Adaptive Near-Minimal Rank Approximation for High Dimensional Operator Equations

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joint work with Markus Bachmayr

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Contents

High Dimensional PDEs

Examples:

(i) Electronic Schrödinger equation: $d = 3n$, $n = \#$ of particles
(ii) Fokker-Planck equations: $d = 3K$, $K =$ length of bead string for polymer model
(iii) Parameter dependent (stochastic) PDEs: $d = \infty$

Core task for (i), (ii): solution of high dimensional elliptic PDEs

Curse of Dimensionality, Tractability - Novak,Woźniakowski

$$u(x_1,\ldots,x_d), \quad N(\varepsilon,d) := \#\text{lin information for accuracy } \varepsilon$$

- Intractable:

$$\liminf_{\varepsilon\to 0, d\to\infty} \frac{\log N(\varepsilon,d)}{\varepsilon^{-1}+d} > 0$$

- Weakly tractable:

$$\lim_{\varepsilon\to 0, d\to\infty} \frac{\log N(\varepsilon,d)}{\varepsilon^{-1}+d} = 0$$

- Polynomially intractable:

$$\nexists\ C,s,q \quad \text{s.t. } N(\varepsilon,d) \le C\varepsilon^{-s}d^q, \quad \forall\ \varepsilon\in(0,1)$$

$u\in C^\infty$, $\|u\|_{C^k}\le M$, $k\in\mathbb{N}$: $\rightsquigarrow$ polynomially intractable

Remedies?...

- "Excessive" regularity (Korobov spaces)
- "Hidden sparsity" with respect to a

  "problem dependent dictionary"

Separation of variables, tensors...

$$u(x) = u(x_1, \ldots, x_d) \in C^s \qquad u(x) \approx \sum_{k=1}^{r(\varepsilon)} u_{k,1}(x_1) \cdots u_{k,d}(x_d)$$

$$\varepsilon \sim n^{-s} \rightsquigarrow N \sim n^d \qquad \varepsilon \sim r(\varepsilon) d n^{-s} \rightsquigarrow N \sim r(\varepsilon) d n$$

$$N(\varepsilon, d) \sim \varepsilon^{-d/s} \text{ or } C^{\alpha d} \varepsilon^{-1/s} \qquad N(\varepsilon, d) \sim r(\varepsilon)^{\frac{1}{s}} d^{1+\frac{1}{s}} \varepsilon^{-1/s}$$

Tractability of High-Dimensional PDEs

- $Au = f$?
- u cannot be queried directly
- “Inversion Complexity” – “Representation Complexity”

THEOREM: [D./DeVore/Grasedyck/Süli]

The inversion complexity of the high-dimensional Poisson problem is computationally polynomially tractable

- Important tools: exponential sums of operators, canonical format
- What about more general diffusion coefficients

$$\mathrm{div}(a\nabla u) = f, \quad a \in \mathbb{R}^{d\times d}?$$

A Nasty Pitfall [de Silva...]

$$\underbrace{u_{n,1} \otimes u_{n,2} \otimes u_{n,3}}_{n(a+\frac{1}{n}e)\otimes b\otimes c} + \underbrace{v_{n,1} \otimes v_{n,2} \otimes v_{n,3}}_{-n\, a\otimes(b-\frac{1}{n}e)\otimes(c+\frac{1}{n}f)}$$

$$\begin{aligned} = \quad & {\color{blue} n\, a \otimes b \otimes c} + e \otimes b \otimes c - {\color{blue} n\, a \otimes b \otimes c} + a \otimes e \otimes c \\ & - a \otimes b \otimes f + \frac{1}{n} a \otimes e \otimes f \\ \to \quad & e \otimes b \otimes c + a \otimes e \otimes c - a \otimes b \otimes f \end{aligned}$$

. . . The limit of rank-2 tensors can have rank 3 ... best approximations don't exist ...

Stable Tensor-Formats

de Silva, Lathauwer, Hackbusch, Falco, Grasedyck, Oseledets, Schneider...

- Subspace based methods (Grassmann manifolds)
- Orthogonal projections, SVD, existence of best approximations...
- But, only in $\mathbb{R}^d$, $\quad f(\nu_1, \ldots, \nu_d)$, $\nu \in \mathcal{J} = \mathcal{J}_1 \times \cdots \mathcal{J}_d$, $\#(\mathcal{J}_j) < \infty$
- Extension to $\ell_2(\mathcal{J}^d)$, $\#(\mathcal{J}) = \infty$, by Hilbert-Schmidt

"background basis" $\rightsquigarrow$ function spaces... But: scaling problem!

Tucker/Hierarchical Tucker Format

View $\mathbf{u} = (u_{\nu_1,\dots,\nu_d})_{(\nu_1,\dots,\nu_d)\in\mathcal{J}^d}$ as order-d-tensor

Mode frames: $\mathbf{U}_k^{(j)} \in \ell_2(\mathcal{J}),\, j = 1,\dots,d,\ \langle \mathbf{U}_k^{(i)}, \mathbf{U}_l^{(i)}\rangle = \delta_{kl},\, k,l \in \mathbb{N}$

$$\mathbf{u} = \sum_{k_1=1}^{\infty}\cdots\sum_{k_d=1}^{\infty} \langle \mathbf{u}, \mathbf{U}_{k_1}^{(1)}\otimes\cdots\otimes\mathbf{U}_{k_d}^{(d)}\rangle\, \mathbf{U}_{k_1}^{(1)}\otimes\cdots\otimes\mathbf{U}_{k_d}^{(d)} =: \sum_{\mathbf{k}\in\mathbb{N}^d} \mathbf{a}_{\mathbf{k}}\mathbb{U}_{\mathbf{k}}$$

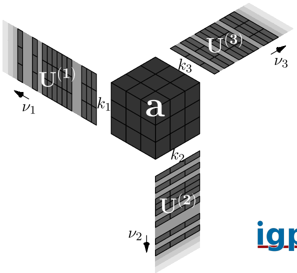
$$\mathbf{U}_k^{(j)} = (\delta_{k,n})_{n\in N} \rightsquigarrow \mathbf{a} = \mathbf{u}$$

Hierarchical Tucker (H-T)-format:

hierarchical factorization of $(\mathbf{a}_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$

rank $r \rightsquigarrow$ rank-vector $\mathrm{r} \in \mathbb{N}^d$

How to find good mode frames?



Workhorse SVD... [DeLathauwer, Hackbusch, Khoromskij...]

• Matricization: $\mathbf{v} = (v_\nu)_{\nu \in \mathcal{J}^d} \rightsquigarrow M_{\mathbf{v}}^{(i)} = (v_{\nu_1,\dots,\nu_{i-1},\nu_i,\nu_{i+1},\dots,\nu_d})_{\nu_i \in \mathcal{J}, \check{\nu}_i \in \mathcal{J}^{d-1}}$

Tucker ranks: $\mathrm{rank}_i(\mathbf{u}) := \dim \mathrm{range}(M_{\mathbf{u}}^{(i)}), \quad i = 1, \dots, d$

• Tucker Format: SVD for $M_{\mathbf{u}}^{(i)} \rightsquigarrow$ left singular vectors $U_k^{(i)} : \rightsquigarrow$ HOSVD

$$\lesssim d\, |\tilde{\mathrm{r}}|_\infty^{d+1} + C\, |\tilde{\mathrm{r}}|_\infty^2 \sum_{i=1}^d \#\mathrm{supp}_i(\mathbf{u}), \quad \mathrm{supp}_i(\mathbf{u}) := \bigcup_{z \in \mathrm{range} M_{\mathbf{u}}^{(i)}} \mathrm{supp}\, z$$

• Hierarchical Tucker Format: $\mathcal{H}$SVD

[Espig, Grasedyck, Hackbusch, Kolda, Khoromskij,Oseledets,...]

Successive SVD for $M_{\mathbf{u}}^{(\alpha)} = (u_{\nu_\alpha,\nu_\beta})_{\nu_\alpha \in \mathcal{J}^{|\alpha|}, \nu_\beta \in \mathcal{J}^{|\beta|}}$, $\alpha \subset \{1, \dots, d\}$

$$\lesssim d\, |\tilde{\mathrm{r}}|_\infty^4 + C\big(\max_i \tilde{r}_i\big)^2 \sum_{i=1}^d \#\mathrm{supp}_i(\mathbf{u})$$

• Projections: $\|\mathbf{u} - \mathrm{P}_{\mathbb{U}(\mathbf{u}),\tilde{\mathrm{r}}}\, \mathbf{u}\| \le \sqrt{2d-3}\, \inf\{\|\mathbf{u} - \mathbf{v}\| : \mathbf{v} \in \mathcal{H}(\tilde{\mathrm{r}})\}$

PDEs on $\Omega := \Omega_1 \times \cdots \times \Omega_d$

Model problem:

$$Au = -\sum_{i,j=1}^{d} \partial_{x_i}(a_{i,j}\partial_{x_j} u) + cu, \quad a(u,v) := \langle v, Au\rangle : \tilde{H}^1(\Omega) \times \tilde{H}^1(\Omega) \to \mathbb{R}$$

For $f \in (\tilde{H}^1(\Omega))'$ find $u \in \mathrm{H} := \tilde{H}^1(\Omega)$ such that

$$a(u,v) = \langle f, v\rangle, \; v \in \mathrm{H}$$

- A has finite (Tucker-) rank
- When $A = -(\partial_{x_1}^2 \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes \partial_{x_d}^2)$

  f "tensor-sparse" $\Rightarrow$ $u = A^{-1} f$ "tensor-sparse"

  [D/DeVore/Grasedyck/Süli]

Some Obstructions

- Stable tensor formats not defined for functions (except perhaps $L_2(D^d)$)
- $A : \mathrm{H} \to \mathrm{H}'$ isomorphism, i.e., $\|u - v\|_{\mathrm{H}} \sim \|f - Av\|_{\mathrm{H}'}$

$$\mathrm{H} = \bigcap_{j=1}^{d} L_2(\Omega_1) \otimes \cdots \otimes L_2(\Omega_{j-1}) \otimes H^1(\Omega_j) \otimes L_2(\Omega_{j+1}) \otimes \cdots \otimes L_2(\Omega_d)$$

  does not have a "cross-norm"
- $A^{-1} : \mathrm{H}' \to \mathrm{H}$ has infinite rank because eigenvalues have the form

$$\lambda_\nu = \lambda_{1,\nu_1} + \cdots + \lambda_{d,\nu_d}, \quad \nu \in \mathbb{N}^d$$

$\rightsquigarrow$ a "scaling trap"

Tensor methods for Opertor equations

So far... [Ehrlache, Falcó, Hackbusch, Khoromskij, Kressner, Mohlenkamp/Beylkin, Nouy, Oseledets, Schneider,...]

- initial reduction to a fixed discrete system
- accuracy considerations detached from continuous solution
- approximation error and residuals are measured in the same (Euclidean) norm - “scaling trap”
- accuracy and rank growth cannot be controlled simultaneously
- PGD...convergence, ranks?... [Falcó, Chinesta, Ladevez, Nouy,...]

What is different here... (building on existing tools $+$...)

- Transformation into an equivalent ∞-dimensional problem on $\ell_2(\mathcal{J}^d)$
- Use stable tensor formats on $\ell_2(\mathcal{J}^d)$
- Establish correct mapping properties by diagonal scaling $\rightsquigarrow$ infinite ranks
- Control ranks by adaptive separable scaling approximations - exponential sums

Reduction to Problem in $\ell_2(\mathcal{J}^d)$

"Universal background" basis: $\Omega := \Omega_1 \times \cdots \times \Omega_d$

$$\{\psi_\nu = \psi_{\nu_1} \otimes \cdots \otimes \psi_{\nu_d} : \nu \in \mathcal{J}^d\} \text{ O.N.B. for } L_2(\Omega) \rightsquigarrow$$

$$\Psi = \Big\{ \Big(\sum_{i=1}^{d} 2^{2|\nu_i|} \Big)^{-\frac{1}{2}} \psi_\nu =: s_\nu^{-1} \psi_\nu \Big\}_{\nu \in \mathcal{J}^d} \text{ Riesz-basis for } \mathrm{H} \subset L_2(\Omega)$$

$$Au = f \;\Leftrightarrow\; \mathbf{A}\mathbf{u} = \mathbf{f}, \quad \mathbf{A} = \underbrace{\left(s_\nu^{-1} a(\psi_\nu, \psi_\mu) s_\mu^{-1}\right)_{\nu,\mu \in \mathcal{J}}}_{\mathbf{S}^{-1}\mathbf{T}\mathbf{S}^{-1}}, \quad \mathbf{f} = \underbrace{\left(\left\langle f, s_\nu^{-1} \psi_\nu \right\rangle\right)_{\nu \in \mathcal{J}}}_{\mathbf{S}^{-1}\mathbf{g}}$$

Theorem:

$$\kappa(\mathbf{A}) := \|\mathbf{A}\| \|\mathbf{A}^{-1}\| \lesssim 1$$

$$u \in \mathrm{H} \quad \leftrightarrow \quad \mathbf{u} = (u_\nu)_{\nu \in \mathcal{J}^d} \in \ell_2(\mathcal{J}^d)$$

Scheme: Perturbed "Ideal Iteration"

Strategy:

$$\mathbf{u}^{k+1} = \mathrm{C}_{\varepsilon_3(k)}\big(\mathrm{P}_{\varepsilon_2(k)}(\mathbf{u}^k + \omega(\mathbf{f} - \boldsymbol{A}\mathbf{u}^k))\big) \rightsquigarrow \|\mathbf{u} - \mathbf{u}^{k+1}\| \leq \rho\|\mathbf{u} - \mathbf{u}^k\|, \;\; \rho < 1$$

- keep the $\mathbf{u}^k$ in hierarchical Tucker format
- $\mathrm{C}_{\varepsilon_3(k)}$ coarsening of mode frames
- $\mathrm{P}_{\varepsilon_2(k)}$ $\mathcal{H}$SVD projection to near-optimal subspaces

  $\rightsquigarrow$ simultaneous control of ranks and mode frame sparsity
- control tolerances so as to ensure convergence

Some New Conceptual Ingredients...

- Tensor-Compression-Coarsening Lemma: $a > 1, b < 1$

$$\|\mathbf{u}-\mathbf{v}\| \le \eta \quad \rightsquigarrow \quad \|\mathrm{P}_{\mathbb{U}(\mathbf{v}),\mathrm{r}(a\eta)}\mathbf{v}-\mathbf{u}\| \le C \inf_{\mathbf{w}\in\mathcal{H}(\mathrm{b}\mathrm{r}(a\eta))} \|\mathbf{u}-\mathbf{w}\|$$

- Contractions:

$$\pi^{(i)}(\mathbf{u}) = \big(\pi^{(i)}_{\nu_i}(\mathbf{u})\big)_{\nu_i\in\mathcal{J}} := \Bigg(\Big(\sum_{\nu_1,\dots,\nu_{i-1},\nu_{i+1},\dots,\nu_d} |u_\nu|^2\Big)^{\frac{1}{2}}\Bigg)_{\nu_i\in\mathcal{J}} \quad \rightsquigarrow$$

$$\pi^{(i)}_\nu(\mathbf{u}) = \Big(\sum_k \big|\mathbf{U}^{(i)}_{\nu,k}\big|^2 \big|\sigma^{(i)}_k\big|^2\Big)^{\frac{1}{2}}, \quad \pi^{(i)}_\nu(P_{\mathbb{U}(\mathbf{u}),\mathbf{r}}\mathbf{u}) \le \pi^{(i)}_\nu(\mathbf{u}), \quad \nu\in\mathcal{J}$$

$\rightsquigarrow$ sparsity of $\pi^{(i)}(\mathbf{u}) \leftrightarrow$ joint sparsity of ith mode-frames

- Exponetial sum approximation to (non-separable) scaling matrices

$\mathbf{S} = (s_\nu\delta_{\nu,\mu})_{\nu,\mu\in\mathcal{J}^d}$ in $\boldsymbol{A} = \mathbf{S}^{-1}\mathbf{T}\mathbf{S}^{-1}$

$$s_\nu^{-1} = (s_{1,\nu_1}+\dots+s_{d,\nu_d})^{-1/2} \approx \sum_{k=1}^r \omega_k e^{-\alpha_k s_\nu^2} = \sum_{k=1}^r \omega_k \prod_{j=1}^d e^{-\alpha_k s_{j,\nu_j}}$$

Controlling Rank Growth

Lemma:

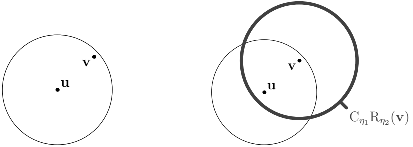
Fix $\alpha > 0$. Let $\|\mathbf{u} - \mathbf{v}\| \le \eta$, set $\mathbf{w}_\eta := \mathrm{R}_{q_d(1+\alpha)\eta}\mathbf{v} = \mathrm{P}_{\mathbb{U}(\mathbf{v}),q_d(1+\alpha)\eta}\,\mathbf{v}$. Then

$$\|\mathbf{u} - \mathbf{w}_\eta\| \le \big(1 + q_d(1+\alpha)\big)\eta$$

and

$$|\mathrm{rank}(\mathbf{w}_\eta)|_\infty \le \min\big\{|\mathrm{r}|_\infty : \ \mathrm{r} \text{ such that } \exists\, \tilde{\mathbf{w}}, \mathrm{rank}(\tilde{\mathbf{w}}) \le \mathrm{r} : \ \|\mathbf{u} - \tilde{\mathbf{w}}\| \le \alpha\eta\big\}\,.$$

- Combine with coarsening: similar result for $\mathrm{C}_{\eta_1}\mathrm{R}_{\eta_2}$



Exponential Sums Braess/Hackbusch

THEOREM:

Let $\alpha(x) := \ln^2(1+e^x)$, $w(x) := 2\pi^{-1/2}(1+e^{-x})^{-1}$, and for given $h > 0$, $n^+, n^- \in \mathbb{N}$, let

$$\varphi_{h,n^+,n^-}(t) := \sum_{k=-n^-}^{n^+} h\, w(kh)\, e^{-\alpha(kh)\, t}. \tag{1}$$

Let $\delta_0 \in (0,1)$, fix $h \in \left(0, \frac{\pi^2}{5(|\ln \delta_0|+4)}\right]$, $n^+ \geq N_0^+(\delta_0) := \lceil h^{-1} \max\{2, \sqrt{|\ln \delta_0|}\}\rceil$, then

$$\left|\frac{1}{\sqrt{t}} - \varphi_{h,n^+,\infty}(t)\right| \leq \frac{\delta_0}{\sqrt{t}} \quad \text{for all } t \in [1,\infty). \tag{2}$$

Furthermore, for any $\varepsilon > 0$ and for all $n^- \geq \lceil h^{-1}(\ln 2\pi^{-\frac{1}{2}} + |\ln \varepsilon|)\rceil$, we have

$$\left|\varphi_{h,n^+,\infty}(t) - \varphi_{h,n^+,n^-}(t)\right| \leq \varepsilon \quad \text{for all } t \in [1,\infty). \tag{3}$$

The Algorithm

Ideal Iteration:

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \omega(\mathbf{f} - \boldsymbol{A}\mathbf{u}^n), \quad n = 0, 1, 2, \ldots, \quad \boldsymbol{A} = \mathbf{S}^{-1}\mathbf{T}\mathbf{S}^{-1}, \; \mathbf{f} = \mathbf{S}^{-1}\mathbf{g}$$

Approximate separable scaling:

$$\tilde{\mathbf{S}}_m^{-1} = \sum_{k=-m}^{N^+} \omega_{m,k} \mathbf{e}_{k,1} \otimes \cdots \otimes \mathbf{e}_{k,d}, \quad (\mathbf{e}_{k,j})_{\nu_j,\mu} = e^{-\alpha_{m,k} 2^{2|\nu_j|}} \delta_{\nu_j,\mu}, \quad \tilde{\mathbf{S}} := \tilde{\mathbf{S}}_\infty$$

$$\|\mathbf{S}\tilde{\mathbf{S}}^{-1}\| \sim 1, \quad \|\mathbf{S}(\tilde{\mathbf{S}}^{-1} - \tilde{\mathbf{S}}_m^{-1})\mathbf{v}\| \leq \eta \|\mathbf{v}\| \text{ if } m \gtrsim |\ln \eta|, C \max_{\nu \in \operatorname{supp} \mathbf{v}} |\nu|$$

Perturbed Iteration:

$$\bar{\mathbf{u}}^{n+1} = \mathrm{C}_{\eta_1(n)} \mathrm{R}_{\eta_2(n)} \Big(\bar{\mathbf{u}}^n + \omega \big(\underbrace{\tilde{\mathbf{S}}_{m_{\eta_4(n)}} \mathbf{g}_{\eta_3(n)}}_{\approx \mathbf{f}} - \underbrace{(\tilde{\mathbf{S}}_{m_{\eta_4(n)}} \mathbf{T}_{\eta_3(n)} \tilde{\mathbf{S}}_{m_{\eta_4(n)}})}_{\approx \boldsymbol{A}} \bar{\mathbf{u}}^n \big) \Big)$$

What is to be shown?...Sparsity Notions...

Benchmarks/Assumptions:

- $\mathbf{u}$ is tensor sparse: $\mathbf{u} \in \mathcal{R}^{\gamma}(\ell_2(\mathcal{J}^d)) =: \mathcal{R}^{\gamma}$, i.e. for $\gamma(r) \nearrow \infty$ (e.g. $\gamma(r) = e^{\alpha r}$)

$$\sup_{r\in\mathbb{N}} \left\{\gamma(r) \inf_{\mathbf{w}\in\mathcal{H}(r)} \|\mathbf{u}-\mathbf{w}\|\right\} := \|\mathbf{u}\|_{\mathcal{R}^{\gamma}} < \infty$$

  i.e. optimal ranks for target accuracy ε satisfy $r(\varepsilon) \lesssim \gamma^{-1}(\|\mathbf{u}\|_{\mathcal{R}^{\gamma}}/\varepsilon)$

- $\pi^{(i)}(\mathbf{u}) \in \mathcal{A}^s = \mathcal{A}^s(\ell_2(\mathcal{J})), 1 \le i \le d,$ i.e.

$$\mathbf{v} \in \mathcal{A}^s \Leftrightarrow \sup_n n^s \left\{ \inf_{\mathrm{supp}\,\mathbf{z} \le n} \|\mathbf{v}-\mathbf{z}\|\right\} =: |\mathbf{v}|_{\mathcal{A}^s} < \infty \}$$

- The low-dimensional components of $\boldsymbol{A}$ are s^*-compressible with $s^* > s$

$\gamma(r) = e^{\alpha r}$, ε-accuracy: $\sim d^b |\log \varepsilon|^{1/\alpha} \varepsilon^{-1/s}$

Show that the approximate solution $\mathbf{u}_{\varepsilon}$ produced by the iteration has this sparsity

Convergence and Complexity

THEOREM:

Assume that problem has some excess regularity:

for $\varepsilon > 0$ the **Algorithm** produces a $\mathbf{u}_\varepsilon$ with $\|\mathbf{u} - \mathbf{u}_\varepsilon\| \le \varepsilon$ s.t.:

$$|\mathrm{rank}\,\mathbf{u}_\varepsilon|_\infty \le C(d)|\log \varepsilon|^b \gamma^{-1}(\|\mathbf{u}\|_{\mathcal{R}^\gamma}/\varepsilon)$$

$$\sum_{i=1}^{d} \#(\mathrm{supp}(\pi^{(i)}(\mathbf{u}_\varepsilon)) \le C(d)\Big(\sum_{i=1}^{d} \|\pi^{(i)}(\mathbf{u})\|_{\mathcal{A}^s}/\varepsilon\Big)^{\frac{1}{s}}$$

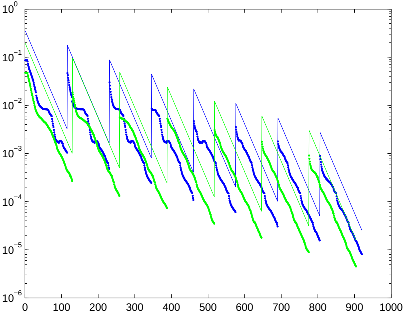
$$\|\mathbf{u}_\varepsilon\|_{\mathcal{R}^\gamma} \le C(d)\|\mathbf{u}\|_{\mathcal{R}^\gamma}, \quad \sum_{i=1}^{d} \|\pi^{(i)}(\mathbf{u}_\varepsilon)\|_{\mathcal{A}^s} \le C(d)\sum_{i=1}^{d} \|\pi^{(i)}(\mathbf{u})\|_{\mathcal{A}^s}$$

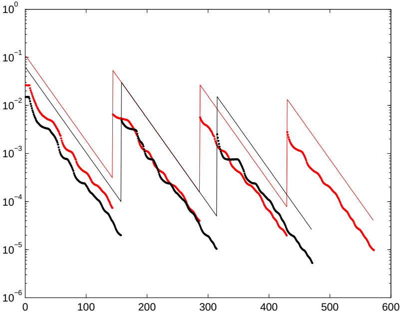
$$\#(\mathbf{ops}) \lesssim C(d)|\log \varepsilon|^b \varepsilon^{-\frac{1}{s}}\Big(\sum_{i=1}^{d} \max\{\|\pi^{(i)}(\mathbf{u})\|_{\mathcal{A}^s}, \|\pi^{(i)}(\mathbf{f})\|_{\mathcal{A}^s}\}\Big)^{\frac{1}{s}}$$

a) High-dim PDE: $C(d) \lesssim d^{\ln d}$, $b \lesssim \ln d$b) Parametric PDE: $C(d) \le d^a$,

Preliminary Numerical Tests

Test problem $-\Delta u = 1$, $u \in H_0^1((0,1)^d)$, $d = \mathbf{4}, \mathbf{8}, \mathbf{16}, \mathbf{32}.$

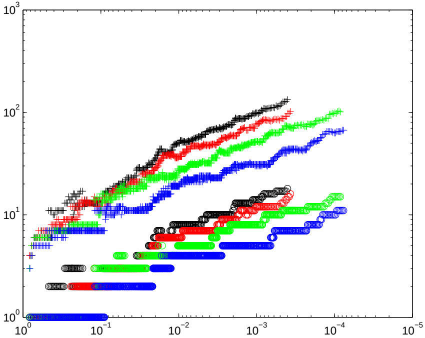




x-axis: iteration number, *y*-axis: residual estimate $\left\|\tilde{\mathbf{f}}_\eta - \tilde{\boldsymbol{A}}_\eta(\mathbf{u}_j)\right\|$ (dots), current error tolerance (lines)

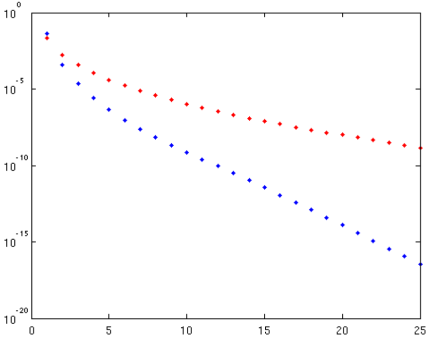
Preliminary Numerical Tests

$d = \mathbf{4}, \mathbf{8}, \mathbf{16}, \mathbf{32}$



x-axis: estimate of relative residual $\|\boldsymbol{A}\mathbf{u} - \mathbf{f}\| \,/\, \|\mathbf{f}\|$,
y-axis: max. ranks of u_j ($\circ$) and of intermediate quantities ($+$)

Concluding Remarks



- wider applicability
- weak tractability
- Poisson: polynomial tractability [D./DeVore/Grasedyck/Süli]
- dependence of $\mathrm{cond}_2(\boldsymbol{A})$ on d

References

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