Weighted reduced basis for the approximation of viscous flows with random coefficients

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Workshop Numerical Methods for High Dimensional Problems

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Acknowledgements: Federico Negri (EPFL)

Outline

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Stochastic Stokes equations with random input data

Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space, where Ω is a set of outcomes $\omega \in \Omega$, $\mathfrak{F}$ is a σ-algebra of events and P is a probability measure defined as $P : \mathfrak{F} \to [0, 1]$ with $P(\Omega) = 1$. We consider a stochastic Stokes equations in physical domain $D \in \mathbb{R}^d$

$$\text{Prob}(\omega) \begin{cases} -\nu(\omega)\triangle \mathbf{u}(\cdot, \omega) + \nabla p(\cdot, \omega) = \mathbf{f}(\cdot, \omega) & \text{in } D, \\ \nabla \cdot \mathbf{u}(\cdot, \omega) = 0 & \text{in } D, \\ \mathbf{u}(\cdot, \omega) = \mathbf{0} & \text{on } \partial D_D, \\ \nu(\omega)\nabla u(\cdot, \omega) \cdot \mathbf{n} - p(\cdot, \omega)\mathbf{n} = \mathbf{h}(\cdot, \omega) & \text{on } \partial D_N, \end{cases} \tag{1}$$

where the uncertainties ω arise from the viscosity ν, force term $\mathbf{f}$ and Neumann BC $\mathbf{h}$.

Finite dimensional noise assumption

The uncertainties depend on N random variables $y = (y_1, \dots, y_N) : \Omega \to \mathbb{R}^N$:

$$\text{e.g. multicomponent fluid:} \quad \nu(y(\omega)) = \nu_0 + \sum_{n=1}^{N} (\nu_n - \nu_0) y_n(\omega); \tag{2}$$

$$\text{e.g. truncated random fields:} \quad \mathbf{f}(x, y(\omega)) = \mathbb{E}[\mathbf{f}](x) + \sum_{n=1}^{N} \sqrt{\lambda_n} \mathbf{f}_n(x) y_n(\omega). \tag{3}$$

Parametrization of the stochastic Stokes equations

... so that the stochastic problem Prob(ω) becomes a parametric problem

$$\text{Prob}(y)\begin{cases} -\nu(y)\triangle\mathbf{u}(\cdot,y)+\nabla p(\cdot,y)=\mathbf{f}(\cdot,y) & \text{in } D,\\ \nabla\cdot\mathbf{u}(\cdot,y)=0 & \text{in } D,\\ \mathbf{u}(\cdot,y)=\mathbf{0} & \text{on } \partial D_D,\\ \nu(y)\nabla u(\cdot,y)\cdot\mathbf{n}-p(\cdot,y)\mathbf{n}=\mathbf{h}(\cdot,y) & \text{on } \partial D_N, \end{cases} \tag{4}$$

Remark: Prob(y) stochastic/parametric problem with random/parameter vector $y:\Omega\to\Gamma:=\otimes_{n=1}^N\Gamma_n\subset\mathbb{R}^N$ and probability density function $\rho:=\otimes_{n=1}^N\rho_n:\Gamma\to\mathbb{R}$.

Stochastic Hilbert Spaces

$$L^2_\rho(\Gamma):=\left\{v:\Gamma\to\mathbb{R}\,\middle|\;\; \mathbb{E}[v^2]:=\int_\Gamma(v(y))^2\rho(y)dy<\infty\right\};$$

$$\mathcal{G}:=(L^2_\rho(\Gamma)\otimes L^2(D))^d;\quad \mathcal{H}:=(L^2_\rho(\Gamma)\otimes L^2(\partial D_N))^d;$$

$$\mathcal{V}:=\left\{\mathbf{v}\in(L^2_\rho(\Gamma)\otimes H^1(D))^d:\mathbf{v}=\mathbf{0}\text{ on }\partial D_D\right\};$$

$$\mathcal{Q}:=L^2_\rho(\Gamma)\otimes Q(D);\quad Q(D):=\left\{q\in L^2(D):\int_D q\,dx=0\right\}.$$

Weak formulation of stochastic Stokes problem

The weak formulation of Prob(y) reads: find $\{\mathbf{u}, p\} \in \mathcal{V} \times \mathcal{Q}$ such that

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) + (\mathbf{h}, \mathbf{v})_{\partial D_N} & \forall \mathbf{v} \in \mathcal{V}, \\ b(\mathbf{u}, q) = 0 & \forall q \in \mathcal{Q}, \end{cases} \tag{5}$$

$$a(\mathbf{w}, \mathbf{v}) := \int_\Gamma \int_D \nu \nabla \mathbf{w} \otimes \nabla \mathbf{v} \rho(y) dx dy \quad \forall \mathbf{w}, \mathbf{v} \in \mathcal{V};$$

$$b(\mathbf{v}, q) := -\int_\Gamma \int_D \nabla \cdot \mathbf{v} q \rho(y) dx dy \quad \forall \mathbf{v} \in \mathcal{V}, q \in \mathcal{Q};$$

$$(\mathbf{f}, \mathbf{v}) := \int_\Gamma \int_D \mathbf{f} \cdot \mathbf{v} \rho(y) dx dy \quad \mathbf{f} \in \mathcal{G}, \mathbf{v} \in \mathcal{V};$$

$$(\mathbf{h}, \mathbf{v})_{\partial D_N} := \int_\Gamma \int_{\partial D_N} \mathbf{h} \cdot \mathbf{v} \rho(y) dx dy \quad \mathbf{h} \in \mathcal{H}, \mathbf{v} \in \mathcal{V}.$$

Remark: d-dimensional deterministic integral and N-dimensional stochastic integral

Assumption on the random input data

$$P(\omega : \nu_{min} \leq \nu(y(\omega)) \leq \nu_{max}) = 1, \quad 0 < \nu_{min} < \nu_{max} < \infty;$$

$||\mathbf{f}||_{\mathcal{G}} < \infty$ and $||\mathbf{h}||_{\mathcal{H}} < \infty$.

Well-posedness of stochastic Stokes problem

Under the assumption above, there exists a unique solution to the stochastic Stokes problem (5). Moreover, the following stability estimate holds (Brezzi, 1974)

$$||\mathbf{u}||_{\mathcal{V}} \leq \frac{1}{\alpha_a}\left(C_P||\mathbf{f}||_{\mathcal{G}} + \frac{\alpha_a+\gamma_a}{\beta_b}C_T||\mathbf{h}||_{\mathcal{H}}\right), \tag{6}$$

and

$$||p||_{\mathcal{Q}} \leq \frac{1}{\beta_b}\left(\left(1+\frac{\gamma_a}{\alpha_a}\right)C_P||\mathbf{f}||_{\mathcal{G}} + \frac{\gamma_a(\alpha_a+\gamma_a)}{\alpha_a\beta_b}C_T||\mathbf{h}||_{\mathcal{H}}\right), \tag{7}$$

where the positive constants $\alpha_a, \gamma_a, \beta_b, \gamma_b$ are defined such that

$$a(\mathbf{w},\mathbf{v}) \leq \gamma_a||\mathbf{w}||_{\mathcal{V}}||\mathbf{v}||_{\mathcal{V}} \quad \forall \mathbf{w},\mathbf{v}\in\mathcal{V} \text{ and } a(\mathbf{v},\mathbf{v}) \geq \alpha_a||\mathbf{v}||^2_{\mathcal{V}} \quad \forall \mathbf{v}\in\mathcal{V}_0, \tag{8}$$

being $\mathcal{V}_0$ the kernel of b given by $\mathcal{V}_0 := \{\mathbf{v}\in\mathcal{V} : b(\mathbf{v},q)=0, \forall q\in\mathcal{Q}\}$, and

$$\inf_{q\in\mathcal{Q}}\sup_{\mathbf{v}\in\mathcal{V}}\frac{b(\mathbf{v},q)}{||\mathbf{v}||_{\mathcal{V}}||q||_{\mathcal{Q}}} \geq \beta_b, \text{ and } b(\mathbf{v},q) \leq \gamma_b||\mathbf{v}||_{\mathcal{V}}||q||_{\mathcal{Q}} \quad \forall \mathbf{v}\in\mathcal{V}, \forall q\in\mathcal{Q}. \tag{9}$$

The constants C_P and C_T are due to Poincaré inequality and trace theorem.

Chen, Quarteroni, Rozza. Multilevel and weighted reduced basis method for stochastic optimal control problems constrained by Stokes equations, submitted, 2013.

Stochastic optimal control problem with Stokes constraint

Cost functional (tracking)

A possible distributed cost functional is defined by discrepancy + regularization

$$\mathcal{J}(\mathbf{u}, p, \mathbf{f}) = \mathbb{E}\left[\frac{1}{2}\int_D (\mathbf{u} - \mathbf{u}_d)^2 dx + \frac{1}{2}\int_D (p - p_d)^2 dx + \frac{\alpha}{2}\int_D \mathbf{f}^2 dx\right]. \tag{10}$$

Remark: may not involve the second term of pressure or more general observation $\mathbf{u}_d$.

Constrained optimal control problem

Find an optimal solution $\{\mathbf{u}^*, p^*, \mathbf{f}^*\} \in \mathcal{V} \times \mathcal{Q} \times \mathcal{G}$ such that

$$\mathcal{J}(\mathbf{u}^*, p^*, \mathbf{f}^*) = \min_{\{\mathbf{u},p,\mathbf{f}\} \in \mathcal{V}\times\mathcal{Q}\times\mathcal{G}} \mathcal{J}(\mathbf{u}, p, \mathbf{f}) \text{ subject to that } \{\mathbf{u}, p, \mathbf{f}\} \text{ solve Prob}(y). \tag{11}$$

Theorem: existence of the stochastic optimal solution

By Lions' argument (Lions, 1971), we have that there exists a stochastic optimal solution $\{\mathbf{u}^*, p^*, \mathbf{f}^*\} \in \mathcal{V} \times \mathcal{Q} \times \mathcal{G}$ of the constrained optimal control problem (11).

Lagrangian formulation - the first order optimality system

Define a compound bilinear form for the weak formulation of Stokes problem as

$$\mathcal{B}(\{\mathbf{u}, p, \mathbf{f}\}, \{\mathbf{v}, q\}) = a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + b(\mathbf{u}, q) - (\mathbf{f}, \mathbf{v}). \tag{12}$$

Associated with this bilinear form, we define the Lagrangian functional as

$$\mathcal{L}(\{\mathbf{u}, p, \mathbf{f}\}, \{\mathbf{u}^a, p^a\}) = \mathcal{J}(\mathbf{u}, p, \mathbf{f}) + \mathcal{B}(\{\mathbf{u}, p, \mathbf{f}\}, \{\mathbf{u}^a, p^a\}) - (\mathbf{h}, \mathbf{u}^a)_{\partial D_N}, \tag{13}$$

where $\{\mathbf{u}^a, p^a\} \in \mathcal{V} \times \mathcal{Q}$ are the adjoint (or dual) variables of the Stokes problem.

First order optimality system

$$\left\{ \begin{array}{llll} (\{\mathbf{u}, p\}, \{\mathbf{v}^a, q^a\}) & + & \mathcal{B}(\{\mathbf{v}^a, q^a, \mathbf{0}\}, \{\mathbf{u}^a, p^a\}) & \\ & & = (\mathbf{u}_d, \mathbf{v}^a) + (p_d, p^a) & \forall \{\mathbf{v}^a, q^a\} \in \mathcal{V} \times \mathcal{Q}, \\ \alpha(\mathbf{f}, \mathbf{g}) - (\mathbf{u}^a, \mathbf{g}) & & = 0 & \forall \mathbf{g} \in \mathcal{G}, \\ \mathcal{B}(\{\mathbf{u}, p, \mathbf{f}\}, \{\mathbf{v}, q\}) & & = (\mathbf{h}, \mathbf{v})_{\partial D_N} & \forall \{\mathbf{v}, q\} \in \mathcal{V} \times \mathcal{Q}, \end{array} \right. \tag{14}$$

$$\left\{ \begin{array}{lll|lllll} (\mathbf{u}, \mathbf{v}^a) & & & +a(\mathbf{u}^a, \mathbf{v}^a) & +b(\mathbf{v}^a, p^a) & = (\mathbf{u}_d, \mathbf{v}^a) & \forall \mathbf{v}^a \in \mathcal{V}, \\ & (p, q^a) & & +b(\mathbf{u}^a, \mathbf{q}^a) & & = (p_d, q^a) & \forall q^a \in \mathcal{Q}, \\ & & \alpha(\mathbf{f}, \mathbf{g}) & -(\mathbf{u}^a, \mathbf{g}) & & = 0 & \forall \mathbf{g} \in \mathcal{G}, \\ \hline a(\mathbf{u}, \mathbf{v}) & +b(\mathbf{v}, p) & -(\mathbf{f}, \mathbf{v}) & & & = (\mathbf{h}, \mathbf{v})_{\partial D_N} & \forall \mathbf{v} \in \mathcal{V}, \\ b(\mathbf{u}, q) & & & & & = 0 & \forall q \in \mathcal{Q}, \end{array} \right. \tag{15}$$

An equivalent stochastic saddle point formulation (Gunzburger, Bochev, 2004)

Let $\mathcal{A} : (\mathcal{V} \times \mathcal{Q} \times \mathcal{G}) \times (\mathcal{V} \times \mathcal{Q} \times \mathcal{G}) \to \mathbb{R}$ be a compound bilinear form defined as

$$\mathcal{A}(\{\mathbf{u}, p, \mathbf{f}\}, \{\mathbf{v}, q, \mathbf{g}\}) = (\mathbf{u}, \mathbf{v}) + (p, q) + \alpha(\mathbf{f}, \mathbf{g}). \tag{16}$$

An equivalent saddle point formulation

Find $\{\mathbf{u}, p, \mathbf{f}\} \in \mathcal{V} \times \mathcal{Q} \times \mathcal{G}$ and $\{\mathbf{u}^a, p^a\} \in \mathcal{V} \times \mathcal{Q}$ such that

$$\begin{cases} \mathcal{A}(\{\mathbf{u}, p, \mathbf{f}\}, \{\mathbf{v}^a, q^a, \mathbf{g}\}) + \mathcal{B}(\{\mathbf{v}^a, q^a, \mathbf{g}\}, \{\mathbf{u}^a, p^a\}) \\ \qquad\qquad = (\{\mathbf{u}_d, p_d, \mathbf{0}\}, \{\mathbf{v}^a, q^a, \mathbf{g}\}) \quad \forall \{\mathbf{v}^a, q^a, \mathbf{g}\} \in \mathcal{V} \times \mathcal{Q} \times \mathcal{G}, \\ \mathcal{B}(\{\mathbf{u}, p, \mathbf{f}\}, \{\mathbf{v}, q\}) = (\mathbf{h}, \mathbf{v})_{\partial D_N} \quad \forall \{\mathbf{v}, q\} \in \mathcal{V} \times \mathcal{Q}. \end{cases} \tag{17}$$

Theorem: there exists a unique optimal solution. Moreover, the optimal solution $\{\mathbf{u}, p, \mathbf{f}\}$ and the adjoint variables $\{\mathbf{u}^a, p^a\}$ satisfy the following stability estimates:

$$||\{\mathbf{u}, p, \mathbf{f}\}||_{\mathcal{V}\times\mathcal{Q}\times\mathcal{G}} \leq \alpha_1 ||\{\mathbf{u}_d, p_d\}||_{\mathcal{L}\times\mathcal{Q}} + \beta_1 ||\mathbf{h}||_{\mathcal{H}} \tag{18}$$

and

$$||\{\mathbf{u}^a, p^a\}||_{\mathcal{V}\times\mathcal{Q}} \leq \alpha_2 ||\{\mathbf{u}_d, p_d\}||_{\mathcal{L}\times\mathcal{Q}} + \beta_2 ||\mathbf{h}||_{\mathcal{H}} \tag{19}$$

where the constants $\alpha_1, \beta_1, \alpha_2, \beta_2$ depends on the data, see more details in *Chen, Quarteroni, Rozza. Multilevel and weighted reduced basis method for stochastic optimal control problems constrained by Stokes equations, submitted, 2013.*

Stochastic regularity of the optimal solution

Assumption on the stochastic regularity of the random input data

For every $y \in \Gamma$, there exists $\mathbf{r} = (r_1, \dots, r_N) \in \mathbb{R}_+^N$ such that the $\mathbf{k}$-th order derivative of the viscosity $\nu : \Gamma \to \mathbb{R}_+$ and the boundary condition $\mathbf{h} : \Gamma \to H$ satisfy

$$C_{\alpha,\beta} \frac{|\partial_y^{\mathbf{k}} \nu(y)|}{\nu(\bar{y})} \leq \mathbf{r}^{\mathbf{k}} =: \prod_{n=1}^{N} r_n^{k_n} \text{ and } \frac{C_\beta ||\partial_y^{\mathbf{k}} \mathbf{h}(y)||_H}{C_\alpha ||\{\mathbf{u}_d, p_d\}||_{L \times Q} + C_\beta ||\mathbf{h}(y)||_H} \leq |\mathbf{k}|! \mathbf{r}^{\mathbf{k}}, \tag{20}$$

where the constants $C_\alpha = \alpha_1 + \alpha_2, C_\beta = \beta_1 + \beta_2, C_{\alpha,\beta} = \max\{\alpha_1 + \alpha_2, \beta_1 + \beta_2\}$.

Theorem: stochastic regularity

Under the above assumption, we have the following stability estimate for the $\mathbf{k}$-th order derivative of the solution $\{\mathbf{u}, p, \mathbf{f}, \mathbf{u}^a, p^a\} : \Gamma \to V \times Q \times G \times V \times Q$

$$\begin{aligned} &||\partial_y^{\mathbf{k}} \{\mathbf{u}(y), p(y), \mathbf{f}(y)\}||_{V \times Q \times G} + ||\partial_y^{\mathbf{k}} \{\mathbf{u}^a(y), p^a(y)\}||_{V \times Q} \\ &\leq C(C_\alpha ||\{\mathbf{u}_d, p_d\}||_{L \times Q} + C_\beta ||\mathbf{h}(y)||_H) |\mathbf{k}|! (r\mathbf{r})^{\mathbf{k}}, \end{aligned} \tag{21}$$

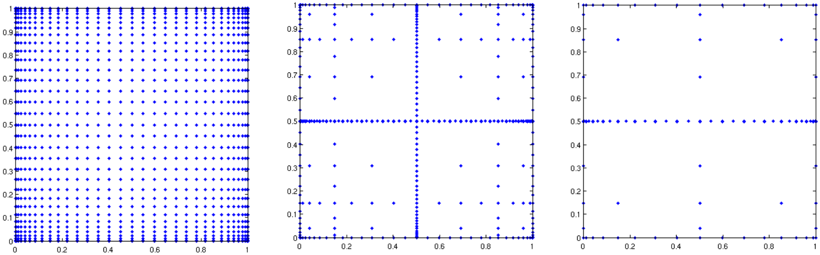
where $r\mathbf{r} = (rr_1, rr_2, \dots, rr_N)$ with the constant rate $r > 1/\log(2)$, and C is a constant. Moreover, the saddle point solution can be analytically extended to the complex region $\Sigma := \left\{ z \in \mathbb{C} : \exists y \in \Gamma \text{ such that } \sum_{n=1}^N rr_n |z_n - y_n| < 1 \right\}$.

Stochastic collocation approximation in probability space

Stochastic collocation methods (SCM) (Griebel, Xiu, Nobile, Hesthaven, etc.)

Choose collocation nodes $y^1, y^2, \ldots, y^M$ (e.g. Clenshaw-Curtis nodes, Gauss quadrature nodes), solve Prob(y) for each of the nodes, evaluate solution at any new $y \in \Gamma$ by multidimensional interpolation and statistics (e.g. mean) by multidimensional quadrature formula. Use sparse-grid SCM to reduce computational effort.
Sparse grid Smolyak formula:

$$\mathcal{S}_q^{\boldsymbol{\alpha}} v(y) = \sum_{\mathbf{i} \in X_{\boldsymbol{\alpha}}(q,N)} (\triangle^{i_1} \otimes \cdots \otimes \triangle^{i_N}) v(y). \tag{22}$$



Finite element approximation in physical space

Given an regular triangulation $\mathcal{T}_h$ of the physical domain $\bar{D} \subset \mathbb{R}^d$ with mesh size h, we define the following finite element space

$$X_h^k := \{v_h \in C^0(\bar{D}) : v_h|_K \in \mathbb{P}_k \quad \forall K \in \mathcal{T}_h\}, \quad k \geq 1, \tag{23}$$

we define $V_h^k := (X_h^k)^d \cap V$, $Q_h^m := X_h^m \cap Q$, and $G_h^l := (X_h^l)^d \cap G$ with $k, m, l \geq 1$ as finite element approximation spaces, e.g. Taylor-Hood $m = k - 1, k \geq 2$.

Finite element problem

For any $y \in \Gamma$, find $\{\mathbf{u}_h(y), p_h(y), \mathbf{f}_h(y)\} \in V_h^k \times Q_h^m \times G_h^l$ and $\{\mathbf{u}_h^a(y), p_h^a(y)\} \in V_h^k \times Q_h^m$

$$\text{s.t.} \begin{cases} \mathcal{A}\left(\{\mathbf{u}_h(y), p_h(y), \mathbf{f}_h(y)\}, \{\mathbf{v}_h^a, q_h^a, \mathbf{g}_h\}\right) + \mathcal{B}(\{\mathbf{v}_h^a, q_h^a, \mathbf{g}_h\}, \{\mathbf{u}_h^a(y), p_h^a(y)\}; y) \\ \qquad = (\mathbf{u}_d, \mathbf{v}_h^a) + (p_d, q_h^a) \quad \forall \{\mathbf{v}_h^a, q_h^a, \mathbf{g}_h\} \in V_h^k \times Q_h^m \times G_h^l, \\ \mathcal{B}(\{\mathbf{u}_h(y), p_h(y), \mathbf{g}_h(y)\}, \{\mathbf{v}_h, q_h\}; y) = (\mathbf{h}(y), \mathbf{v}_h)_{\partial D_N} \quad \forall \{\mathbf{v}_h, q_h\} \in V_h^k \times Q_h^m. \end{cases} \tag{24}$$

Theorem: well-posedness of the finite element problem

There exists a unique finite element saddle point solution to (24). The stability estimates in (18) and (19) hold in the finite element space $V_h^k \times Q_h^m \times G_h^l$.

Algebraic formulation and preconditioning

Let the finite element solution of the saddle point problem (24) be written as

$$\mathbf{u}_h(y) = \sum_{n=1}^{N_v} u_n(y)\boldsymbol{\psi}_n, p_h(y) = \sum_{n=1}^{N_p} p_n(y)\varphi_n, \mathbf{f}_h(y) = \sum_{n=1}^{N_v} f_n(y)\boldsymbol{\psi}_n, \tag{25}$$

we obtain the algebraic formulation of the finite element system as

$$\left(\begin{array}{ccc|cc} M_{v,h} & 0 & 0 & A_h^y & B_h^T \\ 0 & M_{p,h} & 0 & B_h & 0 \\ 0 & 0 & \alpha M_{g,h} & -M_{c,h}^T & 0 \\ \hline A_h^y & B_h^T & -M_{c,h} & 0 & 0 \\ B_h & 0 & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} U_h(y) \\ P_h(y) \\ F_h(y) \\ U_h^a(y) \\ P_h^a(y) \end{pmatrix} = \begin{pmatrix} M_{v,h}U_{d,h} \\ M_{p,h}P_{d,h} \\ 0 \\ M_{n,h}H_h(y) \\ 0 \end{pmatrix}. \tag{26}$$

We solve (26) by MINRES method with a block diagonal preconditioner

$$P(y) = \begin{pmatrix} \hat{M}_{s,h} & 0 & 0 \\ 0 & \alpha\hat{M}_{g,h} & 0 \\ 0 & 0 & \hat{K}_{s,h}^y M_{s,h}^{-1} (\hat{K}_{s,h}^y)^T \end{pmatrix}, \tag{27}$$

where $\hat{M}_{s,h}$ (Gauss-Seidel) and $\hat{K}_{s,h}^y$ (inexact Uzawa, Rees et al., 2011) are approximate of

$$M_{s,h} = \begin{pmatrix} M_{v,h} & 0 \\ 0 & M_{p,h} \end{pmatrix} \text{ and } K_{s,h}^y = \begin{pmatrix} A_h^y & B_h^T \\ B_h & 0 \end{pmatrix}. \tag{28}$$

Multilevel and weighted reduced basis method

Computational challenges

- It is very expensive to solve the full finite element algebraic system (26).
- We need to solve (26) at a large number of samples, e.g. $O(10^5)$.

Computational opportunities

- The finite element optimal solutions live in a low dimensional manifold.
- Model order reduction by adaptive construction and a posteriori certification.

Reduced basis approximation, double saddle point problem (Negri et al., 2011-2012)

The associated reduced basis problem can be formulated as: for any $y \in \Gamma$, find $\{\mathbf{u}_r(y), p_r(y), \mathbf{f}_r(y)\} \in V_{N_r} \times Q_{N_r} \times G_{N_r}$ and $\{\mathbf{u}_r^a(y), p_r^a(y)\} \in V_{N_r} \times Q_{N_r}$ such that

$$\begin{cases} \mathcal{A}\left(\{\mathbf{u}_r(y), p_r(y), \mathbf{f}_r(y)\}, \{\mathbf{v}_r^a, q_r^a, \mathbf{g}_r\}\right) + \mathcal{B}(\{\mathbf{v}_r^a, q_r^a, \mathbf{g}_r\}, \{\mathbf{u}_r^a(y), p_r^a(y)\}; y) \\ \qquad = (\mathbf{u}_d, \mathbf{v}_r^a) + (p_d, q_r^a) \quad \forall \{\mathbf{v}_r^a, q_r^a, \mathbf{g}_r\} \in V_{N_r} \times Q_{N_r} \times G_{N_r}, \\ \mathcal{B}(\{\mathbf{u}_r(y), p_r(y), \mathbf{g}_r(y)\}, \{\mathbf{v}_r, q_r\}; y) = (\mathbf{h}(y), \mathbf{v}_r)_{\partial D_N} \quad \forall \{\mathbf{v}_r, q_r\} \in V_{N_r} \times Q_{N_r}, \end{cases} \tag{29}$$

where $V_{N_r}, Q_{N_r}, G_{N_r}$ are reduced basis spaces constructed from the snapshots at the pre-selected samples $y^1, \ldots, y^{N_r}$.

Construction of RB spaces, double saddle point problem stabilization

Reduced control space

The reduced control space G_{N_r} is constructed by

$$G_{N_r} = \text{span}\{\mathbf{f}_h(y^n), 1 \leq n \leq N_r\}. \tag{30}$$

Reduced pressure space, aggregated approach

As for Q_{N_r}, we take the union of the state and adjoint snapshots of pressure in order to guarantee the approximate stability in the reduced basis space (Negri et al., 2011-12)

$$Q_{N_r} = Q^s_{N_r} \cup Q^a_{N_r} = \text{span}\{p_h(y^n), p^a_h(y^n), 1 \leq n \leq N_r\}. \tag{31}$$

Reduced velocity space (Gerner, Huynh, Manzoni, Patera, Rozza, Veroy, 2003-2014)

To guarantee the the compatibility condition, we enrich the reduced basis velocity space by introducing the supremizer operator $T : Q^m_h \to V^k_h$:

$$(Tq_h, \mathbf{v}_h)_A = b(\mathbf{v}_h, q_h) \quad \forall \mathbf{v} \in V^k_h, \tag{32}$$

where $(\mathbf{u}, \mathbf{v})_A = a(\mathbf{u}, \mathbf{v}; \bar{y})\, \forall \mathbf{u}, \mathbf{v} \in V$, being $\bar{y} \in \Gamma$ a reference value, so we have

$$V_{N_r} = V^s_{N_r} \cup V^a_{N_r} = \text{span}\{\mathbf{u}_h(y^n), Tp_h(y^n), \mathbf{u}^a_h(y^n), Tp^a_h(y^n), 1 \leq n \leq N_r\}. \tag{33}$$

Reduced basis method – basic greedy formulation

It is still not feasible because Γ has infinite elements and $u(y)$ needs expensive solve.

Recipe 1: replace Γ by a finite set $\Xi_{train} \subset \Gamma$ with $|\Xi_{train}| = N_{train}$, known as training set;
Recipe 2: replace $||u_h(y) - u_{N-1}(y)||_X$ by a posteriori error bound $\triangle_{N-1}(y)$, yielding

weak greedy algorithm: $y^N := \arg \max_{y \in \Xi_{train}} \triangle_{N-1}(y),$

$\boxed{\text{Question: how to choose the training set } \Xi_{train} \text{ ?}}$

Criteria for training set

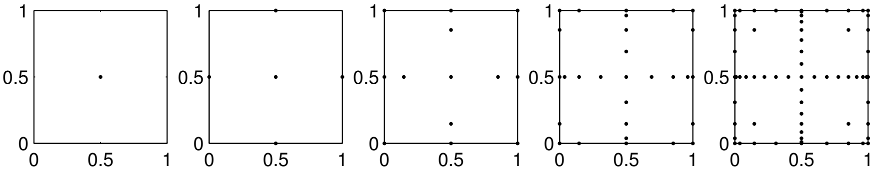
- sufficient to cover a large range of probability domain Γ;
- sparse to alleviate computational effort for reduced basis construction.

Choice of training set

- random sampling according to probability density function [Boyaval et al., 2010];
- adaptively clean and enrich the training set [Hesthaven et al., 2014];
- borrow sparse grid used by stochastic collocation methods [Chen et al., 2012];

A multilevel greedy algorithm

Denote the set of collocation nodes in the qth ($q \geq N$) level of sparse grid as $H(q,N)$



Multilevel greedy algorithm [Elman and Liao, 2013] [Chen et al., 2013]

1. *To start*, we solve a full FE problem at y^1 (e.g. center) and construct RB spaces;
2. *At each level* q, we choose sample y^{N_r+1} to maximize RB error $\mathcal{E}_r = ||u_h - u_r||_X$

$$y^{N_r+1} = \arg \max_{y \in H(q,N) \setminus H(q-1,N)} \mathcal{E}_r(y). \tag{34}$$

   1. Solve a full FE problem at y^{N_r+1} and construct RB spaces;
   2. *If* $\mathcal{E}_r(y^{N_r+1}) < \varepsilon$, $N_r = N_r + 1$, go to step 2 at the next level $q = q + 1$;
   3. *Otherwise*, $N_r = N_r + 1$, choose the next sample y^{N_r+1} at current level.

3. *If* $q > q_{max}$, Stop.

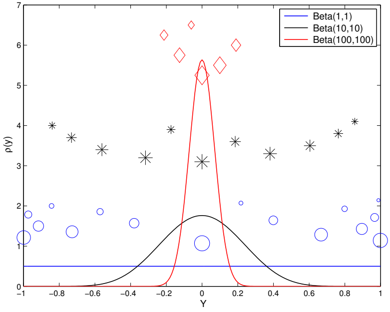
Weighted a posteriori error bound

A full finite element problem has to be solved in order to evaluate the reduced basis approximation error $\mathcal{E}_r$, which is infeasible. Instead, we use a posteriori error bound $\triangle_r$

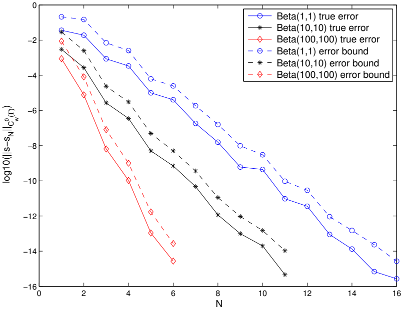
$$c\triangle_r(y) \leq \mathcal{E}_r(y) \leq \triangle_r(y), \tag{35}$$

where $c < 1$. We hope that $c \approx 1$ and $\triangle_r(y)$ is very cheap to compute. We propose

$$\triangle_r^{\rho}(y) = \rho(y)||\mathcal{R}(\mathfrak{u}_r(y))||^2_{\mathfrak{U}}/\beta_r. \tag{36}$$



beta PDF and selected samples



error bound and true error

Error estimates

Stochastic collocation approximation error (stochastic regularity)

$$\mathcal{E}_s^{\boldsymbol{\alpha}} := ||\mathrm{u} - \mathrm{u}_s||_{C(\Gamma;\mathrm{V})} \leq C_s^{\boldsymbol{\alpha}} N_q^{-r(\boldsymbol{\alpha})}, \quad \mathcal{E}_s^e := ||\mathbb{E}[\mathrm{u}] - \mathbb{E}[\mathrm{u}_s]||_{\mathrm{V}} \leq C_s^e N_q^{-r(\boldsymbol{\alpha})}. \tag{37}$$

Finite element approximation error (deterministic regularity & FE polynomial order)

$$\mathcal{E}_h(y) \leq C_h h^k ||\mathrm{u}||_{k+1}. \tag{38}$$

Reduced basis approximation error (stochastic regularity)

$$\mathcal{E}_r := ||\mathrm{u}_h - \mathrm{u}_r||_{C(\Gamma;\mathrm{V})} \leq C_r \exp(-rN_r). \tag{39}$$

Global error estimate

$$||\mathbb{E}[\mathrm{u}] - \mathbb{E}[\mathrm{u}_r]||_{\mathrm{V}} \leq \mathcal{E}_s^e + \max_{y \in H_{\boldsymbol{\alpha}}(q,N)} \mathcal{E}_h(y) + \max_{y \in H_{\boldsymbol{\alpha}}(q,N)} \mathcal{E}_r(y). \tag{40}$$

Chen et al., Multilevel and weighted reduced basis method for stochastic optimal control problems constrained by Stokes equations, submitted, 2013.

Experimental setting

We consider a two dimensional physical domain $D = (0,1)^2$. The observation data is set as (Gunzburger et al., 2000). The random viscosity ν is given as

$$\nu(y^\nu) = \frac{1}{2}\sum_{n=0}^{N_\nu} \nu_n + \frac{1}{2N_\nu}\sum_{n=1}^{N_\nu}(\nu_n - \nu_0)y_n^\nu, \tag{41}$$

where $y^\nu \in \Gamma_\nu = [-1,1]^{N_\nu}$ corresponding to N_ν uniformly distributed random variables. We set $\nu_0 = 0.01$, $\nu_n = \nu_0/2^n$ and use $N_\nu = 3$. We set $\mathbf{h}(x, y^h) = (h_1(x_2, y^h), 0)$ with

$$h_1(x_2, y^h) = \frac{1}{10}\left(\left(\frac{\sqrt{\pi}L}{2}\right)^{1/2} y_1^h + \sum_{n=1}^{N_h}\sqrt{\lambda_n}\left(\sin(n\pi x_2)y_{2n}^h + \cos(n\pi x_2)y_{2n+1}^h\right)\right), \tag{42}$$

which comes from truncation of Karhunen-Loève expansion of a Gauss covariance field with correlation length $L = 1/16$; the eigenvalues $\lambda_n, 1 \le n \le N_h$ are given by

$$\lambda_n = \sqrt{\pi}L\exp\left(-(n\pi L)^2/4\right); \tag{43}$$

$y_n^h, 1 \le n \le 2N_h + 1$ are uncorrelated with zero mean and unit variance, which are independent of y^ν. Therefore, the random inputs are $y = (y^\nu, y^h)$, living in $N = N_\nu + 2N_h + 1$ dimensional probability space. We use P1 element for pressure space and P2 element for velocity and control space with 1342 elements in total.

10 dimensional experiment

Table: The number of samples by multilevel greedy algorithm with different tolerance ϵ_{tol} in each of the sparse grid level; the value in $(\cdot)$ reports the number of samples potential as new bases.

tolerance \ level	$q-N=0$	$q-N=1$	$q-N=2$	$q-N=3$	in total
# nodes	1	21	221	1581	1581
$\epsilon_{tol}=10^{-1}$	1 (1)	6 (14)	1 (21)	0 (0)	8 (36)
$\epsilon_{tol}=10^{-2}$	1 (1)	8 (20)	7 (80)	4 (28)	20 (129)
$\epsilon_{tol}=10^{-3}$	1 (1)	9 (20)	13 (86)	5 (62)	28 (169)
$\epsilon_{tol}=10^{-4}$	1 (1)	9 (20)	18 (90)	9 (67)	37 (178)
$\epsilon_{tol}=10^{-5}$	1 (1)	10 (20)	22 (90)	14 (105)	47 (216)

10 dimensional experiment

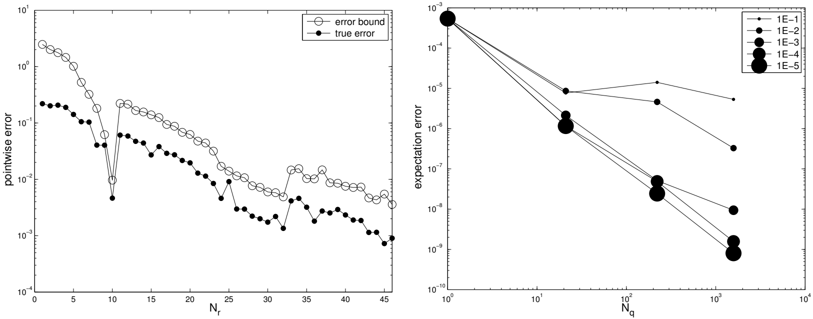


Figure: Left, weighted error bound $\triangle_r^\rho$ and true error of the reduced basis approximation at the selected samples; right, expectation error at different levels with different tolerance ϵ_{tol}.

100 dimensional experiment

Table: The number of samples selected by multilevel greedy algorithm in each of the level with different dimensions; the value in $(\cdot)$ reports the number of samples potential as new bases.

dimension \ level	$q - N = 1$	$q - N = 2$	$q - N = 3$	$q - N = 4$	in total
$N = 10$	5 (10)	13 (40)	19 (85)	10 (100)	48 (236)
# nodes	11	71	401	2141	2141
$N = 20$	5 (10)	21 (60)	36 (205)	15 (204)	78 (480)
# nodes	11	91	1021	12121	12121
$N = 40$	5 (10)	25 (92)	47 (397)	19 (432)	97 (932)
# nodes	11	123	2381	40769	40769
$N = 100$	5 (10)	25 (92)	47 (397)	19 (436)	97 (936)
# nodes	11	123	2393	41349	41349

High dimensional experiments

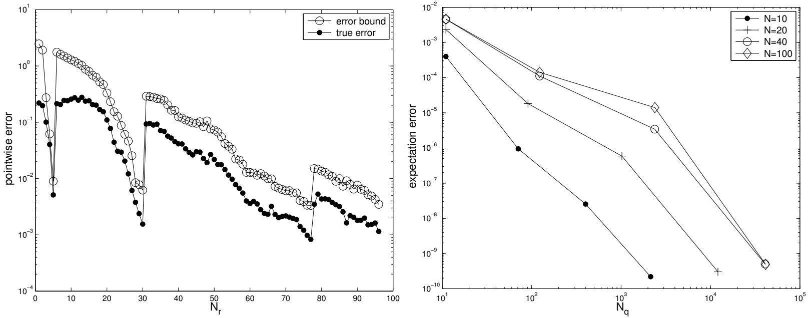


Figure: Weighted error bound $\triangle_r^\rho$ and true error of the reduced basis approximation at the selected samples in the case of stochastic dimension $N = 10$ (left) and high dimensions (right).

Conclusions and perspectives

Conclusions

- We obtained the well-posedness for the stochastic optimal control problem constrained by Stokes equations via stochastic saddle point formulation;
- We developed multilevel and weighted reduced basis method to solve the PDE-constrained stochastic optimization problem, whose numerical error estimates have been verified by numerical experiments of 10 to 100 dimensions.

Perspectives

- Further development of the proposed method for stochastic optimal control problems with more general statistical observation data;
- Application of the proposed method to other stochastic fluid flow control problems, for instance unsteady Stokes or Navier-Stokes constraint.

Thank you for your attention!