A Parametrized-Background Data-Weak Formulation for Variational Data Assimilation:

Dimension Reduction by Experimental Observation

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Numerical Methods for High-Dimensional Problems
École des Ponts Paristech
Marne-la-Vallée, France

April 17, 2014

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Apparatus & Diagnostics
- Conception
- Implementation

Calibration
Data Acquisition
Robotics

$\updownarrow$ Mathematical Modeling and Data Reduction $\updownarrow$

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Mathematical Formulation
Computational Methods
- Conception
- Algorithms
- Implementation

Numerical Analysis

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[0]Thanks also to Debra Blanchard for assistance in the design and preparation of the slides, figures, and graphics.

Chaire d'Excellence,
Fondation Sciences Mathématiques de Paris
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Université Pierre et Marie Curie

AFOSR/Office of Secretary of Defense

Office of Naval Research

MIT-Singapore International Design Center

Objective

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Given

a physical system,

we wish to integrate

a parametrized mathematical model, and
M experimental observations,

to estimate the (assumed) deterministic field

$$u^{\text{true}} \in \mathcal{U}(\Omega), \quad \Omega \subset \mathbb{R}^d,$$

and associated outputs of interest.

Desiderata

We shall *insist* upon

weak formulation $\rightarrow$ actionable theory:

a priori error bounds;

a posteriori error estimates;

$\mathcal{O}(M^{\cdot})$ computational complexity $\rightarrow$ real-time;

simple implementation $\rightarrow$ broad dissemination;

general applicability $\rightarrow$ "industrial" relevance.

We *aspire* to accurate state estimation for modest M.

Our Proposal

We shall *insist* upon

weak formulation $\rightarrow$ actionable theory:

a priori error bounds;

a posteriori error estimates;

$\mathcal{O}(M^3)$ computational complexity $\rightarrow$ real-time;

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We *aspire* to accurate state estimation for modest M.

Parametrized-Background Data-Weak (PBDW)
Formulation for Variational Data Assimilation

Formulation: PBDW

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- Preliminaries
- Unlimited-Observations Statement
- Limited-Observations Statement
- Offline-Online Computational Procedure
- A Priori Error Analysis
- A Posteriori Error Estimates
- Construction of Spaces
- Relation to Prior Work

State Space

Introduce

a spatial domain $\Omega \in \mathbb{R}^d$;

a Hilbert space $\mathcal{U}(\Omega)$ with
inner product (w, v) and norm $\|w\| = \sqrt{(w, w)}$[1];

a dual space $\mathcal{U}'$ and duality pairing $\langle \cdot, \cdot \rangle_{\mathcal{U}' \times \mathcal{U}}$;

a Riesz operator $R_{\mathcal{U}} : \mathcal{U}' \to \mathcal{U}$ such that
for $\ell \in \mathcal{U}'$, $(R_{\mathcal{U}} \ell, v) = \ell(v), \forall v \in \mathcal{U}$.

Assume: real fields; $H_0^1(\Omega) \subset \mathcal{U} \subset H^1(\Omega)$.

[1] In practice, we replace $\mathcal{U}$ by a finite element approximation space $\mathcal{U}^{\mathcal{N}} \subset \mathcal{U}$ of dimension $\mathcal{N}$.

Projection and Complement

Given $\mathcal{Q} \subset \mathcal{U}$, define

projection operator $\Pi_{\mathcal{Q}} : \mathcal{U} \to \mathcal{Q}$

$$(\Pi_{\mathcal{Q}} w, v) = (w, v), \quad \forall v \in \mathcal{Q};$$

orthogonal complement $\mathcal{Q}^{\perp} \subset \mathcal{U}$

$$\mathcal{Q}^{\perp} \equiv \{w \in \mathcal{U} \,|\, (w, v) = 0,\ \forall v \in \mathcal{Q}\}.$$

Objective Précisé

Given

a physical system in configuration $\mathcal{C}$,

we wish to integrate

a parametrized mathematical model, and

M experimental observations,

to estimate the field

$$u^{\mathrm{true}}[\mathcal{C}] \in \mathcal{U}(\Omega), \quad \Omega \subset \mathbb{R}^d,$$

and desired output(s)

$\ell_{\cdot}^{\mathrm{out}}(u^{\mathrm{true}}[\mathcal{C}]) \in \mathbb{R}$, for given $\ell_{\cdot}^{\mathrm{out}} \in \mathcal{U}'$.

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(Prior) Background Spaces $\mathcal{Z}_N$

Introduce hierarchical subspaces

$$\mathcal{Z}_1 \subset \mathcal{Z}_2 \subset \cdots \subset \mathcal{Z}_N \subset \cdots \subset \mathcal{Z}_{N_{\max}} \subset \cdots \subset \mathcal{U};$$

such that

$$\text{as} \quad N \to \infty, \qquad \inf_{w \in \mathcal{Z}_N} \|u^{\text{true}} - w\| \to \epsilon$$

for ϵ an acceptable tolerance.

The spaces $\mathcal{Z}_N$ are constructed from
our (prior) best knowledge of the problem.

Example: Choose $\mathcal{Z}_N$ as the span of N snapshots on a
"best-knowledge-model" parametric manifold.

Minimization Statement

Find $(u_N^* \in \mathcal{U}, z_N^* \in \mathcal{Z}_N, \eta_N^* \in \mathcal{U})$ such that

$$(u_N^*, z_N^*, \eta_N^*) = \arg \inf_{\substack{u_N \in \mathcal{U} \\ z_N \in \mathcal{Z}_N \\ \eta_N \in \mathcal{U}}} \|\eta_N\|^2$$

subject to

$$\begin{aligned} (u_N, v) &= (\eta_N, v) + (z_N, v), \quad \forall v \in \mathcal{U}, \\ (u_N, \phi) &= (u^{\mathrm{true}}, \phi), \quad \forall \phi \in \mathcal{U}. \end{aligned}$$

Minimizer

1. From $(u_N^*, \phi) = (u^{\text{true}}, \phi), \ \forall \phi \in \mathcal{U}$,

   we deduce $u_N^* = u^{\text{true}}$ — "state estimate".

2. From $(u_N^*, v) = (\eta_N^*, v) + (z_N^*, v), \ \forall v \in \mathcal{U}$,

   we deduce $\eta_N^* = u^{\text{true}} - z_N^*$.

3. From $(u_N^*, z_N^*, \eta_N^*) = \arg\inf_{\substack{\eta_N \in \mathcal{U} \\ u_N \in \mathcal{U} \\ z_N \in \mathcal{Z}_N}} \|\eta_N\|^2$,

   we conclude:

   $z_N^* = \Pi_{\mathcal{Z}_N} u^{\text{true}}$ — "deduced background";

   $\eta_N^* = u^{\text{true}} - \Pi_{\mathcal{Z}_N} u^{\text{true}} \equiv \Pi_{\mathcal{Z}_N^\perp} u^{\text{true}}$ — "update."

Note that η_N^* "completes" a deficient prior space $\mathcal{Z}_N$.

Euler-Lagrange Equations: Saddle Problem

Find $(\eta_N^* \in \mathcal{U}, z_N^* \in \mathcal{Z}_N)$ such that

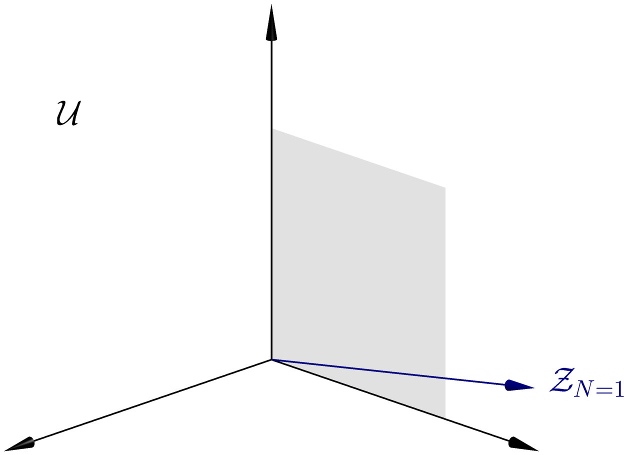
$$\begin{aligned} (\eta_N^*, q) + (z_N^*, q) &= (u^{\mathrm{true}}, q), \quad \forall q \in \mathcal{U} \ , \\ (\eta_N^*, p) \qquad\qquad\quad &= 0, \quad \forall p \in \mathcal{Z}_N \ , \end{aligned}$$

and set $u_N^* = \eta_N^* + z_N^*$.

Solution confirms update-background decomposition

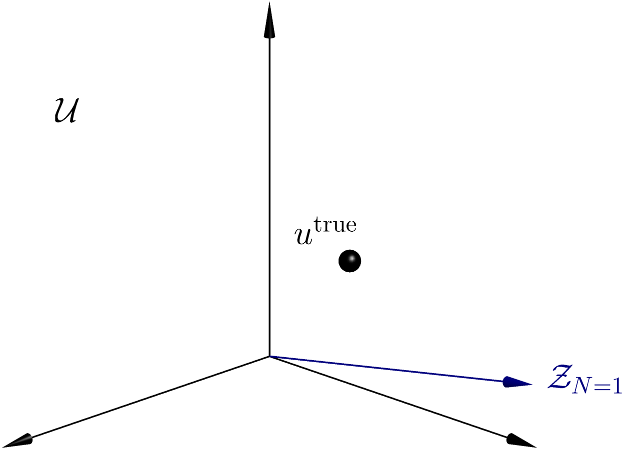
$$u_N^* = \eta_N^* + z_N^* = \underbrace{\Pi_{\mathcal{Z}_N^\perp} u^{\mathrm{true}}}_{\text{update}} + \underbrace{\Pi_{\mathcal{Z}_N} u^{\mathrm{true}}}_{\text{deduced background}} = u^{\mathrm{true}}.$$

Pictorial Projections, $\dim(\mathcal{U}) = 3$: Unlimited Observations



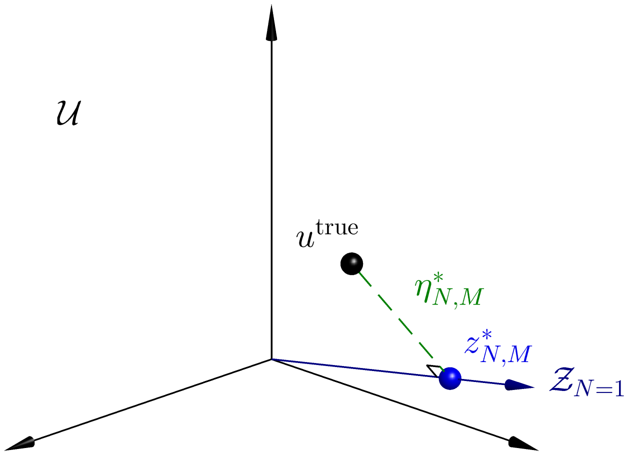
$\mathcal{Z}_N \subset \mathcal{U}$

Pictorial Projections, $\dim(\mathcal{U}) = 3$: Unlimited Observations



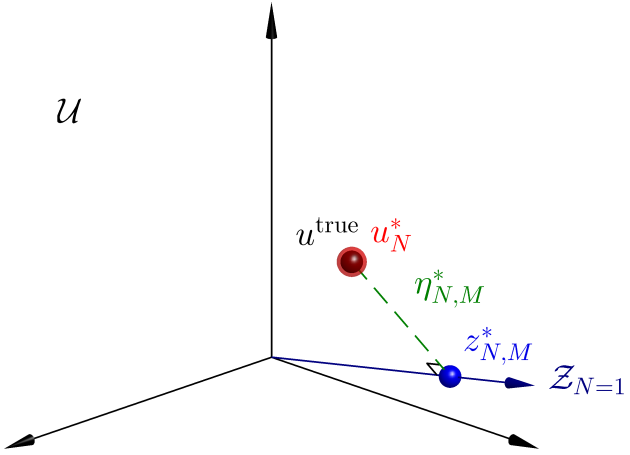
$u^{\text{true}} \in \mathcal{U}$

Pictorial Projections, $\dim(\mathcal{U}) = 3$: Unlimited Observations



$$z^*_N = \Pi_{\mathcal{Z}_N} u^{\text{true}}, \quad \eta^*_N = \Pi_{\mathcal{Z}_N^\perp} u^{\text{true}}$$

Pictorial Projections, $\dim(\mathcal{U}) = 3$: Unlimited Observations



$$u_N^* = z_N^* + \eta_N^*$$

Euler-Lagrange Equations: Saddle

Find $(\eta_N^* \in \mathcal{U}, z_N^* \in \mathcal{Z}_N)$ such that

$$
\begin{aligned}
(\eta_N^*, q) + (z_N^*, q) &= (u^{\text{true}}, q), \quad \forall q \in \mathcal{U}\ , \\
(\eta_N^*, p) &= 0, \quad \forall p \in \mathcal{Z}_N\ ,
\end{aligned}
$$

and set $u_N^* = \eta_N^* + z_N^*$.

Solution confirms update-background decomposition

$$u_N^* = \eta_N^* + z_N^* = \underbrace{\Pi_{\mathcal{Z}_N^\perp} u^{\text{true}}}_{\text{update}} + \underbrace{\Pi_{\mathcal{Z}_N} u^{\text{true}}}_{\text{deduced background}} = u^{\text{true}}.$$

We achieve $u_N^* = u^{\text{true}}$, but we cannot *experimentally* evaluate $(u^{\text{true}}, q),\ \forall q \in \mathcal{U}$.

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Introduce general observation functionals

$$\ell^{\mathrm{o}}_m \in \mathcal{U}', \quad m = 1, \ldots, M_{\max},$$

such that we interpret

$$O_m[\mathcal{C}] : \text{\textit{perfect} experimental observation } m$$
$$\equiv \ell^{\mathrm{o}}_m(u^{\mathrm{true}}[\mathcal{C}]) \; .$$

(Formulation stable *perfect* $\rightarrow$ *imperfect*: inf-sup.)

Local observation functionals: $\{\ell^{\mathrm{o}}_m\}_{m=1}^{M_{\max}}$ defined by

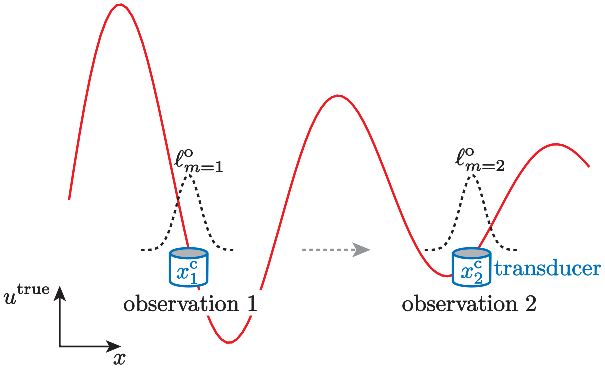
a "center" parameter: $x^{\mathrm{c}}_m \in \Omega$,

a "spread" parameter: $\varphi_m = \varphi \in \mathbb{R}_{\geq 0}$,

for $m = 1, \ldots, M_{\max}$.

We consider $m = 1, \ldots, M_{\max}$

$$\ell^{\mathrm{o}}_m(v) = \mathsf{Gauss}(v; x^{\mathrm{c}}_m, \varphi)$$
$$\equiv \int_\Omega \frac{1}{(2\pi)^{d/2}\varphi^d} \exp\left(-\frac{(x - x^{\mathrm{c}}_m)^2}{2\varphi^2}\right) v(x)dx.$$



Update Spaces: $\{\mathcal{U}_M\}_{M=1}^{M_{\max}}$ — Experimentally Observable

Introduce hierarchical spaces $\qquad M = 1\ldots, M_{\max}, \ldots$

$$\mathcal{U}_M = \text{span}\{q_m \equiv R_{\mathcal{U}}\ell_m^{\text{o}}\}_{m=1}^{M};$$

recall $R_{\mathcal{U}}\ell \in \mathcal{U}$ is the Riesz representation of $\ell \in \mathcal{U}'$.

Then, for $q_m (\in \mathcal{U}_M)$,

$$(u^{\text{true}}, q_m) = (u^{\text{true}}, R_{\mathcal{U}}\ell_m^{\text{o}}) = \ell_m^{\text{o}}(u^{\text{true}}) = O_m$$

is an experimental observation; hence, $\forall q \in \mathcal{U}_M$,

$$\begin{aligned}(u^{\text{true}}, q) = (u^{\text{true}}, \textstyle\sum_{m=1}^{M} \alpha_m q_m) &= \textstyle\sum_{m=1}^{M} \alpha_m \ell_m^{\text{o}}(u^{\text{true}}) \\ &= \textstyle\sum_{m=1}^{M} \alpha_m O_m\end{aligned}$$

is a weighted sum of experimental observations.

Constrained Minimization: Statement

Find $(u^*_{N,M} \in \mathcal{U}, z^*_{N,M} \in \mathcal{Z}_N, \eta^*_{N,M} \in \mathcal{U})$ such that

$$(u^*_{N,M}, z^*_{N,M}, \eta^*_{N,M}) = \underset{\substack{u_{N,M} \in \mathcal{U} \\ z_{N,M} \in \mathcal{Z}_N \\ \eta_{N,M} \in \mathcal{U}_M}}{\arg\inf} \|\eta_{N,M}\|^2$$

subject to

$$(u_{N,M}, v) = (\eta_{N,M}, v) + (z_{N,M}, v), \quad \forall v \in \mathcal{U},$$
$$(u_{N,M}, \phi) = (u^{\text{true}}, \phi), \quad \forall \phi \in \mathcal{U}_M.$$

Euler-Lagrange Equations: Discrete Saddle Problem

Find $(\eta^*_{N,M} \in \mathcal{U}_M, z^*_{N,M} \in \mathcal{Z}_N)$ such that

$$(\eta^*_{N,M}, q) + (z^*_{N,M}, q) = \underbrace{(u^{\text{true}}, q), \quad \forall q \in \mathcal{U}_M}_{\text{weighted sum of observations}},$$

$$(\eta^*_{N,M}, p) = 0, \quad \forall p \in \mathcal{Z}_N;$$

then set

$$\underbrace{u^*_{N,M}}_{\text{state estimate}} = \underbrace{\eta^*_{N,M}}_{\substack{\text{update estimate} \\ \in\, \mathcal{U}_M \cap \mathcal{Z}_N^\perp \\ \text{“CAP SPACE”}}} + \underbrace{z^*_{N,M}}_{\substack{\text{deduced-background estimate} \\ \in\, \mathcal{Z}_N}} .$$

Discrete observation–optimality saddle is of size $M + N$.

(Equivalent) Least Squares Formulation

Define deduced background estimate $z^*_{N,M} \in \mathcal{Z}_N$ by

$$z^*_{N,M} = \arg \inf_{z \in \mathcal{Z}_N} \|\Pi_{\mathcal{U}_M}(u^{\mathrm{true}} - z)\|^2 \ ,$$

which yields normal equations

$$(\Pi_{\mathcal{U}_M} z^*_{N,M}, v) = (\Pi_{\mathcal{U}_M} u^{\mathrm{true}}, v), \forall v \in \mathcal{Z}_N \ ;$$

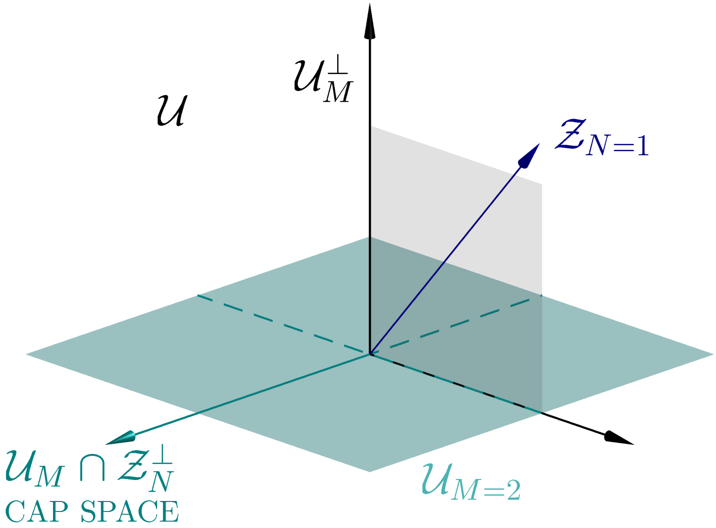
define update estimate $\eta^*_{N,M} \in \mathcal{U}_M$ by

$$\eta^*_{N,M} = \Pi_{\mathcal{U}_M}(u^{\mathrm{true}} - z^*_{N,M}) \ ;$$

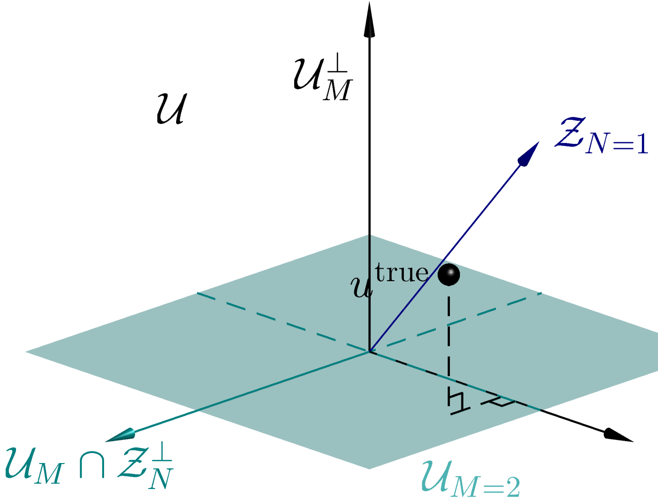
form state estimate $u^*_{N,M} \in \mathcal{U}$ as

$$u^*_{N,M} = \underbrace{\eta^*_{N,M}}_{\text{update estimate}} + \underbrace{z^*_{N,M}}_{\text{deduced-background estimate}} \ .$$

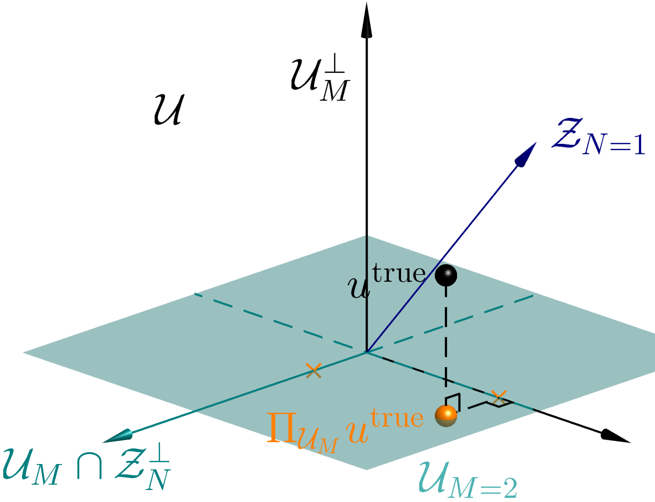
Pictorial Projections, $\dim(\mathcal{U}) = 3$: State Estimation



Pictorial Projections, $\dim(\mathcal{U}) = 3$: State Estimation

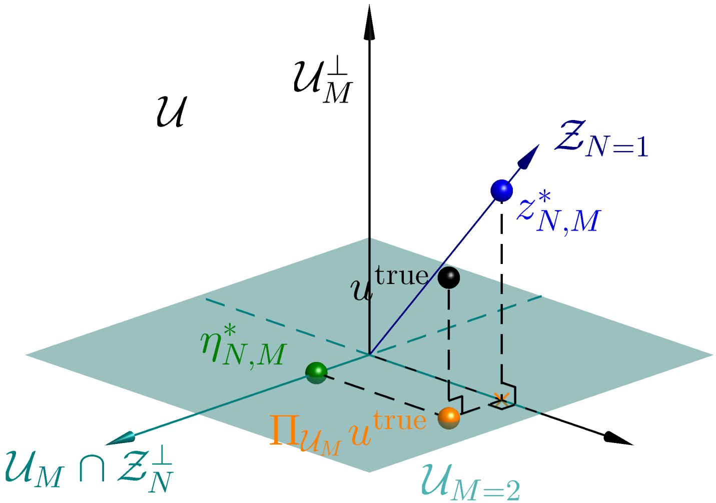


Pictorial Projections, $\dim(\mathcal{U}) = 3$: State Estimation



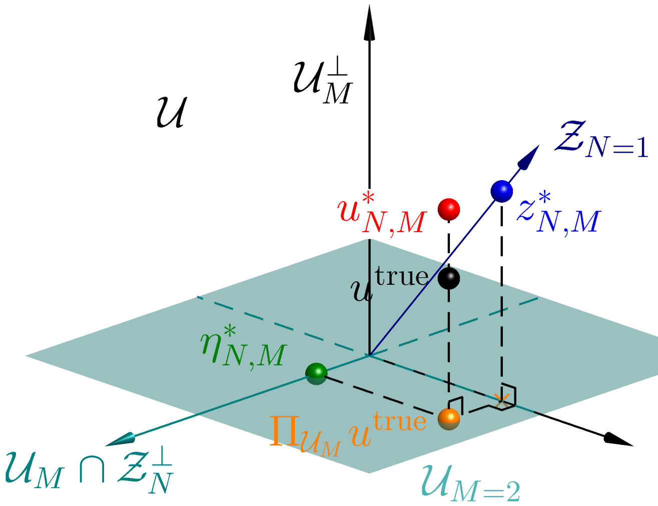
(data) $= \Pi_{\mathcal{U}_M} u^{\text{true}}$

Pictorial Projections, $\dim(\mathcal{U}) = 3$: State Estimation



$$\eta^*_{N,M} = \Pi_{\mathcal{U}_M \cap \mathcal{Z}_N^\perp} u^{\text{true}}, \; z^*_{N,M} \in \mathcal{Z}_N$$

Pictorial Projections, $\dim(\mathcal{U}) = 3$: State Estimation



$$u^*_{N,M} = z^*_{N,M} + \eta^*_{N,M}$$

Role of Mathematical Model

There is *no* reference to any mathematical model in the PBDW saddle problem;

connection to the mathematical model[2] is through the background spaces $\mathcal{Z}_N, 1 \le N \le N_{\max}$.

Implications:

applicability to wide class of problems;

simplicity of implementation.

[2] The model might be a (deterministic) partial differential equation, or we might also consider particle or stochastic descriptions.

Spatial Domain and Regularity

The PBDW formulation is furthermore

a problem in (constrained) approximation

effected as

a *projection* with respect to observations;

no boundary conditions required over $\partial\Omega$.

Implications: flexibility in choice of data-assimilation

spatial domain $\Omega \subset \mathbb{R}^d$ (or manifold);

regularity hypotheses on $u^{\mathrm{true}} \Rightarrow$ space $\mathcal{U}$.

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Construct spaces

$$\mathcal{Z}_{N_{\max}} \equiv \mathrm{span}\{\zeta_n,\ n = 1,\ldots,N_{\max}\} \quad (\Rightarrow\ \mathbf{Z})$$
$$\mathcal{U}_{M_{\max}} \equiv \mathrm{span}\{q_m,\ m = 1,\ldots,M_{\max}\} \quad (\Rightarrow\ \mathbf{U}).$$

SADDLE.Offline$_{N_{\max},M_{\max}}$: Form

$$\mathrm{ONLINEMATRICES}_{N_{\max},M_{\max}} \equiv \{\mathbf{A}, \mathbf{B}, \mathbf{l}^{\mathrm{out},\mathbf{U}}, \mathbf{l}^{\mathrm{out},\mathbf{Z}}\}$$

where

$$\mathbf{A}_{mm'} \equiv (q_{m'}, q_m), \qquad \mathbf{B}_{mn} \equiv (\zeta_n, q_m),$$
$$\mathbf{l}_m^{\mathrm{out},\mathbf{U}} \equiv \ell^{\mathrm{out}}(q_m), \qquad \mathbf{l}_n^{\mathrm{out},\mathbf{Z}} \equiv \ell^{\mathrm{out}}(\zeta_n),$$

for $m, m' = 1,\ldots,M_{\max}$, $n = 1,\ldots,N_{\max}$.

Online Stage: Procedure $\quad \mathcal{C} \to u^*_{N,M}[\mathcal{C}], \ell^{\text{out}}(u^*_{N,M}[\mathcal{C}])$

Collect observations: $\mathbf{l}^{\text{obs}}[\mathcal{C}] \in \mathbb{R}^M$ such that

$$\mathbf{l}^{\text{obs}}_m[\mathcal{C}] = O_m[\mathcal{C}] \equiv \ell^{\text{o}}_m(u^{\text{true}}[\mathcal{C}]), \quad m = 1, \ldots, M.$$

SADDLE.Online$_{N,M}$: $\quad \Leftarrow$ ONLINEMATRICES$_{N,M}$

Find $\boldsymbol{\eta}^*[\mathcal{C}] \in \mathbb{R}^M$ and $\mathbf{z}^*[\mathcal{C}] \in \mathbb{R}^N$ such that

$$\begin{bmatrix} \mathbf{A}_{1:M,1:M} & \mathbf{B}_{1:M,1:N} \\ \mathbf{B}^H_{1:M,1:N} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}^*[\mathcal{C}] \\ \mathbf{z}^*[\mathcal{C}] \end{bmatrix} = \begin{bmatrix} \mathbf{l}^{\text{obs}}[\mathcal{C}] \\ \mathbf{0} \end{bmatrix};$$

compute state and output as

$$u^*_{N,M}[\mathcal{C}] = \mathbf{U}_{:,1:M}\boldsymbol{\eta}^*[\mathcal{C}] + \mathbf{Z}_{:,1:N}\mathbf{z}^*[\mathcal{C}],$$

$$\ell^{\text{out}}(u^*_{N,M}[\mathcal{C}]) = \mathbf{l}^{\text{out},\mathbf{U}}_{1:M}\boldsymbol{\eta}^*[\mathcal{C}] + \mathbf{l}^{\text{out},\mathbf{Z}}_{1:N}\mathbf{z}^*[\mathcal{C}],$$

respectively.

Online Stage: Operation Count

Data acquisition $\mathcal{C} \rightarrow \mathbf{l}^{\text{obs}}[\mathcal{C}]$: M observations.

Solution of saddle for $\boldsymbol{\eta}^*[\mathcal{C}]$, $\mathbf{z}^*[\mathcal{C}]$:

$$\mathcal{O}((N+M)^3) \text{ FLOPS.}$$

Rendering of full state $u^*_{N,M}[\mathcal{C}]$ (if desired):

$$\mathcal{O}(\mathcal{N}) \text{ FLOPS,}$$

where $\mathcal{N}$ is the dimension of $\mathcal{U}^{\mathcal{N}}(\subset \mathcal{U})$.

Evaluation of output(s) $\ell^{\text{out}}_{\cdot}(u^*_{N,M}[\mathcal{C}])$:

$$\mathcal{O}(N+M) \text{ FLOPS}$$

(for each output of interest).

Formulation: PBDW

Problem Statements: Unlimited, Limited

Unlimited observations: find $(\eta_N^* \in \mathcal{U}, z_N^* \in \mathcal{Z}_N)$ s.t.

$$\begin{aligned} (\eta_N^*, q) + (z_N^*, q) &= (u^{\mathrm{true}}, q), \quad \forall q \in \mathcal{U} \ , \\ (\eta_N^*, p) \qquad\qquad &= 0, \quad \forall p \in \mathcal{Z}_N \ . \end{aligned}$$

Limited observations: find $(\eta_{N,M}^* \in \mathcal{U}_M, z_{N,M}^* \in \mathcal{Z}_N)$ s.t.

$$\begin{aligned} (\eta_{N,M}^*, q) + (z_{N,M}^*, q) &= (u^{\mathrm{true}}, q), \quad \forall q \in \mathcal{U}_M \ , \\ (\eta_{N,M}^*, p) \qquad\qquad &= 0, \quad \forall p \in \mathcal{Z}_N \ . \end{aligned}$$

Standard saddle in weak form

$\Rightarrow$ apply variational PDE analysis techniques.

Proposition 1.

The error in the state satisfies

$$\|u^{\text{true}} - u^*_{N,M}\| \le \left(1 + \frac{1}{\beta_{N,M}}\right) \inf_{\eta \in \mathcal{U}_M \cap \mathcal{Z}_N^\perp} \|\Pi_{\mathcal{Z}_N^\perp} u^{\text{true}} - \eta\|$$

for

$$\beta_{N,M} \equiv \inf_{z \in \mathcal{Z}_N} \sup_{v \in \mathcal{U}_M} \frac{(z, v)}{\|z\| \|v\|};$$

recall $\|\Pi_{\mathcal{Z}_N^\perp} u^{\text{true}}\| = \inf_{z \in \mathcal{Z}_N} \|u^{\text{true}} - z\|$.

Contributions to Error Bound

The bound for the state error

$$\left(1 + \frac{1}{\beta_{N,M}}\right) \inf_{\eta \in \mathcal{U}_M \cap \mathcal{Z}_N^{\perp}} \|\Pi_{\mathcal{Z}_N^{\perp}} u^{\text{true}} - \eta\|$$

depends on

1. the stability constant: $\beta_{N,M}$;
2. the background *primary* approximation:

   $\inf_{z \in \mathcal{Z}_N} \|u^{\text{true}} - z\| = \|\Pi_{\mathcal{Z}_N^{\perp}} u^{\text{true}}\|$;
3. the update *secondary* approximation:

   $\inf_{\eta \in \mathcal{U}_M \cap \mathcal{Z}_N^{\perp}} \|\Pi_{\mathcal{Z}_N^{\perp}} u^{\text{true}} - \eta\|$.

Proposition 2. The inf-sup constant (singular value)

$$\beta_{N,M} \equiv \inf_{w \in \mathcal{Z}_N} \sup_{v \in \mathcal{U}_M} \frac{(w, v)}{\|w\| \|v\|}$$

is

a non-increasing function of background span (N),

a non-decreasing function of observable span (M);

furthermore,

$\beta_{N,M} = 0$ for $M < N$ $(\mathcal{Z}_N \cap \mathcal{U}_M^\perp \neq 0)$.

The inf-sup constant $\beta_{N,M}$ is related to

the observability

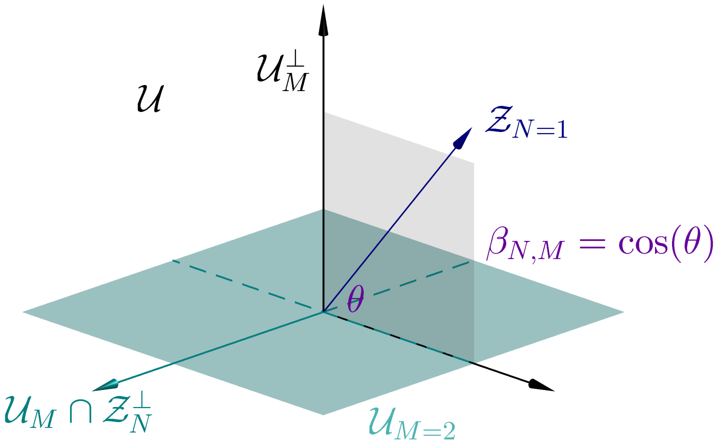
of *our estimation* of the physical system $\mathcal{C}$.

Note that

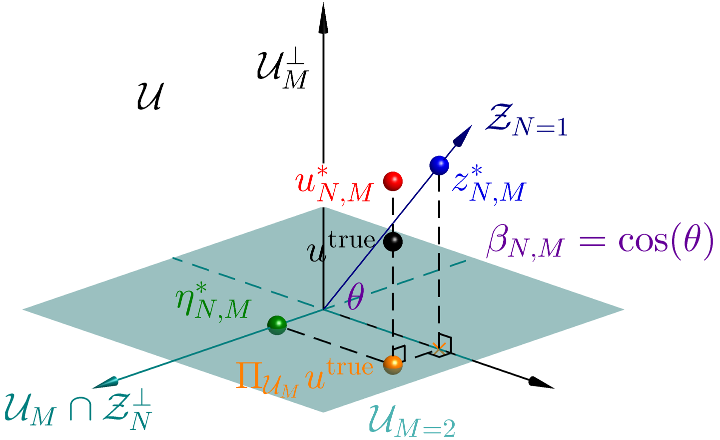
$\beta_{N,M}$ large $\not\Rightarrow$ good (primary) state estimate:
$\mathcal{Z}_N$ must also retain approximation properties;

$\beta_{N,M}$ small $\not\Rightarrow$ unobservable *physical system*:
$\mathcal{Z}_N$ may contain spurious (unstable) elements.

Pictorial Projections: $\dim(\mathcal{U}) = 3$: Inf-Sup



Pictorial Projections: $\dim(\mathcal{U}) = 3$: Inf-Sup



The space $\mathcal{U}_M$ provides stability

— and secondary approximation.

The RoleS of $\mathcal{U}_M$: Stability and Approximation

Space $\mathcal{Z}_N$ must provide

primary approximation: $\inf_{z \in \mathcal{Z}_N} \|u^{\text{true}} - z\|$.

Space $\mathcal{U}_M$ must provide, for given $\mathcal{Z}_N$,

primary stability: $\beta_{N,M} \equiv \inf_{w \in \mathcal{Z}_N} \sup_{v \in \mathcal{U}_M} \dfrac{(w,v)}{\|w\| \|v\|} > 0;$

secondary approximation: $\inf_{\eta \in \mathcal{U}_M \cap \mathcal{Z}_N^\perp} \|\Pi_{\mathcal{Z}_N^\perp} u^{\text{true}} - \eta\|;$

recall also $\mathcal{U}_M$ must be *experimentally observable*.

Online Cost Considerations

Note that for

good background spaces $\{\mathcal{Z}_N\}_{N=1}^{N_{\max}}$

we may choose

N small (for approximation)

and hence subsequently choose

$M (\geq N)$ small (for stability[3]).

Implication: faster `Online` response $\mathcal{C} \to \mathbf{l}^{\text{obs}}[\mathcal{C}]$.

[3] Note if secondary approximation is important, M is not dictated solely by stability considerations.

A Priori Analysis: Output Functional

Introduce linear (or nonlinear)

output functional: $\ell^{\mathrm{out}} \in \mathcal{U}'$.

Proposition 3. Output error satisfies

$$|\ell^{\mathrm{out}}(u^{\mathrm{true}}) - \ell^{\mathrm{out}}(u^*_{N,M})| = |(u^{\mathrm{true}} - u^*_{N,M}, \psi - \Pi_{\mathcal{U}_M}\psi)|$$

$$\leq \|u^{\mathrm{true}} - u^*_{N,M}\| \|\psi - \Pi_{\mathcal{U}_M}\psi\|$$

for $\psi = R_{\mathcal{U}}\ell^{\mathrm{out}} \in \mathcal{U}$.

Output error “superconverges” with M.

Formulation: PBDW

Define assessment centers

$$\{\xi_j^{\mathrm{c}} \in \Omega, 1 \le j \le J\}$$

distinct from observation centers

$$\{x_m^{\mathrm{c}} \in \Omega, 1 \le m \le M\}$$

such that we interpret $\ell^{\mathrm{a}} \sim \ell^{\mathrm{o}}$

$A_j[\mathcal{C}]$: *perfect* experimental assessment j

$$\equiv \mathsf{Gauss}(u^{\mathrm{true}}[\mathcal{C}]; \xi_j^{\mathrm{c}}, \varphi)\ .$$

Then define, for given N, M, and J,

$$E_{\mathrm{avg}}[\mathcal{C}] \equiv \sqrt{\tfrac{1}{J} \textstyle\sum_{j=1}^{J} (A_j[\mathcal{C}] - \mathsf{Gauss}(u^*_{N,M}[\mathcal{C}]; \xi_j^{\mathrm{c}}, \varphi))^2}$$

as an ($L^2(\Omega)$-ish) estimate of the error in the state.

Formulation: PBDW

Formulation: PBDW

Best-Knowledge (bk) Model: Parametrization

Introduce

parameter P–tuple μ, and

parameter domain $\mathcal{D} \subset \mathbb{R}^P$,

and associated bk parametrized form

$$\mu \in \mathcal{D} \;\rightarrow\; G^\mu : \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}$$

(linear in second argument).

In principle, μ, $\mathcal{D}$, and G^μ need not admit any

physical or mechanistic interpretation;

in practice, we benefit from disciplinary knowledge.

Best-Knowledge (bk) Manifold

Define the bk field

$$\mu \in \mathcal{D} \to u^{\mathtt{bk},\mu} \in \mathcal{U}$$

as solution of

$$G^\mu(u^{\mathtt{bk},\mu}, v) = 0, \quad \forall v \in \mathcal{U} \ ;$$

introduce bk parametric manifold

$$\mathcal{M}^{\mathtt{bk}} \equiv \{u^{\mathtt{bk},\mu} \,|\, \mu \in \mathcal{D}\}$$

to characterize the set of bk fields.

Best-Knowledge ($\mathtt{bk}$) Model Error

Introduce best-fit-over-manifold operator

$$F_{\mathcal{M}^{\mathtt{bk}}} : \mathcal{U} \to \mathcal{M}^{\mathtt{bk}}$$

such that

$$F_{\mathcal{M}^{\mathtt{bk}}}\, w = \arg \inf_{v \in \mathcal{M}^{\mathtt{bk}}} \|w - v\| \; .$$

Define model error as

$$\begin{aligned} \epsilon^{\mathtt{bk}}_{\text{mod}}(u^{\text{true}}) &\equiv \|u^{\text{true}} - F_{\mathcal{M}^{\mathtt{bk}}} u^{\text{true}}\| \\ &\equiv \inf_{w \in \mathcal{M}^{\mathtt{bk}}} \|u^{\text{true}} - w\|. \end{aligned}$$

Goal: minimize model error $\epsilon^{\mathtt{bk}}_{\text{mod}}(u^{\text{true}})$

through choice of parametrized model $[\mathcal{D}, G^{\mu}]$.

Three feasibility considerations

available information:	conservation laws;
	constitutive relations;
	constitutive "constants";
experimental cost:	calibration of G^{μ};
computational cost:	solution of $G^{\mu}(u^{\mathtt{bk},\mu}, v) = 0$;

constrain our choice of best-knowledge model.

In practice, u^{true} may be quite far from $\mathcal{M}^{\mathtt{bk}}$.

Formulation: PBDW

- Preliminaries
- Unlimited-Observations Statement
- Limited-Observations Statement
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- A Priori Error Analysis
- A Posteriori Error Estimates
- Construction of Spaces
  - Best-Knowledge Model
  - Background Spaces $\mathcal{Z}_N$
  - Update Spaces $\mathcal{U}_M$
  - Convergence Scenario
- Relation to Prior Work

Manifold $\textsc{Process}_N^{\mathcal{Z}}$

Invoke $\textsc{Process}_N^{\mathcal{Z}}(\mathcal{M}^{\mathrm{bk}})$:

1. $\textsc{Process}_N^{\mathcal{Z}} \equiv \texttt{POD}_N(\mathcal{M}^{\mathrm{bk}})$; or
2. $\textsc{Process}_N^{\mathcal{Z}} \equiv \texttt{WeakGreedy}_N(\mathcal{M}^{\mathrm{bk}})$; or
3. $\textsc{Process}_N^{\mathcal{Z}} \equiv \texttt{Taylor}_N^{\mu_0}(\mathcal{M}^{\mathrm{bk}})$; or

$\vdots$

to form spaces $\mathcal{Z}_N$ for $1 \le N \le N_{\max}$.

Goal: minimize *best-fit-over-*$\mathcal{Z}_N$ error,

$$\epsilon_N^{\mathrm{bk}}(u^{\mathrm{true}}) \equiv \inf_{w \in \mathcal{Z}_N} \|u^{\mathrm{true}} - w\| \, ,$$

for given N ($\rightarrow$ cost).

Model Error and Discretization Error — Contributions

The $\mathcal{Z}_N$ best-fit error may be bounded as

$$\epsilon_N^{\mathsf{bk}}(u^{\mathrm{true}}) \equiv \inf_{w \in \mathcal{Z}_N} \|u^{\mathrm{true}} - w\|$$

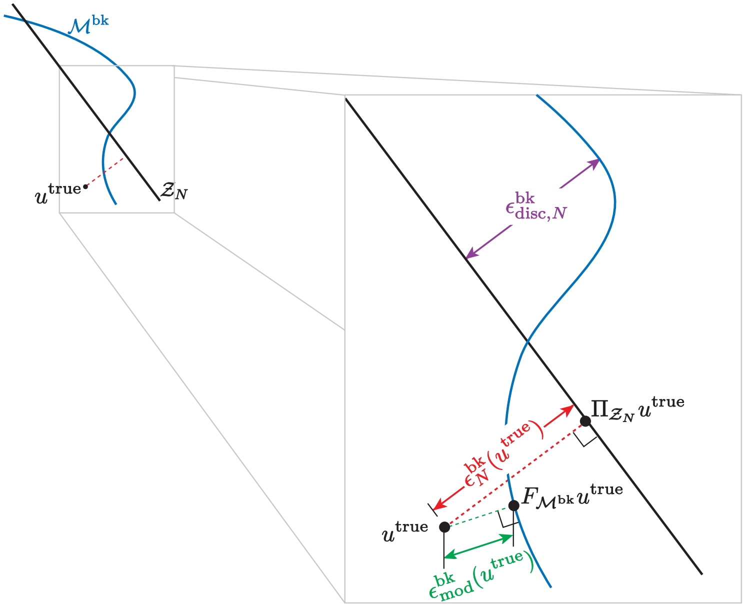
$$\leq \|u^{\mathrm{true}} - \Pi_{\mathcal{Z}_N} F_{\mathcal{M}^{\mathsf{bk}}}(u^{\mathrm{true}})\|$$

$$\leq \|u^{\mathrm{true}} - F_{\mathcal{M}^{\mathsf{bk}}}(u^{\mathrm{true}})\| + \|F_{\mathcal{M}^{\mathsf{bk}}}(u^{\mathrm{true}}) - \Pi_{\mathcal{Z}_N} F_{\mathcal{M}^{\mathsf{bk}}}(u^{\mathrm{true}})\|$$

$$\leq \underbrace{\inf_{w \in \mathcal{M}^{\mathsf{bk}}} \|u^{\mathrm{true}} - w\|}_{\text{model error} \Leftarrow [\mathcal{D}, G^{\mu}]} + \underbrace{\sup_{w \in \mathcal{M}^{\mathsf{bk}}} \|w - \Pi_{\mathcal{Z}_N} w\|}_{\text{discretization error} \Leftarrow \textsc{Process}_N^{\mathcal{Z}}}$$

$$\leq \underbrace{\epsilon_{\mathrm{mod}}^{\mathsf{bk}}(u^{\mathrm{true}})}_{\text{best fit of } u^{\mathrm{true}} \textit{ over} \text{ manifold}} + \underbrace{\epsilon_{\mathrm{disc},N}^{\mathsf{bk}}}_{\text{best fit } \textit{of} \text{ manifold over } \mathcal{Z}_N} .$$

Model and Discretization Errors — Picture



Role of Parameter

Parametrization of $\texttt{bk}$ model

$$\mu \in \mathcal{D} \quad \Rightarrow \quad G^{\mu}$$

induces

manifold $\mathcal{M}^{\mathrm{bk}}$, then
background spaces $\{\mathcal{Z}_N\}_{N=1}^{N_{\max}}$,

and ultimately (with $\mathcal{U}_M$)

state estimate $u^*_{N,M}$.

Note we provide *no* parameter estimate $\mu^*_{N,M}$:
parametrization μ, $\mathcal{D}$ serves only in $\textsc{Process}_N^{\mathcal{Z}}(\mathcal{M}^{\mathrm{bk}})$.

Introduce reduced basis (RB) approximation

$$\mu \to u_{N,\mathrm{Galerkin}}^{\mathtt{bk},\mu}$$

solution of

$$G^\mu(u_{N,\mathrm{Galerkin}}^{\mathtt{bk},\mu}, v) = 0, \quad \forall v \in \mathcal{Z}_N \, ,$$

and *a posteriori* error estimate $\Delta_N^{\mathtt{bk},\mu}(u_{N,\mathrm{Galerkin}}^{\mathtt{bk},\mu})$ such that

$$\underbrace{\|u^{\mathtt{bk},\mu} - \Pi_{\mathcal{Z}_N} u^{\mathtt{bk},\mu}\|}_{\text{discretization error } (\mu)} \leq \|u^{\mathtt{bk},\mu} - u_{N,\mathrm{Galerkin}}^{\mathtt{bk},\mu}\|$$

$$\lessapprox \Delta_N^{\mathtt{bk},\mu}, \quad \forall \mu \in \mathcal{D} \, .$$

Both $u_{N,\mathrm{Galerkin}}^{\mathtt{bk},\mu}$, $\Delta_N^{\mathtt{bk},\mu}$ admit rapid many-query evaluation.

Role of Reduced Basis Approximation

The reduced basis approximation

$$\mu \in \mathcal{D} \to u_{N,\mathrm{Galerkin}}^{\mathtt{bk},\mu}$$

only serves in the `Offline` stage

to define a residual

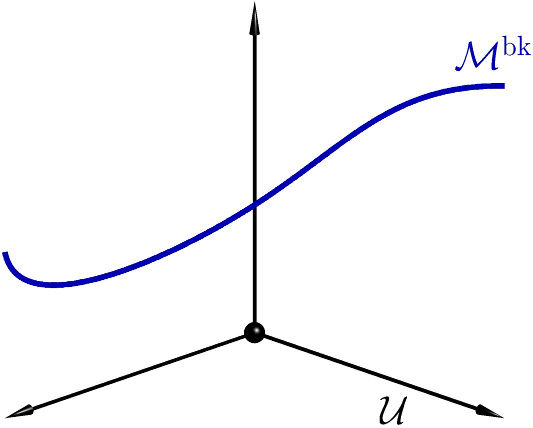
which then serves to evaluate

the error estimator $\Delta_N^{\mathtt{bk},\mu}$

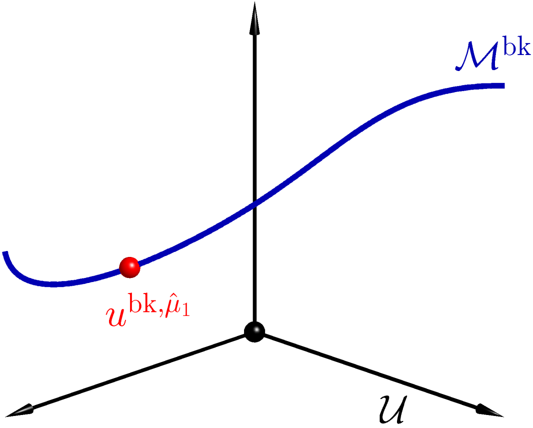
required by $\texttt{WEAKGREEDY}_N \to \mathcal{Z}_N$.

In PBDW, the reduced basis approximation does not appear in the `Online` stage.

$\text{PROCESS}_N^{\mathcal{Z}}$ Example: WEAKGREEDY_N — Picture

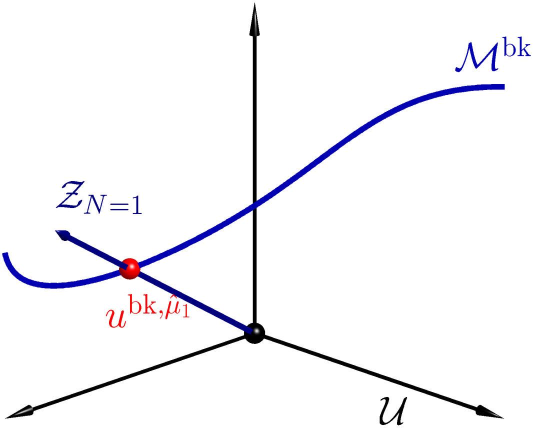


$\mathcal{M}^{\mathrm{bk}}$

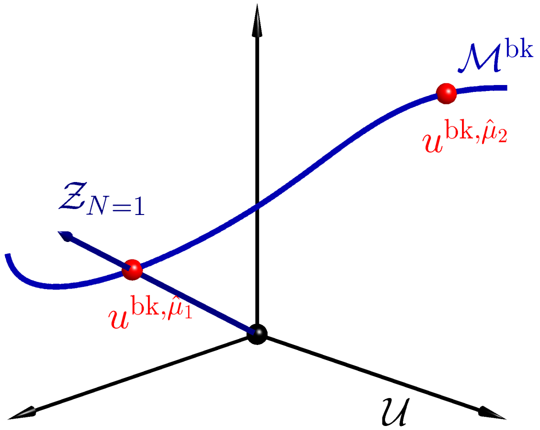


$u^{\mathrm{bk},\hat{\mu}_1} \in \mathcal{M}^{\mathtt{bk}}$

$\text{PROCESS}_N^{\mathcal{Z}}$ Example: WEAKGREEDY_N — Picture

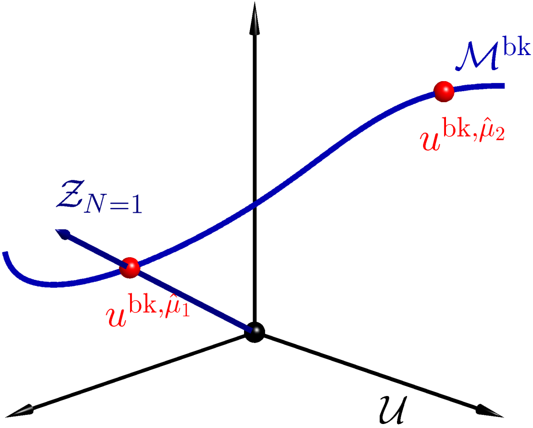


$$\mathcal{Z}_{N=1} \equiv \text{span}\{u^{\text{bk},\hat{\mu}_n}, n = 1\}$$



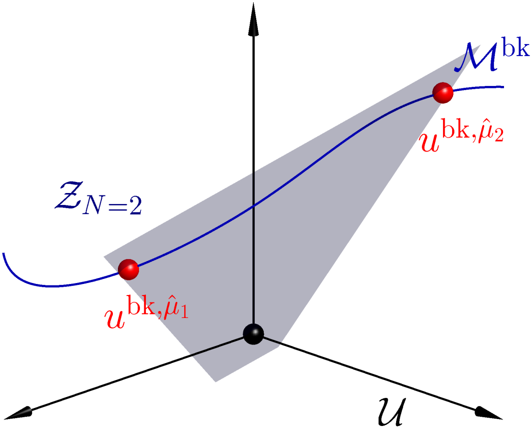
$u^{\mathrm{bk},\hat{\mu}_2}$: element of $\mathcal{M}^{\mathrm{bk}}$ least well represented by $\mathcal{Z}_{N=1}$

$\text{PROCESS}_N^{\mathcal{Z}}$ Example: WEAKGREEDY_N — Picture



$u^{\mathtt{bk},\hat{\mu}_2} : \hat{\mu}_2 = \arg\sup_{\mu\in\mathbb{D}_{\mathrm{train}}} \Delta_{N=1}^{\mathtt{bk},\mu}$ (projection error *estimator*)

$\text{PROCESS}_N^{\mathcal{Z}}$ Example: WEAKGREEDY_N — Picture



$$\mathcal{Z}_{N=2} \equiv \text{span}\{u^{\text{bk},\hat{\mu}_n}, n = 1, 2\}$$

$\text{PROCESS}_N^{\mathcal{Z}}$ Example: WEAKGREEDY_N — Algorithm

WEAKGREEDY_N: $[\mathcal{D}, G^\mu] \to \{\mathcal{Z}_N\}_{N=1}^{N_{\max}}$

For $N = 1, \ldots, N_{\max} - 1$,

1. $\hat{\mu}_{N+1} = \arg\sup_{\mu \in \mathbb{D}_{\text{train}} \subset \mathcal{D}} \Delta_N^{\mathtt{bk},\mu}$
2. $\zeta_{N+1} \equiv u^{\mathtt{bk},\hat{\mu}_{N+1}}$
3. $\mathcal{Z}_{N+1} \equiv \text{span}\{\mathcal{Z}_N, \zeta_{N+1}\}$.

Note that

$$\sup_{\mu \in \mathbb{D}_{\text{train}} \subset \mathcal{D}} \Delta_N^{\mathtt{bk},\mu} \approx \epsilon_{\text{disc},N}^{\mathtt{bk}} \quad :$$

WEAKGREEDY_N "minimizes" discretization error.[4]

[4] Recent theory demonstrates comparable convergence of discretization error $\epsilon_{\text{disc},N}^{\mathtt{bk}}$ and Kolmogorov N-width.

In the case of many parameters,

$$\mu \in \mathcal{D} \subset \mathbb{R}^P,\ P \gg 1 \quad ,$$

the (standard) $\texttt{WEAKGREEDY}_N$ algorithm may be

inefficient — large training set $\mathbb{D}_{\text{train}}$
$\Rightarrow$ unacceptable `Offline` cost;

ineffective — large N ($\Rightarrow$ large M) for desired $\epsilon^{\texttt{bk}}_{\text{disc},N}$
$\Rightarrow$ unacceptable `Online` cost.

In some cases, the failure may be fundamental.

Introduce

bk domain $\Omega^{\mathrm{bk}} \supset \Omega$[5]

bk space $\mathcal{U}^{\mathrm{bk}} = \mathcal{U}^{\mathrm{bk}}(\Omega^{\mathrm{bk}})$.

Form bk background space $\mathcal{Z}_N^{\mathrm{bk}} \subset \mathcal{U}^{\mathrm{bk}}$

$$\texttt{WEAKGREEDY}_N(\mathcal{M}^{\mathrm{bk}}) \to \mathcal{Z}_N^{\mathrm{bk}};$$

then form background space

$$\mathcal{Z}_N = \{z \in \mathcal{U} \mid z = z^{\mathrm{bk}}|_{\Omega},\ z^{\mathrm{bk}} \in \mathcal{Z}_N^{\mathrm{bk}}\}.$$

Focus data assimilation on $\Omega \subset \Omega^{\mathrm{bk}}$

even if bk *model is only well posed on* $\Omega^{\mathrm{bk}} \supset \Omega$.

[5] Note Ω may be a manifold of dimension d in $\Omega^{\mathrm{bk}} \subset \mathbb{R}^{d'}, d' > d$.

Design-of-Experiment $\textsc{Process}_M^{\mathcal{U}}$

Recall $\mathcal{U}_M = \mathrm{span}\{q_m \equiv R_{\mathcal{U}} \ell_m^{\mathrm{o}}\}_{m=1}^{M}$ for

$$\ell_m^{\mathrm{o}}(v) = \mathrm{Gauss}(v; x_m^{\mathrm{c}}, \varphi), \quad m = 1, \ldots, M.$$

Choose

inner product $(\cdot, \cdot) \;\Rightarrow\; R_{\mathcal{U}}$, and

centers $\{x_m^{\mathrm{c}} \in \Omega\}_{m=1}^{M}$,

to

a) maximize $\beta_{N,M} \approx$ "E"-optimality, for

stability of primary (background) approximation;

b) minimize $\inf_{\eta \in \mathcal{U}_M \cap \mathcal{Z}_N^{\perp}} \|\Pi_{\mathcal{Z}_N^{\perp}} u^{\mathrm{true}} - \eta\|$, for

secondary approximation — of unmodeled physics.

Objectives: Approximation *vs* Stability

Algorithms with objective stability

$$\textsc{SGreedy}_M, \ldots$$

often also provide reasonable secondary approximation.

Algorithms with objective secondary approximation,

$$\textsc{Uniform}, \textsc{RandomUniform}, \textsc{Max-Min}, \ldots$$

often do *not* provide reasonable stability.

In any event, for $w \in H^2(\Omega \subset \mathbb{R}^d)$,

$$\inf_{q \in \mathcal{U}_M} \|w - q\|_{H^r(\Omega)} \approx \left(M^{-(2-r)}\right)^{1/d}, \; r = 0, 1 :$$

secondary convergence is slow.

$\textsc{Process}_M^{\mathcal{U}}$ Example: Stability as Principal Objective

$\texttt{SGreedy}_M$: $\{\mathcal{Z}_N\}_{N=1}^{N_{\max}}, (\cdot,\cdot) \to \{\mathcal{U}_M\}_{M=1}^{M_{\max}}$

For $M = 1, \ldots, M_{\max}$,

1. Set $N = \min\{N_{\max}, M\}$.
2. Compute the least-stable mode
$$w_{\inf} \equiv \arg\inf_{w \in \mathcal{Z}_N} \sup_{v \in \mathcal{U}_{M-1}} \frac{(w,v)}{\|w\|\|v\|}.$$
3. Compute the associated supremizer
$$v_{\sup} = \Pi_{\mathcal{U}_{M-1}} w_{\inf}.$$
4. Identify the least well-approximated point
$$x^* = \arg\sup_{x\in\Omega} |(w_{\inf} - v_{\sup})(x)|.$$
5. Set $\mathcal{U}_M \equiv \mathrm{span}\{\mathcal{U}_{M-1}, R_{\mathcal{U}}\mathrm{Gauss}(\cdot; x^*, \varphi_m)\}$.

$\text{PROCESS}_M^{\mathcal{U}}$ Example: Approximation as Principal Objective

Proposition 4. Consider

$$\Omega \equiv]0,1[\subset \mathbb{R}^1,$$
$$\mathcal{U} \equiv H_0^1(\Omega),\ \|\cdot\| \equiv |\cdot|_{H^1(\Omega)},$$
$$\ell_m^{\mathrm{o}}(\cdot) \equiv \delta(\cdot; x_m^{\mathrm{c}}), \quad m = 1, \ldots, M$$
$\{x_m^{\mathrm{c}}\}_{m=1}^M$ equidistributed;

for $w \in H^2(\Omega)$, we obtain for $r = 0, 1$

$$\inf_{q \in \mathcal{U}_M} \|w - q\|_{H^r(\Omega)} \leq C M^{-(2-r)} \|w\|_{H^2(\Omega)}$$

for C independent of M and w.

Local observation functionals $\Rightarrow$ *low-order* convergence.

Formulation: PBDW

Roles of $N, \mathcal{Z}_N$ and $M, \mathcal{U}_M$

Primary: As N increases for fixed M ($\geq N$) expect

$\epsilon^{\mathtt{bk}}_{\mathrm{disc},N} \to 0$ rapidly, and

$\|u^{\mathrm{true}} - u^*_{N,M}\| \to \epsilon^{\mathtt{bk}}_{\mathrm{mod}}(u^{\mathrm{true}})$ rapidly;

$\mathcal{Z}_N$ provides approximation, and $\mathcal{U}_M$ provides stability.

Secondary: As M increases

for fixed $N = N_{\mathrm{plateau}} \equiv \{N \mid \epsilon^{\mathtt{bk}}_{\mathrm{disc},N} \ll \epsilon^{\mathtt{bk}}_{\mathrm{mod}}(u^{\mathrm{true}})\}$

expect

$\epsilon^{\mathtt{bk}}_{\mathrm{mod}}(u^{\mathrm{true}}) \to 0$ slowly, and

$\|u^{\mathrm{true}} - u^*_{N,M}\| \to 0$ slowly;

$\mathcal{U}_M$ provides approximation (of unmodeled physics, . . .).

Formulation: PBDW

- Preliminaries
- Unlimited-Observations Statement
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- Construction of Spaces
- Relation to Prior Work

0. PBDW $\sim$ Data-Projection Reduced Basis (RB)

   PBDW — $\mathcal{Z}_N \oplus \mathcal{U}_M$; RB — $\mathcal{Z}_N$ and G^μ, Galerkin.

1. PBDW $\supset$ GEIM   inf-sup $\equiv$ Lebesgue

   for any given $\mathcal{Z}_N$, and $N = M$.

2. PBDW $\supset$ Gappy-POD

   $\mathtt{POD}_N(\mathcal{M}^{\text{bk}}) \to \mathcal{Z}_N$ and $u^*_{N,M} \equiv z^*_{N,M} \in \mathcal{Z}_N$.

3. PBDW $\supset$ Stable Least Squares Estimation

   $\mu \equiv \mu_{\texttt{DIRICHLET}}$ and $u^*_{N,M} \equiv z^*_{N,M} \in \mathcal{Z}_N$.

4. PBDW $\supset$ linearized Structured Total Least Squares

   $\texttt{TAYLOR}_N^{\mu_0}(\mathcal{M}^{\mathrm{bk}}) \to \mathcal{Z}_N$.

5. PBDW $\subset$ Variational Data Assimilation (3d-VAR)

   background (prior) covariance $\leftarrow (I - \Pi_{\mathcal{Z}_N})^{-1}$.

Contributions

PBDW provides

1. rigorous error estimation:

    a priori bounds;

    a posteriori estimates;

2. computational and experimental efficiency:

    optimal background spaces ($\texttt{WeakGreedy}_N, \ldots$);

    optimal sensor locations ($\texttt{SGreedy}_M, \ldots$);

    $\mathcal{O}(M^3)$ `Online` complexity;

3. simplicity and generality:

    bk model restricted to `Offline` stage ($\rightarrow \mathcal{Z}_N$).

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Raised-Box Acoustic Resonator

- Physical System
- Robotic Observation Platform
- Best-Knowledge Model
- PBDW Formulation
- Real-Time *In Situ* State Estimation
- Error Analysis

Raised-Box Acoustic Resonator

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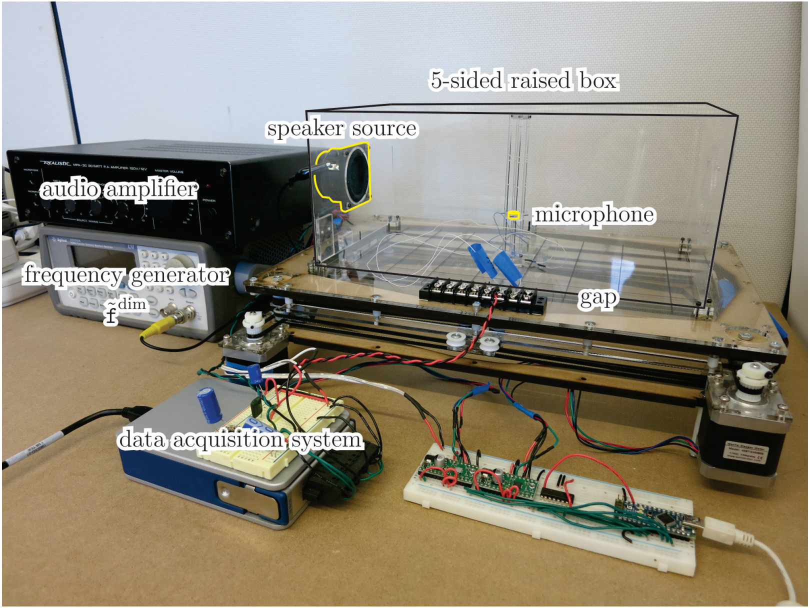
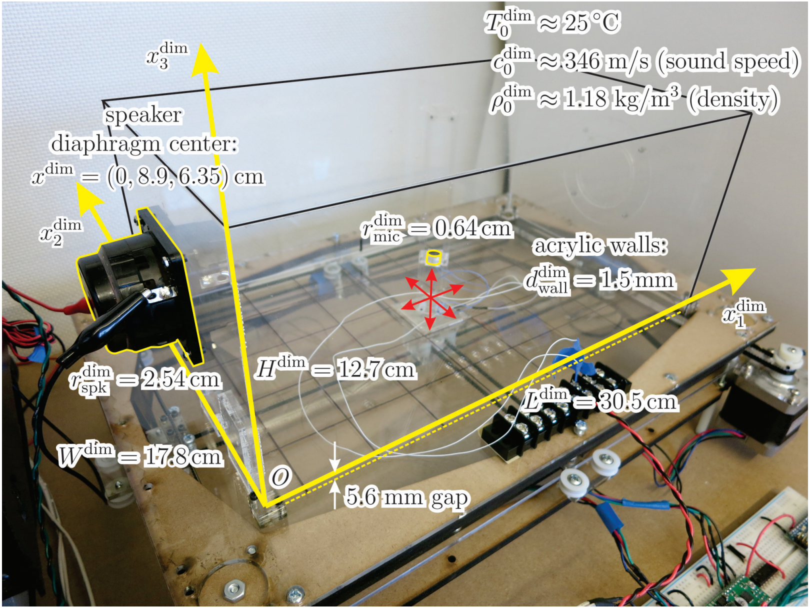
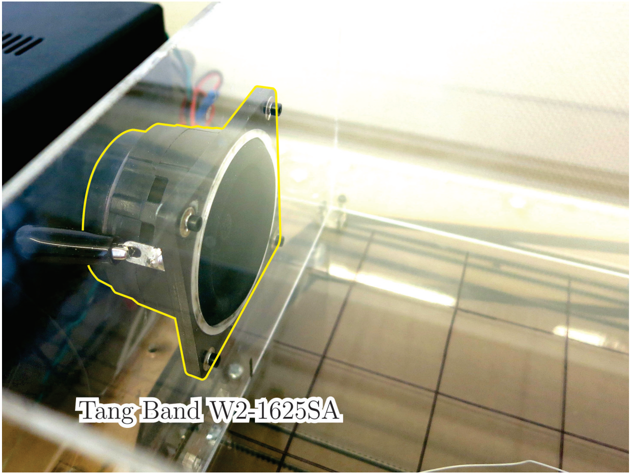
Overview


5-sided raised box
speaker source
audio amplifier
microphone
frequency generator
$\tilde{f}^{dim}$
gap
data acquisition system

Raised-Box Specifications



Speaker Detail



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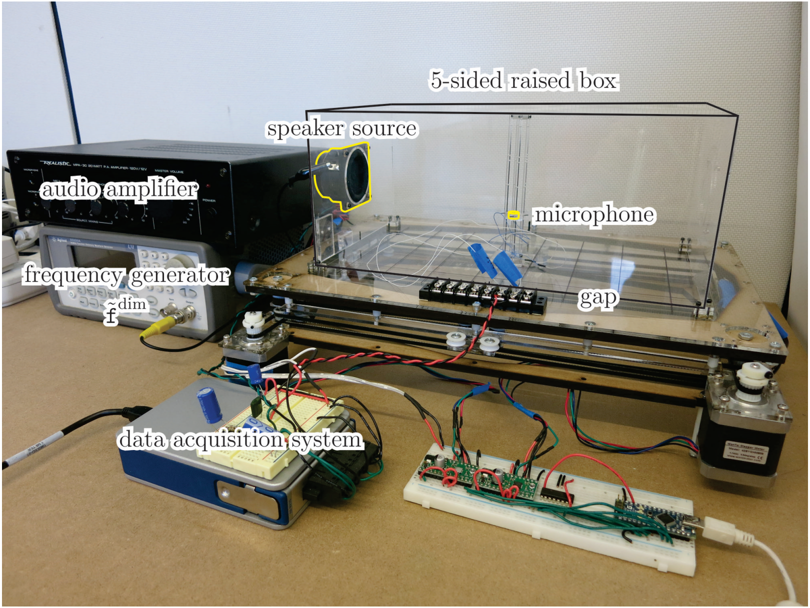
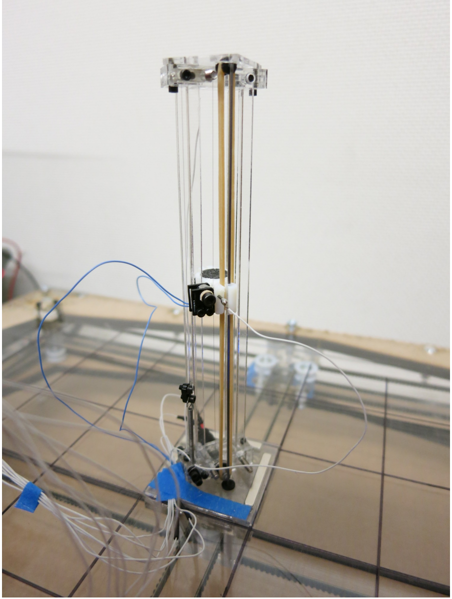
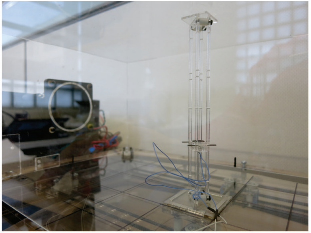
Experimental Apparatus



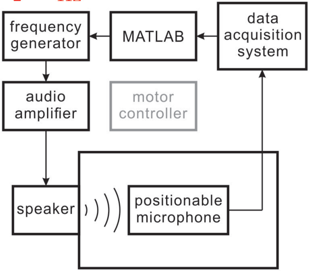
Robotic Microphone



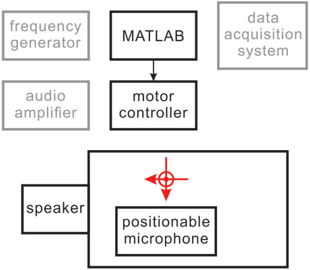
Data Acquisition Protocol



monochromatic input:
$\tilde{f}^{dim}$ Hz



Step 1: Take data at desired frequency(s)
at single spatial position: .3s .



Step 2: Move microphone
to new spatial position: 3s .

Experimental ingredients[6]:

mic calibration relative to reference;

control of environmental conditions: $T_0^{\mathrm{dim}}, \ldots$;

specification of mic location (observation center);

REGRESSION on

time-periodic mic signal (voltage) $\tilde{p}_t^{\mathrm{dim}}(x)$;

introduce relative error in complex pressure

$$\tilde{p}^{\mathrm{dim}}(x) \in \mathbb{C}$$

of magnitude $\approx 5\%$: effectively "exact."

[6]Note $\tilde{}$ denotes an *experimental measurement*.

System Configuration

We define the measured wavenumber as

$$\tilde{k} \equiv \frac{2\pi \tilde{\mathtt{f}}^{\mathrm{dim}} \tilde{r}_{\mathrm{spk}}^{\mathrm{dim}}}{\tilde{c}_0^{\mathrm{dim}}}$$

(equivalently, measured nondimensional frequency).

We then denote our system configuration as

$$\mathcal{C}(\tilde{k}, t_0, \ldots) \approx \mathcal{C}_{\tilde{k}} \; ;$$

we assume the system configuration is

sensibly *constant*

for associated set of observations and assessments, $\cdot[\mathcal{C}_{\tilde{k}}]$.

We normalize our observations and assessments as

$$O_m[\mathcal{C}_{\tilde{k}}] = \frac{\tilde{p}^{\mathrm{dim}}(x_m^{\mathrm{c}})}{\tilde{\rho}_0^{\mathrm{dim}}\tilde{c}_0^{\mathrm{dim}}V_{\mathrm{spk}}^{\mathrm{dim},\mathtt{bk}}(\tilde{k})}[\mathcal{C}_{\tilde{k}}]$$

$$\equiv \mathsf{Gauss}(u^{\mathrm{true}}[\mathcal{C}_{\tilde{k}}]; x_m^{\mathrm{c}}, \varphi = \cdot)$$

$$A_j[\mathcal{C}_{\tilde{k}}] = \frac{\tilde{p}^{\mathrm{dim}}(\xi_j^{\mathrm{c}})}{\tilde{\rho}_0^{\mathrm{dim}}\tilde{c}_0^{\mathrm{dim}}V_{\mathrm{spk}}^{\mathrm{dim},\mathtt{bk}}(\tilde{k})}[\mathcal{C}_{\tilde{k}}]$$

$$\equiv \mathsf{Gauss}(u^{\mathrm{true}}[\mathcal{C}_{\tilde{k}}]; \xi_j^{\mathrm{c}}, \varphi = \cdot)$$

for $1 \leq m \leq M$ and $1 \leq j \leq J$, respectively.

Note $V^{\mathrm{dim},\mathtt{bk}}$ is the speaker diaphragm bk model.

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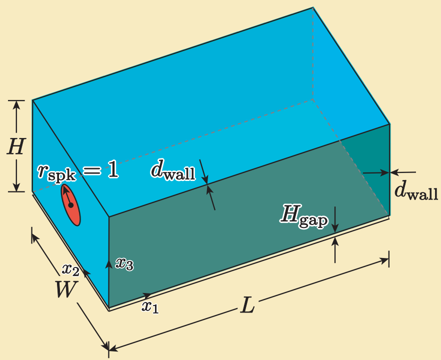
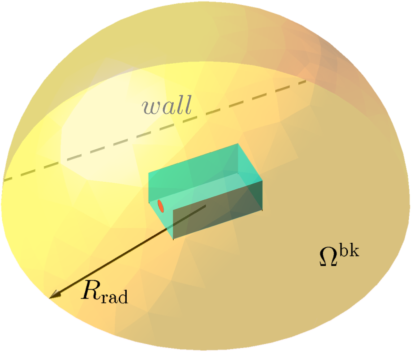
Spatial Domains



Ω: raised box



Ω^{bk}: full domain

Note lengths non-dimensionalized by speaker radius, $r_{\mathrm{spk}}^{\mathrm{dim}}$.

Parametrization: $\mu \equiv (k, \gamma) \in \mathcal{D}_k \times \mathcal{D}_\gamma \equiv \mathcal{D} \subset \mathbb{R} \times \mathbb{C}$

Introduce $\mu_1 \equiv k$, nondimensional wavenumber,

$$\mu \equiv k \equiv \frac{2\pi \mathtt{f}^{\mathrm{dim}} r_{\mathrm{spk}}^{\mathrm{dim}}}{c_0^{\mathrm{dim}}}$$

and associated domain

$$\mathcal{D}_k \equiv [0.3, 0.7]$$

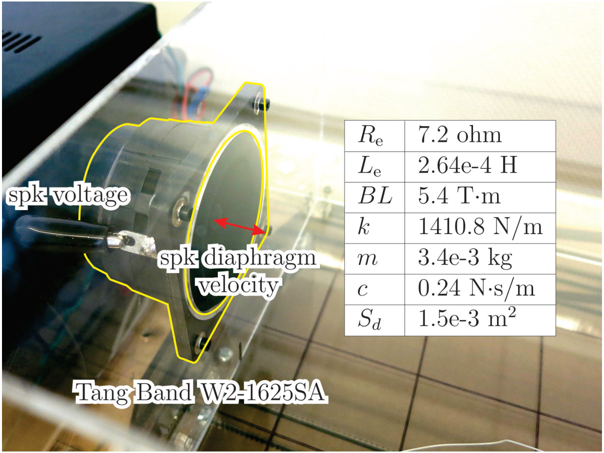
equivalent in dimensional terms to

in $648\mathrm{Hz} \lessapprox \mathtt{f}^{\mathrm{dim}} \lessapprox 1512\mathrm{Hz}$ at $T_0^{\mathrm{dim}} \approx 25°\mathrm{C}$.

Introduce $\mu_2 \equiv \gamma$, speaker velocity correction factor,

for $\gamma \in \mathcal{D}_\gamma \equiv \mathbb{C}$ (amplitude and phase).

bk Speaker Model (*Calibrated*): $V_{\mathrm{spk}}^{\mathrm{dim,bk}}$



R_{e}	7.2 ohm
L_{e}	2.64e-4 H
BL	5.4 T·m
k	1410.8 N/m
m	3.4e-3 kg
c	0.24 N·s/m
S_d	1.5e-3 m^2

Electromechanical Harmonic Oscillator:

Inputs: spk voltage — amplitude, phase, frequency.

Output: spk diaphragm velocity (uniform).

$\mathtt{bk}$ Acoustic Model (Air): Parametrized Helmholtz Equation

Given $\mu \equiv (k, \gamma) \in \mathcal{D}$, find complex field over $\Omega^{\mathtt{bk}}$

$$u^{\mathtt{bk},\mu} \equiv \frac{p^{\mathrm{dim}}}{\rho_0^{\mathrm{dim}} c_0^{\mathrm{dim}} V_{\mathrm{spk}}^{\mathrm{dim},\mathtt{bk}}(k)}$$

solution of

$$G^\mu(u^{\mathtt{bk},\mu}, v) = 0, \quad \forall v \in \mathcal{U}^{\mathtt{bk}},$$

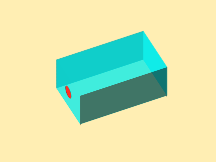
for weak form

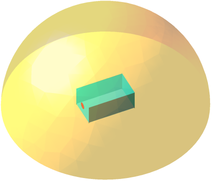
$$\begin{aligned} G^\mu(w, v) = \; & ik\gamma \int_{\Gamma_{\mathrm{spk}}} 1 \; \bar{v} ds - \int_\Omega \nabla w \cdot \nabla \bar{v} dx \\ & + k^2 \int_\Omega w \bar{v} dx - \left(ik + \frac{1}{R_{\mathrm{rad}}} \right) \int_{\Gamma_{\mathrm{rad}}} w \bar{v} ds \end{aligned}$$

and space $\mathcal{U}(\Omega^{\mathtt{bk}}) \equiv H^1(\Omega^{\mathtt{bk}})$.

bk Model Imperfections

There are many sources of bk model error





speaker

imprecision in location: *asymmetry*
non-rigid diaphragm motion
nonlinearity in response
pressure loading ...

wall

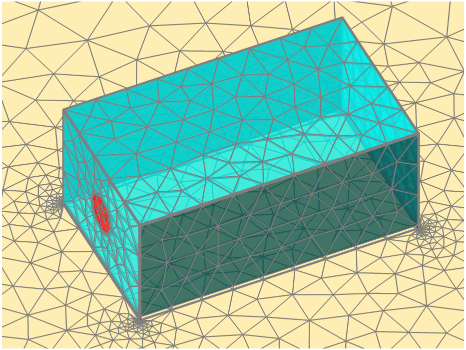
elastic modes
Rayleigh damping
fasteners and joints ...

acoustics

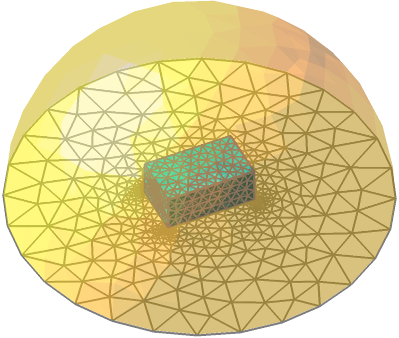
farfield (radiation) effects ...

some of which shall prove significant.

Helmholtz Discretization: $\mathcal{U}^{\mathrm{bk}} \to (\mathcal{U}^{\mathrm{bk}})^{\mathcal{N}}$



raised box



full domain

(Continuous) Galerkin: $\mathbb{P}^3$ finite elements.

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Recall

$$\Omega \subset \Omega^{\mathrm{bk}}$$

is the interior of raised-box acoustic chamber.

Define (over complex fields)

$$\mathcal{U}(\Omega) \equiv H^1(\Omega)$$

with inner product

$$(w, v) \equiv \int_\Omega \nabla w \cdot \nabla \bar{v} dx + \kappa^2 \int_\Omega w \bar{v} dx$$

and associated induced norm; choose $\kappa = 0.5$.

Background Spaces: $\{\mathcal{Z}_N\}_{N=1}^{N_{\max}}$ — Definition

Introduce bk manifold

$$\mathcal{M}^{\texttt{bk}} = \{u^{\texttt{bk},\mu} \mid \mu \in \mathcal{D}\}$$

and invoke

$$\text{WEAKGREEDY}_{N_{\max}} \to \mathcal{Z}_N^{\texttt{bk}}, \quad N = 1, \ldots, N_{\max},$$

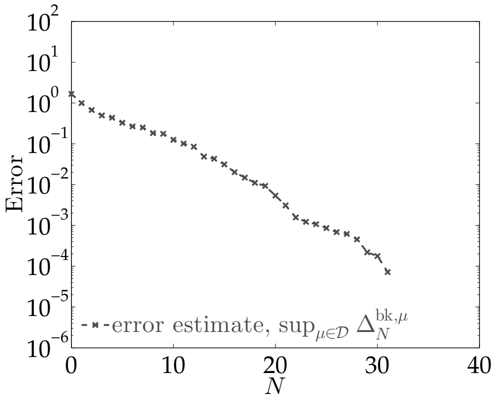
to form

$$\mathcal{Z}_N = \{z \in \mathcal{U} \mid z = z^{\texttt{bk}}|_\Omega, z^{\texttt{bk}} \in \mathcal{Z}_N^{\texttt{bk}}\},$$

for $N = 1, \ldots, N_{\max} = 8$.

Note from linearity we may perform

WEAKGREEDY_N over $(k, \gamma) \in \mathcal{D}_k \times \{1\}$.



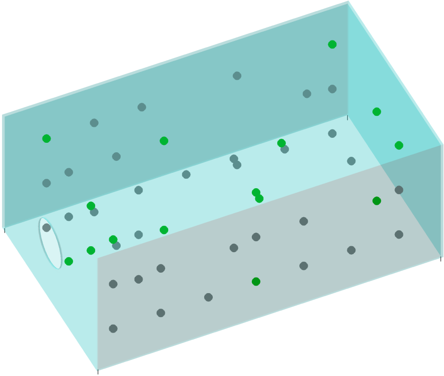
Discretization error (estimate) $\gamma = 1$

$$\sup_{\mu\in\mathcal{D}} \Delta_N^{\mathsf{bk},\mu} \gtrapprox \epsilon_{\mathrm{disc},N}^{\mathsf{bk}} \equiv \sup_{w\in\mathcal{M}^{\mathsf{bk}}} \|w - \Pi_{\mathcal{Z}_N} w\|$$

decreases rapidly with N.

Update Spaces $\{\mathcal{U}_M\}_{M=1}^{M_{\max}}$ — Definition

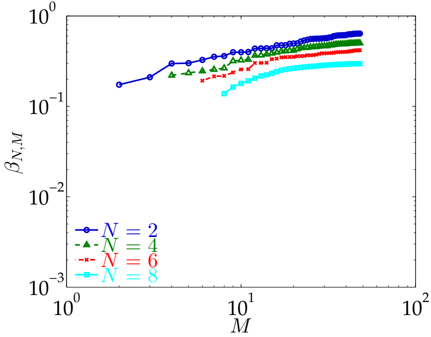
$\texttt{SGreedy}_M^{\text{discrete}}(\mathcal{Z}_{N_{\max}})$ identifies $\{x_m^{\mathrm{c}}\}_{m=1}^{M_{\max}}$



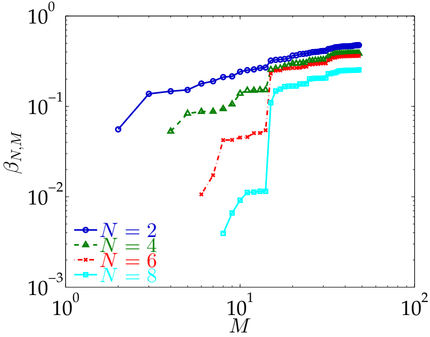
to construct update spaces $\qquad M = 1, \ldots, 48 = M_{\max}$

$$\mathcal{U}_M \equiv \mathsf{span}\{R_{\mathcal{U}}\mathsf{Gauss}(\cdot; x_m^{\mathrm{c}}, \varphi = 0.2)\}_{m=1}^{M}.$$

Update Spaces $\{\mathcal{U}_M\}_{M=1}^{M_{\max}}$ — $\textsc{Process}^{\mathcal{U}}_{M_{\max}}$



$\textsc{SGreedy}^{\text{discrete}}_M$



$\textsc{RandomUniform}$

Stability: effect of $\textsc{Process}^{\mathcal{U}}_M$ on $\beta_{N,M}$.

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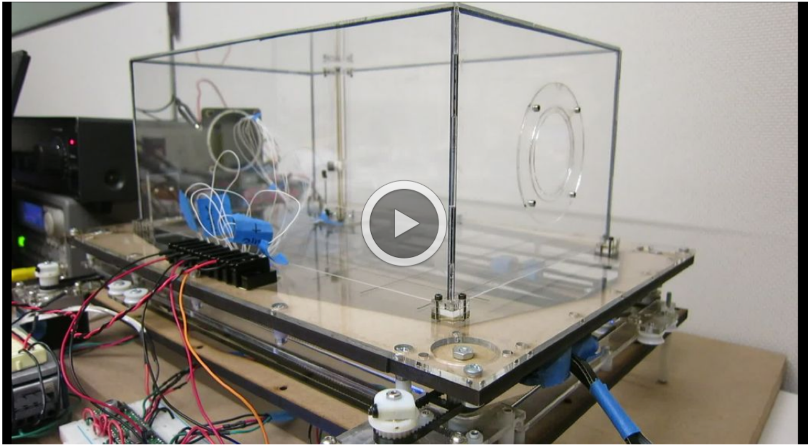
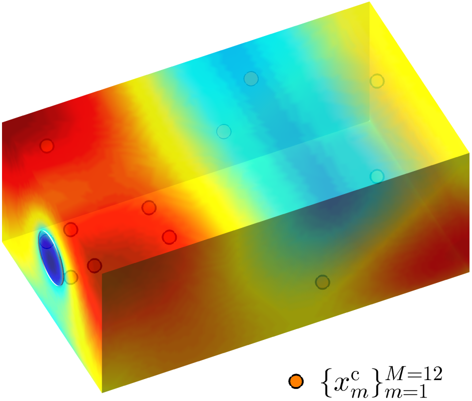
Elapsed time: $3.3M$ s (M observations).



(In video we observe 10 frequencies at each mic center.)

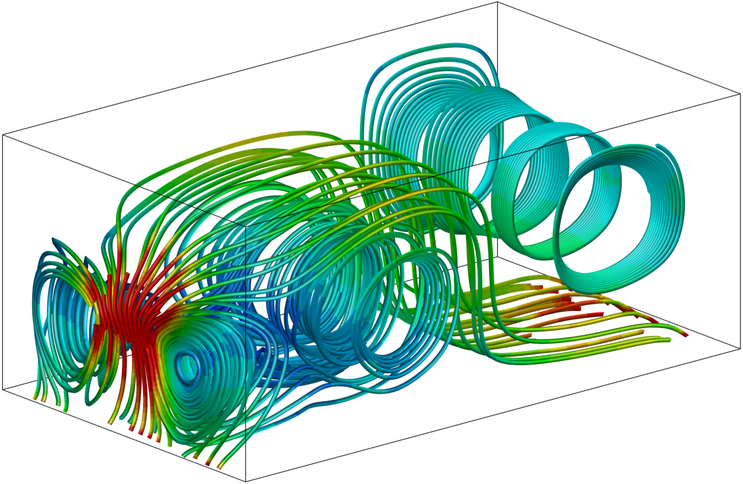
PBDW Data Assimilation: $\mathbf{l}^{\rm obs}[\mathcal{C}_{\tilde{k}=.69}] \to u^*_{N=7,M=12}[\mathcal{C}_{\tilde{k}=.69}]$

Elapsed time: 0.1 ms (assimilation) + 0.8 s (rendering).[7]



[7]We may un-normalize our state estimate to obtain the pressure: $p^{*\,\rm dim}_{N,M}[\mathcal{C}_{\tilde{k}}] = (\, u^*_{N,M}\, (\tilde{\rho}\,\tilde{c})_0^{\rm dim}\, V_{\rm spk}^{\rm dim,\mathbf{bk}}(\tilde{k})\ )[\mathcal{C}_{\tilde{k}}]$.

Engineering Analysis: $u^*_{N=7,M=12}[\mathcal{C}_{\tilde{k}=.69}] \rightarrow I_{\text{avg}}[\mathcal{C}_{\tilde{k}=.69}]$



Sound Intensity: $I_{\text{avg}}(x) \equiv \Re\{ \frac{-i}{4\pi\rho_0^{\text{dim}}\mathtt{f}^{\text{dim}}}\, p^{*\,\text{dim}}_{N,M}\ \nabla \overline{p}^{*\,\text{dim}}_{N,M} \}$.

Response Time (Online)

Best-knowledge model (12 core, 64GB RAM ws):

$$\mu \equiv (k, \gamma) = (\tilde{k}, 1), (\rho, c)_0^{\text{dim}} = (\tilde{\rho}, \tilde{c})_0^{\text{dim}}$$

$\to u^{\mathtt{bk},\mu}$ 20 s.

PBDW state estimation: $N = 7$, $M = 12$

data acquisition: ROP

$\mathcal{C}_{\tilde{k}} \to \mathbf{l}^{\text{obs}}[\mathcal{C}_{\tilde{k}}] = \{O_m[\mathcal{C}_{\tilde{k}}]\}_{m=1}^{M}$ 40 s

data assimilation: PBDW (laptop)

$\textsc{Saddle}.\mathtt{Online}_{N,M} : \mathbf{l}^{\text{obs}}[\mathcal{C}_{\tilde{k}}] \to u_{N,M}^{*}[\mathcal{C}_{\tilde{k}}]$ 0.0001 s

---

Total: 40 s.

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    - Preliminaries
    - Accuracy
    - Convergence: Case I — x_2-Symmetric Resonance, $\mathcal{C}_{\tilde{k}=0.557}$
    - Convergence: Case II — x_2-Antisymmetric Resonance, $\mathcal{C}_{\tilde{k}=0.479}$

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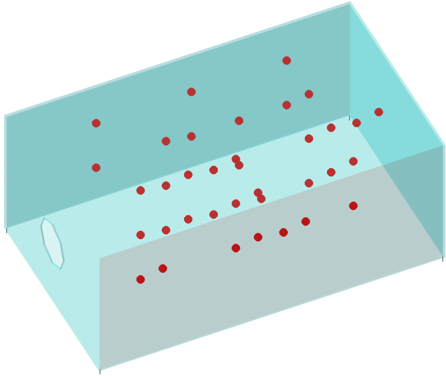
Choose assessment centers $\{\xi^{c}_{m}\}_{j=1}^{J=36}$:



Recall observations and assessments are

mutually exclusive: $\xi^{c}_{j} \notin \{x^{c}_{m}\}_{m=1}^{M_{\max}}, \quad j = 1, \ldots, J.$

We shall compare $j = 1, \ldots, J$

$$
\begin{aligned}
P^{\mathtt{bk}}(j;\tilde{k}) &\equiv \mathsf{Gauss}(u^{\mathtt{bk},\mu=(\tilde{k},1)};\xi^{\mathrm{c}}_j, 0.2)\ , \\
P^{*}_{N,M}(j;\tilde{k}) &\equiv \mathsf{Gauss}(u^{*}_{N,M}[\mathcal{C}_{\tilde{k}}];\xi^{\mathrm{c}}_j, 0.2)\ , \\
P^{\mathrm{true}}(j;\tilde{k}) &\equiv A_j[\mathcal{C}_{\tilde{k}}] \\
&\equiv \mathsf{Gauss}(u^{\mathrm{true}}[\mathcal{C}_{\tilde{k}}];\xi^{\mathrm{c}}_j, \cdot)\ ,
\end{aligned}
$$

where $\mathcal{C}_{\tilde{k}}$ specifies the experimental configuration.

We also evaluate for given N, M, and J,

$$
E_{\mathrm{avg}}[\mathcal{C}_{\tilde{k}}] \equiv \sqrt{\tfrac{1}{J}\textstyle\sum_{j=1}^{J} \mid P^{\mathrm{true}}(j;\tilde{k}) - P^{*}_{N,M}(j;\tilde{k})\mid^2}
$$

as an estimate of the error in the state.

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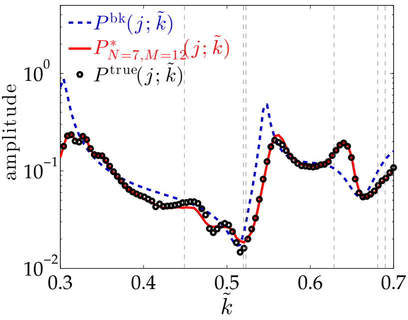
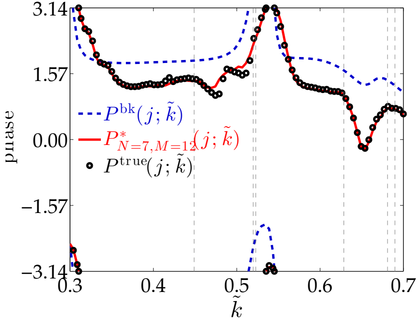
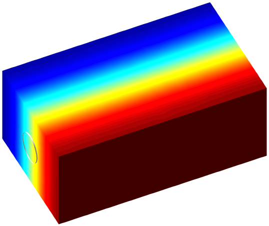
Frequency Response: $\xi_j^c = (2.67, 2.67, 4.50)$

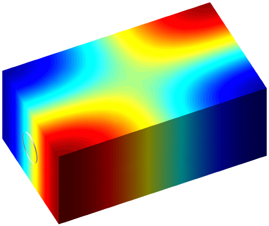




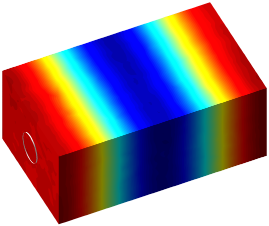
Resonances: Simple Dirichlet Box



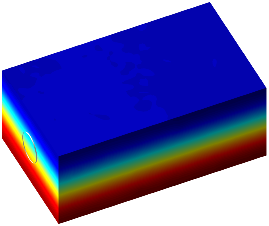
$k^{\text{resonance}} = 0.449$



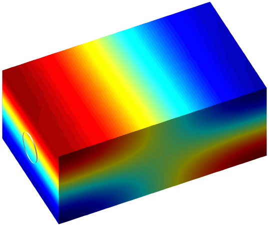
$k^{\text{resonance}} = 0.520$



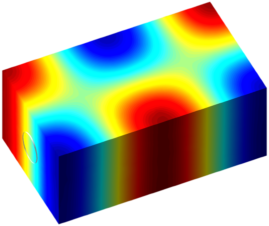
$k^{\text{resonance}} = 0.523$



$k^{\text{resonance}} = 0.629$

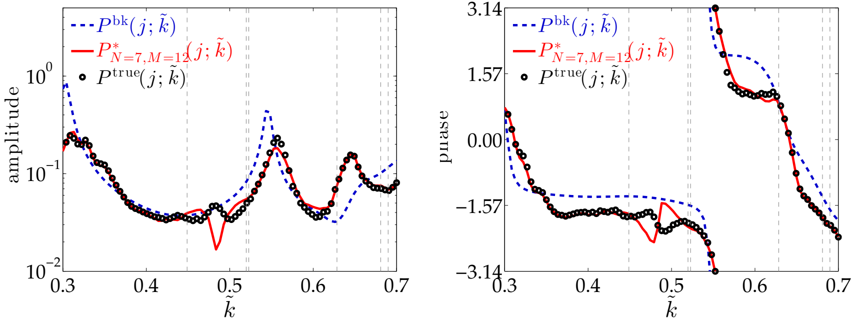


$k^{\text{resonance}} = 0.681$

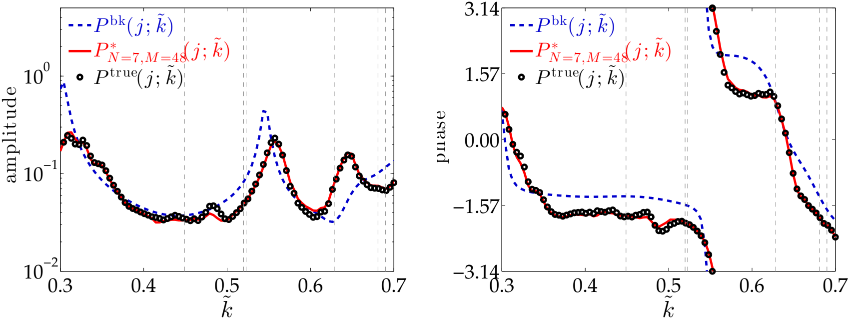


$k^{\text{resonance}} = 0.690$

Frequency Response: $\xi_j^c = (9.33, 2.67, 4.50)$ $(M = 12)$



Frequency Response: $\xi_j^c = (9.33, 2.67, 4.50)$ $(M = 48)$



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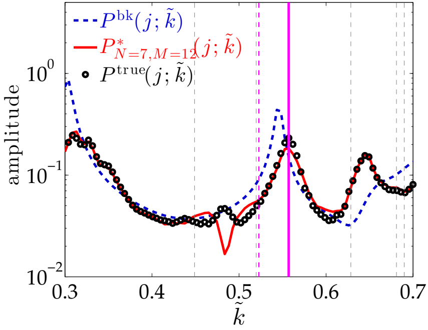
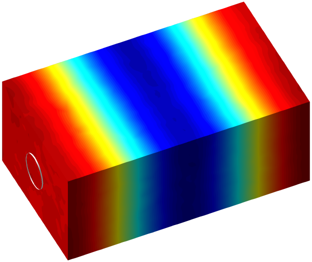
Frequency Response: Amplitude





Inevitable actual speaker *asymmetry* unimportant:

bk model *symmetric* Neumann condition on $\Gamma_{\rm spk}$

does excite relevant symmetric resonance;

$u^{\rm true}$ close to $\mathcal{M}^{\tt bk}$, model error $\epsilon^{\tt bk}_{\rm mod}(u^{\rm true})$ small.

Primary: As N increases for fixed M $(\geq N)$ expect

$\epsilon^{\mathtt{bk}}_{\mathrm{disc},N} \to 0$ rapidly, and

$\|u^{\mathrm{true}} - u^*_{N,M}\| \to \epsilon^{\mathtt{bk}}_{\mathrm{mod}}(u^{\mathrm{true}}) \approx 0$ rapidly;

$\mathcal{Z}_N$ provides approximation, and $\mathcal{U}_M$ provides stability.

Secondary: As M increases

for fixed

$$N = N_{\mathrm{plateau}} \equiv \{N \,|\, \epsilon^{\mathtt{bk}}_{\mathrm{disc},N} \ll \epsilon^{\mathtt{bk}}_{\mathrm{mod}}(u^{\mathrm{true}})\} \approx N_{\max}$$

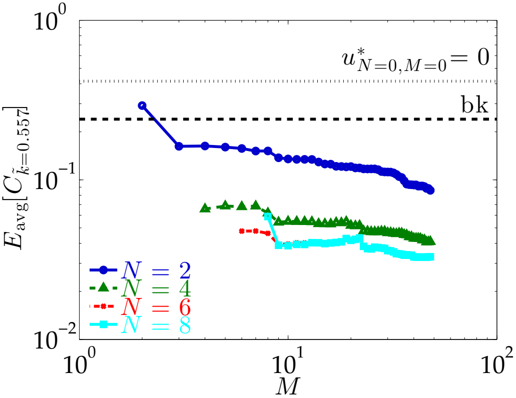
expect

$\epsilon^{\mathtt{bk}}_{\mathrm{mod}}(u^{\mathrm{true}}) \to 0$ slowly, and

$\|u^{\mathrm{true}} - u^*_{N,M}\| \to 0$ slowly;

$\mathcal{U}_M$ provides approximation (of unmodeled physics).

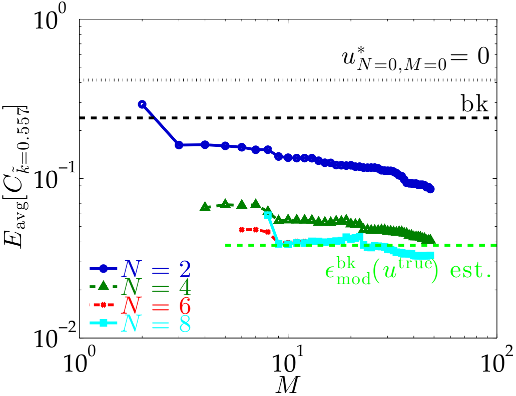
$$E_{\text{avg}}[\mathcal{C}_{\tilde{k}}] \equiv \sqrt{\frac{1}{J}\sum_{j=1}^{J} \mid P^{\text{true}}(j;\tilde{k}) - P^{*}_{N,M}(j;\tilde{k}) \mid^{2}}$$ [8]



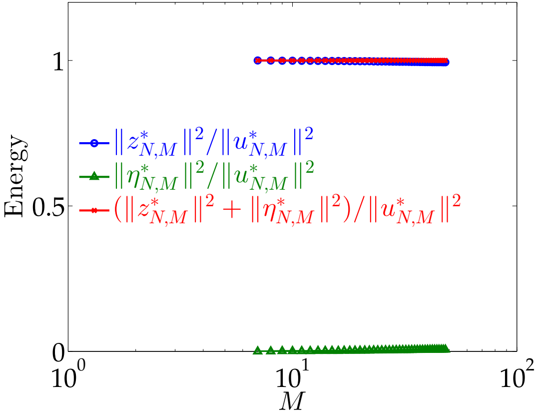
[8]Note $\left(\frac{1}{J}\sum_{j=1}^{J} |P^{\text{true}}(j;\tilde{k}=0.557)|^2 \right)^{1/2} = 0.415$.

Best-Fit-Over-Manifold: $\epsilon_{\mathrm{mod}}^{\mathtt{bk}}(u^{\mathrm{true}}) = \| u^{\mathrm{true}} - \mathcal{F}_{\mathcal{M}^{\mathtt{bk}}} u^{\mathrm{true}} \|$

$$E_{\mathrm{avg}}[\mathcal{C}_{\tilde{k}}] \equiv \sqrt{\frac{1}{J} \sum_{j=1}^{J} \mid P^{\mathrm{true}}(j;\tilde{k}) - P^{*}_{N,M}(j;\tilde{k}) \mid^2}$$



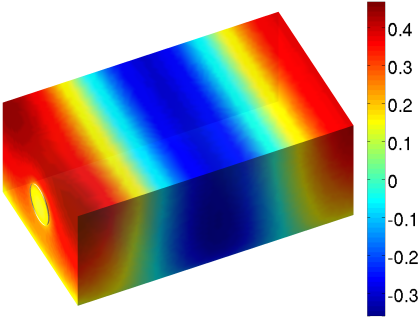
Explanation of State: Modeled vs Unmodeled — Energy



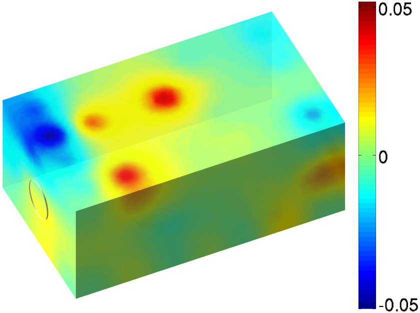
$N = 7$

Little energy is contained in update field $\eta^*_{N,M}$.

Explanation of State: Modeled vs Unmodeled — Fields



$z^*_{N,M}$ $(\|z^*_{N,M}\|^2 = 46.13)$



$\eta^*_{N,M}$ $(\|\eta^*_{N,M}\|^2 = 0.29)$

$$N = 7, M = 48$$

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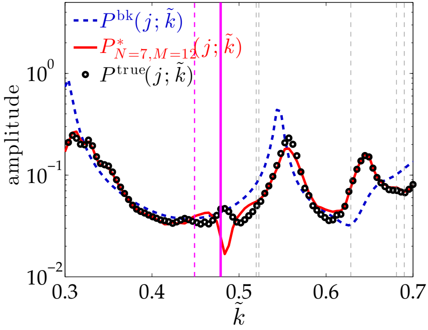
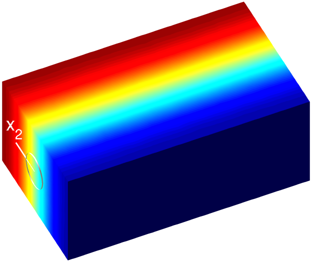
  - Convergence: Case II — x_2-Antisymmetric Resonance, $\mathcal{C}_{\tilde{k}=0.479}$





Inevitable actual speaker *asymmetry* important:

bk model *symmetric* Neumann condition on Γ_{spk}

does not excite relevant x_2–*antisymmetric* resonance;

u^{true} not close to $\mathcal{M}^{\mathtt{bk}}$, model error $\epsilon^{\mathtt{bk}}_{\mathrm{mod}}(u^{\mathrm{true}})$ not small.

Convergence Scenario

Primary: As N increases for fixed M $(\geq N)$ expect

$$\epsilon^{\texttt{bk}}_{\text{disc},N} \to 0 \text{ rapidly, and}$$

$$\|u^{\text{true}} - u^*_{N,M}\| \to \epsilon^{\texttt{bk}}_{\text{mod}}(u^{\text{true}}) \not\approx 0 \text{ rapidly;}$$

$\mathcal{Z}_N$ provides approximation, and $\mathcal{U}_M$ provides stability.

Secondary: As M increases for fixed

$$N = N_{\text{plateau}} \equiv \{N \,|\, \epsilon^{\texttt{bk}}_{\text{disc},N} \ll \epsilon^{\texttt{bk}}_{\text{mod}}(u^{\text{true}})\} \approx 1$$

expect

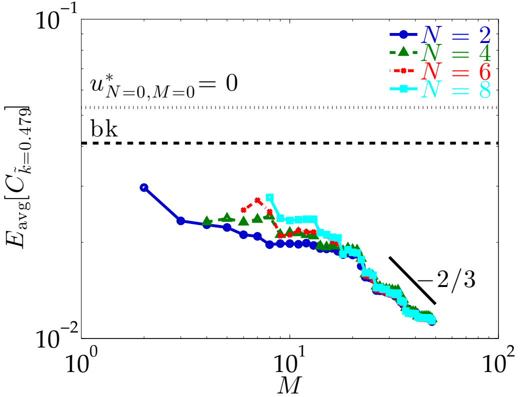
$$\epsilon^{\texttt{bk}}_{\text{mod}}(u^{\text{true}}) \to 0 \text{ slowly, and}$$

$$\|u^{\text{true}} - u^*_{N,M}\| \to 0 \text{ slowly;}$$

$\mathcal{U}_M$ provides approximation of unmodeled physics.

A Posteriori Error Indicators

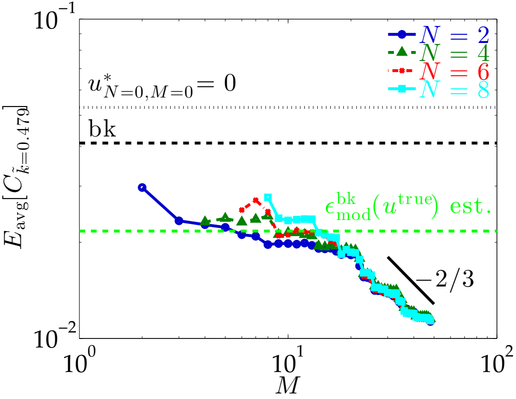
$$E_{\text{avg}}[\mathcal{C}_{\tilde{k}}] \equiv \sqrt{\frac{1}{J}\sum_{j=1}^{J} \mid P^{\text{true}}(j;\tilde{k}) - P^{*}_{N,M}(j;\tilde{k}) \mid^2}$$ [9]



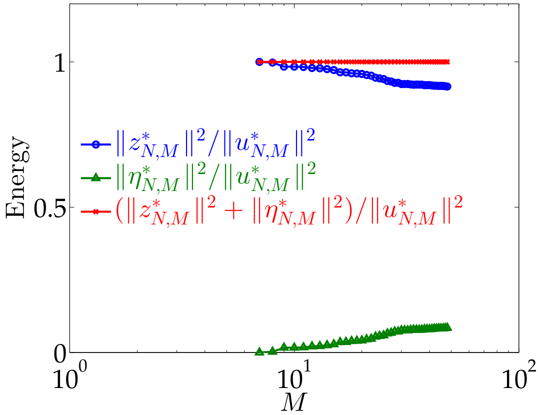
[9]Note $\left(\frac{1}{J}\sum_{j=1}^{J} |P^{\text{true}}(j;\tilde{k}=0.479)|^2\right)^{1/2} = 0.0529$.

Best-Fit-Over-Manifold: $\epsilon^{\mathtt{bk}}_{\mathrm{mod}}(u^{\mathrm{true}}) = \|u^{\mathrm{true}} - \mathcal{F}_{\mathcal{M}^{\mathtt{bk}}} u^{\mathrm{true}}\|$

$$E_{\mathrm{avg}}[\mathcal{C}_{\tilde{k}}] \equiv \sqrt{\frac{1}{J} \sum_{j=1}^{J} \mid P^{\mathrm{true}}(j;\tilde{k}) - P^{*}_{N,M}(j;\tilde{k}) \mid^2}$$



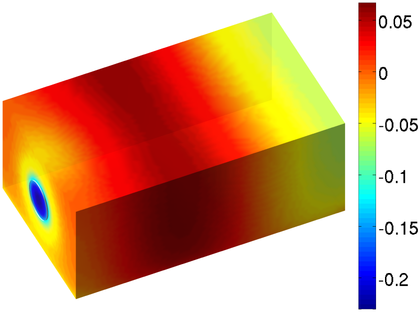
Explanation of State: Modeled vs Unmodeled — Energy

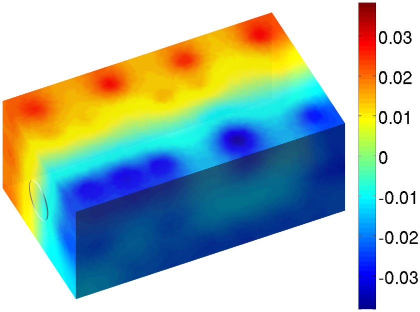


$N = 7$

Significant energy is contained in update field $\eta^*_{N,M}$.

Explanation of State: Modeled vs Unmodeled — Fields



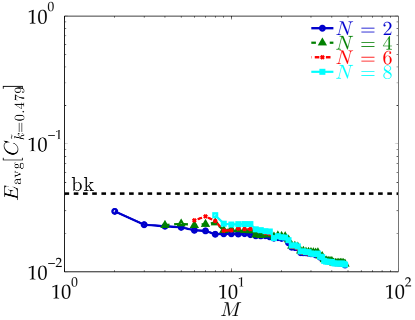


$z^*_{N,M}$ $(\|z^*_{N,M}\|^2 = 0.730)$ $\eta^*_{N,M}$ $(\|\eta^*_{N,M}\|^2 = 0.068)$

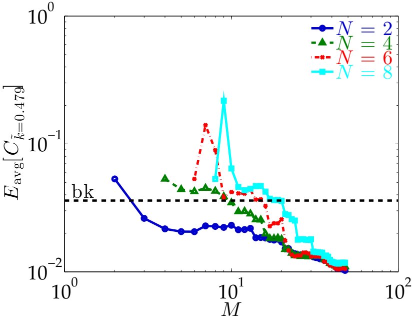
$$N = 7, M = 48$$

Update field $\eta^*_{N,M}$ captures x_2-antisymmetric resonance — or in any event makes a courageous effort.

Design of Experiment: $\{\mathcal{U}_M\}_{M=1}^{M_{\max}}$



$\texttt{SGreedy}_M^{\text{discrete}}$



$\texttt{RandomUniform}_M$

Stability: effect of $\texttt{Process}_M^{\mathcal{U}}$ on error in state.

PBDW for Infinite-Dimensional Parametrizations: An Extracted Domain Approach

- Synthetic Truths
- Best-Knowledge Model
- PBDW Formulation
- State Estimation: 11-Dimensional Truth
- State Estimation: Infinite-Dimensional Truth

Acknowledgement

This more recent development

is indebted to

prior work of, and discussions with,

Albert Cohen, UPMC LJLL,

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Markus Noisternig, Gerard Assayag, IRCAM, Paris.

(ATP: And super-human efforts by Dr Masa Yano.)

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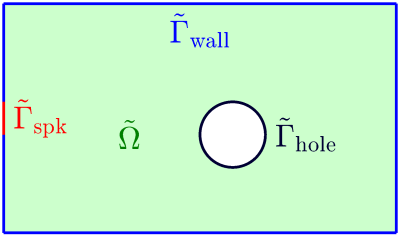


Consider configurations

$$\mathcal{C} \equiv \{k, Z_{\text{wall}}, Z_{\text{hole}}\}$$

characterized by

$$\begin{aligned} \text{wavenumber: } & k \in \mathbb{R}_{>0} \\ \text{wall impedance field: } & Z_{\text{wall}} \in L^{\infty}(\tilde{\Gamma}_{\text{wall}}) \\ \text{hole impedance: } & Z_{\text{hole}} \in \mathbb{C}. \end{aligned}$$

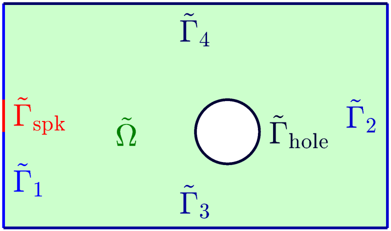
The synthetic truth $u^{\text{true}}[\mathcal{C}] \in H^1(\tilde{\Omega})$ satisifes

$$G^{\text{true}}[\mathcal{C}](u^{\text{true}}[\mathcal{C}], v) = 0, \quad \forall v \in H^1(\tilde{\Omega})$$

for the weak form

$$\begin{aligned} G^{\text{true}}[\mathcal{C}](w, v) \equiv ik \int_{\tilde{\Gamma}_{\text{spk}}} \bar{v} ds - \int_{\tilde{\Omega}} \nabla w \cdot \nabla \bar{v} dx \\ + k^2 \int_{\tilde{\Omega}} w \bar{v} dx - \int_{\tilde{\Gamma}_{\text{wall}}} \frac{ik}{Z_{\text{wall}}(s)} w \bar{v} ds \\ - \int_{\tilde{\Gamma}_{\text{hole}}} \frac{ik}{Z_{\text{hole}}} w \bar{v} ds. \end{aligned}$$

Case 1. Piecewise-Constant $Z \in L^\infty(\tilde{\Gamma}_{\text{wall}})$ Truth

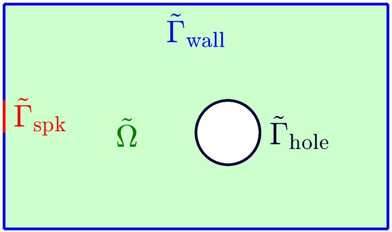


Random piecewise-constant wall-impedance field:

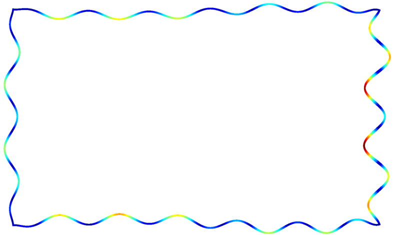
<table>
<tr><th></th><th>Case 1A</th><th>Case 1B</th></tr>
<tr><td>Z_1</td><td colspan="2">$-1.40 - 0.98i$</td></tr>
<tr><td>Z_2</td><td colspan="2">$1.11 - 6.16i$</td></tr>
<tr><td>Z_3</td><td colspan="2">$0.10 + 0.03i$</td></tr>
<tr><td>Z_4</td><td colspan="2">$-2.28 + 0.62i$</td></tr>
<tr><td>Z_{hole}</td><td>∞</td><td>$-1.34 + 0.31i$</td></tr>
</table>

High-dimensional parametrization: 11 (real) parameters.

Case 2. Vibroacoustics $Z \in L^\infty(\tilde{\Gamma}_{\text{wall}})$ Truth



domain configuration



$V_{\text{wall},n}(\cdot; k = 1.0)$, Case 2

Elastodynamics-induced wall-impedance field:

	Case 2A	Case 2B
$Z_{\text{wall}}(\cdot; k)$	$p^{\text{true}}(\cdot; k)/V_{\text{wall},n}(\cdot; k)$	
Z_{hole}	∞	$-1.34 + 0.31i$

Infinite-dimensional parametrization: $L^\infty(\tilde{\Gamma}_{\text{wall}})$ field.

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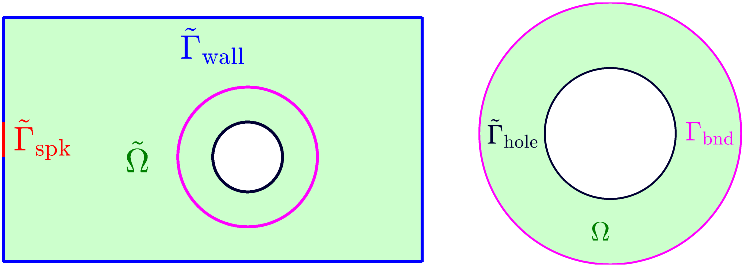
Spatial and Parameter Domains



Spatial Domain: $(u^{\mathtt{bk},\mu},\, u^*_{N,M})\ \Omega^{\mathtt{bk}} \equiv \Omega\ \subset\ \tilde{\Omega}\ (u^{\rm true})$.

Parameter domain:

$$\mu \equiv (k, g) \in \mathcal{D}_k \times \mathcal{D}_g \equiv \mathcal{D} \quad ,$$

for wavenumber $k \in \mathcal{D}_k \equiv \mathbb{R}_{>0}$, and

boundary trace: $g \in \mathcal{D}_g \equiv H^{1/2}(\Gamma_{\rm bnd})$.

Parametrized Best-Knowledge Solutions

Given $\mu \equiv (k, g) \in \mathcal{D}$, we seek $u^{\texttt{bk},(k,g)}$ in space

$$H^1_{(g)}(\Omega) \equiv \{w \in H^1(\Omega) \mid w|_{\Gamma_{\text{bnd}}} = g\},$$

such that

$$G^{(k,g)}(u^{\texttt{bk},(k,g)}, v) = 0, \quad \forall v \in H^1_{(0)}(\Omega)$$

for weak form

$$G^{(k,g)}(w, v) \equiv -\int_\Omega \nabla w \cdot \nabla \bar{v} dx + k^2 \int_\Omega w \bar{v} dx$$

$$- \frac{ik}{Z^{\texttt{bk}}_{\text{hole}} \to \infty} \int_{\tilde{\Gamma}_{\text{hole}}} w \bar{v} ds \;;$$

we take $Z^{\texttt{bk}}_{\text{hole}} \to \infty$ in the best-knowledge model.

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Background Space Dirichlet Boundary Representation

Express the trace $g \in L^\infty(\Gamma_{\mathrm{bnd}})$ as

$$g(\theta) \equiv \sum_{n=1}^{\infty} \alpha_n g_n(\theta)$$

for $\alpha_n \in \mathbb{C}$, $n = 1, \ldots,$, and

$$g_n(\theta) = \begin{cases} \cos(\lfloor n/2 \rfloor \pi\theta), & n = 1, 3, 5, \ldots, \\ \sin(\lfloor n/2 \rfloor \pi\theta), & n = 2, 4, 6, \ldots. \end{cases}$$

We presume g sufficiently smooth, say

$$\alpha_n \lesssim \exp(\lfloor n/2 \rfloor)$$

(or more generally, smooth family of functions).

$\mathcal{Z}_N$ Design Criterion

We wish to construct

$$\mathcal{Z}_N^{\text{ideal}} \equiv \underset{\substack{\mathcal{W} \\ \dim(\mathcal{W})=N}}{\arg\inf} \int_{\mathcal{D}_k} \sum_{n=1}^{\infty} \inf_{w \in \mathcal{W}} \|u^{\mathtt{bk},(k,\gamma_n g_n)} - w\|^2 dk,$$

for Dirichlet boundary weights

$$\gamma_n \equiv \exp(\lfloor n/2 \rfloor)$$

and the norm $\|\cdot\|$ induced by $\kappa = 1.0$

$$(w, v) \equiv \int_\Omega \nabla w \cdot \nabla \bar{v} dx + \kappa^2 \int_\Omega w \bar{v} dx.$$

Algorithmic embodiment: POD or POD-Greedy.

$\mathcal{Z}_N$ Construction: POD-Greedy

Introduce an error estimate $\Delta^{\mathtt{bk},(k,g)}_{N_k N_{\mathrm{bc}}}$ such that

$$\|u^{\mathtt{bk},(k,g)} - \Pi_{\mathcal{W}_{N_k N_{\mathrm{bc}}}} u^{\mathtt{bk},(k,g)}\| \lesssim \Delta^{\mathtt{bk},(k,g)}_{N_k N_{\mathrm{bc}}}.$$

$\mathtt{POD\text{-}Greedy}_{N_{\max}}$: $(\Xi_k \subset \mathcal{D}_k), (N_{\mathrm{bc}} < \infty) \to \{\mathcal{Z}_N\}_{N=1}^{N_{\max}}$

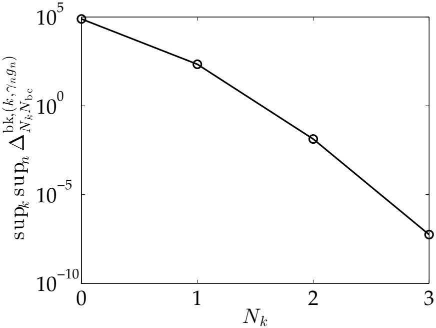
Construct $\mathcal{W}_{N_k N_{\mathrm{bc}}}$: for $N_k = 1, \ldots$

1. $\hat{k}_{N_k} = \arg\sup\limits_{k \in \Xi_k} \sup\limits_{n=1,\ldots,N_{\mathrm{bc}}} \Delta^{\mathtt{bk},(k,\gamma_n g_n)}_{(N_k-1)N_{\mathrm{bc}}}$
2. $\mathcal{W}_{N_k N_{\mathrm{bc}}} \equiv$ $\mathrm{span}\{\mathcal{Z}_N, u^{\mathtt{bk},(\hat{k}_{N_k},\gamma_1 g_1)}, \ldots, u^{\mathtt{bk},(\hat{k}_{N_k},\gamma_{N_{\mathrm{bc}}} g_{N_{\mathrm{bc}}})}\}$

Apply POD:

$$\mathtt{POD}_N(\mathcal{W}_{N_k N_{\mathrm{bc}}}) \to \mathcal{Z}_N, \quad N = 1, \ldots, N_{\max}$$

Greedy Convergence: $\mathcal{D}_k \equiv [0.5, 1.0]$

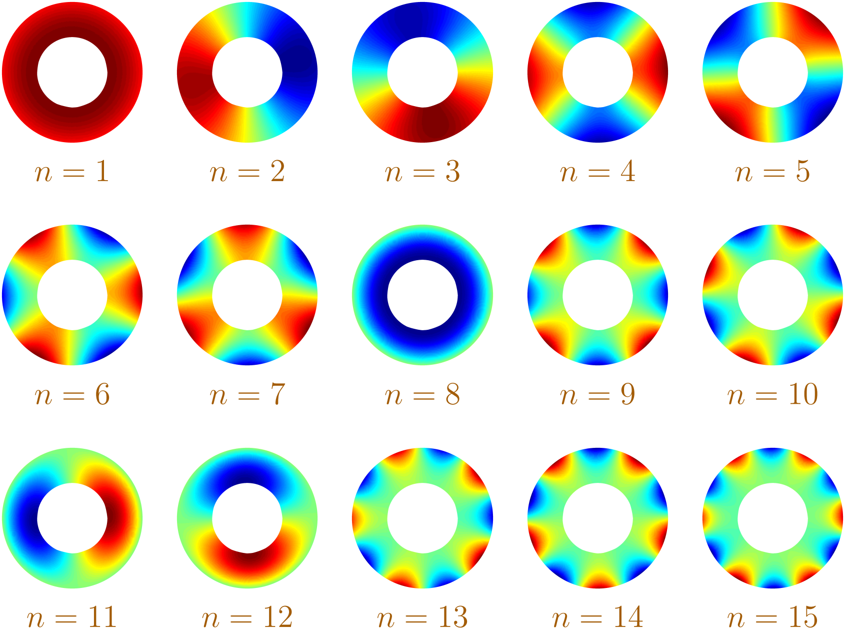


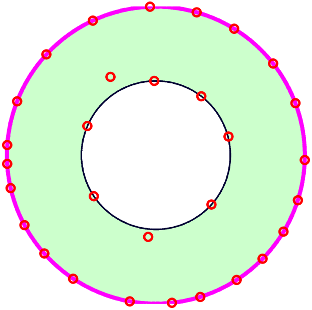
Training set:

wavenumber: $\Xi_k = \{0.50, 0.51, 0.51, \ldots, 1.00\}$
boundary conditions: $N_{\mathrm{bc}} = 20$.

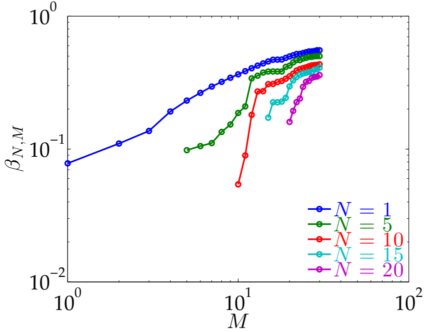
Selected parameter values: $\hat{k} = \{1.0, 0.5, 0.81\}$.

POD-Greedy $\mathcal{Z}_N$: $\mathcal{D}_k \equiv [0.5, 1.0]$





$\{x_m^{\mathrm{c}}\}_{m=1}^{M_{\max}=30}$



stability constant

Invoke

$$\mathtt{SGREEDY}_M(\mathcal{Z}_{N_{\max}}) \to \mathcal{U}_M, \quad M = 1, \ldots, M_{\max} = 30.$$

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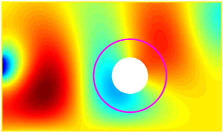
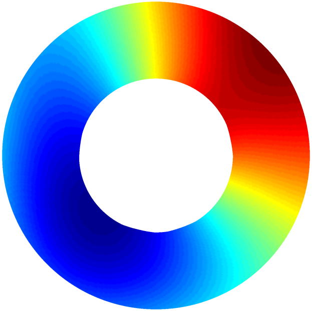
Case 1A: Field $\tilde{k} = 1.0$



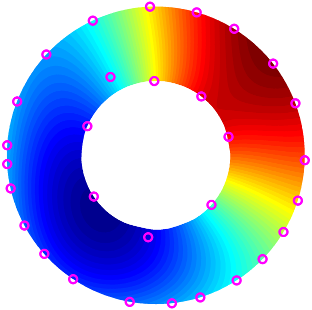
Perfect μ-$\texttt{bk}$ model since $Z_{\text{hole}} = Z_{\text{hole}}^{\texttt{bk}}$:

$$u^{\text{true}} = u^{\texttt{bk},(k,g)} \quad \text{for some} \quad (k,g) \in \mathcal{D};$$

model error $\epsilon_{\text{mod}}^{\texttt{bk}}(u^{\text{true}}) = 0$, discretization error $\epsilon_{\text{disc},N}^{\texttt{bk}} \neq 0$.

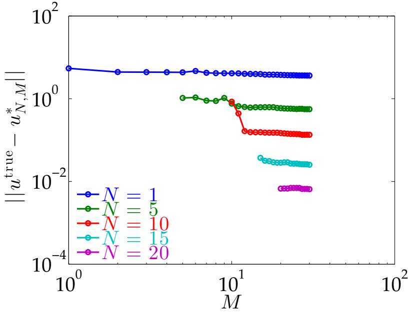


$\Im(u^{\text{true}})$

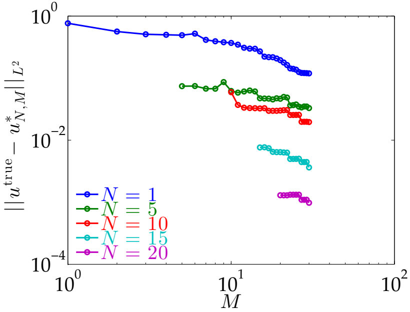


$\Im(u^{*}_{N=20,M=30})$

Case 1A: Convergence



$H^1(\Omega)$ error

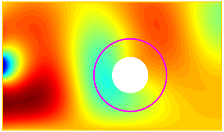


$L^2(\Omega)$ error

Model error $\epsilon^{\mathtt{bk}}(u^{\text{true}}) = 0$ since $u^{\text{true}} \in \mathcal{M}^{\mathtt{bk}}$:

$\mathcal{Z}_N$ provides rapid convergence for small N
even though the truth is high-dimensional.

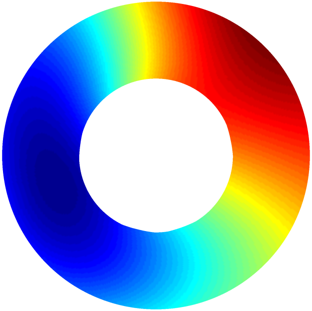
Case 1B: Field $\tilde{k} = 1.0$



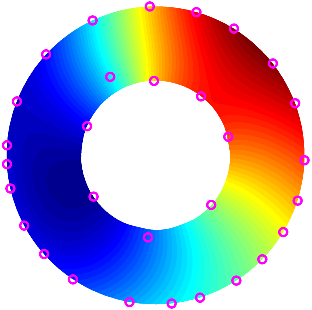
Imperfect μ-$\mathtt{bk}$ model since $Z_{\mathrm{hole}} \neq Z_{\mathrm{hole}}^{\mathtt{bk}}$:

$$u^{\mathrm{true}} \neq u^{\mathtt{bk},(k,g)} \quad \text{for any} \quad \tilde{(k, g)} \in \mathcal{D};$$

model error $\epsilon_{\mathrm{mod}}^{\mathtt{bk}}(u^{\mathrm{true}}) \neq 0$, discretization error $\epsilon_{\mathrm{disc},N}^{\mathtt{bk}} \neq 0$.

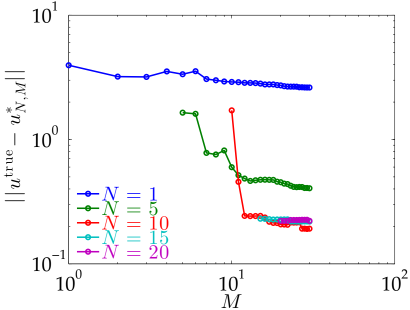


$\Im(u^{\mathrm{true}})$

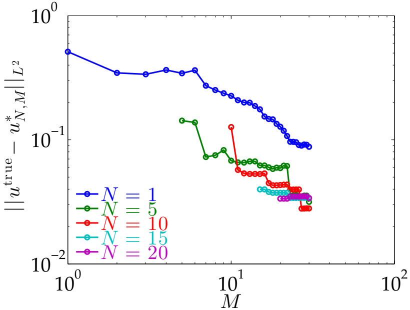


$\Im(u^{*}_{N=20,M=30})$

Case 1B: Convergence



$H^1(\Omega)$ error



$L^2(\Omega)$ error

Finite model error $\epsilon^{\texttt{bk}}_{\text{mod}}(u^{\text{true}})$ since $u^{\text{true}} \notin \mathcal{M}^{\texttt{bk}}$:

$\mathcal{Z}_{N\to\infty}$ does *not* provide convergence;

$\mathcal{U}_M (= \mathcal{U}_{M\geq N})$ required for stability *and* convergence.

Role of Observations

The set of (synthetic) observations

effectively perform projection $u^{\mathrm{true}} \to u^*_{N,M}$;

implicitly identify Dirichlet conditions;

ensure stability of primary ($z^*_{N,M} \in \mathcal{Z}_N$) approximation;

provide secondary approximation ($\eta^*_{N,M} \in \mathcal{U}_M$):

deficiency in best-knowledge model (Z_{hole});

deficiency in background space $\mathcal{Z}_N$.

In short: observations effect dimension reduction.

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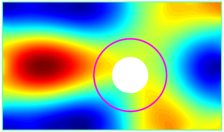
Case 2A: Field $\tilde{k} = 1.0$



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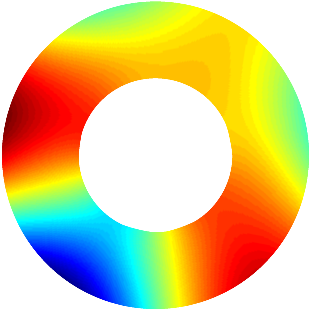
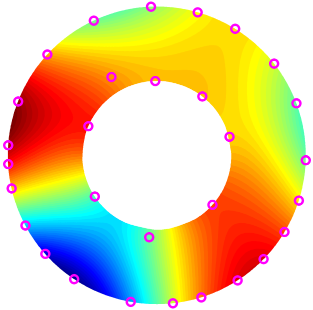
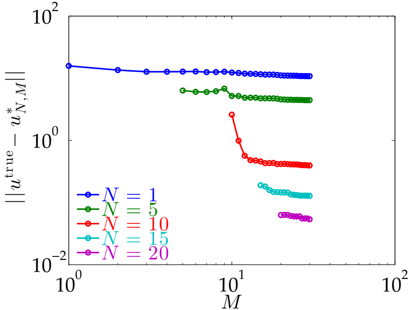


$\Im(u^{\text{true}})$

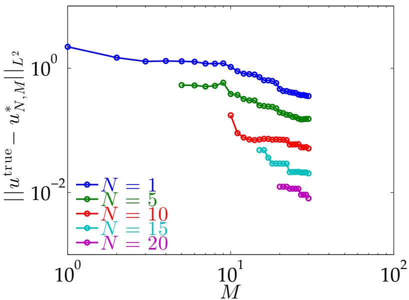


$\Im(u^{*}_{N=20,M=30})$

Case 2A: Convergence



$H^1(\Omega)$ error



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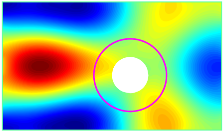
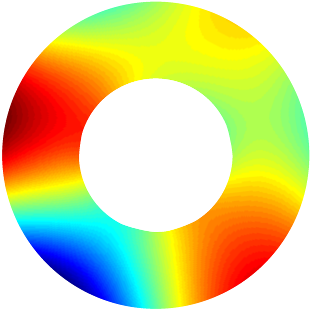
Case 2B: Field $\tilde{k} = 1.0$



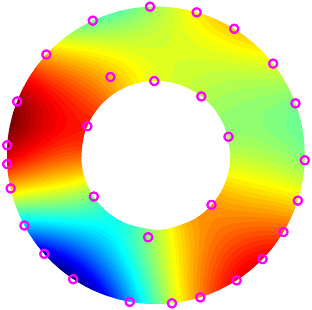
Imperfect μ-$\texttt{bk}$ model since $Z_{\text{hole}} \neq Z_{\text{hole}}^{\texttt{bk}}$:

$$u^{\text{true}} \neq u^{\texttt{bk},(k,g)} \quad \text{for any} \quad \tilde{(}k, g) \in \mathcal{D};$$

model error $\epsilon_{\text{mod}}^{\texttt{bk}}(u^{\text{true}}) \neq 0$, discretization error $\epsilon_{\text{disc},N}^{\texttt{bk}} \neq 0$.

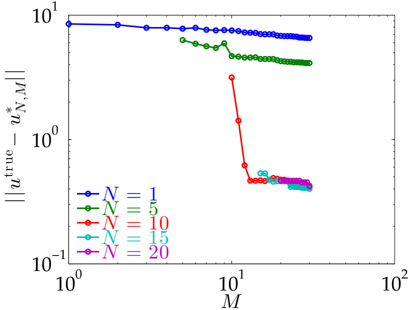


$\Im(u^{\text{true}})$

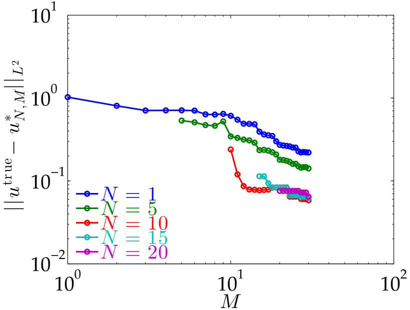


$\Im(u^{*}_{N=20,M=30})$

Case 2B: Convergence



$H^1(\Omega)$ error



$L^2(\Omega)$ error

Finite model error $\epsilon^{\texttt{bk}}_{\text{mod}}(u^{\text{true}})$ since $u^{\text{true}} \notin \mathcal{M}^{\texttt{bk}}$:

$\mathcal{Z}_{N\to\infty}$ does *not* provide convergence;

$\mathcal{U}_M (= \mathcal{U}_{M \geq N})$ required for stability *and* convergence.

Ongoing and Future Work

Methodology

Treatment of

- time-dependent problems (parabolic, hyperbolic);
- nonlinear problems.

Consideration of

- spatial and temporal filters;
- adaptive methods;
- noisy experimental observations;
- advanced greedy procedures;
- parameter estimation;
- domain decomposition frameworks;
- non-variational (e.g., stochastic) bk models.

Physical Phenomena

Acoustics

Elasticity

Conduction heat transfer

Fluid flow

Electromagnetism

Multi-field phenomena:

vibroacoustics, natural convection,...

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(methodology/Seminar Presentations)