_s ds\right\}.$$

Recall that $\mathcal{Z}_0 = 0$ under $\mathbb{N}^\psi$. For any non-negative measurable functional F defined on $\mathbb{T}$, we have for $\theta \in \Theta$ and $a \geq 0$:

$$\mathbb{E}_r^{\psi_\theta}[F(\mathcal{T}^{(a)})] = \mathbb{E}_r^{\psi}\left[F(\mathcal{T}^{(a)})M_a^{\psi,\theta}\right] \quad \text{and} \quad \mathbb{N}^{\psi_\theta}[F(\mathcal{T}^{(a)})] = \mathbb{N}^{\psi}\left[F(\mathcal{T}^{(a)})M_a^{\psi,\theta}\right]. \tag{18}$$

Furthermore, if $\theta \geq \theta^*$, then we have:

$$\mathbb{E}_r^{\psi_\theta}[F(\mathcal{T})] = \mathbb{E}_r^{\psi}\left[F(\mathcal{T})\mathrm{e}^{\theta r - \psi(\theta)\sigma}\mathbf{1}_{\{\sigma<+\infty\}}\right], \tag{19}$$

$$\mathbb{N}^{\psi_\theta}[F(\mathcal{T})] = \mathbb{N}^{\psi}\left[F(\mathcal{T})\mathrm{e}^{-\psi(\theta)\sigma}\mathbf{1}_{\{\sigma<+\infty\}}\right], \tag{20}$$

$$\mathbf{N}^{\psi_\theta}[F(\mathcal{T})] = \mathbf{N}^{\psi}\left[F(\mathcal{T})\mathrm{e}^{-\psi(\theta)\sigma}\mathbf{1}_{\{\sigma<+\infty\}}\right]. \tag{21}$$

We have that under $\mathbb{P}_r^\psi(d\mathcal{T})$, the random measure $\mathcal{N}_0^{\mathcal{T}}(dx, d\mathcal{T}')$, defined by (4) with $a = 0$, is a Poisson point measure on $\{\emptyset\} \times \mathbb{T}$ with intensity $r\delta_\emptyset(dx)\mathbb{N}^\psi[d\mathcal{T}']$. Then, using the first equality in (18) with $F = 1$, we get that for $\theta \geq \theta^*$ and $a > 0$,

$$\mathbb{N}^{\psi_\theta}\left[1 - \exp\left\{\theta \mathcal{Z}_a + \psi(\theta)\int_0^a \mathcal{Z}_s ds\right\}\right] = -\theta. \tag{22}$$

2.10. *Pruning Lévy trees and CRT-valued processes*

A general pruning of a Lévy tree has been defined in [5]. Under $\mathbb{N}^\psi[d\mathcal{T}]$ and conditionally on $\mathcal{T}$, we consider a mark process $M^{\mathcal{T}}(d\theta, dy)$ on the tree which is a Poisson point measure on $\mathbb{R}_+ \times \mathcal{T}$ with intensity:

$$\mathbf{1}_{[0,+\infty)}(\theta)d\theta\left(2\beta\ell^{\mathcal{T}}(dy) + \sum_{x\in \mathrm{Br}_\infty(\mathcal{T})} \Delta_x \delta_x(dy)\right).$$

The atoms $(\theta_i, y_i)_{i\in I}$ of this measure can be seen as marks that arrive on the tree, y_i being the location of the mark and θ_i the "time" at which it appears. There are two kinds of marks: some are "uniformly" distributed on the skeleton of the tree (they correspond to the term $2\beta\ell^{\mathcal{T}}$ in the intensity) whereas the others lay on the infinite branching points of the tree: an infinite branching point y being first marked after an exponential time with parameter Δ_y.

We define the pruned tree at time q as:

$$\mathcal{T}_q = \{x \in \mathcal{T},\ M^{\mathcal{T}}([0, q] \times [\![\emptyset, x[\![) = 0\}$$

with the induced metric, root $\emptyset$ and mass measure the restriction of the mass measure $\mathbf{m}^{\mathcal{T}}$. For $\theta_i \leq q$, if one cuts the tree $\mathcal{T}$ at time θ_i at point y_i, then $\mathcal{T}_q$ corresponds to the resulting sub-tree of $\mathcal{T}$ containing the root at time q. According to [5, Theorem 1.1], for fixed $q > 0$, the distribution of $\mathcal{T}_q$ under $\mathbb{N}^\psi$ is $\mathbb{N}^{\psi_q}$. We set:

$$\sigma_q = \mathbf{m}^{\mathcal{T}_q}(\mathcal{T}_q). \tag{23}$$

Because of the pruning procedure, we have $\mathcal{T}_\theta \subset \mathcal{T}_q$ for $0 \le q \le \theta$. The tree-valued process $(\mathcal{T}_q, q \ge 0)$ is a Markov process under $\mathbb{N}^\psi$, see [1]. The process $(\mathcal{T}_q, q \ge 0)$ is a non-increasing process (for the inclusion of trees), and is càdlàg. We recall the transition probabilities for the time reversed process which are given by the so-called special Markov property (see [5, Theorem 4.2] or [1, Theorem 5.6]).

Theorem 2.3. *Let ψ be a branching mechanism satisfying (6). Let $0 \le q \le \theta$ and $\mathcal{T}_\theta$ distributed according to $\mathbb{N}^{\psi_\theta}$. Conditionally on $\mathcal{T}_\theta$, let $\sum_{i \in I^{\theta,q}} \delta_{(x_i, \mathcal{T}_q^i)}$ be a Poisson point measure on $\mathcal{T}_\theta \times \mathbb{T}$ with intensity:*

$$\mathbf{m}^{\mathcal{T}_\theta}(dx)\mathcal{N}_{\theta-q}^{\psi_q}[d\mathcal{T}].$$

Then, under $\mathbb{N}^\psi$, $(\mathcal{T}_\theta, \mathcal{T}_q)$ is distributed as:

$$\left(\mathcal{T}_\theta,\ \mathcal{T}_\theta \circledast_{i \in I^{\theta,q}} (\mathcal{T}_q^i, x_i)\right).$$

According to (17), the intensity $\mathcal{N}_{\theta-q}^{\psi_q}$ is given by (12) with ψ replaced by ψ_q and $\pi(dr)$ replaced by $\mathrm{e}^{-qr}\pi(dr)$, that is:

$$\mathcal{N}_{\theta-q}^{\psi_q}[d\mathcal{T}] = 2\beta(\theta - q)\mathbb{N}^{\psi_q}[d\mathcal{T}] + \int_{(0,+\infty)} \mathrm{e}^{-qr}\pi(dr)(1 - \mathrm{e}^{-(\theta-q)r})\mathbb{P}_r^{\psi_q}(d\mathcal{T}). \tag{24}$$

The time-reversed process is a Markov process and its infinitesimal transitions are described in [3].

3. Sub-tree processes

3.1. Sub-trees of the Lévy tree

Following [14], we define a sub-tree process obtained from pruned CRTs and Poissonian selection of leaves. Let ψ be a branching mechanism satisfying (6). Recall the definition of ψ^{-1} in Section 2.8. We set:

$$\eta = \psi^{-1}(\lambda) \quad \text{for } \lambda \ge 0. \tag{25}$$

Notice that $\psi(\eta) = \lambda$ and, with q_0 defined by (15), $\eta > q_0$ if $\lambda > 0$.

Conditionally on the tree-valued process $(\mathcal{T}_\theta, \theta \in \Theta)$, let

$$\mathcal{P}^0(dtdx) = \sum_{i \in I_0} \delta_{(t_i, x_i)}(dtdx) \tag{26}$$

be a Poisson point measure on $\mathbb{R}_+ \times \mathcal{T}_0$ with intensity measure $dt\, \mathbf{m}^{\mathcal{T}_0}(dx)$. We shall refer to x_i as the marked leaves. We denote by

$$M_\lambda = \mathcal{P}^0([0, \lambda] \times \mathcal{T}_0), \quad \text{for } \lambda > 0, \quad \text{and} \quad M_0 = \lim_{\lambda \to 0} M_\lambda$$

the number of marked leaves in $\mathcal{T}_0$. Notice that $M_0 \in \{0, +\infty\}$. We shall be working on $\{M_\lambda \ge 1\}$ and consider the probability measure:

$$\mathbb{P}^{\psi,\lambda}(d\mathcal{T}) = \mathbb{N}^\psi[d\mathcal{T} \mid M_\lambda \ge 1], \quad \lambda > 0. \tag{27}$$

When ψ is super-critical, we define the probability measure

$$\mathbb{P}^{\psi,0}(d\mathcal{T}) = \mathbb{N}^{\psi}[d\mathcal{T} \mid M_0 = +\infty]. \tag{28}$$

Notice that $\eta = \mathbb{N}^{\psi}[M_\lambda \geq 1]$. We might write $\mathcal{P}^{\theta}(dt, dx) = \sum_{i \in I_\theta} \delta_{(t_i, x_i)}(dt, dx)$ for the restriction of $\mathcal{P}^0$ to $\mathbb{R}_+ \times \mathcal{T}_\theta$ for $\theta \geq 0$. On $\{M_\lambda \geq 1\}$, for $\theta \geq 0$, we define the pruned sub-tree $\tau_\theta(\lambda)$ containing the root and all the ancestors in $\mathcal{T}_\theta$ of the marked leaves of $\mathcal{T}_0$:

$$\tau_0(\lambda) = \bigcup_{i \in I_0, t_i \leq \lambda} [\![\emptyset, x_i]\!] \quad \text{and} \quad \tau_\theta(\lambda) = \tau_0(\lambda) \cap \mathcal{T}_\theta, \quad \text{if } \lambda > 0, \tag{29}$$

and if $\lambda = 0$, we set:

$$\tau_\theta(0) = \bigcap_{\lambda > 0} \tau_\theta(\lambda) = \tau_0(0) \cap \mathcal{T}_\theta. \tag{30}$$

Notice that $\tau_\theta(0) = \emptyset$ if $\mathcal{T}_0$ has finite mass measure (and $M_0 = 0$), whereas $\tau_\theta(0) \neq \emptyset$ if $\mathcal{T}_0$ has infinite mass (and $M_0 = +\infty$). By construction, we have a.s. that $\tau_\theta(\lambda)$ is compact if and only if $\mathcal{T}_\theta$ is compact (that is $\mathcal{T}_\theta$ has finite mass measure). The sub-tree $\tau_\theta(\lambda)$ of $\mathcal{T}_\theta$ and thus of $\mathcal{T}_0$ is endowed with the obvious metric. We shall consider the following mass measure on $\tau_\theta(\lambda)$:

$$\mathbf{m}^{\tau_\theta(\lambda)} = \frac{1}{\psi_\theta(\eta)} \sum_{x \in \mathrm{Lf}(\tau_\theta(\lambda))} \delta_x, \tag{31}$$

where we make the convention that $\mathbf{m}^{\tau_\theta(\lambda)} = 0$ if $\psi_\theta(\eta) = 0$. As θ varies, we obtain a sub-tree process with parameter λ: $\tau(\lambda) = (\tau_\theta(\lambda), \theta \geq 0)$ which is a non-decreasing tree-valued stochastic process, that is for $q < \theta$, $\tau_\theta(\lambda) \subset \tau_q(\lambda)$.

Remark 3.1. One may want to define the subtree process on Θ. However, for our purpose, $\tau_\theta(\lambda)$ may not make sense for some $\theta \in \Theta$ ($\psi_\theta(\eta)$ may be negative.) We will discuss this problem in Section 6.2.

3.2. Reconstruction of the Lévy tree

Let g be the generating function of a distribution $p = (p(n), n \in \mathbb{N})$ such that $g'(0) = 0$ (i.e. $p(1) = 0$) and let $c > 0$. We shall define by recursion a Galton–Watson real tree with reproduction distribution p and branch length distributed according to an exponential random variable with mean $1/c$.

Recall Notation (2) for the grafting procedure of trees without mass measure.

Definition 3.2. We say that a discrete rooted real tree $\mathcal{G}$ is a (g, c)-Galton–Watson real tree if $\mathcal{G}$ is distributed as:

$$[\![\emptyset, x]\!] \tilde{\circledast}_{1 \leq k \leq K} (\mathcal{G}_k, x),$$

with:

- $[\![\emptyset, x]\!]$ a real tree rooted at $\emptyset$ with no branching point and such that $E_\emptyset = d(\emptyset, x)$ is a random exponential variable with parameter c,
- K has generating function g and is independent of $E_\emptyset$,
- $(\mathcal{G}_k, k \in \mathbb{N}^*)$ is a sequence of independent rooted real trees which have the same distribution as $\mathcal{G}$ and are independent of $E_\emptyset$ and K.

Let $\lambda \geq 0$ and $\eta = \psi^{-1}(\lambda)$ such that $\eta > 0$. We consider the following generating function:

$$g_{(\psi,\lambda)}(r) = r + \frac{\psi((1-r)\psi^{-1}(\lambda))}{\psi^{-1}(\lambda)\psi'(\psi^{-1}(\lambda))} = r + \frac{\psi((1-r)\eta)}{\eta\psi'(\eta)}\cdot \tag{32}$$

Notice that:

$$g'_{(\psi,\lambda)}(0) = 0 \quad \text{and} \quad g'_{(\psi,\lambda)}(1) = 1 - \frac{\psi'(0)}{\psi'(\eta)}\cdot \tag{33}$$

We write $\mathcal{G}(\psi, \lambda)$ for the $(g_{\psi,\lambda}, \psi'(\eta))$-Galton–Watson real tree. According to Theorem 3.2.1 in [12], if ψ is (sub)critical, then the discrete tree $\tau_0(\lambda)$ under $\mathbb{P}^{\psi,\lambda}$ is distributed as a Galton–Watson tree $\mathcal{G}(\psi, \lambda)$ with mass measure given by (31). Furthermore, we can reconstruct the Lévy tree $\mathcal{T}$ from $\tau_0(\lambda)$, thanks to [14]. For this, recall Definition (13) of $\mathbf{N}^\psi$ and define the following probability measure on $\mathbb{R}_+$:

$$\Gamma^{\psi}_{d,\lambda}(dr) = \mathbf{1}_{\{d=2\}} \frac{2\beta}{\psi''(\eta)} \delta_0(dr) + \frac{r^d \mathrm{e}^{-r\eta}}{|\psi^{(d)}(\eta)|} \pi(dr). \tag{34}$$

Theorem 3.3 (*Theorem 5.6 of [14]*)**.** *Assume that ψ is (sub)critical and (6) holds. Let $\lambda > 0$ and $\eta = \psi^{-1}(\lambda)$. Under $\mathbb{P}^{\psi,\lambda}$ and conditionally on $\tau_0(\lambda)$, $\mathcal{T}_0$ is distributed as:*

$$\tilde{\tau}_0(\lambda) \circledast_{i\in I} (\mathcal{T}_i, x_i) \circledast_{x\in \mathrm{Br}(\tau_0(\lambda))} (\mathcal{T}'_x, x),$$

with:

- *$\tilde{\tau}_0(\lambda)$ as $\tau_0(\lambda)$ but with 0 as mass measure,*
- *$\sum_{i\in I} \delta_{(x_i,\mathcal{T}_i)}$ is a Poisson point measure on $\tau_0(\lambda) \times \mathbb{T}$ with intensity $\ell^{\tau_0(\lambda)}(dx)\, \mathbf{N}^{\psi_\eta}[d\mathcal{T}]$,*
- *conditionally on $\sum_{i\in I} \delta_{(x_i,\mathcal{T}_i)}$, the trees $\left(\mathcal{T}'_x, x \in \mathrm{Br}(\tau_0(\lambda))\right)$ are independent with $\mathcal{T}'_x$ distributed as*
$$\int \Gamma^{\psi}_{\kappa(x),\lambda}(dr)\, \mathbb{P}^{\psi_\eta}_r[d\mathcal{T}].$$

Remark 3.4. In fact, in Theorem 5.6 of [14], ψ can be super-critical and $\lambda \geq 0$ with $\eta = \psi^{-1}(\lambda) > 0$. But it is not obvious that in this case the super-critical Lévy tree distribution defined in [14] and the super-critical Lévy tree distribution defined in [1] and recalled here in Section 2.6, are in fact the same. However, we deduce from Remark 5.2 that this equality indeed holds.

4. Marginal distributions

The main goal of this section is to study the one-dimensional distribution of the sub-tree process $\tau(\lambda) = (\tau_\theta(\lambda), \theta \geq 0)$, see Corollary 4.5. We first give an application of the special Markov property.

Proposition 4.1. *Let ψ be a branching mechanism satisfying (6). Let $\lambda \geq 0$ and $\eta = \psi^{-1}(\lambda)$. Under $\mathbb{N}^\psi$, the couple of trees $(\mathcal{T}_\theta, \tau_\theta(\lambda))$ on $\{M_\lambda \geq 1\}$ is distributed as $(\mathcal{T}_0, \tau_0(\psi_\theta(\eta)))$ under $\mathbb{N}^{\psi_\theta}$ on $\{M_{\psi_\theta(\eta)} \geq 1\}$.*

Proof. We first assume $\lambda > 0$. From the special Markov property of Theorem 2.3 for the process $(\mathcal{T}(\theta), \theta \geq 0)$ under $\mathbb{N}^\psi$, we get:

$$\mathcal{T}_0 = \mathcal{T}_\theta \circledast_{j\in J^{\theta,0}} (\mathcal{T}^j_0, y_j), \tag{35}$$

where $\sum_{j\in J^{\theta,0}} \delta_{(y_j,\mathcal{T}_0^j)}$ is, conditionally on $\mathcal{T}_\theta$, a Poisson point measure on $\mathcal{T}_\theta \times \mathbb{T}$ with intensity $\mathbf{m}^{\mathcal{T}_\theta}(dy)\mathcal{N}_\theta^\psi[d\mathcal{T}]$.

Recall that $\mathcal{P}^0(dtdx) = \sum_{i\in I_0} \delta_{(t_i,x_i)}(dtdx)$ is a Poisson point measure on $\mathbb{R}_+ \times \mathcal{T}_0$ defined in (26) and $\mathcal{P}^\theta$ is the restriction to $\mathbb{R}_+ \times \mathcal{T}_\theta$ of $\mathcal{P}^0$. Thus $\mathcal{P}_1^\theta = \sum_{i\in I_\theta} \delta_{x_i}\mathbf{1}_{\{t_i\leq\lambda\}}$ is a Poisson point measure on $\mathcal{T}_\theta$ with intensity $\lambda\mathbf{m}^{\mathcal{T}_\theta}(dx)$.

For $j \in J^{\theta,0}$, let $s_j = \inf\{t_i; x_i \in \mathcal{T}_0^j \text{ for } i \in I_0\}$. Notice that conditionally on $\mathcal{T}_0^j$, s_j has an exponential distribution with parameter $\lambda\mathbf{m}^{\mathcal{T}_0}(\mathcal{T}_0^j)$. We deduce that, conditionally on $\mathcal{T}_\theta$, $\mathcal{P}_2^\theta = \sum_{j\in J^{\theta,0}} \delta_{y_j}\mathbf{1}_{\{s_j\leq\lambda\}}$ is a Poisson point measure on $\mathcal{T}_\theta \times \mathbb{R}_+$ with intensity:

$$\mathbf{m}^{\mathcal{T}_\theta}(dx)\mathcal{N}_\theta^\psi\left[1 - \mathrm{e}^{-\lambda\sigma}\right] = [\psi(\theta+\eta) - \psi(\theta) - \psi(\eta)]\, \mathbf{m}^{\mathcal{T}_\theta}(dx),$$

where we use (14) to get the equality. By construction $\mathcal{P}_1^\theta$ and $\mathcal{P}_2^\theta$ are independent Poisson point measures. Therefore, $\mathcal{P}_1^\theta + \mathcal{P}_2^\theta$ is a Poisson point measure with intensity:

$$\mathbf{m}^{\mathcal{T}_\theta}(dx)\,[\lambda + \psi(\theta+\eta) - \psi(\theta) - \psi(\eta)] = \psi_\theta(\eta)\mathbf{m}^{\mathcal{T}_\theta}(dx).$$

To conclude, notice that $\tau_\theta(\lambda)$ is the sub-tree generated by the marked leaves before time λ of $\mathcal{T}_\theta$, which are given by the atoms of $\mathcal{P}_1^\theta$, and the roots x_j of the trees $\mathcal{T}_0^j$ having marked leaves before time λ, that is the atoms of $\mathcal{P}_2^\theta$. Then use that $\mathcal{T}_\theta$ under $\mathbb{N}^\psi$ is distributed as $\mathcal{T}_0$ under $\mathbb{N}^{\psi_\theta}$ to conclude.

If $\lambda = 0$, then $\eta = q_0$. We have $\mathcal{P}_1^\theta = 0$ and $\mathcal{T}_0^j$ contributes to $\tau_\theta(0)$ if and only if it has infinite mass. So, in the previous argument, one has to replace $\mathcal{P}_2^\theta$ by $\sum_{j\in J^{\theta,0}} \delta_{y_j}\mathbf{1}_{\{\sigma^{\mathcal{T}_0^j}=+\infty\}}$ which is a Poisson point measure with intensity:

$$\mathbf{m}^{\mathcal{T}_\theta}(dx)\mathcal{N}_\theta^\psi\,[\sigma = +\infty] = \psi_\theta(\eta)\,\mathbf{m}^{\mathcal{T}_\theta}(dx).$$

Hence the conclusion follows. $\square$

Remark 4.2. Assume $\lambda > 0$. Using the notation from the previous proof, for $k \in \mathbb{N}^*$, we let:

$$Y_k = \mathrm{Card}\,\{j \in J^{\theta,0};\ \mathrm{Card}\,\left(\mathrm{Lf}(\tau_0(\lambda)) \cap \mathcal{T}_0^j\right) = k\}$$

be the number of trees grafted on $\mathcal{T}_\theta$ having exactly k leaves marked at time λ and $Y_0 = \langle\mathcal{P}_1^\theta, \mathbf{1}\rangle = \mathrm{Card}\ (\mathrm{Lf}(\tau_\theta(\lambda)) \cap \mathrm{Lf}(\mathcal{T}))$ be the number of marked leaves on $\mathcal{T}_\theta$. We get that under $\mathbb{N}^\psi$, conditionally on $\mathcal{T}_\theta$, the random variables $(Y_k, k \in \mathbb{N})$ are independent, Y_0 is Poisson with parameter $\lambda\sigma_\theta$, and for $k \in \mathbb{N}^*$, Y_k is Poisson with parameter $\sigma_\theta\mathcal{N}_\theta^\psi\left[(\lambda\sigma)^k\mathrm{e}^{-\lambda\sigma}\right]/k!$, where $\sigma_\theta = \mathbf{m}^{\mathcal{T}_\theta}(\mathcal{T}_\theta)$.

Using the Girsanov transformation from Section 2.9, we will give a Girsanov transformation for $\theta \mapsto \tau_\theta(\lambda)$.

Recall first Notation (3) for the truncated tree at height a. For $\mathcal{T} \in \mathbb{T}$, let $L(\mathcal{T}) = \mathrm{Card}\,\mathrm{Lf}(\mathcal{T})$ be the number of leaves of the tree $\mathcal{T}$ and let

$$L(a, \mathcal{T}) = L(a, \mathcal{T}^{(a)}) = \mathrm{Card}\,\{x \in \mathcal{T}; d^{\mathcal{T}}(\emptyset, x) = a\} \tag{36}$$

be the number of elements of $\mathcal{T}$ at distance a from the root. Note that:

$$\eta = \psi^{-1}(\lambda) = q_0 + \psi_{q_0}^{-1}(\lambda), \quad \text{and} \quad \psi'(\eta) = \psi'(\psi^{-1}(\lambda)) = \psi_{q_0}'(\psi_{q_0}^{-1}(\lambda)). \tag{37}$$

We first state a preliminary Lemma. Let $\mathrm{P}^{\psi,\lambda}(d\mathcal{G})$ denote the distribution of the Galton–Watson tree $\mathcal{G}(\psi, \lambda)$ defined in Section 3.2.

Lemma 4.3. *Let ψ be a branching mechanism satisfying (6). Let $\lambda \geq 0$ and $\eta = \psi^{-1}(\lambda) > 0$. For any non-negative measurable function F on $\mathbb{T}$ and $a \geq 0$, we have:*

$$\mathrm{E}^{\psi,\lambda}[F(\mathcal{G}^{(a)})] = \mathrm{E}^{\psi_{q_0},\lambda}\left[F(\mathcal{G}^{(a)})\left(\frac{\eta}{\eta - q_0}\right)^{L(a,\mathcal{G})-1}\right].$$

Proof. Let $(p_{(\psi,\lambda)}(n), n \in \mathbb{N})$ be the probability measure determined by $g_{(\psi,\lambda)}$ defined by (32). Then $p_{(\psi,\lambda)}(1) = 0$ and for $n \neq 1$, we have:

$$p_{(\psi,\lambda)}(n) = \frac{g^{(n)}_{(\psi,\lambda)}(0)}{n!} = \frac{|\psi^{(n)}(\eta)|\,\eta^{n-1}}{\psi'(\eta)n!}. \tag{38}$$

Thanks to (37), we have $\psi_{q_0}^{-1}(\lambda) = \eta - q_0$ and for $n \geq 0, \psi^{(n)}(\eta) = \psi_{q_0}^{(n)}(\eta - q_0)$. Set $u = (\eta - q_0)/\eta$. Then, we have for $n \in \mathbb{N}$:

$$\begin{aligned} p_{(\psi_{q_0},\lambda)}(n) &= \frac{|\psi_{q_0}^{(n)}(\psi_{q_0}^{-1}(\lambda))|\,(\psi_{q_0}^{-1}(\lambda))^{n-1}}{\psi'_{q_0}(\psi_{q_0}^{-1}(\lambda))n!}\mathbf{1}_{\{n\neq 1\}} \\ &= u^{n-1}\frac{|\psi^{(n)}(\eta)|\,\eta^{n-1}}{\psi'(\eta)n!}\mathbf{1}_{\{n\neq 1\}} = u^{n-1}p_{(\psi,\lambda)}(n). \end{aligned} \tag{39}$$

$\mathrm{P}^{\psi_{q_0},\lambda}(d\mathcal{G})$-a.s. for fixed a, $\mathrm{Card}(\mathrm{Lf}(\mathcal{G}^{(a)}) \cap \mathrm{Lf}(\mathcal{G})) = \mathrm{Card}\{x \in \mathrm{Lf}(\mathcal{G}^{(a)}); d^{\mathcal{G}}(\emptyset, x) < a\} = L(\mathcal{G}^{(a)}) - L(a, \mathcal{G})$. Thanks to (37), the individual lifetimes under $\mathrm{P}^{\psi,\lambda}$ and $\mathrm{P}^{\psi_{q_0},\lambda}$ have the same distribution. Recall that κ_x is the number of children of x. Therefore, we have:

$$\begin{aligned} \mathrm{E}^{\psi,\lambda}[F(\mathcal{G}^{(a)})] &= \mathrm{E}^{\psi_{q_0},\lambda}\left[F(\mathcal{G}^{(a)})\left(\frac{p_{(\psi,\lambda)}(0)}{p_{(\psi_{q_0},\lambda)}(0)}\right)^{L(\mathcal{G}^{(a)})-L(a,\mathcal{G})}\prod_{x\in\mathrm{Br}(\mathcal{G}^{(a)})}\frac{p_{(\psi,\lambda)}(\kappa_x)}{p_{(\psi_{q_0},\lambda)}(\kappa_x)}\right] \\ &= \mathrm{E}^{\psi_{q_0},\lambda}\left[F(\mathcal{G}^{(a)})u^{L(\mathcal{G}^{(a)})-L(a,\mathcal{G})-\sum_{x\in\mathrm{Br}(\mathcal{G}^{(a)})}(\kappa_x-1)}\right] \\ &= \mathrm{E}^{\psi_{q_0},\lambda}\left[F(\mathcal{G}^{(a)})u^{1-L(a,\mathcal{G})}\right], \end{aligned}$$

where the last equality is a consequence of the following fact for finite discrete trees $\mathcal{G}$:

$$1 + \sum_{x\in\mathrm{Br}(\mathcal{G})}(\kappa_x - 1) = L(\mathcal{G}). \quad \square$$

Recall (36). We shall consider the following processes:

$$\tau^{(a)}_{\theta,\lambda} = \{\tau^{(a)}_\theta(z), z \geq \lambda\}, \qquad L(a, \tau_{\theta,\lambda}) = L(a, \tau^{(a)}_{\theta,\lambda}) = \{L(a, \tau^{(a)}_\theta(z)), z \geq \lambda\}.$$

We have the following Girsanov theorem.

Theorem 4.4. *Let ψ be a branching mechanism satisfying (6). Let $\lambda > 0$ and $\eta = \psi^{-1}(\lambda)$. If ψ is super-critical, then for any non-negative measurable functional H on the Skorokhod space $\mathbb{D}([\lambda, +\infty), \mathbb{T})$, we have:*

$$\mathbb{N}^{\psi}\left[H(\tau^{(a)}_{0,\lambda})\mathbf{1}_{\{M_\lambda\geq 1\}}\right] = \mathbb{N}^{\psi_{q_0}}\left[\left(\frac{\eta}{\eta - q_0}\right)^{L(a,\tau_0(\lambda))} H(\tau^{(a)}_{0,\lambda})\mathbf{1}_{\{M_\lambda\geq 1\}}\right]. \tag{40}$$

Proof. Recall that the random measure $\mathcal{N}_a^{\mathcal{T}}$ defined in (4) which, according to the branching property (iv), conditionally on $\mathcal{T}^{(a)}$, is a Poisson point measure with intensity $\ell^a(dx)\mathbb{N}^\psi[dT]$. Since the sub-trees are determined by marked leaves of Lévy trees according to a Poisson point measure, we will deduce the desired result by counting the number of marked leaves and using Girsanov transformation (18). We deduce that, conditionally on $\mathcal{T}^{(a)}$, $L(a, \tau_0(\lambda)) = L(a, \tau_0^{(a)}(\lambda))$ is a Poisson random variable with parameter:

$$\mathbb{N}^\psi\left[1 - \mathrm{e}^{-\lambda\sigma}\right]\mathcal{Z}_a = \eta\mathcal{Z}_a.$$

Let $\mathcal{P}^\psi$ be, conditionally on $\mathcal{T}^{(a)}$ and $\tau_0^{(a)}(\lambda)$, a Poisson point measure on $[\lambda, +\infty)$ with intensity $\mathcal{Z}_a\left(\psi^{-1}\right)'(z)\,dz$. We consider the family of random variables:

$$\mathcal{P}_\lambda^\psi = \{\mathcal{P}^\psi([\lambda, z]), z \geq \lambda\}.$$

Using again the branching property (iv), we get that, under $\mathbb{N}^\psi$ and conditionally on $\mathcal{T}^{(a)}$, $L\left(a, \tau_{0,\lambda}\right)$ is distributed as $L\,(a, \tau_0(\lambda)) + \mathcal{P}_\lambda^\psi := \{L\,(a, \tau_0(\lambda)) + \mathcal{P}^\psi([\lambda, z]), z \geq \lambda\}$. Then notice that the first equality of (37) implies that $\mathcal{P}_\lambda^\psi$ under $\mathbb{N}^\psi\left[\,\cdot\,|\mathcal{T}^{(a)}\right]$ is distributed as $\mathcal{P}_\lambda^{\psi_{q_0}}$ under $\mathbb{N}^{\psi_{q_0}}\left[\,\cdot\,|\mathcal{T}^{(a)}\right]$. We set:

$$F(\mathcal{T}^{(a)}, L(a, \tau_{0,\lambda}^{(a)})) = \mathbb{N}^\psi\left[H(\tau_{0,\lambda}^{(a)})\mathbf{1}_{\{M_\lambda \geq 1\}} \mid \mathcal{T}^{(a)}, L(a, \tau_{0,\lambda}^{(a)})\right].$$

We deduce that:

$$\begin{aligned}
\mathbb{N}^\psi\left[H(\tau_{0,\lambda}^{(a)})\mathbf{1}_{\{M_\lambda \geq 1\}}\right] &= \mathbb{N}^\psi\left[F(\mathcal{T}^{(a)}, L(a, \tau_{0,\lambda}^{(a)}))\right] \\
&= \mathbb{N}^\psi\left[F(\mathcal{T}^{(a)}, L(a, \tau_0^{(a)}(\lambda)) + \mathcal{P}_\lambda^\psi)\right] \\
&= \mathbb{N}^\psi\left[\sum_{k=0}^\infty F(\mathcal{T}^{(a)}, k + \mathcal{P}_\lambda^\psi)\frac{(\eta\mathcal{Z}_a)^k}{k!}\mathrm{e}^{-\eta\mathcal{Z}_a}\right] \\
&= \mathbb{N}^{\psi_{q_0}}\left[\sum_{k=0}^\infty F(\mathcal{T}^{(a)}, k + \mathcal{P}_\lambda^{\psi_{q_0}})\frac{(\eta\mathcal{Z}_a)^k}{k!}\mathrm{e}^{-(\eta-q_0)\mathcal{Z}_a}\right],
\end{aligned}$$

where we used the conditional independence of $\mathcal{P}^\psi$ and $\tau_0^{(a)}(\lambda)$ given $\mathcal{T}^{(a)}$ for the third equality, the Girsanov transformation (18) for the last equality (and that $\psi(q_0) = 0$). Using $\psi_{q_0}^{-1}(\lambda) = \eta - q_0$, we notice that $L(a, \tau_0(\lambda))$ is under $\mathbb{N}^{\psi_{q_0}}\left[\,\cdot\,|\mathcal{T}^{(a)}\right]$ a Poisson random variable with parameter:

$$\mathbb{N}^{\psi_{q_0}}\left[1 - \mathrm{e}^{-\lambda\sigma}\right]\mathcal{Z}_a = (\eta - q_0)\mathcal{Z}_a.$$

Therefore, we obtain:

$$\begin{aligned}
\mathbb{N}^\psi\left[H(\tau_{0,\lambda}^{(a)})\mathbf{1}_{\{M_\lambda \geq 1\}}\right] &= \mathbb{N}^{\psi_{q_0}}\left[\sum_{k=0}^\infty\left(\frac{\eta}{\eta - q_0}\right)^k F(\mathcal{T}^{(a)}, k + \mathcal{P}_\lambda^{\psi_{q_0}})\frac{((\eta - q_0)\mathcal{Z}_a)^k}{k!}\mathrm{e}^{-(\eta-q_0)\mathcal{Z}_a}\right] \\
&= \mathbb{N}^{\psi_{q_0}}\left[\left(\frac{\eta}{\eta - q_0}\right)^{L(a,\tau_0(\lambda))} F(\mathcal{T}^{(a)}, L(a, \tau_0(\lambda)) + \mathcal{P}_\lambda^{\psi_{q_0}})\right] \\
&= \mathbb{N}^{\psi_{q_0}}\left[\left(\frac{\eta}{\eta - q_0}\right)^{L(a,\tau_0(\lambda))} F(\mathcal{T}^{(a)}, L(a, \tau_{0,\lambda}^{(0)}))\right],
\end{aligned}$$

where we used for the last equality that under $\mathbb{N}^{\psi_{q_0}}$ and conditionally on $\mathcal{T}^{(a)}$, $L\left(a, \tau_{0,\lambda}^{(a)}\right)$ is distributed as $L(a, \tau_0(\lambda)) + \mathcal{P}_\lambda^{\psi_{q_0}}$.

By construction, the distribution of $\tau_{0,\lambda}^{(a)}$ conditionally on $\mathcal{T}^{(a)}$ and $L(a, \tau_{0,\lambda}^{(a)})$ is the same under $\mathbb{N}^\psi$ and $\mathbb{N}^{\psi_\theta}$ for any $\theta > 0$ and in particular for $\theta = q_0$. We deduce (40). $\square$

We immediately deduce the following corollary.

Corollary 4.5. *Let ψ be a branching mechanism satisfying (6). Let $\lambda > 0$ and $\eta = \psi^{-1}(\lambda)$. Under $\mathbb{P}^{\psi,\lambda}$, for each $\theta \geq 0$, the sub-tree $\tau_\theta(\lambda)$ is distributed as the Galton–Watson real tree $\mathcal{G}(\psi_\theta, \psi_\theta(\eta))$ with mass measure given by (31).*

Recall that $\mathrm{P}^{\psi,\lambda}(d\mathcal{G})$ denotes the distribution of the Galton–Watson tree $\mathcal{G}(\psi, \lambda)$ defined in Section 3.2.

Proof. If ψ is (sub)critical, then this is a consequence of Theorem 3.2.1 in [12] and Proposition 4.1. Now we assume that ψ is super-critical. Notice that:

$$\frac{\eta}{\eta - q_0} = \frac{\mathbb{N}^\psi[M_\lambda \geq 1]}{\mathbb{N}^{\psi_{q_0}}[M_\lambda \geq 1]}.$$

Using Theorem 4.4, this gives that for $a > 0$ and G a non-negative measurable functional defined on $\mathbb{T}$:

$$\mathbb{E}^{\psi,\lambda}\left[G(\tau_0^{(a)}(\lambda))\right] = \mathbb{E}^{\psi_{q_0},\lambda}\left[\left(\frac{\eta}{\eta - q_0}\right)^{L(a,\tau_0(\lambda))-1} G(\tau_0^{(a)}(\lambda))\right].$$

Recall that if $(\mathcal{T}, \emptyset, d, \mathbf{m})$ is a measured rooted real tree, then we denote by $\tilde{\mathcal{T}}$ the real tree $(\mathcal{T}, \emptyset, d)$. Since ψ_{q_0} is sub-critical, thanks to Theorem 3.2.1 in [12], we get that under $\mathbb{P}^{\psi_{q_0},\lambda}$, $\tilde{\tau}_0(\lambda)$ has distribution $\mathrm{P}^{\psi_{q_0},\lambda}$. Then by Lemma 4.3, we get that under $\mathbb{P}^{\psi,\lambda}$, $\tilde{\tau}_0(\lambda)$ has distribution $\mathrm{P}^{\psi,\lambda}$. Then use Proposition 4.1 to get that for each $\theta \geq 0$, $\tilde{\tau}_\theta(\lambda)$ under $\mathbb{P}^{\psi,\lambda}$ has distribution $\mathrm{P}^{\psi_\theta,\psi_\theta(\eta)}$. $\square$

The following corollary is another direct consequence of Theorem 4.4.

Corollary 4.6. *Let $\lambda > 0$ and $a > 0$ be fixed. Under $\mathbb{N}^{\psi_{q_0}}$ on $\{M_\lambda \geq 1\}$, the process $(Q_z, z \geq \lambda)$ defined by:*

$$Q_z = \left(\frac{\psi^{-1}(z)}{\psi^{-1}(z) - q_0}\right)^{L(a,\tau_0(z))}$$

is a backward martingale with respect to the filtration $(\mathcal{Q}_z, z \geq \lambda)$ with $\mathcal{Q}_z = \sigma(\tau_0(z'); z' \geq z)$.

We present another Girsanov transformation for sub-trees.

Remark 4.7. Let ψ be a branching mechanism satisfying (6). For any $q \geq \theta \geq 0$, $a > 0$ and F a non-negative measurable functional, we have:

$$\mathbb{E}^{\psi,\lambda}\left[F(\tilde{\tau}_q^{(a)}(\lambda))\right] = \mathbb{E}^{\psi,\lambda}\left[F(\tilde{\tau}_\theta^{(a)}(\lambda)) N_{a,\lambda}^{\theta,q}(\tau_\theta(\lambda))\right], \tag{41}$$

where $N_{a,\lambda}^{\theta,q}$ is defined for discrete trees by:

$$N_{a,\lambda}^{\theta,q}(T) = \left(\frac{\psi_q(\eta)}{\psi_\theta(\eta)}\right)^{L(T^{(a)})-L(a,T)} \mathrm{e}^{(\psi'(\theta+\eta)-\psi'(q+\eta))\ell^T(T^{(a)})} \prod_{x\in \mathrm{Br}(T^{(a)})} \frac{\psi_q^{(\kappa_x)}(\eta)}{\psi_\theta^{(\kappa_x)}(\eta)},$$

with the convention $\prod_{x\in\emptyset} = 1$. Under $\mathbb{P}^{\psi,\lambda}$, the process $N_\lambda^{\theta,q} = \left(N_{a,\lambda}^{\theta,q}(\tilde{\tau}_\theta(\lambda)), a \geq 0\right)$ is a martingale with respect to the filtration $\left(\sigma(\tilde{\tau}_\theta^{(a)}(\lambda)), a \geq 0\right)$.

5. Convergence of the sub-tree processes

We provide an alternative proof of the convergence of the sub-trees to the Lévy tree from [14] using the Gromov–Hausdorff–Prohorov distance on $\mathbb{T}$ which relies on the Girsanov transformation. Recall that for simplicity, we identify T and $(T, d^T, \emptyset^T, \mathbf{m}^T) \in \mathbb{T}$. And, under $\mathbb{P}_r^\psi$ or $\mathbb{N}^\psi$, the mass measure on $\tau_0(\lambda)$ is given by (31).

Theorem 5.1. *Let ψ be a branching mechanism satisfying (6). We have $\mathbb{N}^\psi$-a.e. or $\mathbb{P}_r^\psi$-a.s.:*

$$\lim_{\lambda\to+\infty} d_{GHP}(\mathcal{T}, \tau_0(\lambda)) = 0. \tag{42}$$

Proof. Under $\mathbb{N}^\psi$, the convergence (42) is a consequence of Lemma 5.4 (see also Proposition 2.8 in [4] to get the d_{GHP} convergence from the d_{GHP}^c convergence) for the (sub)critical case and Lemma 5.5 for the super-critical case. Then the $\mathbb{P}_r^\psi$-a.s. convergence is a consequence of the representation of $\mathbb{P}_r^\psi$ from Section 2.7. $\square$

Remark 5.2. Notice in particular that Theorem 5.1 asserts that $(\mathcal{F}, (\mathcal{F}(\lambda), \lambda \geq 0))$ in [14] and $(\mathcal{T}, (\tau_0(\lambda), \lambda \geq 0))$ have the same distribution. In particular, this implies that the distribution for super-critical Lévy trees defined in [14] based on a coloring leaves process and the one defined in [3] based on a Girsanov transformation are the same. Therefore, Theorem 3.3 is also valid for ψ super-critical and $\lambda > 0$ with $\psi^{-1}(\lambda) > 0$.

Lemma 5.4 is stated in Section 5.2 and Lemma 5.5 is proved in Section 5.3. Section 5.1 presents preliminaries on approximation of trees by discrete sub-trees.

5.1. Distance between trees and discrete sub-trees

In this section, we present an immediate convergence result from sub-trees to trees which could be coded by functions.

Let f be a non-negative continuous function with compact support s.t. $f(0) = 0$. We set $\sigma = \sup\{t;\, f(t) > 0\}$. We define:

$$d^f(x, y) = f(x) + f(y) - 2 \inf_{u\in[x\wedge y, x\vee y]} f(u)$$

and the equivalence relation: $x \sim y$ if $d^f(x, y) = 0$. We set $T^f = [0, \sigma]/\sim$. Let p^f be the projection from $[0, \sigma]$ to T^f, with $p^f(x)$ the equivalent class of x in T^f. Let $\mathbf{m}^f$ be the image of the Lebesgue measure on $[0, \sigma]$ by the projection p^f. Set $\emptyset^f = p^f(0)$ and we still denote by d^f the distance on T, image of d^f by p^f. It is well known that $(T^f, d^f, \emptyset^f, \mathbf{m}^f)$ is a measured rooted compact real tree.

Let $\Delta = \{y_0, \dots, y_{N_\Delta}\}$, with $1 \le N_\Delta < +\infty$ and $0 = y_0 < \dots < y_{N_\Delta} \le \sigma$, be a finite subdivision of $[0, \sigma]$. Let $|\Delta| = \sup_{0 \le i < N_\Delta} y_{i+1} - y_i$ be the mesh of the subdivision. For $0 \le i < N_\Delta$, let $\bar{y}_i \in [y_i, y_{i+1}]$ such that $f(\bar{y}_i) = \inf_{u \in [y_i, y_{i+1}]} f(u)$. We consider f_Δ the linear interpolation of the points $\{(y_i, f(y_i)), (\bar{y}_i, f(\bar{y}_i)); 0 \le i < N_\Delta\} \cup \{(y_{N_\Delta}, f(y_{N_\Delta}))\}$.

Denote T^{f_Δ}, the tree coded by f_Δ. Let $a_\Delta \ge 0$ and $\mathbf{m}^{f,\Delta}$ be the image of the measure $\mu_\Delta = a_\Delta \sum_{y \in \Delta, y \neq 0} \delta_y$ by the projection p^f. We consider the measured rooted real tree $T^{f,\Delta} = (T^{f_\Delta}, d^{f_\Delta}, \emptyset^f, \mathbf{m}^{f,\Delta})$. Then we have

Lemma 5.3.

$$d^c_{GHP}(T^f, T^{f,\Delta}) \le \sup_{|x-y| \le |\Delta|} |f(x) - f(y)| + d_P^{[0,\sigma]}(Leb, \mu_\Delta), \tag{43}$$

where Leb is the Lebesgue measure on $[0, \sigma]$, *and the space* $[0, \sigma]$ *is endowed with the usual distance.*

Proof. By construction T^{f_Δ} is the smallest sub-tree of T^f containing $\{p^f(y_i), 0 \le i \le N_\Delta\}$. That is $T^{f_\Delta} = \bigcup_{y_i \in \Delta} [\![\emptyset^f, p^f(y_i)]\!]$. Note that $f(t) > 0$ for $t \in (0, \sigma)$. Then

$$\begin{aligned} d_H^{T^f}(T^f, T^{f_\Delta}) &\le \max_{0 < i < N_\Delta} \sup\{d^f(p^f(y_i), x), p^f(y_i) \in [\![\emptyset^f, x]\!]\} \\ &\le \sup_{|x-y| \le |\Delta|} |f(x) - f(y)|. \end{aligned}$$

On the other hand, let A be a Borel set of T^f. We can also regard $\mathbf{m}^{f,\Delta}$ as a measure on T^f by $\mathbf{m}^{f,\Delta}(A) = \mathbf{m}^{f,\Delta}(A \cap T^{f_\Delta})$. Set $I = \{t \in [0, \sigma], p^f(t) \in A\}$. By definition of $\mathbf{m}^f$, we have $\mathbf{m}^f(A) = \text{Leb}(I)$. Set $A^\Delta = p^{f_\Delta}(I)$. We have $A^\Delta = A \cap T^{f_\Delta}$ and $\mathbf{m}^{f,\Delta}(A^\Delta) = \mathbf{m}^{f,\Delta}(A) = \mu_\Delta(I)$. Thus $d_P^{T^f}(\mathbf{m}^f, \mathbf{m}^{f,\Delta}) = d_P^{[0,\sigma]}(\text{Leb}, \mu_\Delta)$. Then the desired result follows from the definition of d^c_{GHP}. $\square$

5.2. The (sub)critical case

The main result of this section is the following lemma.

Lemma 5.4. *Let ψ be a (sub)critical branching mechanism satisfying (6). We have $\mathbb{N}^\psi$-a.e. for all $a_0 \ge 0$:*

$$\lim_{\lambda \to +\infty} d^c_{GHP}(\mathcal{T}, \tau_0(\lambda)) = 0 \quad \text{and} \quad \lim_{\lambda \to +\infty} \sup_{a \le a_0} d^c_{GHP}(\mathcal{T}^{(a)}, \tau_0^{(a)}(\lambda)) = 0. \tag{44}$$

Proof. According to [12], there exists a continuous stochastic process h, called the height process, such that under its excursion measure it has compact support $[0, \sigma^h]$ and (T^h, σ^h) is distributed at $(\mathcal{T}, \sigma)$ under $\mathbb{N}^\psi$. Notice that the continuity of the height process is a consequence of (6).

Conditionally on h, let $\mathcal{P} = \sum_{i \in I} \delta_{(y_i, t_i)}$ be a Poisson point measure on $[0, \sigma] \times \mathbb{R}_+$ with intensity $dydt$. For $\lambda > 0$, we set:

$$\Delta_\lambda = \{y_i; i \in I \text{ and } t_i \le \lambda\} \cup \{0\} \quad \text{and} \quad \mu_{\Delta_\lambda} = \frac{1}{\lambda} \sum_{y \in \Delta_\lambda, y \neq 0} \delta_y.$$

By construction, we get the following equality in distribution:

$$(T^h, (T^{h,\Delta_\lambda}, \lambda \ge 0)) \stackrel{(d)}{=} (\mathcal{T}, (\tau_0(\lambda), \lambda \ge 0)).$$

The properties of the Poisson point measures imply that a.e. under the excursion measure of h, $\lim_{\lambda\to+\infty} |\Delta_\lambda| = 0$ and $\lim_{\lambda\to+\infty} d_{\mathrm{P}}^{[0,\sigma^h]}(\mathrm{Leb}, \mu_{\Delta_\lambda}) = 0$. Thus, we deduce from Section 5.1 and (43) that a.e. under the excursion measure of h,

$$\lim_{\lambda\to+\infty} d^c_{\mathrm{GHP}}(T^h, T^{h,\Delta_\lambda}) = 0.$$

Thus, we obtain the first part of (44).

We set $\varepsilon_\lambda = d^c_{\mathrm{GHP}}(\mathcal{T}, \tau_0(\lambda))$. According to the proof of Proposition 2.8 in [4], we have, for $a \geq 0$:

$$d^c_{\mathrm{GHP}}(\mathcal{T}^{(a)}, \tau_0^{(a)}(\lambda)) \leq 3\varepsilon_\lambda + \mathbf{m}^{\mathcal{T}}\left(\mathcal{T}^{(a+2\varepsilon_\lambda)} \setminus \mathcal{T}^{(a-\varepsilon_\lambda)}\right).$$

Using (7) and the definition of $\mathcal{Z}$, we deduce that for $a_0 \geq 0$:

$$\sup_{a\leq a_0} d^c_{\mathrm{GHP}}(\mathcal{T}^{(a)}, \tau_0^{(a)}(\lambda)) \leq 3\left(1 + \sup_{a\leq a_0+2\varepsilon_\lambda} \mathcal{Z}_a\right)\varepsilon_\lambda.$$

We deduce then the second part of (44) from the first part of (44). This ends the proof of the Lemma. □

5.3. The super-critical case

The main result of this section is the following lemma.

Lemma 5.5. *Let ψ be a super-critical branching mechanism satisfying (6). We have $\mathbb{N}^\psi$-a.e.:*

$$\lim_{\lambda\to+\infty} d_{GHP}(\mathcal{T}, \tau_0(\lambda)) = 0.$$

Proof. Recall that ψ_{q_0} is (sub)critical. We deduce from Theorem 4.4 that for $a > 0$:

$$\begin{aligned}
&\mathbb{N}^\psi\left[\liminf_{\lambda\to+\infty} \int_0^a \mathrm{e}^{-r}\left(1 \wedge d^c_{\mathrm{GHP}}\left(\mathcal{T}^{(r)}, \tau_0^{(r)}(\lambda)\right)\right) dr > 0,\ M_\lambda \geq 1\right]\\
&\quad = \mathbb{N}^{\psi_{q_0}}\left[\left(\frac{\psi^{-1}(1)}{\psi^{-1}(1) - q_0}\right)^{L(a,\tau_0(1))}, \liminf_{\lambda\to+\infty} \int_0^a \mathrm{e}^{-r}\right.\\
&\qquad \left.\times \left(1 \wedge d^c_{\mathrm{GHP}}\left(\mathcal{T}^{(r)}, \tau_0^{(r)}(\lambda)\right)\right) dr > 0, M_\lambda \geq 1\right].
\end{aligned}$$

Then use (44) to get that the right hand-side in the previous equality is 0 for all $a > 0$. This implies that $\mathbb{N}^\psi\left[\liminf_{\lambda\to+\infty} d_{\mathrm{GHP}}(\mathcal{T}, \tau_0(\lambda)) > 0,\ M_\lambda \geq 1\right] = 0$. □

6. Pruning and growth of the discrete sub-trees

6.1. The pruning process

Recall that $\eta = \psi^{-1}(\lambda)$ and $\psi^{-1}(0) = q_0$ which is the largest root of $\psi(s) = 0$. We assume that $\eta > 0$ which is equivalent to $\lambda > q_0$.

We define the following pruning procedure for the discrete sub-trees. Under $\mathbb{P}^{\psi,\lambda}$, let $\mathfrak{T}$ be distributed as $\tau_0(\lambda)$. Conditionally on $\mathfrak{T}$, we consider a Poisson point measure $\mathcal{M}^{\mathrm{Sk}}(d\theta, dy)$ on $\mathbb{R}_+ \times \mathfrak{T}$ with intensity:

$$\psi''(\eta+\theta)\mathbf{1}_{[0,+\infty)}(\theta)d\theta\, \ell^{\mathfrak{T}}(dy)$$

and an independent family of independent random variables $(\xi_x, x \in \mathrm{Br}(\mathfrak{T}))$, such that the distribution of ξ_x has density:

$$-\frac{\psi^{(\kappa_x+1)}(\eta+z)}{\psi^{(\kappa_x)}(\eta)}\mathbf{1}_{\{z>0\}}dz.$$

Recall that $\mathcal{M}^{\mathrm{Sk}}(d\theta, dy)$ is defined above. We define the following random measure:

$$\mathcal{M}^{\mathfrak{T}}(d\theta, dy) = \mathcal{M}^{\mathrm{Sk}}(d\theta, dy) + \sum_{x\in \mathrm{Br}(\mathfrak{T})} \delta_{(\xi_x, x)}(d\theta, dy).$$

We define the pruned tree at time $q \geq 0$ as:

$$\mathfrak{T}_q = \{x \in \mathfrak{T},\ \mathcal{M}^{\mathfrak{T}}([0,q] \times [\![\emptyset, x[\![) = 0\}$$

equipped with the induced metric, with the root $\emptyset$ and with the measure

$$\mathbf{m}^{\mathfrak{T}_q} = \frac{1}{\psi_q(\eta)} \sum_{x\in \mathrm{Lf}(\mathfrak{T}_q)} \delta_x,$$

where we make the convention that $\mathbf{m}^{\mathfrak{T}_q} = 0$ if $\psi_q(\eta) = 0$. Then we have the following theorem.

Theorem 6.1. *Let ψ be a branching mechanism satisfying (6). Let $\lambda \geq 0$. We assume that $\psi^{-1}(\lambda) = \eta > 0$. Then under $\mathbb{P}^{\psi,\lambda}$, the two processes $(\tau_\theta(\lambda), \theta \geq 0)$ and $(\mathfrak{T}_\theta, \theta \geq 0)$ have the same distribution.*

Proof. The proof is based on Theorem 3.3 and Remark 5.2. Notice that the processes $(\tau_\theta(\lambda), \theta \geq 0)$ and $(\mathfrak{T}_\theta, \theta \geq 0)$ are by construction Markov and right continuous. Therefore, it is enough to check that the two-dimensional marginals have the same distribution.

Let $\theta \geq q \geq 0$. Recall the pruning procedure defined in Section 2.10. On one hand, a mark appears on the skeleton of $\tau_q(\lambda)$ before time θ is either on the skeleton of $\mathcal{T}_q$ or at a branching point of $\mathcal{T}_q$. Those marks appearing before time θ that are on the skeleton of $\mathcal{T}_q$ are distributed as a Poisson point process with intensity $2\beta(\theta-q)\ell^{\tau_q(\lambda)}(dy)$. A node of $\mathcal{T}_q$ with mass r has a mark before time θ with probability $1-\mathrm{e}^{-(\theta-q)r}$. And the nodes of $\mathcal{T}_q$ with mass r which lie on the skeleton of $\tau_q(\lambda)$ are, thanks to Theorem 3.3, distributed on $\tau_q(\lambda)$ according to a Poisson point measure with intensity $r\mathrm{e}^{-r\eta-rq}\pi(dr)\,\ell^{\tau_q(\lambda)}(dy)$. This implies that the marks on the skeleton of $\tau_q(\lambda)$ before time θ are distributed according to a Poisson point measure with intensity:

$$\begin{aligned}
&\left[2\beta(\theta-q) + \int_{(0,+\infty)} (1-\mathrm{e}^{-(\theta-q)r})r\mathrm{e}^{-r\eta-rq}\pi(dr)\right] \ell^{\tau_q(\lambda)}(dy)\\
&\quad = \left[\psi'_\theta(\eta) - \psi'_q(\eta)\right] \ell^{\tau_q(\lambda)}(dy)\\
&\quad = \left[\int_q^\theta \psi''(\eta+z)dz\right] \ell^{\tau_q(\lambda)}(dy).
\end{aligned}$$

On the other hand, if x is a branching point of $\tau_q(\lambda)$ with number of children κ_x, then a mark appears on it before time θ, if it appears before time θ on $\mathcal{T}_q$. Recall $\mathbb{P}^{\psi,\lambda}$ from (27) and (28). Also recall that we assume $\eta = \psi^{-1}(\lambda) > 0$. Proposition 4.1 applied to $\theta = q$ entails that $(\mathcal{T}_q, \tau_q(\lambda))$ under $\mathbb{P}^{\psi,\lambda}$ has the same distribution as $(\mathcal{T}_0, \tau_0(\psi_q(\eta)))$ under $\mathbb{P}^{\psi_q,\psi_q(\eta)}$. By Remark 5.2, Theorem 3.3 is also valid for ψ super-critical and $\lambda \geq 0$ with $\psi^{-1}(\lambda) > 0$. Then we apply Theorem 3.3 under $\mathbb{P}^{\psi_q,\psi_q(\eta)}$ with ψ and λ replaced by ψ_q and $\psi_q(\eta)$, respectively. This implies that conditionally on $\tau_0(\psi_q(\eta))$ the mass Δ_x is distributed according to $\Gamma^{\psi_q}_{\kappa_x,\psi_q(\eta)}$ defined by (34). Therefore a mark appears on the node x of $\tau_q(\lambda)$ before time θ with probability:

$$\int \Gamma^{\psi_q}_{\kappa_x,\psi_q(\eta)}(dr)(1 - \mathrm{e}^{-(\theta-q)r}) = 1 - \frac{\psi_\theta^{(\kappa_x)}(\eta)}{\psi_q^{(\kappa_x)}(\eta)} = \mathbb{P}(\xi_x < \theta | \xi_x > q).$$

By construction of $\mathfrak{T}_\theta$ from $\mathfrak{T}_q$, we deduce that if $\mathfrak{T}_q$ has the same distribution as $\tau_q(\lambda)$, then $(\mathfrak{T}_q, \mathfrak{T}_\theta)$ has the same distribution as $(\tau_q(\lambda), \tau_\theta(\lambda))$. Use that $\mathfrak{T}_0$ is distributed as $\tau_0(\lambda)$, to deduce that $\mathfrak{T}_\theta$ has the same distribution as $\tau_\theta(\lambda)$. Thus, we get that the processes $(\tau_\theta(\lambda), \theta \geq 0)$ and $(\mathfrak{T}_\theta, \theta \geq 0)$ have the same two-dimensional marginals distribution. □

Remark 6.2. Recall the definition of $\mathbb{P}^{\psi,\lambda}$ (27) and (28). Assume that $\eta = \psi^{-1}(\lambda) > 0$. By construction of $\mathfrak{T}$ and thanks to Proposition 4.1, we get that $(\tau_{\theta+q}(\lambda), \theta \geq 0)$ under $\mathbb{P}^{\psi,\lambda}$ is distributed as $(\tau_\theta(\lambda), \theta \geq 0)$ under $\mathbb{P}^{\psi_q,\psi_q(\eta)}$.

6.2. The growth process

Let $\lambda > 0$. Theorem 6.1 gives the pruning procedure of the sub-tree process. Conversely, we will also give a growth procedure for the time reversed sub-tree process. The process $\theta \mapsto \mathcal{T}_\theta$ can be defined on Θ. However, this is no more the case for $\tau_\theta(\lambda)$ ($\psi_\theta(\eta)$ may be negative). Note that $\theta \mapsto \psi_\theta(\lambda)$ is increasing for all $\lambda > 0$. Recall that $\psi_\theta(\eta) \geq \lambda > 0$ for $\theta \geq 0$. We define:

$$\theta_\lambda = \inf\{\theta \in \Theta; \psi_\theta(\eta) \geq 0\} \quad \text{and} \quad \Theta^{\psi,\lambda} = [\theta_\lambda, +\infty) \cap \Theta. \tag{45}$$

Notice that $\theta_\lambda \leq 0$.

Remark 6.3. If $\theta_\lambda \in \Theta^{\psi,\lambda}$, then we have $\psi_{\theta_\lambda}(\eta) = 0$. And since $\eta > 0$, we further have that ψ_{θ_λ} is super-critical. Then Theorem 6.1 and Remark 6.2 are applicable with ψ and λ replaced by ψ_{θ_λ} and $\psi_{\theta_\lambda}(\eta)$, respectively. Typically, under $\mathbb{P}^{\psi_{\theta_\lambda},0}$, $(\tau_\theta(0), \theta \geq 0)$ is well defined.

Recall Definition 3.2 and (32). Then Theorem 6.1, Remark 6.2, Corollary 4.5 and the Kolmogorov extension theorem ensure the following proposition.

Proposition 6.4. *Let $\lambda > 0$ and let ψ be a branching mechanism satisfying (6). There exists a process $(\tau_\theta(\lambda), \theta \in \Theta^{\psi,\lambda})$ under $\mathbb{P}^{\psi,\lambda}$, such that for all $q \in \Theta^{\psi,\lambda}$ the process $(\tau_{\theta+q}(\lambda), \theta \geq 0)$ is distributed as $(\tau_\theta(\psi_q(\eta)), \theta \geq 0)$ under $\mathbb{P}^{\psi_q,\psi_q(\eta)}$. Furthermore, under $\mathbb{P}^{\psi,\lambda}$, for each $\theta \in \Theta^{\psi,\lambda}$, $\tau_\theta(\lambda)$ is distributed as the $(g_{\psi_\theta,\psi_\theta(\lambda)}, \psi'_\theta(\eta))$-Galton–Watson real tree with mass measure given by (31).*

Remark 6.5. Note that $\Theta^{\psi,\lambda} = [\theta_\lambda, +\infty)$ or $(\theta_\lambda, +\infty)$. In the first case we could define the process $\{\tau_\theta(\lambda), \theta \in \Theta^{\psi,\lambda}\}$ under $\mathbb{P}^{\psi_{\theta_\lambda},0}$ according to the arguments in Section 3.1 (we need to shift the time by $\theta - \theta_\lambda, \theta \in \Theta^{\psi,\lambda}$). However in the second case, $\mathbb{N}^{\psi_{\theta_\lambda}}$ does not make sense. We have to use the Kolmogorov extension theorem to show the existence of a tree-valued Markov process $\{\tau_\theta(\lambda), \theta \in \Theta^{\psi,\lambda}\}$.

We consider the function $g^{q,\theta}_{(\psi,\lambda)}$ defined for $q \in \Theta^{\psi,\lambda}$ and $\theta > q$ by:

$$g^{q,\theta}_{(\psi,\lambda)}(r) = 1 - \frac{\psi_\theta(\eta(1-r)) - \psi_q(\eta(1-r))}{\psi_\theta(\eta)}. \tag{46}$$

Notice that $\psi_\theta(\eta) > 0$ for $\theta > q$ and thus $g^{q,\theta}_{(\psi,\lambda)}(1) = 1$, $g^{q,\theta}_{(\psi,\lambda)}(0) = \psi_q(\eta)/\psi_\theta(\eta)$ and for $k \in \mathbb{N}^*$:

$$\left(g^{q,\theta}_{(\psi,\lambda)}\right)^{(k)}(0) = \frac{(-1)^{k+1}\eta^k}{\psi_\theta(\eta)}(\psi^{(k)}(\theta+\eta) - \psi^{(k)}(q+\eta)) \geq 0.$$

Since ψ is analytic at least on $(\theta_\lambda, +\infty)$, we deduce that $g^{q,\theta}_{(\psi,\lambda)}(r)$ is the generating function of a random variable K taking values in $\mathbb{N}$. Let $(\tau^k, k \in \mathbb{N}^*)$ be independent random trees distributed as $\tau_q(\lambda)$ under $\mathbb{P}^{\psi,\lambda}$ and independent of K. We set:

$$\mathcal{G}_{q,\theta}(\psi,\lambda) = \emptyset \circledast_{1\leq k\leq K} (\tau^k, \emptyset),$$

with the convention that $\emptyset \circledast_{1\leq k\leq K} (\tau^k, \emptyset) = \emptyset$ if $K = 0$.

Theorem 6.6. *Let ψ be a branching mechanism satisfying (6). Let $\lambda > 0$ and $\eta = \psi^{-1}(\lambda)$. Let $\theta > q$ with $q \in \Theta^{\psi,\lambda}$. Then under $\mathbb{P}^{\psi,\lambda}$, conditionally on $\tau_\theta(\lambda)$, $\tau_q(\lambda)$ is distributed as*

$$\tau_\theta(\lambda) \circledast_{x\in \mathrm{Lf}(\tau_\theta(\lambda))} (\mathcal{G}^x_q, x),$$

with mass measure given by (31) (with θ replaced by q) and where $(\mathcal{G}^x_q, x \in \mathrm{Lf}(\tau_\theta(\lambda)))$ are independent and distributed according to $\mathcal{G}_{q,\theta}(\psi,\lambda)$.

We first state a preliminary lemma.

Lemma 6.7. *Under the Hypothesis of Theorem 6.6, the sub-tree $\tau_0(\psi_q(\eta))$ is distributed under $\mathcal{N}^{\psi_q}_{\theta-q}[\,\cdot\,|M_{\psi_q(\eta)} \geq 1]$ as $\mathcal{G}_{q,\theta}(\psi,\lambda)$ conditionally on $\mathcal{G}_{q,\theta}(\psi,\lambda) \neq \emptyset$.*

Proof. By construction of $\mathcal{G}_{q,\theta}(\psi,\lambda)$, the lemma will be proved as soon as we check that the degree of the root of $\tau_0(\psi_q(\eta))$ under $\mathcal{N}^{\psi_q}_{\theta-q}\left[\,\cdot \mid M_{\psi_q(\eta)} \geq 1\right]$ is distributed as K conditionally on $\{K \geq 1\}$.

We only need to prove the lemma for (sub)critical ψ, $\lambda > 0$ and $\theta \geq 0$, which therefore applies to ψ_q, $\psi_q(\eta)$ and $\theta - q$. Let $N_\emptyset$ be the degree of the root $\emptyset$ in $\tau_0(\lambda)$. Notice that $\{M_\lambda \geq 1\} = \{N_\emptyset \geq 1\}$. We set $h(u) = \mathcal{N}^\psi_\theta\left[u^{N_\emptyset}\mathbf{1}_{\{N_\emptyset\geq 1\}}\right]$. Notice that, under $\mathbb{N}^\psi$, $N_\emptyset$ is 0 or 1 and that, under $\mathbb{P}^\psi_r$, $N_\emptyset$ is a Poisson random variable with mean $r\mathbb{N}^\psi[M_\lambda \geq 1] = r\eta$. We deduce that for $u \in [0,1]$:

$$\begin{aligned} h(u) &= 2\beta\theta u \mathbb{N}^\psi[M_\lambda \geq 1] + \int_{(0,+\infty)} \pi(dr)(1-\mathrm{e}^{-\theta r})\mathbb{E}^\psi_r\left[u^{N_\emptyset}\mathbf{1}_{\{N_\emptyset \geq 1\}}\right] \\ &= 2\beta\theta\eta u + \int_{(0,+\infty)} \pi(dr)(1-\mathrm{e}^{-\theta r})(\mathrm{e}^{-r\eta(1-u)} - \mathrm{e}^{-r\eta}). \end{aligned}$$

Let $g_0 = g^{0,\theta}_{(\psi,\lambda)}$ be the generating function of K and g_1 be the generating function of K conditionally on $\{K \geq 1\}$. Elementary computations yield $g_0(u) = g_0(0) + h(u)/\psi_\theta(\eta)$. We deduce that $g_1(u) = h(u)/h(1)$. This readily implies that $N_\emptyset$ under $\mathcal{N}^\psi_\theta[\,\cdot \mid M_\lambda \geq 1]$ is distributed as K conditionally on $\{K \geq 1\}$. $\square$

Proof of Theorem 6.6. The proof is very similar to the proof of Proposition 4.1. From the special Markov property Theorem 2.3, we get:

$$\mathcal{T}_q = \mathcal{T}_\theta \circledast_{j\in J^{\theta,q}} (\mathcal{T}_q^j, x_j),$$

where $\sum_{j\in J^{\theta,q}} \delta_{(x_j,\mathcal{T}_q^j)}$ is, conditionally on $\mathcal{T}_\theta$, a Poisson point measure on $\mathcal{T}_\theta \times \mathbb{T}$ with intensity $\mathbf{m}^{\mathcal{T}_\theta}(dx)\mathcal{N}^{\psi_q}_{\theta-q}[d\mathcal{T}]$. Notice that $\mathcal{T}_q^j$ gives a contribution to $\tau_q(\lambda)$ (that is $\mathcal{T}_q^j \cap \tau_q(\lambda) \neq \emptyset$) if there is at least one marked leaf on $\mathcal{T}_q^j$. Furthermore, if there is a contribution, then $\mathcal{T}_q^j \cap \tau_q(\lambda)$ is distributed as $\tau_0(\psi_q(\eta))$ under $\mathcal{N}^{\psi_q}_{\theta-q}[\,\cdot\,|M_{\psi_q(\eta)} \geq 1]$ (note that the root of $\mathcal{T}_q^j \cap \tau_q(\lambda)$ is x_j). This distribution is given in Lemma 6.7. Thanks to (14), we have:

$$\begin{aligned}\mathcal{N}^{\psi_q}_{\theta-q}[M_{\psi_q(\eta)} \geq 1] &= \mathcal{N}^{\psi_q}_{\theta-q}[1 - \mathrm{e}^{-\psi_q(\eta)\sigma}] \\ &= \psi(\theta+\eta) - \psi(\theta) - \psi_q(\eta) = \psi_\theta(\eta) - \psi_q(\eta).\end{aligned} \tag{47}$$

Standard results on marked Poisson point process imply that the point measure on the leaves of $\tau_q(\lambda)$ which are still in $\tau_\theta(\lambda)$, that is $\sum_{x\in \mathrm{Lf}(\tau_\theta(\lambda))\cap \mathrm{Lf}(\tau_q(\lambda))} \delta_x(dy)$, is, conditionally on $\mathcal{T}_\theta$, a Poisson point process on $\mathcal{T}_\theta$ with intensity $\psi_q(\eta)\mathbf{m}^{\mathcal{T}_\theta}(dy)$, and is also independent of $\sum_{j\in J^{\theta,q}} \delta_{(x_j,\mathcal{T}_q^j)}$.

Using standard results on marked Poisson point measures, we get that $\tau_q(\lambda)$ can be recovered from $\tau_\theta(\lambda)$ by grafting independently on each leaf $x \in \mathrm{Lf}(\tau_\theta(\lambda))$:

- Nothing with probability $\psi_q(\eta)/\psi_\theta(\eta)$.
- A sub-tree distributed as $\mathcal{G}_{q,\theta}(\psi,\lambda)$ conditionally on $\mathcal{G}_{q,\theta}(\psi,\lambda) \neq \emptyset$ with probability $1 - \psi_q(\eta)/\psi_\theta(\eta)$.

Then use that $\mathrm{P}(\mathcal{G}_{q,\theta}(\psi,\lambda) = \emptyset) = \mathrm{P}(K = 0) = \psi_q(\eta)/\psi_\theta(\eta)$ and that the mass measure of $\tau_q(\lambda)$ is given by (31) (with θ replaced by q) to end the proof. $\square$

Remark 6.8. We deduce from Theorem 6.6 that the transition rate (for the backward process) at time θ from $\tau_\theta(\lambda)$ to $\tau_\theta(\lambda) \circledast_{1\leq k\leq k_0} (\tau^k, x)$, with x a leaf of $\tau_\theta(\lambda)$, is given by:

$$\frac{(-1)^{k_0+1}\eta^{k_0}}{k_0!}\,\frac{\psi^{(k_0+1)}(\theta+\eta)}{\psi_\theta(\eta)}\mu_\theta(d\tau^1)\cdots\mu_\theta(d\tau^{k_0}),$$

with μ_θ the distribution of $\tau_\theta(\lambda)$ under $\mathbb{P}^{\psi,\lambda}$. The mass measure process is always defined by (31).

7. Study of leaves

Recall the definition of σ_θ from (23). We first present a martingale based on the total mass of the pruned process.

Proposition 7.1. *Let ψ be a branching mechanism satisfying (6). Then under $\mathbb{P}^\psi_r$ and $\mathbb{N}^\psi$, the process $(R_\theta, \theta > q_0)$, with:*

$$R_\theta = \psi'(\theta)\sigma_\theta,$$

is a backward martingale with respect to the filtration $(\mathcal{F}_\theta, \theta > q_0)$ where $\mathcal{F}_\theta = \sigma(\mathcal{T}_q, q \geq \theta)$.

Proof. Let $q_0 < q \leq \theta$. According to the special Markov property, we have:

$$(\mathcal{T}_\theta, \sigma_q) \stackrel{(d)}{=} \left(\mathcal{T}_\theta, \sigma_\theta + \sum_{i\in I} \sigma^{\mathcal{T}^i}\right),$$

where $\sum_{i\in I} \delta_{\mathcal{T}^i}$ is conditionally on $\mathcal{T}_\theta$ a Poisson point measure on $\mathbb{T}$ with intensity:

$$\mathbf{m}^{\mathcal{T}_\theta}(dx)\mathcal{N}^{\psi_q}_{\theta-q}[d\mathcal{T}].$$

Using (14) with ψ_q and $\theta - q$ instead of ψ and q, respectively, we have:

$$\mathbb{E}^\psi_r[\sigma_q|\mathcal{F}_\theta] = \mathbb{E}^\psi_r\left[\sigma_\theta + \sum_{i\in I}\sigma^{\mathcal{T}^i}\middle|\mathcal{F}_\theta\right] = \sigma_\theta + \sigma_\theta\mathcal{N}^{\psi_q}_{\theta-q}[\sigma] = \frac{\psi'(\theta)}{\psi'(q)}\sigma_\theta.$$

This gives the result under $\mathbb{P}^\psi_r$. The proof is similar under $\mathbb{N}^\psi$. □

Notice that Proposition 7.1 is also a direct consequence of the infinitesimal transitions of the time-reversed process $(\mathcal{T}_\theta, \theta \in \Theta)$ given in [3].

Now we present a result on the number of leaves for the sub-tree process. Let $\lambda \geq 0$. We consider the leaves process of the sub-trees $L(\lambda) = \{L_\theta(\lambda), \theta \in \Theta^{\psi,\lambda}\}$:

$$L_\theta(\lambda) = L(\tau_\theta(\lambda)) = \mathrm{Card}\,(\mathrm{Lf}(\tau_\theta(\lambda))).$$

Proposition 7.2. *Let ψ be a branching mechanism satisfying (6). Let $\lambda \geq 0$ and $\eta = \psi^{-1}(\lambda) > 0$. Under $\mathbb{P}^{\psi,\lambda}$, the process $(R_\theta(\lambda), \theta > q_0)$ with:*

$$R_\theta(\lambda) = \frac{\psi'(\theta)}{\psi_\theta(\eta)}L_\theta(\lambda),$$

is a backward martingale with respect to the filtration $(\mathcal{H}_\theta, \theta > q_0)$, where $\mathcal{H}_\theta = \sigma(\tau_q(\lambda), q \geq \theta)$.

Remark 7.3. Notice that $L_\theta(\lambda)/\psi_\theta(\eta)$ is the total mass of $\mathbf{m}^{\tau_\theta(\lambda)}$ which converges to the total mass of $\mathbf{m}^{\mathcal{T}_\theta}$ that is σ_θ as λ goes to infinity. Thus Proposition 7.1 appears as a consequence of Proposition 7.2.

Recall the definition of $g_{(\psi,\lambda)}$ (32) and of $g^{q,\theta}_{(\psi,\lambda)}$ (46). For $\theta \geq q$ and $q \in \Theta^{\psi,\lambda}$, we set:

$$g_q(r) = g_{(\psi_q,\psi_q(\eta))}(r) \quad \text{and} \quad g(r) = g^{q,\theta}_{(\psi,\lambda)}(r). \tag{48}$$

Proof of Proposition 7.2. We write L_θ for $L_\theta(\lambda)$. Let $q_0 < q \leq \theta$. By Theorem 6.6, we have:

$$\mathbb{E}^{\psi,\lambda}\left[L_q|\tau_\theta(\lambda)\right] = L_\theta g(0) + L_\theta g'(1)\mathbb{E}^{\psi,\lambda}\left[L_q\right]. \tag{49}$$

Thanks to Corollary 4.5 and the branching property, we have:

$$\mathbb{E}^{\psi,\lambda}\left[L_q\right] = g_q(0) + \mathbb{E}^{\psi,\lambda}\left[L_q\right]g'_q(1). \tag{50}$$

This gives:

$$\mathbb{P}^{\psi,\lambda}\left[L_q\right] = \frac{g_q(0)}{1-g'_q(1)} = \frac{\psi_q(\eta)}{\eta\psi'(q)}.$$

Then use that:

$$g(0) = \frac{\psi_q(\eta)}{\psi_\theta(\eta)}, \qquad g'(1) = \frac{\eta}{\psi_\theta(\eta)} \left(\psi'(\theta) - \psi'(q)\right),$$

and (49) to get that:

$$\mathbb{E}^{\psi,\lambda}\left[L_q|\tau_\theta(\lambda)\right] = \frac{\psi_q(\eta)}{\psi'(q)} \frac{\psi'(\theta)}{\psi_\theta(\eta)} L_\theta.$$

This gives the result. $\square$

Remark 7.4. A similar result for the leaves process of the discrete time Galton–Watson tree-valued process was proved in Corollary 3.4 of [2] using a quantity similar to $(1 - g_q'(1))/g_q(0)$ which comes from (50).

For $\theta > \theta_\lambda$, the function g_θ is convex positive with $g_\theta(0) > 0$ and $g_\theta(1) = 1$. Hence, for $\zeta \in [0, 1)$, the equation:

$$x = g_\theta(x) + g_\theta(0)(\zeta - 1)$$

has a unique solution $x \in [0, 1]$, which we denote by $h_\theta(\zeta)$. By construction the backward process $(L_\theta(\lambda), \theta > \theta_\lambda)$ is Markov under $\mathbb{P}^{\psi,\lambda}$. The next proposition gives its one and two-dimensional marginals.

Proposition 7.5. *Let ψ be a branching mechanism satisfying (6). For $\theta \geq q > \theta_\lambda$ and $\zeta, z \in [0, 1)$, we have:*

$$\mathbb{E}^{\psi,\lambda}\left[\zeta^{L_\theta(\lambda)}\right] = h_\theta(\zeta), \quad \text{and} \quad \mathbb{E}^{\psi,\lambda}\left[\zeta^{L_\theta(\lambda)} z^{L_q(\lambda)}\right] = h_\theta(\zeta w^{q,\theta}(z)), \tag{51}$$

with $w^{q,\theta}(z) = g(h_q(z)) + g(0)(z - 1)$.

Proof. We write L_θ for $L_\theta(\lambda)$. Conditioning on the number of children of the lowest branching point and using the branching property of the Galton–Watson trees $\tau_\theta(\lambda)$, we get:

$$\mathbb{E}^{\psi,\lambda}\left[\zeta^{L_\theta}\right] = g_\theta(0)\zeta + \sum_{k=1}^{\infty} \mathbb{E}^{\psi,\lambda}\left[\zeta^{L_\theta}\right]^k \frac{g_\theta^{(k)}(0)}{k!} = g_\theta\left(\mathbb{E}^{\psi,\lambda}\left[\zeta^{L_\theta}\right]\right) + g_\theta(0)(\zeta - 1).$$

This gives the first part of (51). Recall $\mathcal{G}_{q,\theta}(\psi, \lambda)$ defined in Section 6. Using again the branching property, we have:

$$\mathbb{E}\left[z^{L(\mathcal{G}_{q,\theta}(\psi,\lambda))}\right] = g(0)z + g(h_q(z)) - g(0).$$

Then, by Theorem 6.6, we have:

$$\begin{aligned}
\mathbb{E}^{\psi,\lambda}\left[\zeta^{L_\theta} z^{L_q}\right] &= \mathbb{E}^{\psi,\lambda}\left[\zeta^{L_\theta} z^{\sum_{x\in \mathrm{Lf}(\tau_\theta(\lambda))} L(\mathcal{G}^x_{q,\theta}(\psi,\lambda))}\right] \\
&= \mathbb{E}^{\psi,\lambda}\left[\zeta^{L_\theta} (g(h_q(z)) + g(0)(z - 1))^{L_\theta}\right] \\
&= h_\theta\Big(\zeta (g(h_q(z)) + g(0)(z - 1))\Big).
\end{aligned}$$

This ends the proof. $\square$

Example 7.6. Assume $\psi(u) = \beta u^2$, with $\beta > 0$, so that $\Theta = \mathbb{R}$ and $q_0 = 0$. Let $\lambda > 0$. We have $\eta = \sqrt{\lambda/\beta}$, $\theta_\lambda = -\eta/2$ and $\Theta^{\psi,\lambda} = [\theta_\lambda, +\infty)$. For $\theta > \theta_\lambda$ and $\zeta \in [0, 1)$, we have:

$$\mathbb{E}^{\psi,\lambda}\left[\zeta^{L_\theta(\lambda)}\right] = \frac{\eta + \theta - \sqrt{\theta^2\zeta + (1-\zeta)(\theta+\eta)^2}}{\eta}$$

and for $\theta > 0$:

$$\mathbb{E}^{\psi,\lambda}\left[L_\theta(\lambda)\right] = \frac{\eta + 2\theta}{2|\theta|}\cdot$$

When $\theta_\lambda < \theta < 0$, $\mathbb{P}^{\psi,\lambda}(L_\theta(\lambda) = +\infty) > 0$. Then we deduce in (59) below for a related result. For $\theta = \theta_\lambda$, we have $g_{\theta_\lambda}(0) = 0$, and the tree $\tau_{\theta_\lambda}(\lambda)$ is a Yule tree which has no leaf (formally, we have $\mathbb{E}^{\psi,\lambda}\left[\zeta^{L_{\theta_\lambda}(\lambda)}\right] = 0$).

8. Ascension time and tree at the ascension time

For convenience, we assume in this section that ψ is a critical branching mechanism satisfying (6). Then $\tau_\theta(\lambda)$ (or $\mathcal{T}_\theta$) is super-critical, critical or subcritical for $\theta < 0, \theta = 0$ or $\theta > 0$, respectively.

8.1. Ascension process and ascension time

Let $\lambda > 0$. Recall θ_λ and $\Theta^{\psi,\lambda}$ defined in Section 6.2. From now on, we shall always assume that

$$\theta_\lambda < 0,$$

which implies that $\inf \Theta < 0$. Define the ascension time on $\{M_\lambda \geq 1\}$:

$$A_\lambda = \inf\{\theta \in \Theta^{\psi,\lambda};\ \tau_\theta(\lambda) \text{ is a compact tree}\}. \tag{52}$$

$\mathbb{P}^{\psi,\lambda}$-a.s., we have $A_\lambda \leq 0$. Since, by construction, $\tau_\theta(\lambda)$ is a compact tree if and only if $\mathcal{T}_\theta$ is a compact tree, we have $A_\lambda = \inf\{\theta \in \Theta^{\psi,\lambda} : \sigma_\theta < \infty\}$. For $\theta \in \Theta$, we set $\bar\theta = \psi^{-1}(\psi(\theta))$, so that $\bar\theta$ is the unique positive number such that:

$$\psi(\bar\theta) = \psi(\theta). \tag{53}$$

By Theorem 6.5 of [1] and its proof, we have for all $\theta \in \Theta$:

$$\bar\theta - \theta = \psi_\theta^{-1}(0). \tag{54}$$

Recall $g_\theta(r)$ from (32) and (48). Notice that $1 - \frac{\bar\theta - \theta}{\eta}$ is the minimal solution of the equation $r = g_\theta(r)$. Since $\tau_\theta(\lambda)$ is under $\mathbb{P}^{\psi,\lambda}$ a Galton–Watson tree whose reproduction law has generating function g_θ, we deduce that for $\theta \in (\theta_\lambda, 0)$:

$$\mathbb{P}^{\psi,\lambda}(A_\lambda < \theta) = \mathbb{P}^{\psi,\lambda}(\tau_\theta(\lambda) \text{ is compact}) = 1 - \frac{\bar\theta - \theta}{\eta}\cdot \tag{55}$$

Since $d\bar\theta/d\theta = \psi'(\theta)/\psi'(\bar\theta)$, we have for $\theta_\lambda < \theta < 0$:

$$\mathbb{P}^{\psi,\lambda}(A_\lambda \in d\theta) = \frac{1}{\eta}\left(1 - \frac{\psi'(\theta)}{\psi'(\bar\theta)}\right) d\theta. \tag{56}$$

Thanks to Corollary 4.5 and Theorem 6.1, for $\theta \in \Theta^{\psi,\lambda}$, the process $(\tau_{\theta+q}(\lambda), q \geq 0)$ under $\mathbb{P}^{\psi,\lambda}$ is distributed as the process $(\tau_q(\psi_\theta(\eta)), q \geq 0)$ under $\mathbb{P}^{\psi_\theta,\psi_\theta(\eta)}$.

The next proposition gives the distribution of the sub-tree process starting from the ascension time. The result will be used to deduce the representation of the ascension process whose conditional law given $\{A_\lambda = \theta\}$ is absolutely continuous with respect to the law of the initial process starting from $\tau_\theta(\lambda)$.

Recall that $L_\theta(\lambda)$ denotes the number of leaves of the tree $\tau_\theta(\lambda)$ and that A_λ is the ascension time of the process $(\tau_\theta(\lambda), \theta \in \Theta^{\psi,\lambda})$.

Proposition 8.1. *Assume that ψ is a critical branching mechanism satisfying (6). Let $\lambda > 0$ and $\eta = \psi^{-1}(\lambda)$. Suppose that $\theta_\lambda < 0$. Then for $\theta_\lambda < \theta < 0$ and any non-negative measurable function F, we have:*

$$\begin{aligned}
&\mathbb{E}^{\psi,\lambda}[F(\tau_{A_\lambda+q}(\lambda), q \geq 0) | A_\lambda = \theta] \\
&\quad = \frac{\eta\psi'(\bar{\theta})}{\psi_\theta(\eta)} \mathbb{E}^{\psi,\lambda}\left[F(\tau_{\theta+q}(\lambda), q \geq 0)\, L_\theta(\lambda)\, \mathbf{1}_{\{L_\theta(\lambda)<\infty\}}\right].
\end{aligned}$$

Proof. By considering $\mathbb{E}^{\psi,\lambda}\left[F(\tau_{\theta+q}(\lambda), q \geq 0) | \tau_\theta(\lambda)\right]$ instead of $F(\tau_{\theta+q}(\lambda), q \geq 0)$, one can assume that F is a measurable functional defined on $\mathbb{T}$ (instead of the set of $\mathbb{T}$-valued processes). Assume that $F(T) = 0$ if T is non compact. For $\theta_\lambda < q < \theta < 0$, we have:

$$\mathbb{E}^{\psi,\lambda}\left[F(\tau_\theta(\lambda))\mathbf{1}_{\{A_\lambda \geq q\}}\right] = \mathbb{E}^{\psi,\lambda}\left[F(\tau_\theta(\lambda))\mathbb{P}^{\psi,\lambda}\left(\tau_q(\lambda) \text{ is non compact} | \tau_\theta(\lambda)\right)\right]. \tag{57}$$

We write L_θ for $L_\theta(\lambda)$. On $\{\tau_\theta(\lambda)$ is compact$\}$ that is $\{L_\theta < \infty\}$, we get that $\tau_q(\lambda)$ is compact if and only if the trees grafted on $\tau_\theta(\lambda)$ to get $\tau_q(\lambda)$ are compact, see Theorem 6.6. Using (55), (46) and Notation (48), we get on $\{L_\theta < \infty\}$:

$$\mathbb{P}^{\psi,\lambda}(\tau_q(\lambda) \text{ is non compact} | \tau_\theta(\lambda)) = 1 - g\left(1 - \frac{\bar{q} - q}{\eta}\right)^{L_\theta}. \tag{58}$$

A simple calculation (recall that g depends on q) based on the computation of (56) yields on $\{L_\theta < \infty\}$:

$$\begin{aligned}
\frac{d}{dq} g\left(1 - \frac{\bar{q}-q}{\eta}\right)^{L_\theta}_{|q=\theta} &= L_\theta\, g\left(1 - \frac{\bar{q}-q}{\eta}\right)^{L_\theta - 1}_{|q=\theta} \frac{d}{dq} g\left(1 - \frac{\bar{q}-q}{\eta}\right)_{|q=\theta} \\
&= L_\theta \left[\frac{dg}{dq}\left(1 - \frac{\bar{q}-q}{\eta}\right) - g'\left(1 - \frac{\bar{q}-q}{\eta}\right)\frac{1}{\eta}\left(1 - \frac{\psi'(q)}{\psi'(\bar{q})}\right)\right]_{|q=\theta} \\
&= L_\theta\, \frac{\psi'(\bar{\theta})}{\psi_\theta(\eta)}.
\end{aligned}$$

Then by (57) and (58) and thanks to the regularity of g and $\bar{q}$ in q, we have:

$$\begin{aligned}
\frac{\mathbb{E}^{\psi,\lambda}[F(\tau_\theta(\lambda)), A_\lambda \in d\theta]}{d\theta} &= -\frac{d}{dq}\mathbb{E}^{\psi,\lambda}[F(\tau_\theta(\lambda))\mathbf{1}_{\{A_\lambda \geq q\}}]_{|q=\theta} \\
&= \mathbb{E}^{\psi,\lambda}\left[F(\tau_\theta(\lambda)) L_\theta\, \frac{\psi'(\bar{\theta})}{\psi_\theta(\eta)} \mathbf{1}_{\{L_\theta < \infty\}}\right].
\end{aligned}$$

Meanwhile, by Proposition 7.5, we have:

$$\mathbb{E}^{\psi,\lambda}[L_\theta \mathbf{1}_{\{L_\theta<\infty\}}] = \lim_{\zeta \to 1-} \frac{\partial}{\partial \zeta} h_\theta(\zeta) = \frac{g_\theta(\eta)(0)}{1 - g_\theta'(h_\theta(1-))} = \frac{\psi_\theta(\eta)}{\eta\psi'(\bar{\theta})}, \tag{59}$$

where we use the fact that $h_\theta(1-) = 1 - \frac{\bar\theta-\theta}{\eta}$ which is the minimal solution of the equation $r = g_\theta(r)$. Thus, we get:

$$\begin{aligned}\mathbb{E}^{\psi,\lambda}[F(\tau_\theta(\lambda))|A_\lambda \in d\theta] &= \frac{\mathbb{E}^{\psi,\lambda}[F(\tau_\theta(\lambda)), A_\lambda \in d\theta]}{\mathbb{P}^{\psi,\lambda}(A_\lambda \in d\theta)}\\ &= \frac{\mathbb{E}^{\psi,\lambda}[F(\tau_\theta(\lambda))L_\theta \mathbf{1}_{\{L_\theta<+\infty\}}]}{\mathbb{E}^{\psi,\lambda}[L_\theta \mathbf{1}_{\{L_\theta<+\infty\}}]}\\ &= \frac{\eta\psi'(\bar\theta)}{\psi_\theta(\eta)}\,\mathbb{E}^{\psi,\lambda}\left[F(\tau_\theta(\lambda))L_\theta\mathbf{1}_{\{L_\theta<\infty\}}\right].\end{aligned}$$

This ends the proof. □

We give an immediate corollary.

Corollary 8.2. *Assume that ψ is a critical branching mechanism satisfying (6). Let $\lambda > 0$ and $\eta = \psi^{-1}(\lambda)$. Suppose that $\theta_\lambda < 0$. For $\theta_\lambda < \theta < 0$ and any non-negative measurable function F, we have, with $\eta_\theta = \eta - \bar\theta + \theta$:*

$$\begin{aligned}&\mathbb{E}^{\psi,\lambda}[F(\tau_{A_\lambda+q}(\lambda), q \geq 0)|A_\lambda = \theta]\\ &\quad= \frac{\eta_\theta\psi'(\bar\theta)}{\psi_{\bar\theta}(\eta_\theta)}\mathbb{E}^{\psi,\lambda}\left[F(\tau_{\bar\theta+q}(\psi(\eta_\theta)), q\geq 0)L_{\bar\theta}(\psi(\eta_\theta))\right].\end{aligned}$$

Proof. Let us denote $\mathcal{S}_\theta(\lambda) = (\tau_{\theta+q}(\lambda), q \geq 0)$. Then similarly to Proposition 4.6 of [2], we have:

$$\begin{aligned}\mathbb{E}^{\psi,\lambda}[F(\mathcal{S}_{A_\lambda}(\lambda))|A_\lambda = \theta] &= \frac{\eta\psi'(\bar\theta)}{\psi_\theta(\eta)}\mathbb{E}^{\psi,\lambda}\left[F(\mathcal{S}_\theta(\lambda))L_\theta(\lambda)\mathbf{1}_{\{L_\theta(\lambda)<\infty\}}\right]\\ &= \frac{\psi'(\bar\theta)}{\psi_\theta(\eta)}\mathbb{N}^{\psi}\left[F(\mathcal{S}_\theta(\lambda))L_\theta(\lambda)\mathbf{1}_{\{1\leq L_\theta(\lambda)<\infty\}}\right]\\ &= \frac{\psi'(\bar\theta)}{\psi_\theta(\eta)}\mathbb{N}^{\psi_\theta}\left[F(\mathcal{S}_0(\psi_\theta(\eta)))L_0(\psi_\theta(\eta))\mathbf{1}_{\{1\leq L_0(\psi_\theta(\eta))<\infty\}}\right]\\ &= \frac{\psi'(\bar\theta)}{\psi_\theta(\eta)}\mathbb{N}^{\psi_{\bar\theta}}\left[F(\mathcal{S}_0(\psi_\theta(\eta)))L_0(\psi_\theta(\eta))\mathbf{1}_{\{L_0(\psi_\theta(\eta))\geq 1\}}\right]\\ &= \frac{\psi'(\bar\theta)}{\psi_\theta(\eta)}\mathbb{N}^{\psi}\left[F(\mathcal{S}_{\bar\theta}(\psi(\eta-\bar\theta+\theta)))L_{\bar\theta}(\psi(\eta-\bar\theta+\theta))\mathbf{1}_{\{L_{\bar\theta}(\psi(\eta-\bar\theta+\theta))\geq 1\}}\right]\\ &= \frac{(\eta-\bar\theta+\theta)\psi'(\bar\theta)}{\psi_{\bar\theta}(\eta-\bar\theta+\theta)}\mathbb{E}^{\psi,\lambda}\left[F(\mathcal{S}_{\bar\theta}(\psi(\eta-\bar\theta+\theta)))L_{\bar\theta}(\psi(\eta-\bar\theta+\theta))\right],\end{aligned}$$

where we used Proposition 8.1 for the first equality; definition (27) of $\mathbb{P}^{\psi,\lambda}$ and $\{M_\lambda \geq 1\} = \{L_\theta(\lambda) \geq 1\}$ for the second; Proposition 4.1 for the third; Girsanov transformation (20) and $\psi_\theta(\bar\theta - \theta) = 0$ as well as the fact that the number of the leaves are finite under $\mathbb{N}^{\psi_{\bar\theta}}$ as $\psi_{\bar\theta}$ is sub-critical for the fourth; Proposition 4.1 as well as the equality $\psi_{\bar\theta}(\eta - \bar\theta + \theta) = \psi_\theta(\eta)$ for the fifth and sixth equalities. □

8.2. An infinite CRT and its pruning

Kesten in [18] constructed an infinite Galton–Watson tree which arises as a (sub)-critical Galton–Watson tree conditioned to be infinite; see also [9]. Inspired by the structure of the infinite Galton–Watson tree in [18], Abraham and Delmas constructed an infinite CRT $\mathcal{T}^*$ in [1] which could be regarded as a (sub)critical Lévy CRT conditioned to have infinite height. It can also be obtained as the genealogical tree associated with a CSBP with immigration, see [11]. Furthermore, in [1], motivated by the result on Galton–Watson trees in [9], Abraham and Delmas showed that, under a mild condition on ψ, the law of the process $\{\mathcal{T}_\theta, \theta \in \Theta\}$ after the ascension time can be represented by another tree-valued process obtained by pruning $\mathcal{T}^*$. In the following part of this work, we are devoted to proving that a similar result also holds for $\{\tau_\theta(\lambda), \theta \in \Theta^{\psi,\lambda}\}$ and the subtree process obtained from $\mathcal{T}^*$.

We first recall the construction of $\mathcal{T}^*$. Assume that ψ is critical satisfying (6). Notice that since ψ is critical the event of infinite height is of measure zero. Before recalling its construction, we stress that under $\mathbb{P}_r^\psi$, the root $\emptyset$ belongs to Br_∞ and has mass $\Delta_\emptyset = r$. We identify the half real line $[0, +\infty)$ with a real tree denoted by $[\![0, \infty[\![$ with the null mass measure. We denote by dx the length measure on $[\![0, \infty[\![$. Let $\sum_{i\in I^*} \delta_{(x_i^*, T^{*,i})}$ be a Poisson point measure on $[\![0, \infty[\![\times\mathbb{T}$ with intensity $dx\, \mathbf{N}^\psi[d\mathcal{T}]$, with $\mathbf{N}^\psi[d\mathcal{T}]$ defined in (13). The infinite CRT from [1] is defined as:

$$\mathcal{T}^* = [\![\emptyset, \infty[\![\circledast_{i\in I^*}(T^{*,i}, x_i^*). \tag{60}$$

We denote by $\mathbb{P}^{*,\psi}(d\mathcal{T}^*)$ the distribution of $\mathcal{T}^*$. Following [1] and similarly to the setting in Section 2.10, we consider on $\mathcal{T}^*$ a mark process $M^{\mathcal{T}^*}(d\theta, dy)$ which is a Poisson point measure on $\mathbb{R}_+ \times \mathcal{T}^*$ with intensity:

$$\mathbf{1}_{[\emptyset,+\infty)}(\theta)d\theta \left(2\beta\ell^{\mathcal{T}^*}(dy) + \sum_{i\in I^*}\sum_{x\in \mathrm{Br}_\infty(T^{*,i})} \Delta_x\delta_x(dy)\right),$$

with the identification of x_i^* as the root of $\mathcal{T}^{*,i}$. In particular nodes in $[\![\emptyset, \infty[\![$ with infinite degree will be charged by $M^{\mathcal{T}^*}$. Then we define the pruned tree at time q as

$$\mathcal{T}_q^* = \{x \in \mathcal{T}^*, M^{\mathcal{T}^*}([0, q] \times [\![\emptyset, x[\![) = 0\}$$

with the induced metric, root $\emptyset$ and mass measure the restriction to $\mathcal{T}_q^*$ of the mass measure $\mathbf{m}^{\mathcal{T}^*}$. Since $\{T^{*,i}\}$ are all compact trees and for any $q > 0$, $\mathbb{P}^{\psi,*}\left(M^{\mathcal{T}^*}([0, q] \times [\![\emptyset, \infty[\![) = 0\right) = 0$, then for any $q > 0$, $\mathcal{T}_q^*$ is compact.

Given $\mathcal{T}^*$, let $\mathcal{P}^*(dtdx) = \sum_{j\in J^*} \delta_{(t_j^*, y_j^*)}$ be a Poisson point measure on $[0, \infty) \times \mathcal{T}^*$ with intensity $dt\, \mathbf{m}^{\mathcal{T}^*}(dx)$. For $\theta \ge 0$ and $\lambda > 0$, define the pruned sub-tree $\tau_\theta^*(\lambda)$ containing the root and all the ancestors in $\mathcal{T}_\theta^*$ of the marked leaves of $\mathcal{T}^*$:

$$\tau_0^*(\lambda) = \bigcup_{j\in J^*, t_j^*\le\lambda} [\![\emptyset, y_j^*]\!] \quad \text{and} \quad \tau_\theta^*(\lambda) = \tau_0^*(\lambda) \bigcap \mathcal{T}_\theta^*, \quad \lambda > 0. \tag{61}$$

We define $\tau_\theta^*(0) = \bigcap_{\lambda>0} \tau_\theta^*(\lambda)$, and since ψ is critical, we have that $\tau_0^*(0) = [\![\emptyset, \infty[\![$ which has no leaf. Similarly to (31), we define the mass measure of $\tau_\theta^*(\lambda)$ by:

$$\mathbf{m}^{\tau_\theta^*(\lambda)} = \frac{1}{\psi_\theta(\eta)} \sum_{x\in \mathrm{Lf}(\tau_\theta^*(\lambda))} \delta_x, \tag{62}$$

with $\eta = \psi^{-1}(\lambda)$ and the convention the mass measure is zero if $\lambda = 0$.

We have a similar convergence result as Theorem 5.1.

Theorem 8.3. *Assume that ψ is critical satisfying (6). Then for all $\theta \geq 0$, we have $\mathbb{P}^{*,\psi}$-a.s.:*

$$\lim_{\lambda\to+\infty} d_{GHP}(\mathcal{T}^*_\theta, \tau^*_\theta(\lambda)) = 0.$$

Proof. According to [1], there exists a family of random continuous functions $(H^{(a)}, a > 0)$ with compact support such that: $H^{(a)}$ takes values in $[0, a]$; for all $0 < b < a$ and $t \geq 0$, we have:

$$H^{(b)}(t) = H^{(a)}(C^{-1}_{b,a}(t)) \quad \text{with } C_{b,a}(s) = \int_0^s \mathbf{1}_{\{H^{(a)}(r)\leq b\}}\, dr;$$

and $\left(\left(\mathcal{T}^*_\theta\right)^{(a)}, a > 0\right)$ under $\mathbb{P}^{*,\psi}$ is distributed as $\left(\mathcal{T}^{H^{(a)}}, a > 0\right)$. Following the proof of Lemma 5.4, we get that for all $a > 0$, $\mathbb{P}^{*,\psi}$ a.s.

$$\lim_{\lambda\to+\infty} d^c_{\text{GHP}}\left(\left(\mathcal{T}^*_\theta\right)^{(a)}, \left(\tau^*_\theta(\lambda)\right)^{(a)}\right) = 0.$$

This and the definition of d_{GHP} give the result. □

Remark 8.4. Similarly to Theorem 3.3, according to the argument in Section 3.2.2 in [10], we could reconstruct $\mathcal{T}^*$ from $\tau^*_0(\lambda)$. Recall (34). Conditionally on $\tau^*_0(\lambda)$, $\mathcal{T}^*_0$ is distributed as:

$$\tilde{\tau}^*_0(\lambda) \circledast_{i\in I} (\mathcal{T}^*_i, x^*_i) \circledast_{x\in \text{Br}(\tilde{\tau}^*_0(\lambda))} (\mathcal{T}^*_x, x),$$

with:

- $\tilde{\tau}^*_0(\lambda)$ as $\tau^*_0(\lambda)$ but with 0 as mass measure,
- $\sum_{i\in I} \delta_{(x^*_i, \mathcal{T}^*_i)}$ is a random Poisson point measure on $\tilde{\tau}^*_0(\lambda) \times \mathbb{T}$ with intensity given by $\ell^{\tilde{\tau}^*_0(\lambda)}(dx)\, \mathbf{N}^{\psi_\eta}[d\mathcal{T}]$,
- conditionally on $\sum_{i\in I} \delta_{(x^*_i, \mathcal{T}^*_i)}$, the trees $\left(\mathcal{T}^*_x, x \in \text{Br}(\tilde{\tau}^*_0(\lambda))\right)$ are independent with $\mathcal{T}^*_x$ distributed as:
$$\int \Gamma^{\psi}_{\kappa(x),\lambda}(dr)\, \mathbb{P}^{\psi_\eta}_r[d\mathcal{T}].$$

8.3. Distribution of the sub-tree of the infinite CRT

Recall that $\mathrm{P}^{\psi,\lambda}$ is defined before Lemma 4.3 and recall that $\tilde{\tau}_0(\lambda)$ is defined in Theorem 3.3. Note that $\tilde{\tau}_0(\lambda)$ is under $\mathbb{P}^{\psi,\lambda}$, a Galton–Watson tree with distribution $\mathrm{P}^{\psi,\lambda}$. We shall now describe the distribution of $\tilde{\tau}^*_0(\lambda)$ under $\mathbb{P}^{\psi,\lambda}$, which can be seen as a Galton–Watson tree with distribution $\mathrm{P}^{\psi,\lambda}$ conditionally on the non extinction event.

Let K be an integer-valued random variable with generating function $g_{(\psi,\lambda)}$ defined by (32). Since ψ is critical, we have $g'_{(\psi,\lambda)}(1) = 1$, which implies that $g'_{(\psi,\lambda)}$ itself is the generating function of an integer-valued random variable, say K^*. Since $g'_{(\psi,\lambda)}(0) = 0$, K^* is a.s. positive. Notice that the distribution of $K^* + 1$ is the size-biased distribution of K. Let $(\tau^{k,*}, k \in \mathbb{N}^*)$ be independent random trees distributed as $\tau_0(\lambda)$ under $\mathbb{P}^{\psi,\lambda}$ (that is with distribution $\mathrm{P}^{\psi,\lambda}$ and mass measure given by (31)) independent of K^*. We set:

$$\mathcal{G}^* = \emptyset \circledast_{1\leq k\leq K^*} (\tau^{k,*}, \emptyset).$$

Theorem 8.5. *Assume that ψ is critical satisfying (6). Let $\lambda > 0$ and $\eta = \psi^{-1}(\lambda)$. Under $\mathbb{P}^{*,\psi}$, $\tau_0^*(\lambda)$ is a rooted real tree distributed as:*

$$[\![\emptyset, \infty[\![\circledast_{i\in I_0^*} (\mathcal{G}^{*,i}, x_i^*),$$

where $\sum_{i\in I_0^*} \delta_{x_i^*}$ is a Poisson point measure on $[\![\emptyset, \infty[\![$ with intensity $\psi'(\eta)dx$ and conditionally on this Poisson point measure, the real trees $(\mathcal{G}^{*,i}, i \in I_0^*)$ are independent and distributed as $\mathcal{G}^*$.

Remark 8.6. If we do not consider the edge-lengths of $\tau_0^*(\lambda)$ and $\tau_0(\lambda)$, then $\tau_0^*(\lambda)$ is just $\tau_0(\lambda)$ conditioned to be infinite; see Proposition 2 in [9] for related results on discrete Galton–Watson trees.

Proof. By construction, thanks to (60), we have:

$$\tau_0^*(\lambda) = [\![\emptyset, \infty[\![\circledast_{i\in I^*} (\tau^{*,i}(\lambda), x_i^*),$$

with $\tau^{*,i}(\lambda) = \bigcup_{j\in J^*, t_j^*\leq\lambda, x_i^*\preccurlyeq y_j^*} [\![x_i^*, y_j^*]\!]$ distributed as $\tau_0(\lambda)$ under $\mathbf{N}^\psi[d\mathcal{T}]$. The marked Poisson point measure $\sum_{i\in I^*} \mathbf{1}_{\{\tau^{*,i}(\lambda)\neq\emptyset\}} \delta_{x_i^*}$ is a Poisson point measure on $[\![\emptyset, \infty[\![$ with intensity $\mathbf{N}^\psi[M_\lambda \geq 1]\,dx = \psi'(\eta)\,dx$.

Let $I_0^* = \{i \in I^*; \tau^{*,i}(\lambda) \neq \emptyset\}$. The sub-trees $(\tau^{*,i}(\lambda), i \in I_0^*)$ are independent and distributed as $\tau_0(\lambda)$ under $\mathbf{N}^\psi[\,\cdot\,|M_\lambda \geq 1]$. Let $N_\emptyset$ be the degree of the root of $\tau_0(\lambda)$. The theorem will be proved once we check that $N_\emptyset$ under $\mathbf{N}^\psi[\,\cdot\,|M_\lambda \geq 1]$ is distributed as K^*. Following the proof of Lemma 6.7, we set $h^*(u) = \mathbf{N}^\psi\left[u^{N_\emptyset}\mathbf{1}_{\{N_\emptyset\geq 1\}}\right]$, and we have for $u \in [0, 1]$:

$$\begin{aligned} h^*(u) &= 2\beta\mathbb{N}^\psi[M_\lambda \geq 1]u + \int_{(0,+\infty)} r\pi(dr)\,\mathbb{E}_r^\psi\left[u^{N_\emptyset}\mathbf{1}_{\{N_\emptyset\geq 1\}}\right] \\ &= 2\beta\eta u + \int_{(0,+\infty)} r\pi(dr)\left(\mathrm{e}^{-r\eta(1-u)} - \mathrm{e}^{-r\eta}\right). \end{aligned}$$

Elementary computations yield $g'_{(\psi,\lambda)}(u) = h^*(u)/h^*(1)$. Thus $N_\emptyset$ under $\mathbf{N}^\psi[\,\cdot\,|M_\lambda \geq 1]$ is distributed as K^*. This ends the proof. $\square$

We give a similar representation formula for $\tau_\theta^*(\lambda)$. Let K_θ^* be an integer-valued random variable with generating function $g'_\theta/g'_\theta(1)$, see definitions (48) and (32). Since $g'_\theta(0) = 0$, K_θ^* is a.s. positive. Notice that the distribution of $K_\theta^* + 1$ is the size-biased distribution of K_θ with generating function g_θ. Let $(\tau_\theta^{k,*}, k \in \mathbb{N}^*)$ be independent random trees distributed as $\tau_\theta(\lambda)$ under $\mathbb{P}^{\psi,\lambda}$ (that is with distribution $\mathrm{P}^{\psi_\theta,\psi_\theta(\eta)}$ and mass measure given by (31)) independent of K_θ^*. We set:

$$\mathcal{G}_\theta^* = \emptyset \circledast_{1\leq k\leq K_\theta^*} (\tau_\theta^{k,*}, \emptyset).$$

Theorem 8.7. *Assume that ψ is critical satisfying (6). Let $\lambda > 0$ and $\eta = \psi^{-1}(\lambda)$. For $\theta > 0$, under $\mathbb{P}^{*,\psi}$, $\tau_\theta^*(\lambda)$ is a rooted real tree distributed as:*

$$[\![\emptyset, E_\theta]\!] \circledast_{i\in I_\theta^*} (\mathcal{G}_\theta^{*,i}, x_i^*),$$

where

- *$[\![\emptyset, E_\theta]\!]$ is a real tree rooted at $\emptyset$ with no branching point and zero mass measure and such that $d(\emptyset, E_\theta)$ is an exponential random variable with parameter $\psi'_\theta(0)$,*

- *$\sum_{i\in I_\theta^*}\delta_{x_i^*}$ is an independent Poisson point measure on $[\![\emptyset, E_\theta]\!]$ with intensity $[\psi_\theta'(\eta) - \psi_\theta'(0)]\,dx$,*
- *conditionally on E_θ and $\sum_{i\in I_\theta^*}\delta_{x_i^*}$, the real trees $(\mathcal{G}^{*,i}, i \in I_0^*)$ are independent and distributed as $\mathcal{G}_\theta^*$.*

Proof. Recall notations of the proof of Theorem 8.5. The distribution of $d(\emptyset, E_\theta)$ is given in [1]. By construction, thanks to (60), we have:

$$\tau_0^*(\lambda) = [\![\emptyset, E_\theta]\!] \circledast_{i\in I^*} (\tau_\theta^{*,i}(\lambda), x_i^*),$$

with $\tau_\theta^{*,i}(\lambda) = \tau^{*,i}(\lambda)\bigcap \mathcal{T}_\theta^*$. Let $N_{\emptyset,\theta}$ (resp. $N'_\emptyset$) be the degree of the root of $\tau_\theta(\lambda)$ (resp. $\tau_0(\psi_\theta(\eta))$). Notice that $\tau_\theta^{*,i}(\lambda)$ is distributed as $\tau_\theta(\lambda)$ under $\mathbf{N}^\psi[d\mathcal{T}, N_\emptyset \geq 1]$ that is as $\tau_0(\psi_\theta(\eta))$ under $\mathbf{N}^{\psi_\theta}[d\mathcal{T}]$. The rate at which sub-trees are grafted on the spine $[\![\emptyset, E_\theta]\!]$ is given by:

$$\mathbf{N}^{\psi_\theta}\left[N'_\emptyset \geq 1\right] = \psi_\theta'(\eta) - \psi_\theta'(0).$$

Then to end the proof, it is enough to check that $N'_\emptyset$ under $\mathbf{N}^{\psi_\theta}[\,\cdot\mid N_\emptyset \geq 1]$ is distributed as K_θ^*. Elementary computations give:

$$h_\theta^*(u) = \mathbf{N}^{\psi_\theta}\left[u^{N'_\emptyset}\mathbf{1}_{\{N'_\emptyset\geq 1\}}\right] = \psi_\theta(\eta) - \psi_\theta'(\eta(1-u)),$$

so that $h_\theta^*(u)/h_\theta^*(1) = g_\theta'(u)/g_\theta'(1)$. Thus, $N'_\emptyset$ under $\mathbf{N}^{\psi_\theta}[\,\cdot\mid N_\emptyset \geq 1]$ is distributed as K_θ^*. □

We also provide a recursive distribution of the tree $\tau_\theta^*(\lambda)$. Let $a_\theta(\lambda) = \psi_\theta'(0)/\psi_\theta'(\eta) = 1 - g_\theta'(1)$.

Corollary 8.8. *Assume that ψ is critical satisfying (6). Let $\lambda > 0$ and $\eta = \psi^{-1}(\lambda)$. For $\theta > 0$, under $\mathbb{P}^{*,\psi}$, $\tau_\theta^*(\lambda)$ is a rooted real tree distributed as $[\![\emptyset, E_\theta(\lambda)]\!]$ with probability $a_\theta(\lambda)$ and with probability $1 - a_\theta(\lambda)$ as:*

$$[\![\emptyset, E_\theta(\lambda)]\!] \circledast_{0\leq i\leq 1} (\mathcal{G}_\theta^{*,i}, E_\theta(\lambda)),$$

where

- *$[\![\emptyset, E_\theta(\lambda)]\!]$ is a real tree rooted at $\emptyset$ with no branching point and zero mass measure and such that $d(\emptyset, E_\theta(\lambda))$ is an exponential random variable with parameter $\psi_\theta'(\eta)$,*
- *conditionally on $E_\lambda(\theta)$, $\mathcal{G}_\theta^{*,0}$ and $\mathcal{G}^{*,1}$ are independent and distributed respectively as $\mathcal{G}_\theta^*$ and $\tau_\theta^*(\lambda)$.*

Notice that the number of children of $E_\theta(\lambda)$ has generating function $1 - g_\theta'(1) + ug_\theta'(u)$.

Proof. This is a direct consequence of Theorem 8.7, when considering the decomposition of $\tau_\theta^*(\lambda)$ with respect to the lowest branching point and using the branching property. Notice that there is no such branching point (and then $\tau_\theta^*(\lambda)$ is reduced to a spine) if the point measure $\sum_{i\in I_\theta^*}\delta_{x_i^*}$ defined in Theorem 8.7 is zero. This happens with probability $a_\theta(\lambda)$. □

Remark 8.9. Notice that $\tau_\theta^*(\lambda)$ could be obtained from $\tau_0^*(\lambda)$ by a similar pruning procedure as the one defined in Section 6.1.

8.4. Sub-tree process from the ascension time

We prove in this subsection that the ascension process can be represented using the process obtained by pruning the infinite CRT.

We first need an absolute continuity property between the laws of the processes $(\tau_{\theta+q}, q \geq 0)$ and $(\tau^*_{\theta+q}, q \geq 0)$.

Proposition 8.10. *Assume that ψ is a critical branching mechanism satisfying (6). For $\theta > 0, \lambda > 0$ and non-negative measurable functionals F, we have:*

$$\frac{\eta\psi'(\theta)}{\psi_\theta(\eta)}\mathbb{E}^{\psi,\lambda}[F(\tau_{\theta+q}(\lambda), q \geq 0)L_\theta(\lambda)] = \mathbb{E}^{*,\psi}[F(\tau^*_{\theta+q}(\lambda), q \geq 0)]. \tag{63}$$

Proof. We denote $\mathcal{S}^*_\theta(\lambda) = (\tau^*_{\theta+q}(\lambda), q \geq 0)$. Recall that for $\theta \in \Theta^{\psi,\lambda}$, $\mathcal{S}_\theta(\lambda) = (\tau_{\theta+q}(\lambda), q \geq 0)$.

By considering $\mathbb{E}^{\psi,\lambda}[F(\mathcal{S}_\theta)|\tau_\theta(\lambda)]$ instead of $F(\mathcal{S}_\theta(\lambda))$ and $\mathbb{E}^{*,\psi}\left[F(\mathcal{S}^*_\theta(\lambda))|\tau^*_\theta(\lambda)\right]$ instead of $F(\mathcal{S}^*_\theta(\lambda))$, one can assume that F is a measurable functional defined on $\mathbb{T}$. Since the life times of all individuals in $\tau_\theta(\lambda)$ and $\tau^*_\theta(\lambda)$ have the same distribution, we only need to consider the distribution of the number of offsprings. This is equivalent to considering the corresponding discrete (or size-biased) Galton–Watson tree described below.

Let $\mathcal{G}$ be a discrete sub-critical Galton–Watson tree starting with one root and with g as the generating function of the reproduction law. Let L be the number of leaves of $\mathcal{G}$. We have $\mathbb{E}[L] = g(0)/[1 - g'(1)]$. Let $\mathcal{G}^*$ be distributed as the size-biased distribution of $\mathcal{G}$ with respect to L, that is for any non-negative measurable function:

$$\mathbb{E}[F(\mathcal{G}^*)] = \frac{\mathbb{E}[LF(\mathcal{G})]}{\mathbb{E}[L]}.$$

The distribution of $\mathcal{G}^*$ is characterized as follows:

$$\mathcal{G}^* \overset{(d)}{=} \{\emptyset\} \circledast_{1\leq i<N} (\mathcal{G}_i, \emptyset) \circledast (\tilde{\mathcal{G}}^*, \emptyset)$$

where N has generating function $u \to 1 - g'(1) + ug'(u)$ (N is the number of children of the root, if $N = 0$, then $\mathcal{G}^*$ is reduced to the root), the trees $(\mathcal{G}^*_i)$ are i.i.d., independent of N, and distributed as $\mathcal{G}$, and $\tilde{\mathcal{G}}^*$ is a tree independent of the previous variables and distributed as $\mathcal{G}^*$. This result can be proved inductively by decomposing the tree with respect to the children of the root.

The result follows then directly from this description. □

Recall that the function $\theta \mapsto \bar{\theta}$ is defined by (53). If $\theta_\lambda \in \Theta$, then we deduce from (54) that:

$$\bar{\theta}_\lambda - \theta_\lambda = \eta.$$

In particular the function f defined by:

$$f_\lambda(r) = \frac{1}{\eta}\left(1 - \frac{\psi'(r)}{\psi'(\bar{r})}\right)\mathbf{1}_{\{r\in(\theta_\lambda,0)\}}$$

is a probability density. The corresponding cumulative distribution is F_λ defined on $[\theta_\lambda, 0)$ by:

$$F_\lambda(r) = 1 - \frac{\bar{r} - r}{\eta} = \mathbb{P}^{\psi,\lambda}(A_\lambda < r).$$

Proposition 8.11. *Assume that ψ is critical satisfying (6). Let $\lambda > 0$ and $\eta = \psi^{-1}(\lambda)$. Assume that $\theta_\lambda \in \Theta$ and $\theta_\lambda < 0$. Under $\mathbb{P}^{*,\psi}$, let U be a random variable with density f_λ and independent of $\mathcal{S}_0^*(\lambda)$. Recall the notation $\bar{U} = \psi^{-1}(\psi(U))$. Then the process $(\tau_{A_\lambda+q}(\lambda), q \geq 0)$ under $\mathbb{P}^{\psi,\lambda}$ has the same distribution as the process $\left((\tau^*_{\bar{U}+q}\left(\psi\left(\eta F_\lambda(U)\right)\right), q \geq 0\right)$ under $\mathbb{P}^{*,\psi}$.*

Remark 8.12. This proposition can be viewed as a counterpart on Galton–Watson real trees of Corollary 8.2 in [1]. It implies that the law of $\{\tau_{A_\lambda+\theta}(\lambda) : \theta \geq 0\}$ can be represented in terms of $\{\tau^*_{\bar{\theta}}(\lambda), \theta \geq 0\}$. Similar results for discrete Galton–Watson tree-valued processes have also been explored in [9,2] for different pruning procedures.

Proof. Using Corollary 8.2, with $\eta_\theta = \eta - \bar{\theta} + \theta$, we get for any non-negative measurable functionals F,

$$\begin{aligned}\mathbb{E}^{\psi,\lambda}[F(\mathcal{S}_{A_\lambda}(\lambda))|A_\lambda = \theta] &= \frac{\eta_\theta \psi'(\bar{\theta})}{\psi_{\bar{\theta}}(\eta_\theta)}\mathbb{E}^{\psi,\lambda}\left[F(\mathcal{S}_{\bar{\theta}}(\psi(\eta_\theta)))L_{\bar{\theta}}(\psi(\eta_\theta))\right]\\ &= \mathbb{E}^{*,\psi}[F(\mathcal{S}^*_{\bar{\theta}}(\psi(\eta_\theta)))].\end{aligned}$$

Then by (56) and $\eta_\theta = \eta F_\lambda(\theta)$, we have

$$\begin{aligned}\mathbb{E}^{\psi,\lambda}[F(\mathcal{S}_{A_\lambda}(\lambda))] &= \int_{\theta_\lambda}^0 \mathbb{E}^{\psi,\lambda}[F(\mathcal{S}_{A_\lambda}(\lambda))|A_\lambda = \theta]\,\mathbb{P}^{\psi,\lambda}(A_\lambda \in d\theta)\\ &= \int_{\theta_\lambda}^0 \mathbb{E}^{*,\psi}[F(\mathcal{S}^*_{\bar{\theta}}(\psi(\eta_\theta)))]f_\lambda(\theta)\,d\theta\\ &= \int_{\theta_\lambda}^0 \mathbb{E}^{*,\psi}[F(\mathcal{S}^*_{\bar{\theta}}(\psi(\eta F_\lambda(\theta))))]f_\lambda(\theta)\,d\theta\\ &= \mathbb{E}^{*\lambda}[F(\mathcal{S}^*_{\bar{U}}(\psi(\eta F_\lambda(U))))].\end{aligned}$$

We have completed the proof. □

Acknowledgments

We would like to give our sincere thanks to two anonymous referees for their careful reading and valuable comments which improved significantly the presentation of the paper. H. He is also supported by Fundamental Research Funds for the Central Universities (2013YB59). This work is partially supported by the "Agence Nationale de la Recherche", ANR-BLAN06-3-146282 and ANR-08-BLAN-0190, and by NSFC (Nos. 11201030, and 11371061).

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