Geometric structures and tensor based algorithms

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WORKSHOP NUMERICAL METHODS FOR HIGH-DIMENSIONAL PROBLEMS

1. Algebraic and topological tensor spaces

2. Tensor based Banach manifolds
   - Algebraic and Topological Tree Based Tensors (TBT)
   - Main Results

- A. Falcó, W. Hackbusch and A. Nouy. Geometric Structures in Tensor Representations. Preprint 9/2013 at Max Planck Institute for Mathematics in the Sciences (2013).
- A. Falcó and W. Hackbusch. Minimal subspaces in tensor representations. Foundations of Computational Mathematics, Volume 12, Issue 6 (2012), pp 765-803 .

Algebraic tensor spaces

As a first example

$$ {}_a\bigotimes_{j=1}^{d} H^{N,p}(I_j) = \operatorname{span} \underbrace{\{f_1(x_1)\cdots f_d(x_d) : f_i \in H^{N,p}(I_i)\}}_{\Sigma} $$

is a tensor space. In particular, we have

$$ {}_a\bigotimes_{j=1}^{d} L^{p}(I_j) = \operatorname{span} \underbrace{\{f_1(x_1)\cdots f_d(x_d) : f_i \in L^{p}(I_i)\}}_{\Sigma} $$

for $1 \leq p < \infty$.

Topological tensor spaces

$$H^{N,p}(I_1 \times \cdots \times I_d) = {}_{\|\cdot\|_{N,p}} \bigotimes_{j=1}^{d} H^{N,p}(I_j)$$

is a Banach tensor space $p \neq 2$ and a Hilbert tensor space for $p = 2$. In particular, we have

$$L^p(I_1 \times \cdots \times I_d) = {}_{\|\cdot\|_{0,p}} \bigotimes_{j=1}^{d} L^p(I_j)$$

for $1 \leq p < \infty$. In general, for a norm $\|\cdot\|$ defined over an algebraic tensor space we will write:

$${}_{\|\cdot\|} \bigotimes_{j=1}^{d} V_j = \overline{{}_a \bigotimes_{j=1}^{d} V_j}^{\|\cdot\|}, \quad V_j \text{ is a vector space !}$$

Definition

Take an algebraic tensor space $\mathbf{V}_D := {}_a\bigotimes_{j=1}^d V_j$ and fix some d-tuple $\mathbf{r} \in \mathbb{N}^d$ The *set of tensors of bounded rank* $\mathbf{r}$ is defined by

$$\mathcal{T}_{\mathbf{r}}(\mathbf{V}_D) := \left\{ \mathbf{v} \in \mathbf{V}_D : \mathbf{v} \in {}_a\bigotimes_{j=1}^{d} U_j \quad \dim U_j \leq r_\alpha \text{ for all } j \right\}, \tag{1}$$

and *the set of tensors of fixed rank* $\mathbf{r}$ is defined by

$$\mathcal{M}_{\mathbf{r}}(\mathbf{V}_D) := \{\mathbf{v} \in \mathcal{T}_{\mathfrak{r}}(\mathbf{V}_D) : \dim U_j = r_j \text{ for all } j\}\,. \tag{2}$$

The good,the bad and the ugly (Sergio Leone-1966)

Take $\mathbf{V}_{123} := V_1 \otimes_a V_2 \otimes_a V_3$ and a norm $\|\cdot\|_{23}$ defined over $\mathbf{V}_{23} := V_2 \otimes_a V_3$. Let $\|\cdot\|_{123}$ be a norm defined over $V_1 \otimes_a \overline{\mathbf{V}_{23}}^{\|\cdot\|_{23}}$ Then we have either

$$\mathcal{M}_{\mathbf{r}}(\mathbf{V}_{123}) \subset \mathcal{T}_{\mathbf{r}}(\mathbf{V}_{123}) \subset \mathbf{V}_{123} \subset \overline{V_1 \otimes_a \overline{\mathbf{V}_{23}}^{\|\cdot\|_{23}}}^{\|\cdot\|_{123}}$$

or

$$\mathcal{M}_{\mathbf{r}}(\mathbf{V}_{123}) \subset \mathcal{T}_{\mathbf{r}}(\mathbf{V}_{123}) \subset \mathbf{V}_{123} \subset \overline{V_1 \otimes_a \mathbf{V}_{23}}^{\|\cdot\|_{123}}$$

The best approximation and the geometric structure are norm dependent problems. The question is: Who is the good/bad/ugly?

The answer my friend is blowing in the wind (Bob Dylan)

- If V_i is a normed space with norm $\|\cdot\|_i$, then $\overline{V_i}^{\|\cdot\|_i}$ is always a Banach space, and then we have

$$ {}_a\bigotimes_{j=1}^{d} V_j \subset {}_a\bigotimes_{j=1}^{d} \overline{V_i}^{\|\cdot\|_i} \quad \text{then} \quad \overline{{}_a\bigotimes_{j=1}^{d} V_j}^{\|\cdot\|} \subset \overline{{}_a\bigotimes_{j=1}^{d} \overline{V_i}^{\|\cdot\|_i}}^{\|\cdot\|} $$

- The equality (that always holds in finite dimension)

$$ \overline{{}_a\bigotimes_{j=1}^{d} V_j}^{\|\cdot\|} = \overline{{}_a\bigotimes_{j=1}^{d} \overline{V_i}^{\|\cdot\|_i}}^{\|\cdot\|} $$

is true when the tensor product is continuous.

A desirable property

The equality is also true when $\|\cdot\| \gtrsim \|\cdot\|_{\vee(\overline{V}_1^{\|\cdot\|_1},\ldots,\overline{V}_d^{\|\cdot\|_d})}$. Clearly, the tensor product is continuous.

Definition (Injective norm)

Let V_i be a Banach spaces with norm $\|\cdot\|_i$ for $1 \leq i \leq d$. Then for $\mathbf{v} \in \mathbf{V} = {}_a\bigotimes_{j=1}^d V_j$ define $\|\cdot\|_\vee = \|\cdot\|_{\vee(V_1,\ldots,V_d)}$ by

$$\|\mathbf{v}\|_\vee := \sup \left\{ \frac{|(\varphi_1 \otimes \varphi_2 \otimes \ldots \otimes \varphi_d)(\mathbf{v})|}{\prod_{j=1}^d \|\varphi_j\|_j^*} : 0 \neq \varphi_j \in V_j^*, 1 \leq j \leq d \right\}.$$

Theorem (Best approximation)

Let V_i be a Banach spaces with norm $\|\cdot\|_i$ for $1 \leq i \leq d$. and let $\|\cdot\|$ be a norm on the algebraic tensor space $\mathbf{V}_D := {}_a\bigotimes_{j=1}^{d} V_j$. If $\|\cdot\| \gtrsim \|\cdot\|_{\vee(V_1,\ldots,V_d)}$ holds and $\overline{V_D}^{\|\cdot\|}$ is a reflexive Banach space. Then the set $\mathcal{T}_{\mathbf{r}}(\mathbf{V}_D)$ is (weakly closed) proximinal in $\overline{V_D}^{\|\cdot\|}$.

An optimization problem

Assume that there exists a manifold $\mathbb{M} = \mathcal{M}_{\mathbf{r}}(\mathbf{V}_D) \subset \Sigma = \mathcal{T}_{\mathbf{r}}(\mathbf{V}_D)$ such that

$$\min_{\mathbf{w}\in\Sigma} J(\mathbf{w}) = \min_{\mathbf{w}\in\mathbb{M}} J(\mathbf{w})$$

holds. Then $\mathbf{v} \in \arg\min_{\mathbf{w}\in\Sigma} J(\mathbf{w})$ satisfies the following first order condition (Euler-Lagrange Equation):

$$\langle J'(\mathbf{v}), \dot{\mathbf{w}}\rangle = 0 \text{ for all } \dot{\mathbf{w}} \in \mathbb{T}_{\mathbf{v}}\mathbb{M}.$$

Question

Is $\mathcal{M}_{\mathbf{r}}(\mathbf{V}_D)$ a manifold? and a more important thing: Where is? (ambient space).

Tensor Based manifolds: Hartree Banach manifold

$$\mathbb{M}_{\text{Hartree}} = \left\{ f_1(x_1)\cdots f_d(x_d) : f_i \in H^{N,p}(I_i) \setminus \{0\} \right\} \subset H^{N,p}(I_1 \times \cdots \times I_d)$$

The natural coordinates of $\mathbb{M}_{\text{Hartree}} = \mathcal{M}_{(1,\ldots,1)}\left({}_a \bigotimes_{j=1}^{d} H^{1,p}(I_j) \right)$

$$\mathbf{v} = \lambda\, f_1(x_1)\cdots f_d(x_d) = \lambda\, f_1 \otimes \cdots \otimes f_d,$$

are given as follows: Let $W_i(f_i) : H^{N,p}(I_i) = \mathrm{span}\{f_i\} \oplus W_i(f_i)$ for $1 \le i \le d$. Then $\mathbf{v} + \delta\mathbf{v}$ is in a "natural neighborhood" of $\mathbf{v}$ if and only if

$$\mathbf{v} + \delta\mathbf{v} = \eta\, (f_1 + \delta f_1)(x_1)\cdots(f_d + \delta f_d)(x_d) \quad \delta f_i \in W_i(f_i) \quad 1 \le i \le d.$$

So the "coordinates" of $\mathbf{v} + \delta\mathbf{v}$ are $(\eta, \delta f_1, \ldots, \delta f_d)$ (for $\mathbf{v}$ are $(\lambda, 0, \ldots, 0)$).

Tangent space

Under this coordinates the "natural tangent space" at $\mathbf{v} = \lambda f_1 \otimes \cdots \otimes f_d$ is

$$\mathbb{T}_{\mathbf{v}}\mathbb{M}_{\text{Hartree}} = \mathbb{R} \times W_1(f_1) \times \cdots W_d(f_d),$$

that is, a velocity $\dot{v}$ in $\mathbb{T}_{\mathbf{v}}\mathbb{M}_{\text{Hartree}}$ is given by

$$\dot{v} \equiv (\eta, \delta f_1, \ldots, \delta f_d).$$

Rank-one minimization

Since $\Sigma = \mathcal{T}_{(1,\ldots,1)}({}_a\bigotimes_{j=1}^d H^{1,p}(I_j)) = \overline{\mathbb{M}_{\text{Hartree}}}^{\|\cdot\|_{N,p}} \subset H^{N,p}(I_1 \times \cdots \times I_d)$ is weakly closed, then the problem

$$\min_{\mathbf{w}\in\Sigma} J(\mathbf{w})$$

is well-posed. If $\mathbf{0} \neq \mathbf{v} \in \arg\min_{\mathbf{w}\in\Sigma} J(\mathbf{w})$, then

$$\min_{\mathbf{w}\in\Sigma} J(\mathbf{w}) = \min_{\mathbf{w}\in\mathbb{M}_{\text{Hartree}}} J(\mathbf{w}).$$

Find $\lambda f_1(x_1)\cdots f_d(x_d) \in \mathbb{M}_{\text{Hartree}}$:

$$\langle J'(\lambda f_1(x_1)\cdots f_d(x_d)), \dot{\mathbf{v}}\rangle = 0 \quad \dot{\mathbf{v}} \in \mathbb{R}\times W_1(f_1)\times\cdots W_d(f_d).$$

We need “to embed” $\dot{\mathbf{v}} \in \mathbb{R}\times W_1(f_1)\times\cdots W_d(f_d)$ into $H^{N,p}(I_1\times\cdots\times I_d)$.

Embedding manifold

Let us consider the standard inclusion map (the identity)

$$i : \mathbb{M}_{\text{Hartree}} \to H^{N,p}(I_1 \times \cdots \times I_d), \quad \lambda\, f_1 \cdots f_d \mapsto \lambda \bigotimes_{i=1}^{d} f_i.$$

In local coordinates is a map $(i \circ \varphi_{\mathbf{v}}^{-1})$:
$(\mathbb{R} \setminus \{0\}) \times W_1(f_1) \times \cdots W_d(f_d) \to H^{N,p}(I_1 \times \cdots \times I_d)$ given by

$$(\eta, \delta f_1, \ldots, \delta f_d) \mapsto \eta\,(f_1 + \delta f_1) \cdots (f_d + \delta f_d).$$

Its derivative $\mathrm{T}_{\mathbf{v}} i := (i \circ \varphi_{\mathbf{v}}^{-1})'(\lambda, 0, \cdots, 0)$ is a linear map given by

$$\mathrm{T}_{\mathbf{v}} i(\gamma, \delta f_1, \ldots, \delta f_d) = \gamma \bigotimes_{i=1}^{d} f_j + \sum_{j=1}^{d} \lambda\, \delta f_j \otimes \bigotimes_{k \neq j} f_k.$$

Is an embedding manifold ?

- Is $\mathrm{T}_{\mathbf{v}} i : \mathbb{R} \times W_1(f_1) \times \cdots W_d(f_d) \to H^{N,p}(I_1 \times \cdots \times I_d)$ injective ?
- Is the linear subspace $\mathrm{T}_{\mathbf{v}} i(\mathbb{R} \times W_1(f_1) \times \cdots W_d(f_d)) = \mathbf{Z}(\mathbf{v})$ where

$$\mathbf{Z}(\mathbf{v}) := \left\{ \gamma \bigotimes_{i=1}^{d} f_j + \sum_{j=1}^{d} \lambda\, \delta f_j \otimes \bigotimes_{k \neq j} f_k : \begin{array}{l} (\gamma, \delta f_1, \ldots, \delta f_d) \\ \in \mathbb{T}_{\mathbf{v}} \mathbb{M}_{\text{Hartree}} \end{array} \right\}$$

closed and complemented in $H^{N,p}(I_1 \times \cdots \times I_d)$?

- Observe that the subspace

$$\mathbf{Z}(\mathbf{v}) = \bigotimes_{i=1}^{d} \operatorname{span}\{f_i\} \oplus \left(\bigoplus_{j=1}^{d} W_j(f_j) \otimes_a \operatorname{span}\{\lambda \bigotimes_{k \neq j} f_k\} \right).$$

The answer my friend is blowing in the wind (Bob Dylan)

- If the tensor product $\otimes$ is continuous then $\mathrm{T}_{\mathbf{v}} i$ is well defined and it is also injective.

Theorem

Let V_i be a Banach spaces with norm $\|\cdot\|_i$ for $1 \leq i \leq d$. and let $\|\cdot\|$ be a norm on the algebraic tensor space $\mathbf{V}_D := {}_a\bigotimes_{j=1}^{d} V_j$. If $\|\cdot\| \gtrsim \|\cdot\|_{\vee(V_1,\ldots,V_d)}$ holds then for each $\mathbf{v} \in \mathcal{M}_{\mathbf{r}}(\mathbf{V}_D)$ the linear subspace $\mathbf{Z}(\mathbf{v})$ is closed and complemented in $\overline{\mathbf{V}_D}^{\|\cdot\|}$, and hence $\mathcal{M}_{\mathbf{r}}(\mathbf{V}_D)$ is a submanifold of $\overline{\mathbf{V}_D}^{\|\cdot\|}$.

Comments

- The manifold of tensors of fixed rank is an analytical Banach manifold even if the tensor product map is not continuous.
- If the tensor product map is continuous then (i)we can compute $\mathrm{T}i$ in order to transport velocities and (ii)the tangent space can be identify with a linear space $\mathbf{Z}(\mathbf{v})$ inside the tensor Banach space.
- If $\|\cdot\| \gtrsim \|\cdot\|_{\vee(V_1,\ldots,V_d)}$ holds then the manifold of tensor of fixed rank is a submanifold inside the tensor Banach space.

The moral of the tale

$$\overline{{}_a\bigotimes_{j=1}^d V_j}^{\|\cdot\|}$$

$$V_1 \qquad V_2 \qquad \cdots \qquad V_d$$

$$\overline{{}_a\bigotimes_{j=1}^d \overline{V_i}^{\|\cdot\|_i}}^{\|\cdot\|}$$

$$\overline{V_1}^{\|\cdot\|_1} \qquad \overline{V_2}^{\|\cdot\|_2} \qquad \cdots \qquad \overline{V_d}^{\|\cdot\|_d}$$

both tensor representations are the same when $\|\cdot\| \gtrsim \|\cdot\|_\vee$and the existence of a best approximation holds. We assume a tensor representation like

$$\mathbf{v} = \sum_{i_1=1}^{r_1} \cdots \sum_{i_d=1}^{r_d} C_{i_1\cdots i_d}\, \mathbf{u}_{i_1} \otimes \cdots \otimes \mathbf{u}_{i_d}$$

where $C_{i_1\cdots i_d} \in \mathbb{R}^{r_1\times\cdots\times r_d}$ and $\{\mathbf{u}_{i_k}\}_{i_k=1}^{i_k=r_k}$ is a basis of a subspace U_k in V_k for $1 \le k \le d$. So $\mathbf{v} \in {}_a\bigotimes_{k=1}^d U_k$ and $\operatorname{rank} \mathbf{v} = (r_1, \dots, r_d)$.

Algebraic Tree Based Tensors

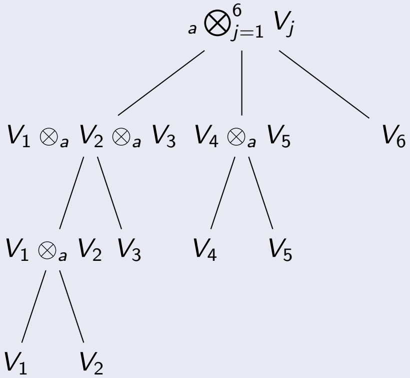
Take $D = \{1, 2, \ldots, d\}$ be the root then a tree T_D is defined by

$$\mathbf{V}_D := {}_a \bigotimes_{\alpha \in S(D)} \mathbf{V}_\alpha = {}_a \bigotimes_{\alpha \in S(D)} \left({}_a \bigotimes_{\beta \in S(\alpha)} \mathbf{V}_\beta \right) = \cdots = {}_a \bigotimes_{j \in D} V_j$$

where for $\delta \in 2^D \setminus \{\emptyset\}$ we put

$$\mathbf{V}_\delta := {}_a \bigotimes_{j \in \delta} V_j$$

Algebraic Tree Based Representation



Tensor representation

$$\mathbf{v} = \sum_{i_{123}=1}^{r_{123}} \sum_{i_{45}=1}^{r_{45}} \sum_{i_6=1}^{r_6} C_{i_{123} i_{45} i_6} \mathbf{u}_{i_{123}} \otimes \mathbf{u}_{i_{45}} \otimes u_{i_6}$$

where

$$\mathbf{u}_{i_{123}} = \sum_{i_{12}=1}^{r_{12}} \sum_{i_3=1}^{r_3} C_{i_{123}; i_{12} i_3} \mathbf{u}_{i_{12}} \otimes u_{i_3},$$

$$\mathbf{u}_{i_{12}} = \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} C_{i_{12}; i_1 i_2} u_{i_1} \otimes u_{i_2},$$

and

$$\mathbf{u}_{i_{45}} \sum_{i_4=1}^{r_4} \sum_{i_5=1}^{r_5} C_{i_{45}; i_4 i_5} u_{i_4} \otimes u_{i_5}.$$

Now the $\mathrm{rank}\,\mathbf{v} = (r_{123}, r_{45}, r_6, r_{12}, r_3, r_4, r_5, r_1, r_2)$

Definition

Let T_D be a given dimension partition tree and fix some tuple $\mathfrak{r} \in \mathbb{N}^{T_D}$ for T_D. The *set of TBF tensors of bounded TB rank* $\mathfrak{r}$ is defined by

$$\mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D) := \left\{ \mathbf{v} \in \mathbf{V}_D : \ \dim U_\alpha^{\min}(\mathbf{v}) \leq r_\alpha \text{ for all } \alpha \in T_D \ \right\}, \tag{3}$$

and *the set of TBF tensors of fixed TB rank* $\mathfrak{r}$ is defined by

$$\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) := \left\{ \mathbf{v} \in \mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D) : \dim U_\alpha^{\min}(\mathbf{v}) = r_\alpha \text{ for all } \alpha \in T_D \right\}. \tag{4}$$

Remark

$$\mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D) = \cup_{\mathfrak{s} \leq \mathfrak{r}} \mathcal{FT}_{\mathfrak{s}}(\mathbf{V}_D)$$

Property

Assume that $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ then for each $\alpha \in T_D \setminus \{\{1\}, \ldots, \{d\}\}$ it can be show that

$$C_{i_\alpha;(i_\beta)_{\beta \in S(\alpha)}} \in \mathbb{R}_*^{r_\alpha \times \left(\bigtimes_{\beta \in S(\alpha)} r_\beta \right)},$$

that is, $\operatorname{rank} \mathcal{M}_\alpha(C_{i_\alpha;(i_\beta)_{\beta \in S(\alpha)}}) = r_\alpha$ and $\operatorname{rank} \mathcal{M}_\beta(C_{i_\alpha;(i_\beta)_{\beta \in S(\alpha)}}) = r_\beta$ for all $\beta \in S(\alpha)$. Here rank means matrix rank and $\mathcal{M}_\beta$ is the matrization of the tensor with respect the index β.

Topological Tree Based representation

- $\left({}_a \bigotimes_{j=1}^{6} V_j \,, \|\cdot\|_D\right)$
  - $(V_1 \otimes_a V_2 \otimes_a V_3, \|\cdot\|_{123})$
    - $(V_1 \otimes_a V_2, \|\cdot\|_{12})$
      - $(V_1, \|\cdot\|_1)$
      - $(V_2, \|\cdot\|_2)$
    - $(V_3, \|\cdot\|_3)$
  - $(V_4 \otimes_a V_5, \|\cdot\|_{45})$
    - $(V_4, \|\cdot\|_4)$
    - $(V_5, \|\cdot\|_5)$
  - $(V_6, \|\cdot\|_6)$

Example

Representation on topological tree based format

$$
\begin{tikzcd}
 & \overline{L^p(I_1) \otimes_a H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)}^{\|\cdot\|_{123}} \arrow[dl, dash] \arrow[dr, dash] & & \\
L^p(I_1) & & \overline{H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)}^{\|\cdot\|_{23}} \arrow[dl, dash] \arrow[dr, dash] & \\
 & H^{N,p}(I_2) & & H^{N,p}(I_3)
\end{tikzcd}
$$

Example

No representation on topological tree based format

$$\begin{array}{ccc}
 & \overline{L^p(I_1) \otimes_a \overline{H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)}^{\|\cdot\|_{23}}}^{\|\cdot\|_{123}} & \\
L^p(I_1) & & \overline{H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)}^{\|\cdot\|_{23}} \\
 & H^{N,p}(I_2) & H^{N,p}(I_3)
\end{array}$$

However, if $\| \cdot \|_{123} \gtrsim \| \cdot \|_{\vee(L^p(I_1), \overline{H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)}^{\|\cdot\|_{23}})}$ holds both trees give the same representation.

Banach-Grassmann Manifold (A. Douady-1961)

For each $\alpha \in T_D \setminus \{D\}$, there exists $W_\alpha^{\min}(\mathbf{v})$ such that

$$\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = U_\alpha^{\min}(\mathbf{v}) \oplus W_\alpha^{\min}(\mathbf{v})$$

Every U such that

$$\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = U \oplus W_\alpha^{\min}(\mathbf{v})$$

is characterized by the existence of a unique $L_\alpha \in \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v}))$ such that

$$U = \operatorname{span}\{\mathbf{u}_{i_\alpha} + L_\alpha(\mathbf{u}_{i_\alpha}) : 1 \leq i_\alpha \leq r_\alpha\},$$

where

$$U_\alpha^{\min}(\mathbf{v}) = \operatorname{span}\{\mathbf{u}_{i_\alpha} : 1 \leq i_\alpha \leq r_\alpha\}.$$

Theorem

The set $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ of TBF tensors with fixed TB rank is an analytical Banach manifold. **This geometric structure is independent of the choice of the norm** $\|\cdot\|_D$.

Example

Let $V_{1_{\|\cdot\|_1}} := H^{1,p}(I_1)$ and $V_{2_{\|\cdot\|_2}} = H^{1,p}(I_2)$. Take $\mathbf{V}_D := H^{1,p}(I_1) \otimes_a H^{1,p}(I_2)$, from Theorem 8 we obtain that $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ is a Banach manifold. However, we can consider as ambient manifold either $\overline{\mathbf{V}_D}^{\|\cdot\|_{D,1}} := H^{1,p}(I_1 \times I_2)$ or $\overline{\mathbf{V}_D}^{\|\cdot\|_{D,2}} = H^{1,p}(I_1) \otimes_{\|\cdot\|_{(0,1),p}} H^{1,p}(I_2)$, where $\|\cdot\|_{(0,1),p}$ is the norm given by

$$\|f\|_{(0,1),p} := \left(\|f\|_p^p + \left\| \frac{\partial f}{\partial x_1} \right\|_p^p \right)$$

for $1 \leq p < \infty$.

Assumption

$$\|\cdot\|_{\alpha} \gtrsim \|\cdot\|_{\vee(S(\alpha))} \text{ for each } \alpha \in T_D \setminus \mathcal{L}(T_D), \tag{5}$$

Theorem

Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_{\alpha}}}\}_{\alpha \in T_D \setminus \{D\}}$ *be a representation of a tensor Banach space* $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j \in D} V_j$ *, in topological tree based format and assume that (5) holds. Then* $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ *is an embedded submanifold of* $\mathbf{V}_{D_{\|\cdot\|_D}}$*. Moreover, we can construct a complemented subspace* $\mathbf{Z}^{(D)}(\mathbf{v})$ *such that* $\mathbf{Z}^{(D)}(\mathbf{v}) = \mathrm{T}_{\mathbf{v}} i\left(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))\right)$ *holds for* $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$*).*

If (5) holds then both trees are the same

$$
\begin{tikzcd}[column sep=small]
 & \overline{L^p(I_1) \otimes_a H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)}^{\|\cdot\|_{123}} \arrow[dl, dash] \arrow[dr, dash] & & \\
L^p(I_1) & & \overline{H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)}^{\|\cdot\|_{23}} \arrow[dl, dash] \arrow[dr, dash] & \\
 & H^{N,p}(I_2) & & H^{N,p}(I_3)
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & \overline{L^p(I_1) \otimes_a \overline{H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)}^{\|\cdot\|_{23}}}^{\|\cdot\|_{123}} \arrow[dl, dash] \arrow[dr, dash] & & \\
L^p(I_1) & & \overline{H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)}^{\|\cdot\|_{23}} \arrow[dl, dash] \arrow[dr, dash] & \\
 & H^{N,p}(I_2) & & H^{N,p}(I_3)
\end{tikzcd}
$$

Theorem

Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha \in T_D \setminus \{D\}}$ *be a representation of a reflexive Banach tensor space* $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D} \bigotimes_{j \in D} V_j$ *, in topological tree based format and assume that (5) holds. Then for each* $\mathbf{v} \in \mathbf{V}_{D_{\|\cdot\|_D}}$ *there exists* $\mathbf{u}_{best} \in \mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D)$ *such that*

$$\|\mathbf{v} - \mathbf{u}_{best}\|_D = \min_{\mathbf{u} \in \mathcal{BT}_{\mathfrak{r}}(\mathbf{V}_D)} \|\mathbf{v} - \mathbf{u}\|_D,$$

here $\mathbf{V}_D = {}_a \bigotimes_{j \in D} V_j$ *.*

Thank you for your attention !