A Reduced Basis Method for Multiple Electromagnetic Scattering in Three Dimensions

"Numerical methods for high-dimensional problems”
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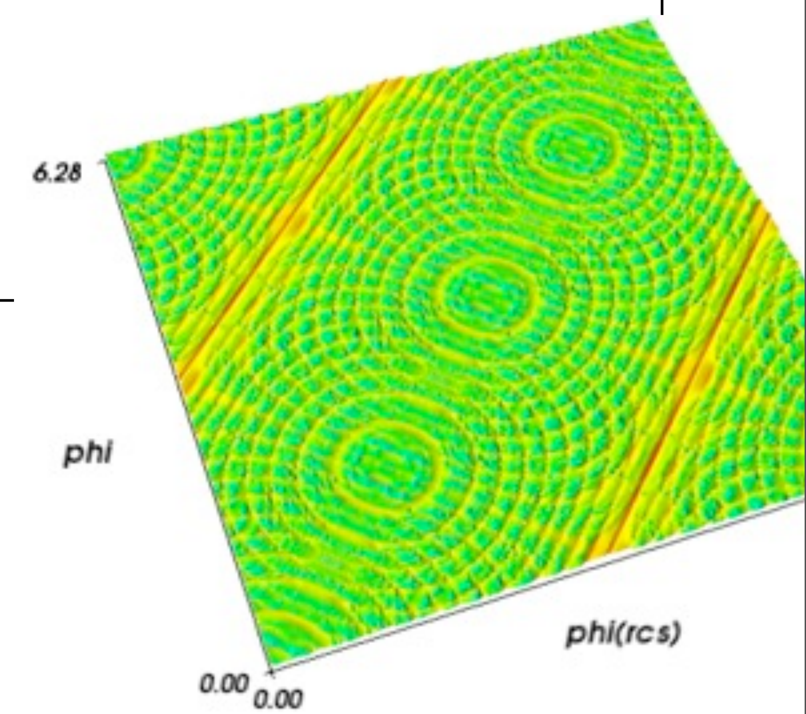
Benjamin Stamm
LJLL, Paris 6 and CNRS



Outline

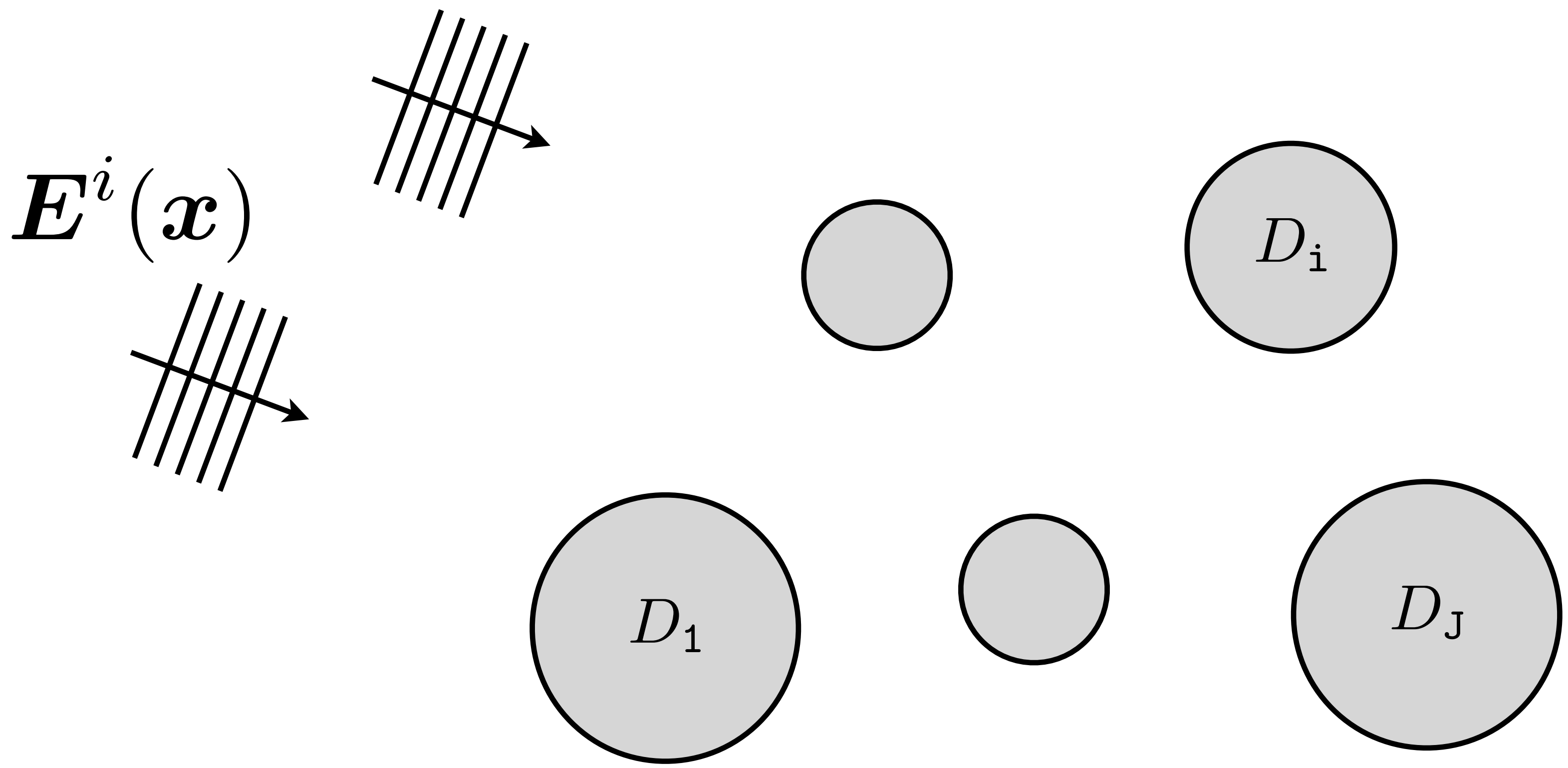
- Problem setting
- Single scatterer RBM
- RBM for multiple scattering problem
- Numerical results

In collaboration with

- M. Fares (CERFACS, Toulouse)
- M. Ganesh (Colorado School of Mines)
- J. Hesthaven (EPFL)
- Y. Maday (Paris 6 and Brown University)
- S. Zhang (City University of Hong Kong)

Problem setting

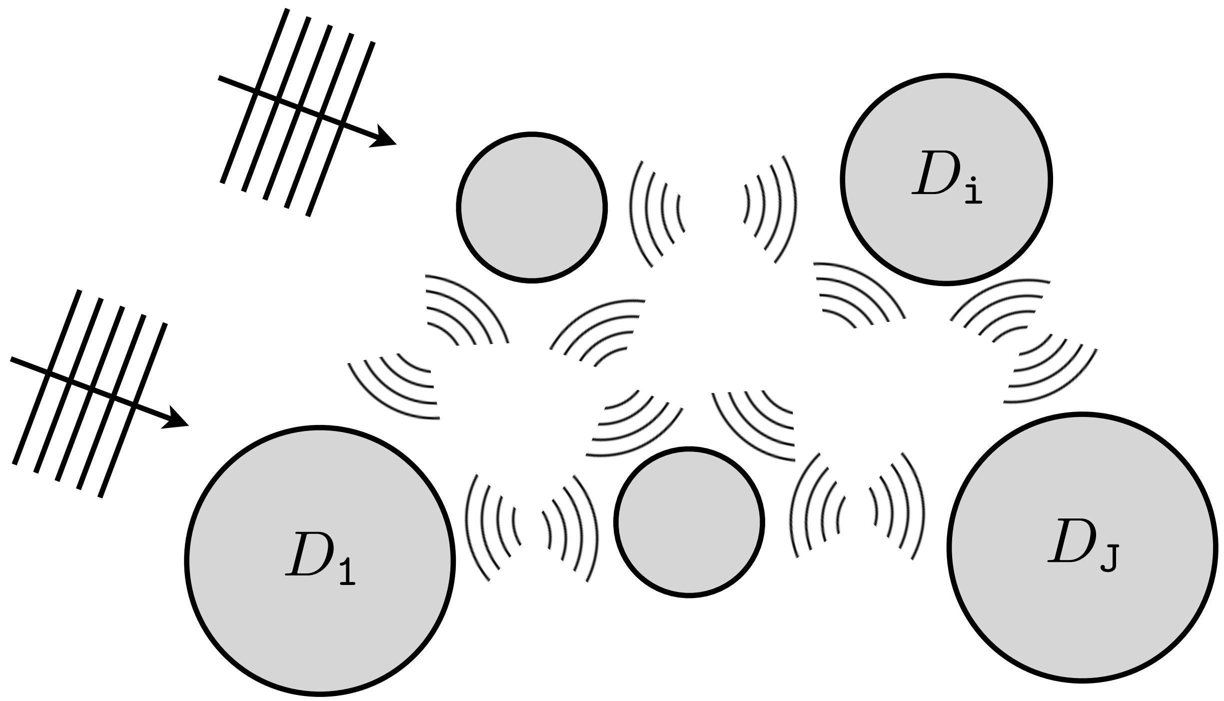
Physical problem/Geometrical Configuration [in 3D]



Incident plane wave impinging onto collection of $\mathtt{J}$ perfectly conducting obstacles $D_{\mathtt{1}}, \ldots, D_{\mathtt{J}}$.

Physical problem/Geometrical Configuration [in 3D]

$\boldsymbol{E}^i(\boldsymbol{x})$



$\Rightarrow$ Scattered field $\boldsymbol{E}^s(\boldsymbol{x})$

Incident plane wave impinging onto collection of $\mathtt{J}$ perfectly conducting obstacles $D_{\mathtt{1}}, \ldots, D_{\mathtt{J}}$.

Parametrization

The system is parametrized by:

- The wave number k,
- The angle and polarization of the incident wave $\boldsymbol{E}^i(\boldsymbol{x}; k, \boldsymbol{p}, \hat{\boldsymbol{k}}) = -\boldsymbol{p}\, e^{ik\boldsymbol{x}\cdot\hat{\boldsymbol{k}}_{(\theta,\phi)}}$,
- The location and shape of the obstacles:

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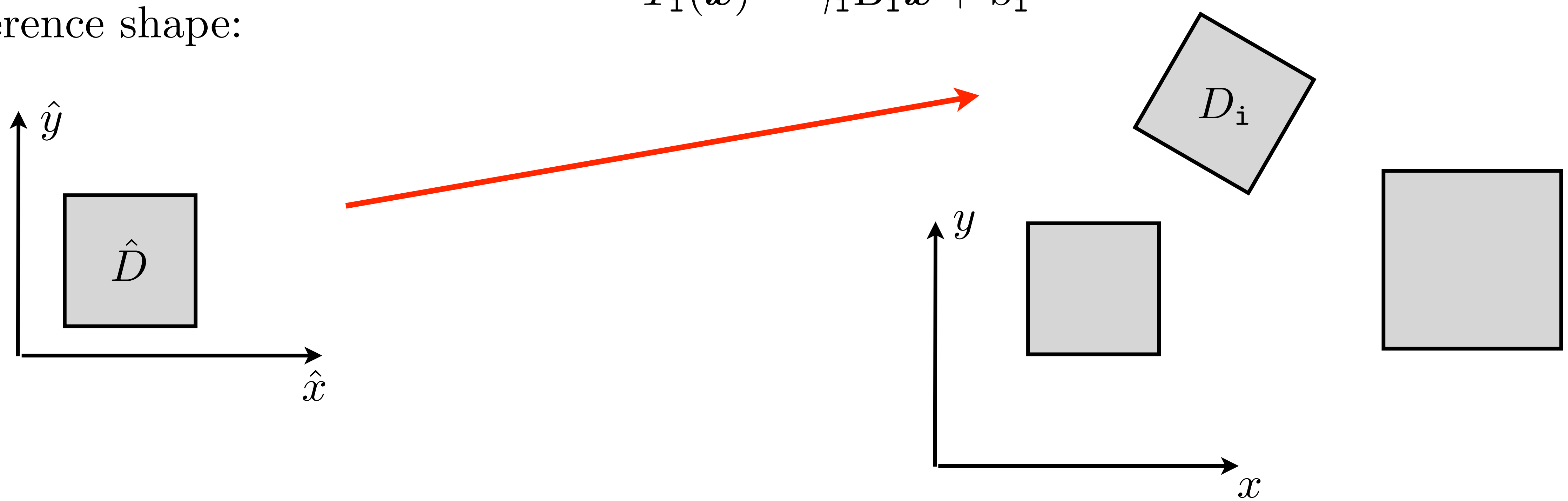
- The location and shape of the obstacles:

$$T_{\mathtt{i}}(\hat{\boldsymbol{x}}) = \gamma_{\mathtt{i}} \mathrm{B}_{\mathtt{i}} \hat{\boldsymbol{x}} + \mathrm{b}_{\mathtt{i}}$$

Reference shape:



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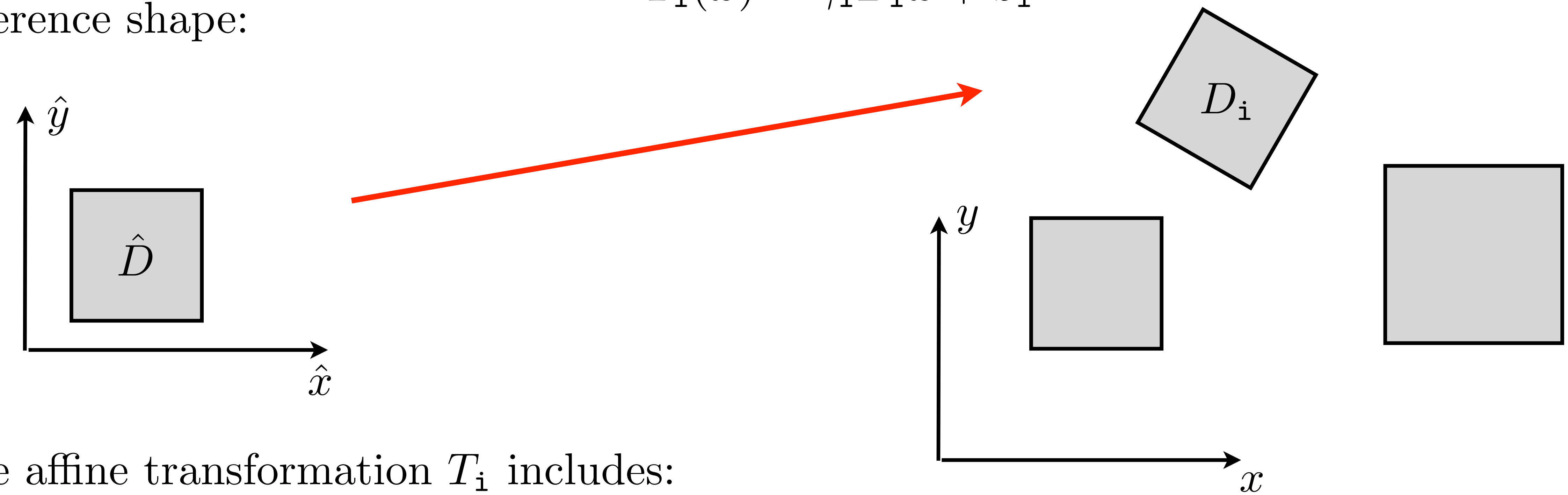

The affine transformation $T_{\mathtt{i}}$ includes:
$\mathrm{B}_{\mathtt{i}} \in SO(3)$: rotation
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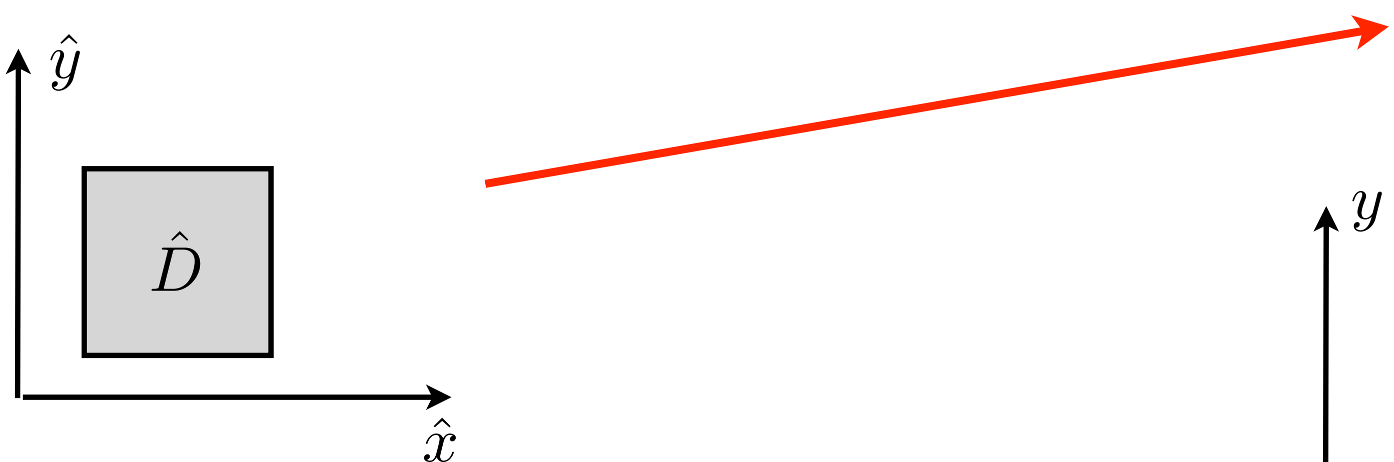

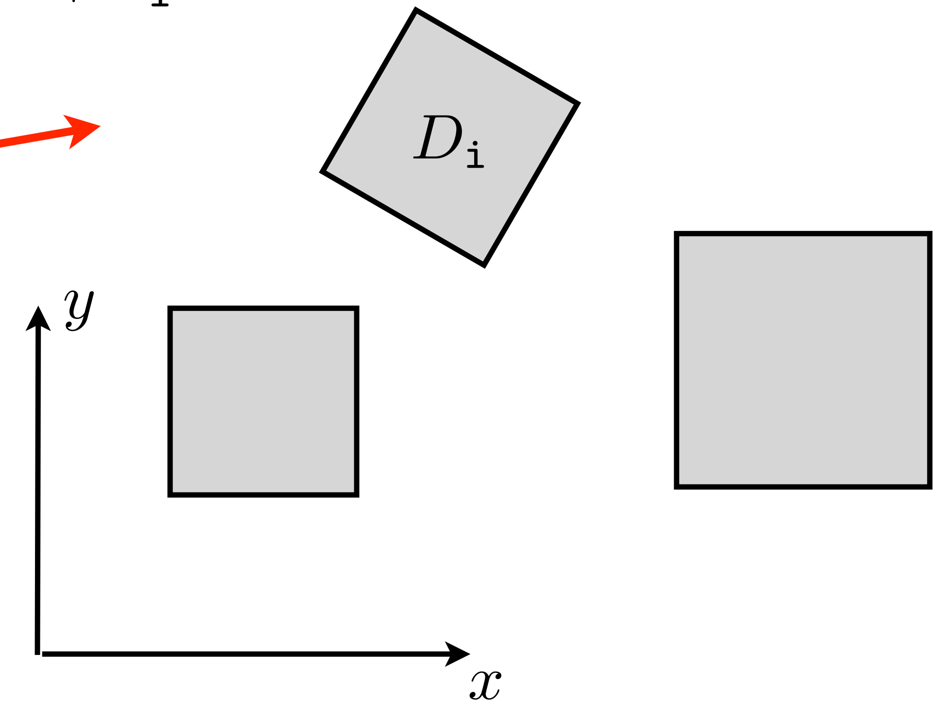

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Parameter: $\boldsymbol{\mu} = (k, \hat{\boldsymbol{k}}, \boldsymbol{p}, \underbrace{\mathrm{b}_1, \mathrm{B}_1, \gamma_1}, \ldots, \underbrace{\mathrm{b}_{\mathtt{J}}, \mathrm{B}_{\mathtt{J}}, \gamma_{\mathtt{J}}}) \in \mathbb{P} \subset \mathbb{R}^{5+7\mathtt{J}}$

Governing equations
(time-harmonic ansatz)

Assume that the free space is a homogenous media with magnetic permeability μ and electrical permittivity ε.

The total electric field $\mathbf{E} = \mathbf{E}^i + \mathbf{E}^s \in \boldsymbol{H}(\mathrm{curl}, \Omega)$ satisfies

$$
\begin{aligned}
\mathrm{curl\ curl\ }\mathbf{E} - k^2\mathbf{E} &= 0 && \text{in } \Omega, && \boxed{\text{Maxwell}} \\
\mathbf{E} \times \boldsymbol{n} &= 0 && \text{on } \Gamma, && \boxed{\text{boundary condition}} \\
\left|\mathrm{curl}\mathbf{E}^s(\boldsymbol{x}) \times \frac{\boldsymbol{x}}{|\boldsymbol{x}|} - ik\mathbf{E}^s(\boldsymbol{x})\right| &= \mathcal{O}\left(\frac{1}{|\boldsymbol{x}|}\right) && \text{as } |\boldsymbol{x}| \to \infty. && \boxed{\text{Silver-Müller radiation condition}}
\end{aligned}
$$

$\Omega = \mathbb{R}^3 \setminus \cup_{\mathtt{i}=1}^{\mathtt{J}} D_{\mathtt{i}}$.
Γ is the collection of all surfaces: $\Gamma = \cup_{\mathtt{i}=1}^{\mathtt{J}} \partial D_{\mathtt{i}}$.

see book of [Colton,Kress], [Nedelec]

Variational formulation of the Electric Field Integral Equation (EFIE)

Change the unknown to be $\boldsymbol{u}$: Electric currant on collection of surfaces.

For any fixed $\boldsymbol{\mu} \in \mathbb{P}$, find $\boldsymbol{u}(\boldsymbol{\mu}) \in \mathbb{V}$ s.t.

$$\boxed{a[\boldsymbol{u}(\boldsymbol{\mu}), \boldsymbol{v}; \boldsymbol{\mu}] = f[\boldsymbol{v}; \boldsymbol{\mu}], \qquad \forall \boldsymbol{v} \in \mathbb{V}}$$

with

$$a[\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{\mu}] = ikZ \int_{\Gamma(\boldsymbol{\mu})} \int_{\Gamma(\boldsymbol{\mu})} \mathrm{G}_k(\boldsymbol{x}, \boldsymbol{y}) \left\{ \boldsymbol{u}(\boldsymbol{y}) \cdot \overline{\boldsymbol{v}(\boldsymbol{x})} - \frac{1}{k^2} \mathrm{div}_{\boldsymbol{y}} \boldsymbol{u}(\boldsymbol{y}) \overline{\mathrm{div}_{\boldsymbol{x}} \boldsymbol{v}(\boldsymbol{x})} \right\} d\boldsymbol{y}\, d\boldsymbol{x}$$

$$f[\boldsymbol{v}; \boldsymbol{\mu}] = -\int_{\Gamma(\boldsymbol{\mu})} \mathbf{E}^i(\boldsymbol{y}; k, \boldsymbol{p}, \hat{\boldsymbol{k}}) \cdot \overline{\boldsymbol{v}(\boldsymbol{y})}\, d\boldsymbol{y}$$

The scattered electric field $\boldsymbol{E}^s$ is then uniquely determined by the electric currant $\boldsymbol{u}$.

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The kernel function is given by

$$\boxed{\mathrm{G}_k(\boldsymbol{x}, \boldsymbol{y}) = \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}}$$

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Output of interest: Radar Cross Section (RCS)

- Describes pattern/energy of electrical field at infinity
- Functional of the current on body

$$A_\infty[\boldsymbol{u};\boldsymbol{\mu},\hat{\boldsymbol{d}}] = \frac{ikZ}{4\pi}\int_\Gamma \hat{\boldsymbol{d}} \times (\boldsymbol{u}(\boldsymbol{x}) \times \hat{\boldsymbol{d}}) e^{-ik\boldsymbol{x}\cdot\hat{\boldsymbol{d}}} d\boldsymbol{x}$$

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where

$\boldsymbol{u}$: current on surface
$\hat{\boldsymbol{d}}$: given directional unit vector

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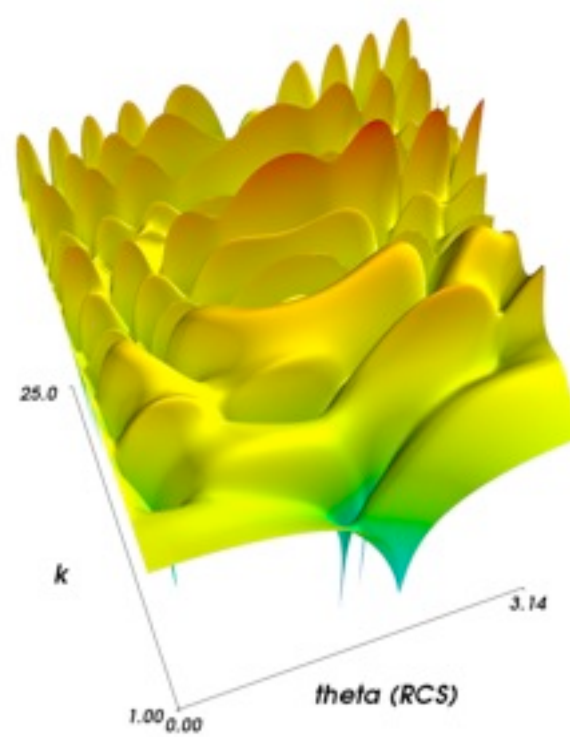

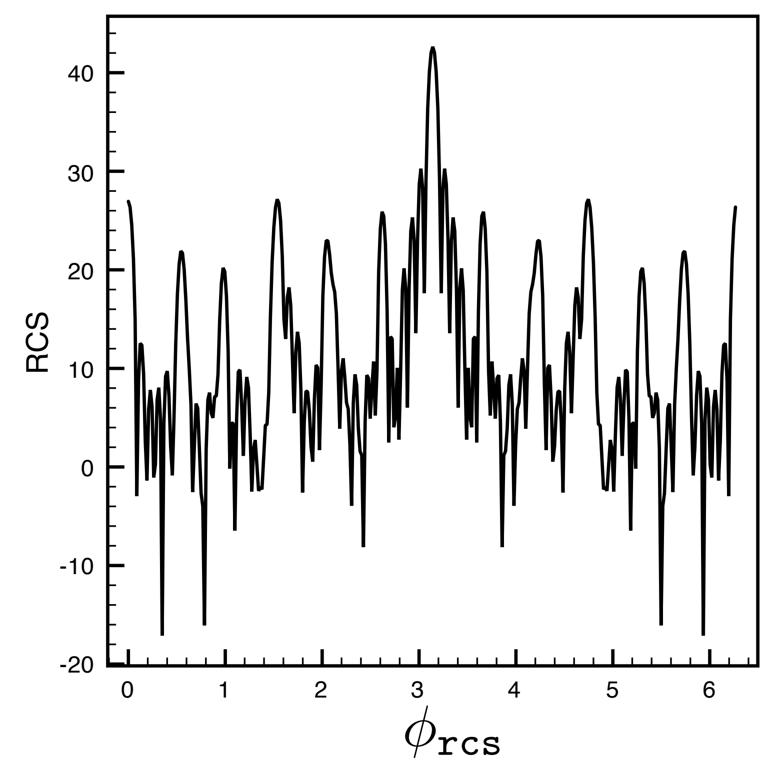

Single obstacle scattering

Reduced Basis Method

Reduced Basis Ansatz:

$$\boxed{\mathbb{V}_N = \mathrm{span}\{\boldsymbol{u}_\delta(\boldsymbol{\mu}_1), \ldots, \boldsymbol{u}_\delta(\boldsymbol{\mu}_N)\}}$$

for some well-chosen sample points $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_N$.

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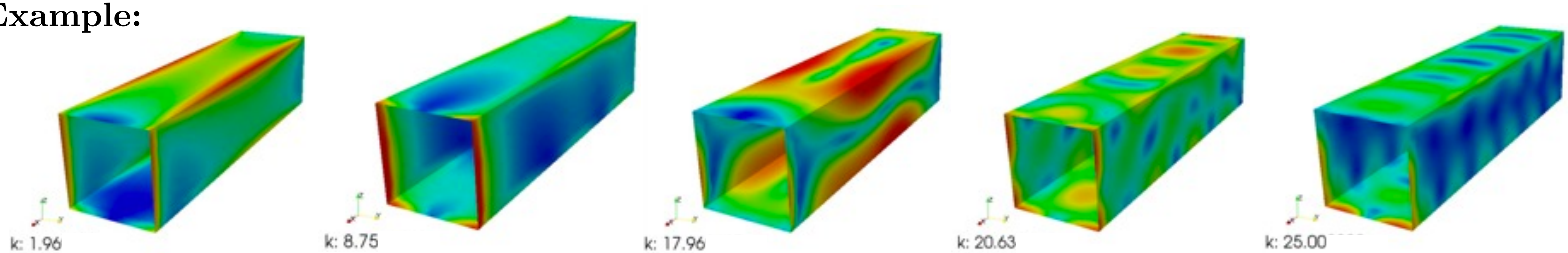

1 parameter: wavenumber k
Different snapshots illustrated

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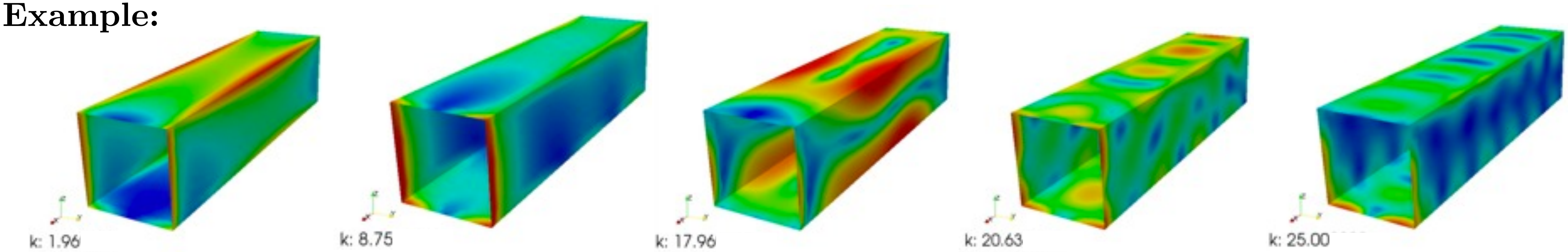

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Question: How to find the sample points $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_N$ such that

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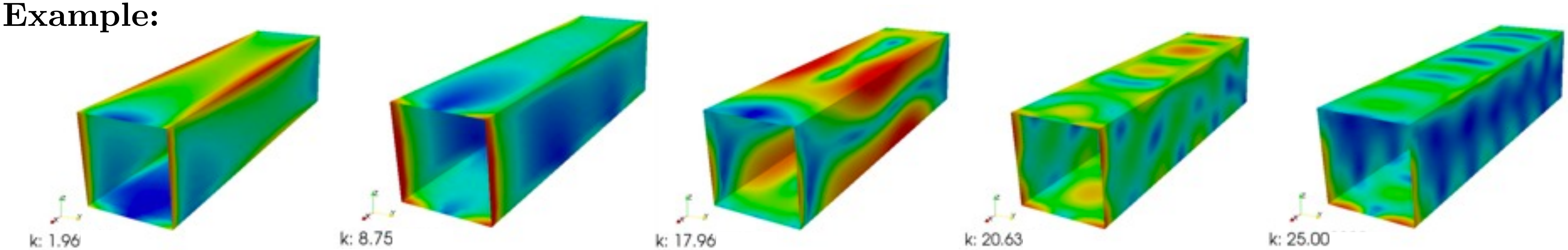

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Answer: **Greedy-algorithm**

Affine decomposition for EFIE

For any parameter value $\boldsymbol{\mu} \in \mathbb{P}$, find $\boldsymbol{u}(\boldsymbol{\mu}) \in \mathbb{V}$ s.t.

$$\boxed{\mathtt{a}(\boldsymbol{u}(\boldsymbol{\mu}), \boldsymbol{v}; \boldsymbol{\mu}) = \mathtt{f}(\boldsymbol{v}; \boldsymbol{\mu}), \qquad \forall \boldsymbol{v} \in \mathbb{V}}$$

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Solution: Empirical Interpolation Method (EIM) (also based on a greedy algorithm)

Given: A parametrized function $g(\boldsymbol{x}; \boldsymbol{\mu})$.

Output: $\{\boldsymbol{\mu}_q\}_{q=1}^Q$ such that

$$g(\boldsymbol{x}; \boldsymbol{\mu}) \approx \mathcal{I}_Q(g)(\boldsymbol{x}; \boldsymbol{\mu}) = \sum_{q=1}^{Q} \alpha_q^{\mathrm{g}}(\boldsymbol{\mu}) \, g(\boldsymbol{x}; \boldsymbol{\mu}_q).$$

Similar problem formulation as for the RBM, but solutions are explicitly known (not solution to PDE)

[Maday et al. 2004] (happy birthday!)

Affine decomposition for the EFIE

Approximating

$$\frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \approx \sum_{q=1}^{Q} \alpha_q^{\mathtt{a}}(k) \frac{e^{ik_q|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}$$

$$e^{ik\boldsymbol{x}\cdot\hat{\boldsymbol{k}}_{(\theta,\phi)}} \approx \sum_{q=1}^{Q} \alpha_q^{\mathtt{f}}(\boldsymbol{\mu}) e^{ik_q\boldsymbol{x}\cdot\hat{\boldsymbol{k}}_{(\theta_q,\phi_q)}}$$

red: parameter-dependent,
blue: parameter-independent.

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results in

$$\begin{aligned}\mathsf{a}(\boldsymbol{v},\boldsymbol{w};\boldsymbol{\mu}) &= ikZ\int_\Gamma\int_\Gamma \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} u(\boldsymbol{x})\cdot\overline{\boldsymbol{v}(\boldsymbol{y})}\,d\boldsymbol{x}\,d\boldsymbol{y} - \frac{iZ}{k}\int_\Gamma\int_\Gamma \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{div}_{\Gamma,\boldsymbol{x}}\boldsymbol{u}(\boldsymbol{x})\,\overline{\mathrm{div}_{\Gamma,\boldsymbol{y}}\boldsymbol{v}(\boldsymbol{y})}\,d\boldsymbol{x}\,d\boldsymbol{y}\\ &\approx \sum_{q=1}^{Q} ikZ\alpha_q^{\mathsf{a}}(k)\int_\Gamma\int_\Gamma \frac{e^{ik_q|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} u(\boldsymbol{x})\cdot\overline{\boldsymbol{v}(\boldsymbol{y})}\,d\boldsymbol{x}\,d\boldsymbol{y}\\ &\quad - \sum_{q=1}^{Q}\frac{iZ\alpha_q^{\mathsf{a}}(k)}{k}\int_\Gamma\int_\Gamma \frac{e^{ik_q|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{div}_{\Gamma,\boldsymbol{x}}\boldsymbol{u}(\boldsymbol{x})\,\overline{\mathrm{div}_{\Gamma,\boldsymbol{y}}\boldsymbol{v}(\boldsymbol{y})}\,d\boldsymbol{x}\,d\boldsymbol{y}\end{aligned}$$

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and for the source term in

$$\mathtt{f}(\boldsymbol{v};\boldsymbol{\mu}) \approx -\sum_{q=1}^{Q} \boldsymbol{p}\,\alpha_q^{\mathsf{f}}(\boldsymbol{\mu})\cdot\int_\Gamma e^{ik_q\boldsymbol{x}\cdot\hat{\boldsymbol{k}}_{(\theta_q,\phi_q)}}\overline{\boldsymbol{v}(\boldsymbol{x})}\,d\boldsymbol{x}$$

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Output functional (RCS)

Recall: an important object of interest for scattering is the RCS:

$$\mathrm{s}_\infty(\boldsymbol{u}, \hat{\boldsymbol{d}}) = \frac{ikZ}{4\pi} \int_\Gamma \hat{\boldsymbol{d}} \times (\boldsymbol{u}(\boldsymbol{x}) \times \hat{\boldsymbol{d}}) e^{-ik\boldsymbol{x}\cdot\hat{\boldsymbol{d}}} d\boldsymbol{x}$$

$$\mathrm{rcs}(\boldsymbol{u}, \hat{\boldsymbol{d}}) = 10 \log_{10} \left(4\pi \frac{|\mathrm{s}_\infty(\boldsymbol{u}, \hat{\boldsymbol{d}})|^2}{|E^i|^2} \right)$$

Rigorous computable error bounds for the output functional can be developed:

Theorem: The error of the functionals are bounded by

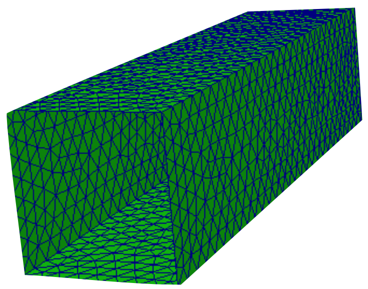
$$|\mathrm{s}_\infty(\boldsymbol{u}_\delta(\boldsymbol{\mu}), \hat{\boldsymbol{d}}) - \mathrm{s}_\infty(\boldsymbol{u}_N(\boldsymbol{\mu}), \hat{\boldsymbol{d}})| \le \boldsymbol{\varepsilon}_{\mathrm{s}} = \frac{kZ\sqrt{|\Gamma|}}{4\pi} \eta_N(\boldsymbol{\mu}),$$

$$\begin{aligned} &|\mathrm{rcs}(\boldsymbol{u}_\delta(\boldsymbol{\mu}), \hat{\boldsymbol{d}}) - \mathrm{rcs}(\boldsymbol{u}_N(\boldsymbol{\mu}), \hat{\boldsymbol{d}})| \\ &\qquad \le 20 \max \left(\log_{10} \left(\frac{|\mathrm{s}_\infty(\boldsymbol{u}_N(\boldsymbol{\mu}), \hat{\boldsymbol{d}})| + \boldsymbol{\varepsilon}_{\mathrm{s}}}{|\mathrm{s}_\infty(\boldsymbol{u}_N(\boldsymbol{\mu}), \hat{\boldsymbol{d}})|} \right), \log_{10} \left(\frac{|\mathrm{s}_\infty(\boldsymbol{u}_N(\boldsymbol{\mu}), \hat{\boldsymbol{d}})|}{|\mathrm{s}_\infty(\boldsymbol{u}_N(\boldsymbol{\mu}), \hat{\boldsymbol{d}})| - \boldsymbol{\varepsilon}_{\mathrm{s}}} \right) \right). \end{aligned}$$

Application to scattering problems

Parameter space: $k \in [10, 20]$, $\theta = \frac{\pi}{2}$, $\phi = 0$.

Scatterer:



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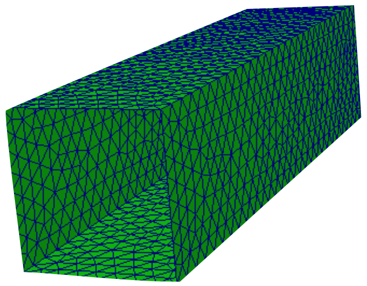
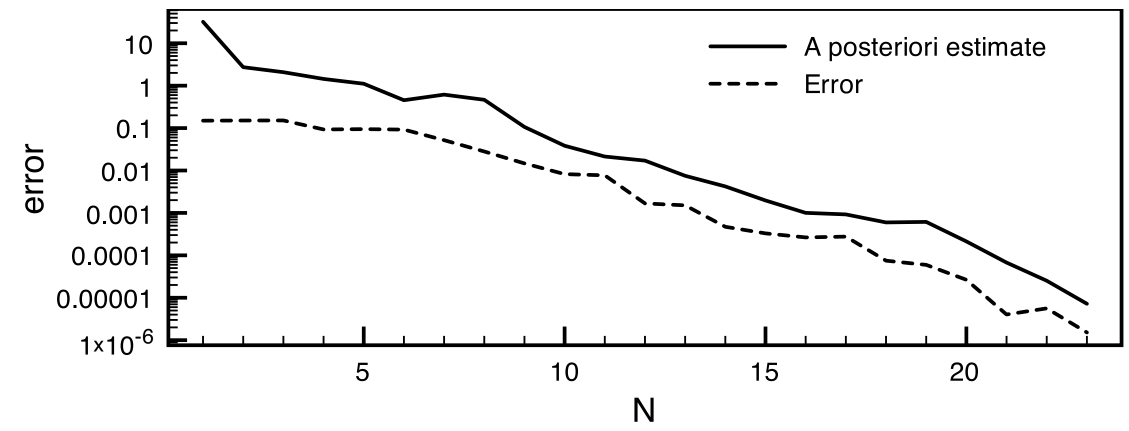


Convergence of greedy algorithm (Offline)



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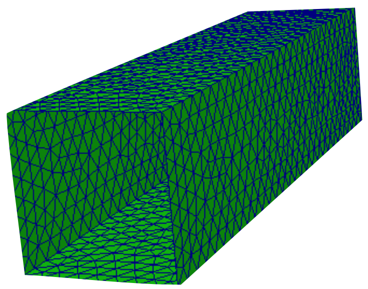

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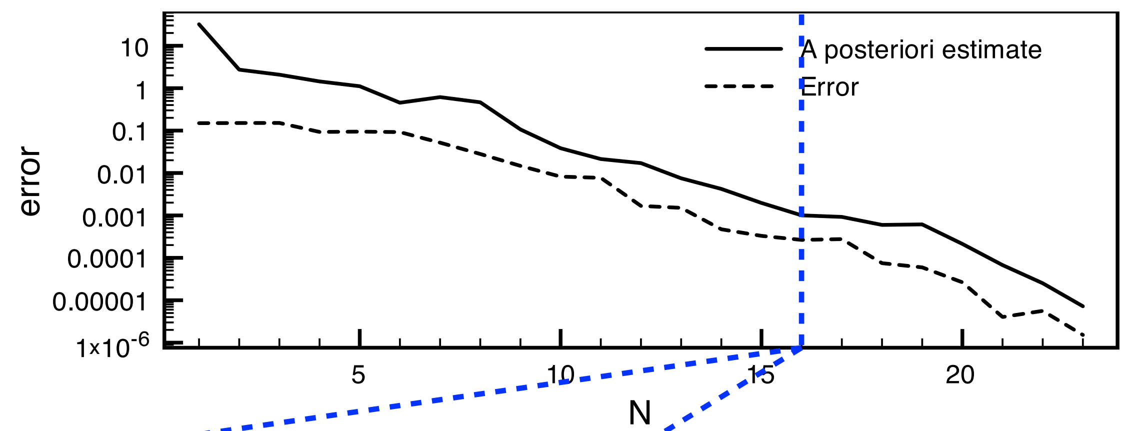
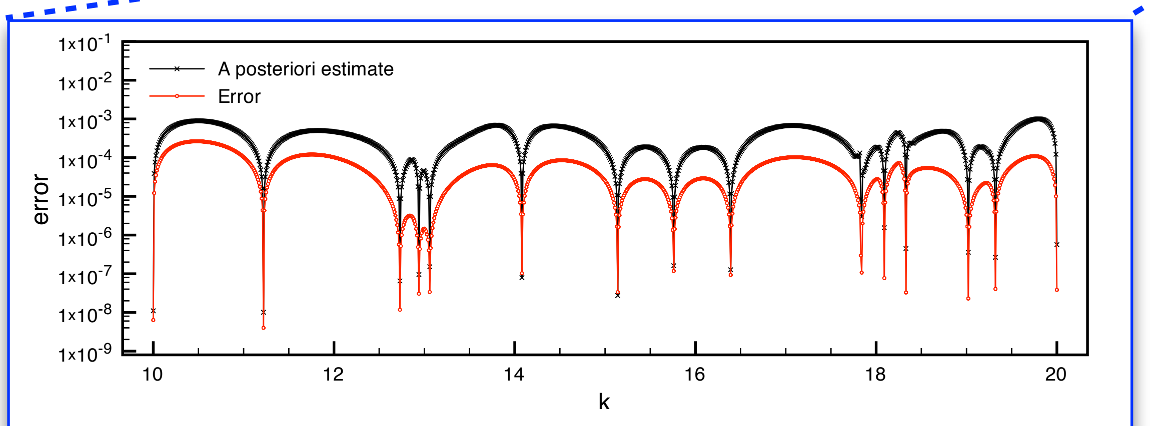


Error-profile:

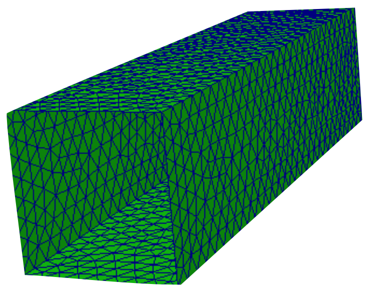


N=16

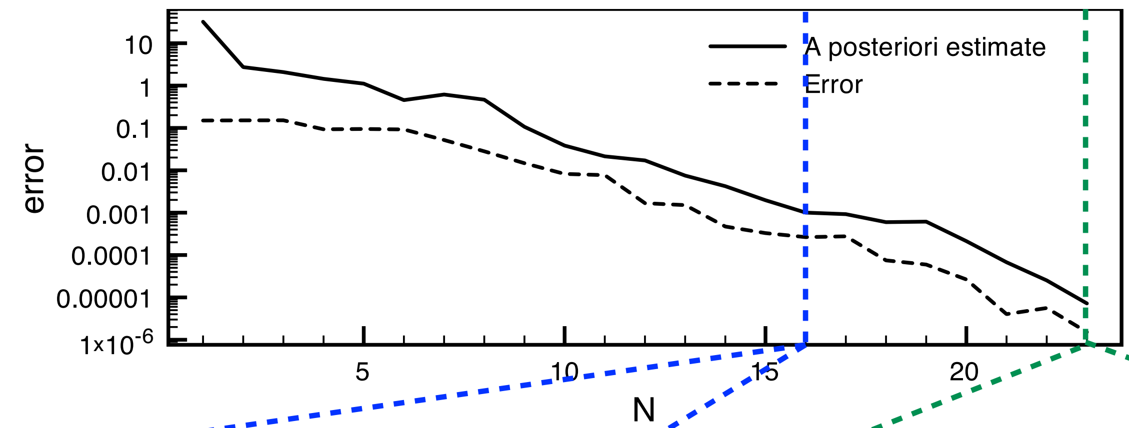
Application to scattering problems

Parameter space: $k \in [10, 20]$, $\theta = \frac{\pi}{2}$, $\phi = 0$.

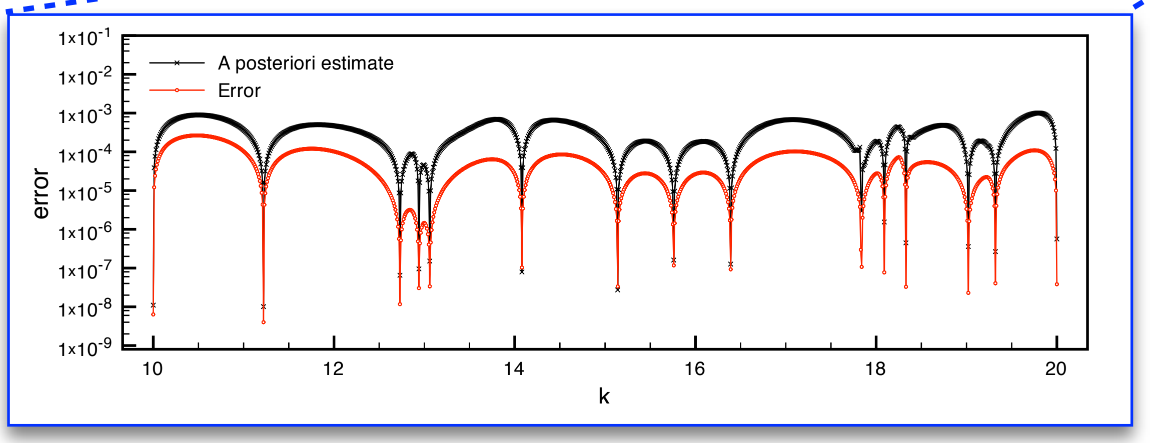
Scatterer:



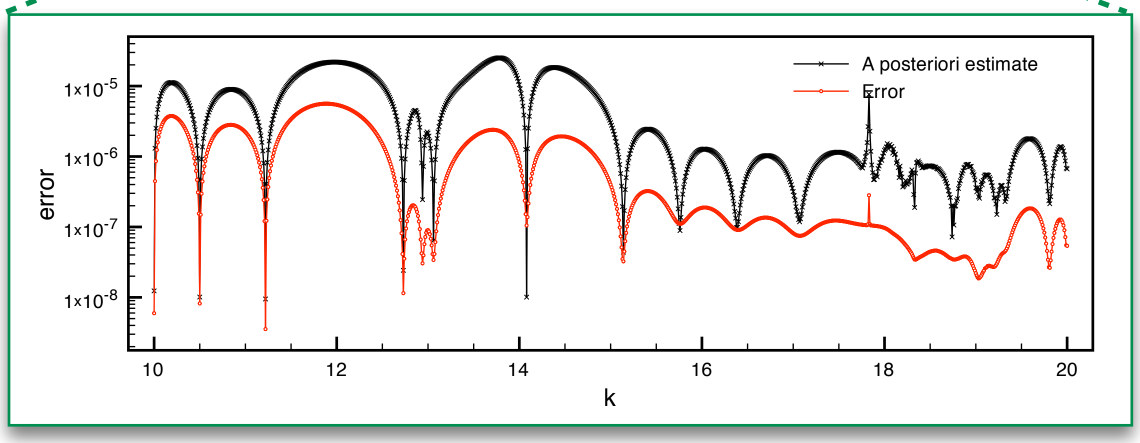
Convergence of greedy algorithm (Offline)



Error-profile:



N=16

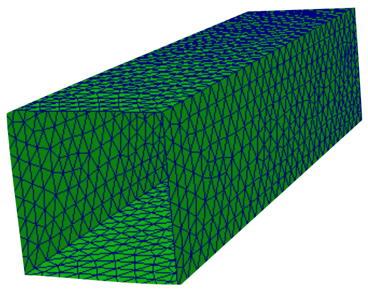


N=23

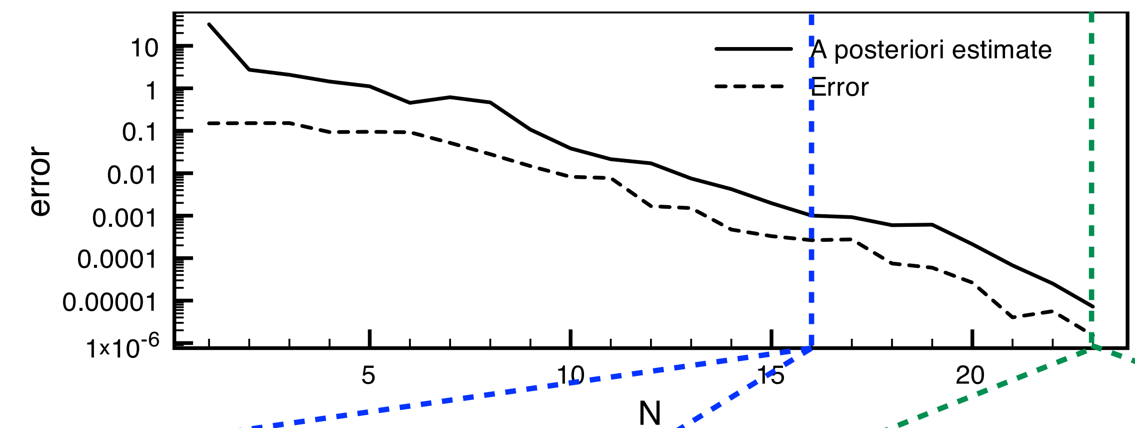
Application to scattering problems

Parameter space: $k \in [10, 20]$, $\theta = \frac{\pi}{2}$, $\phi = 0$.

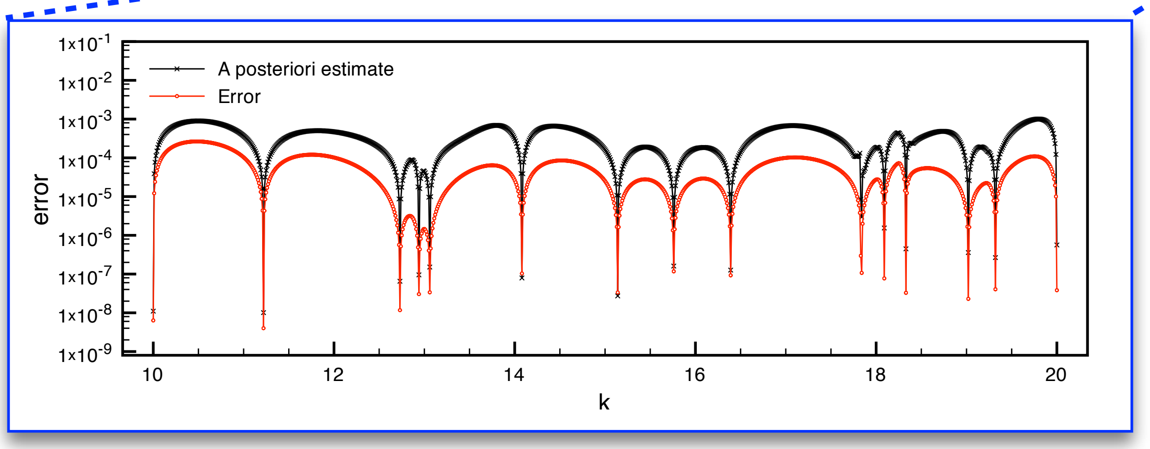
Scatterer:



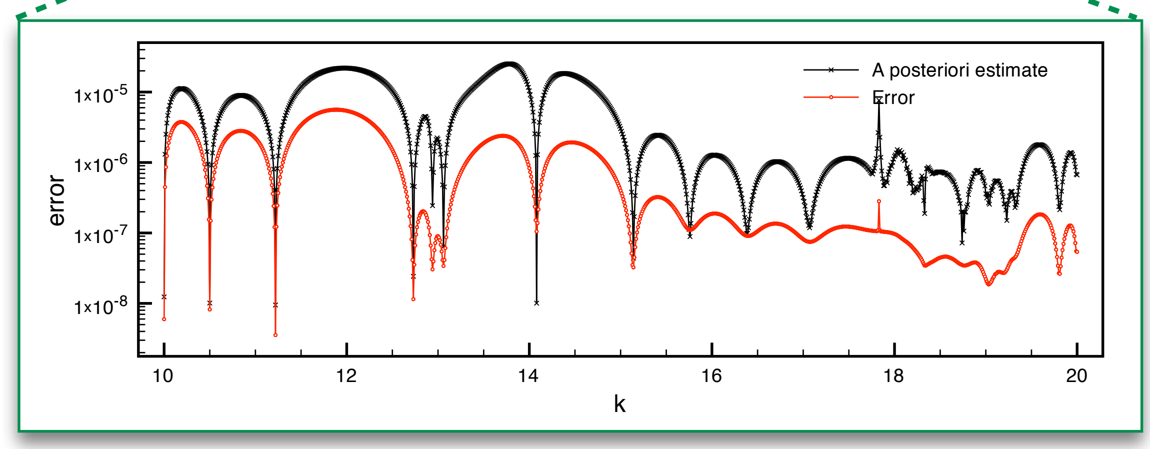
Convergence of greedy algorithm (Offline)



Error-profile:



N=16

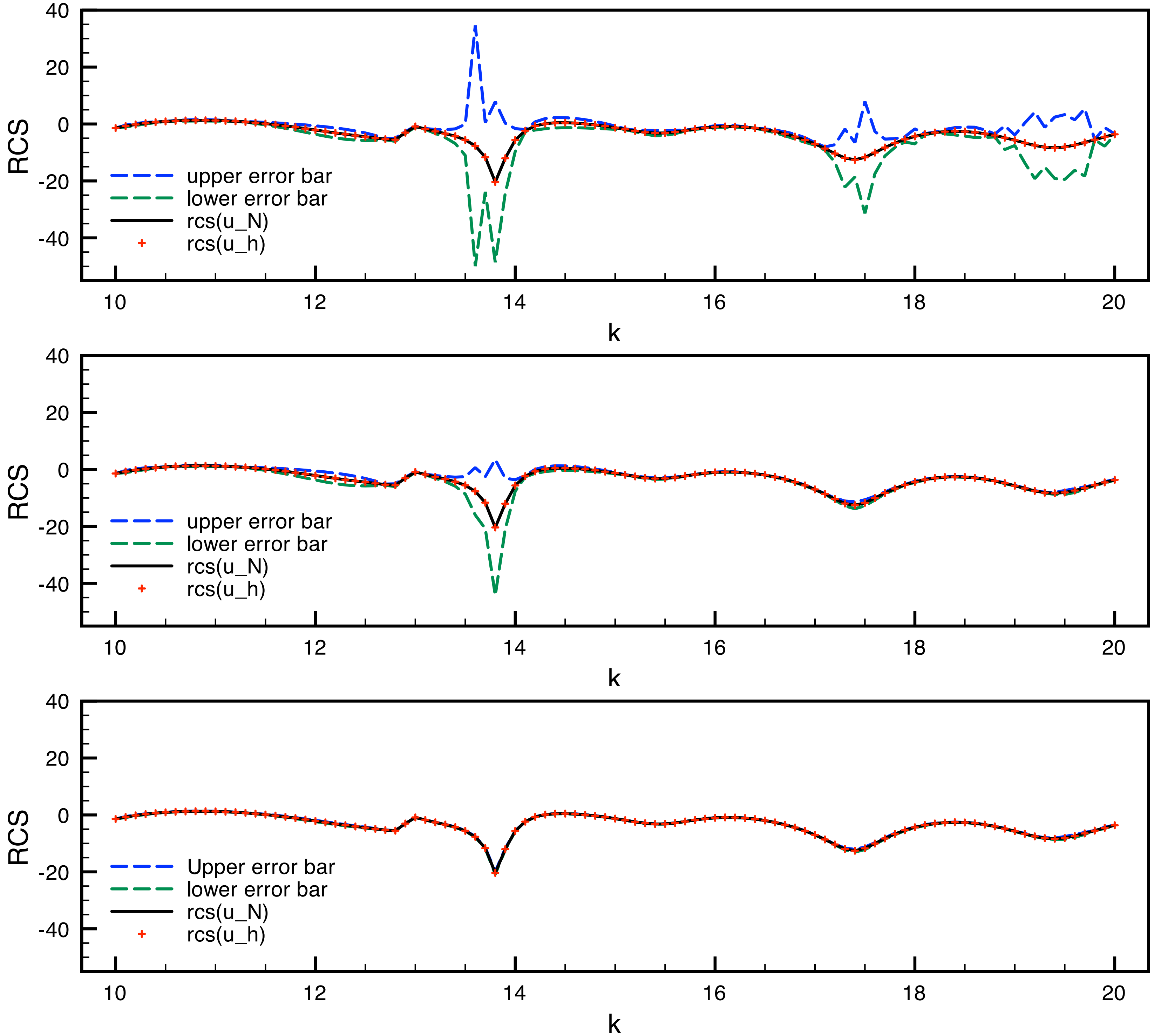


N=23

- Error estimate is a strict upper bound of error as can be shown in theory
- Typical error-profiles for greedy algorithm

Output functional

Monostatic RCS (backscattering) for different wave-numbers:



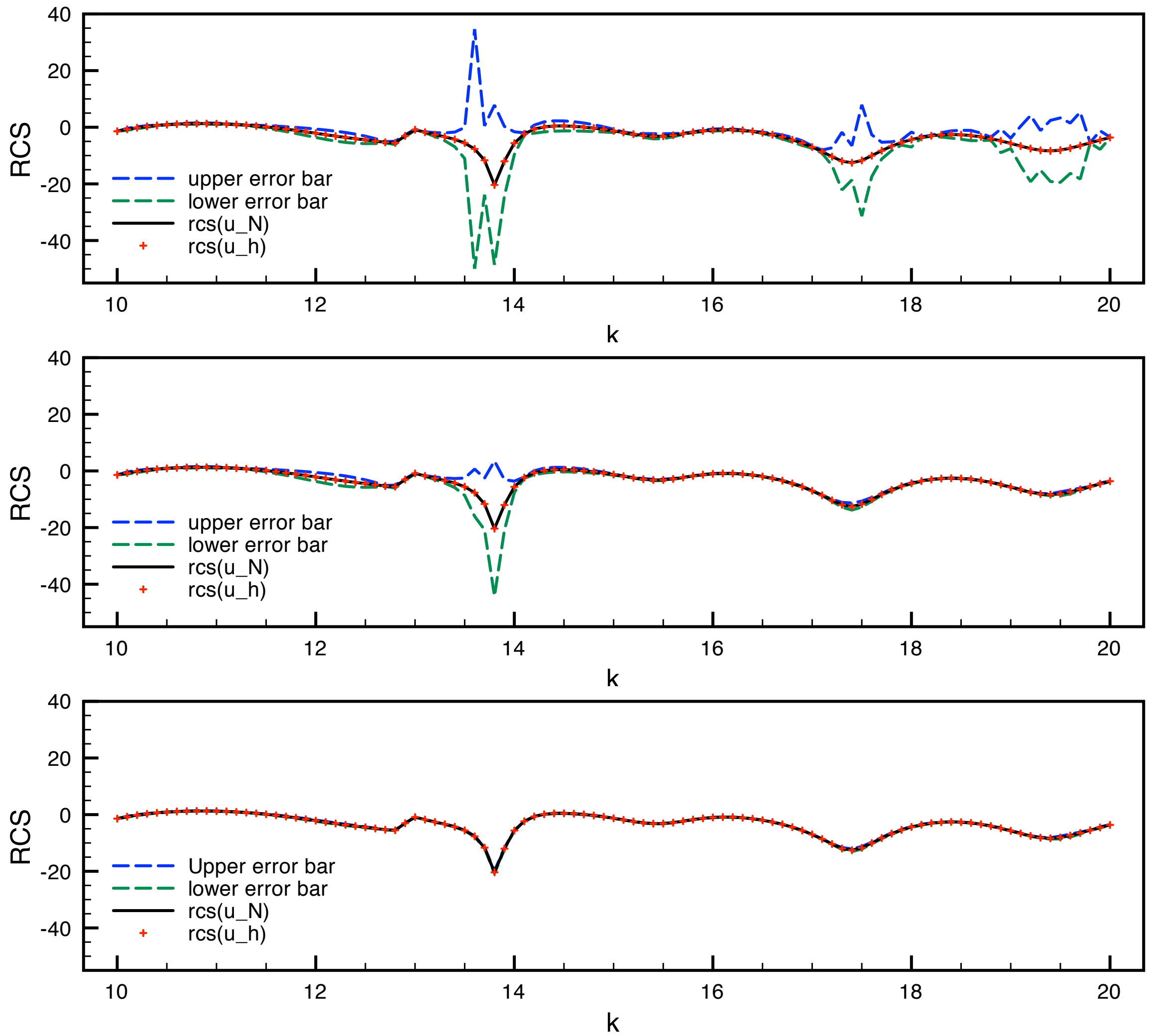
N=21

N=22

N=23

Output functional

Monostatic RCS (backscattering) for different wave-numbers:



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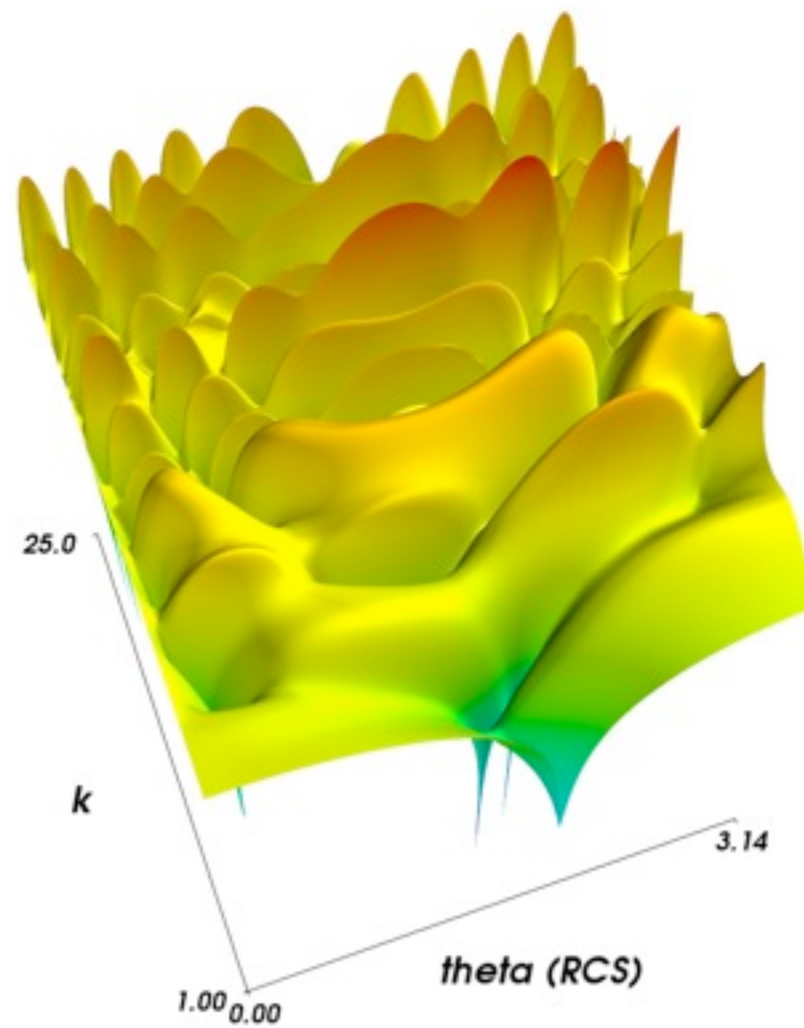
N=23



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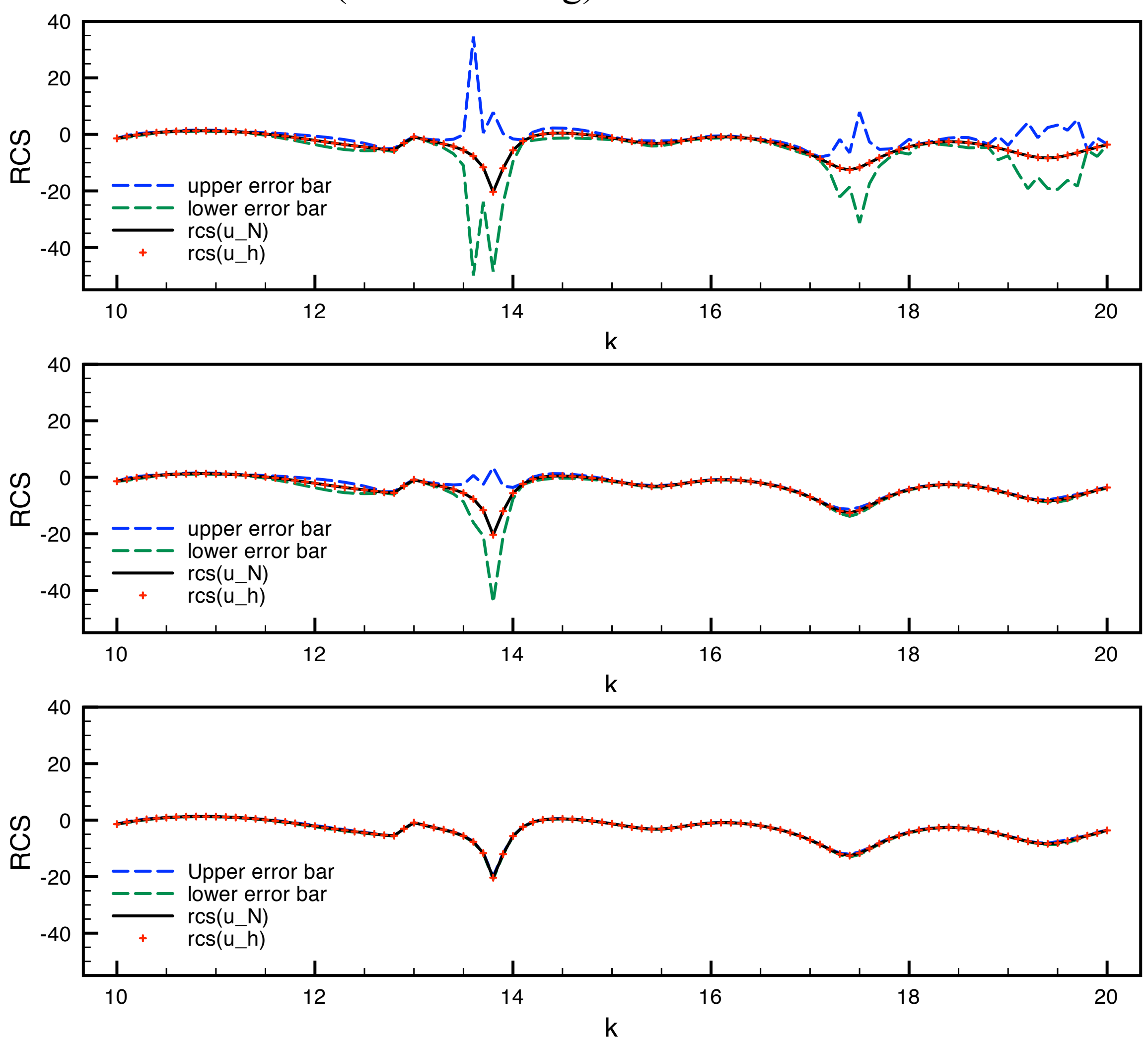

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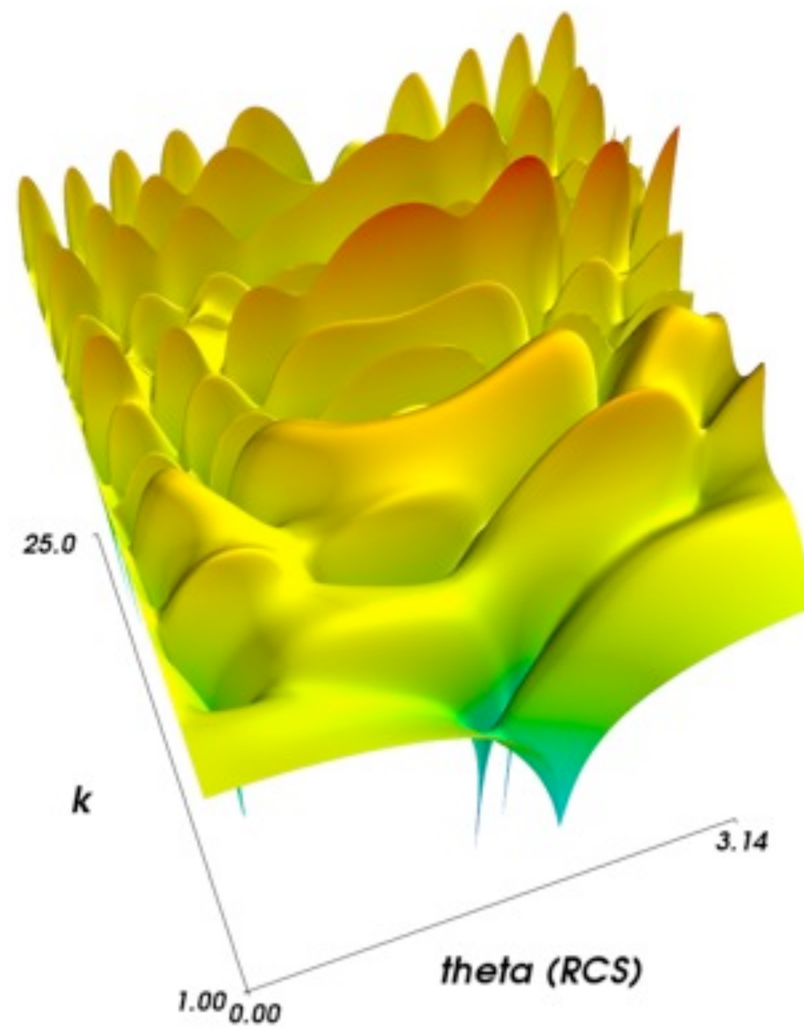

Fast and **reliable** many-query computations with **certified** error control over model reduction!

Multi obstacle scattering

Truth solver (BEM)

Galerkin approach: we replace the functional space $\mathbb{V} = \boldsymbol{H}_{div}^{-\frac{1}{2}}(\Gamma)$ by a finite dimensional subspace $\mathbb{V}_h = \mathbf{RT}_0$.

For any fixed $\boldsymbol{\mu} \in \mathbb{P}$, find $\boldsymbol{u}_h(\boldsymbol{\mu}) \in \mathbb{V}_h$ s.t.

$$a[\boldsymbol{u}_h(\boldsymbol{\mu}), \boldsymbol{v}_h; \boldsymbol{\mu}] = f[\boldsymbol{v}_h; \boldsymbol{\mu}], \qquad \forall \boldsymbol{v}_h \in \mathbb{V}_h$$

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Embed the structure of the $\mathtt{J}$ elements:

$$\mathbb{V}_h = \oplus_{\mathtt{i}=1}^{\mathtt{J}} \mathbb{V}_h(\Gamma_{\mathtt{i}})$$

$$a[\cdot, \cdot; \boldsymbol{\mu}] = \sum_{\mathtt{i},\mathtt{j}=1}^{\mathtt{J}} a^{\mathtt{ij}}[\cdot, \cdot; \boldsymbol{\mu}]$$

where

$V_h(\Gamma_{\mathtt{i}})$: is the Boundary Element space on the surface $\Gamma_{\mathtt{i}}$

$$a^{\mathtt{ij}}[\cdot, \cdot; \boldsymbol{\mu}] = a[\cdot, \cdot; \boldsymbol{\mu}]|_{V_h(\Gamma_{\mathtt{i}}) \times V_h(\Gamma_{\mathtt{j}})}$$

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Integral equation/BEM:
Double integral
$\Rightarrow$ double sum!

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Generalized Born series

In matrix form:

$$\begin{bmatrix} \mathrm{M}^{11} & \ldots & \mathrm{M}^{1\mathtt{J}} \\ \mathrm{M}^{21} & \ldots & \mathrm{M}^{2\mathtt{J}} \\ \vdots & & \vdots \\ \mathrm{M}^{\mathtt{J}1} & \ldots & \mathrm{M}^{\mathtt{J}\mathtt{J}} \end{bmatrix} \begin{bmatrix} \mathrm{u}^1 \\ \mathrm{u}^2 \\ \vdots \\ \mathrm{u}^{\mathtt{J}} \end{bmatrix} = \begin{bmatrix} \mathrm{f}^1 \\ \mathrm{f}^2 \\ \vdots \\ \mathrm{f}^{\mathtt{J}} \end{bmatrix}$$

where $\mathrm{M}^{\mathtt{ij}}$ corresponds to the sesequilinear form $a^{\mathtt{ij}}[\cdot, \cdot; \boldsymbol{\mu}]$.

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Then, the solution $\mathrm{u}^{\mathtt{j}}$ is represented in series as

$$\mathrm{u}^{\mathtt{j}} = \sum_{\mathtt{k}=1}^{\infty} \mathrm{u}^{\mathtt{j}}_{\mathtt{k}}$$

where $\mathrm{u}^{\mathtt{j}}_{\mathtt{k}}$ solves

$$\begin{aligned} &\mathrm{M}^{\mathtt{ii}}\mathrm{u}^{\mathtt{i}}_{1} = \mathrm{f}^{\mathtt{i}}, \\ &\mathrm{M}^{\mathtt{ii}}\mathrm{u}^{\mathtt{i}}_{\mathtt{k}} = -\sum_{\mathtt{i}\neq\mathtt{j}} \mathrm{M}^{\mathtt{ij}}\mathrm{u}^{\mathtt{j}}_{\mathtt{k}-1}, \quad \mathtt{k} > 1. \end{aligned}$$

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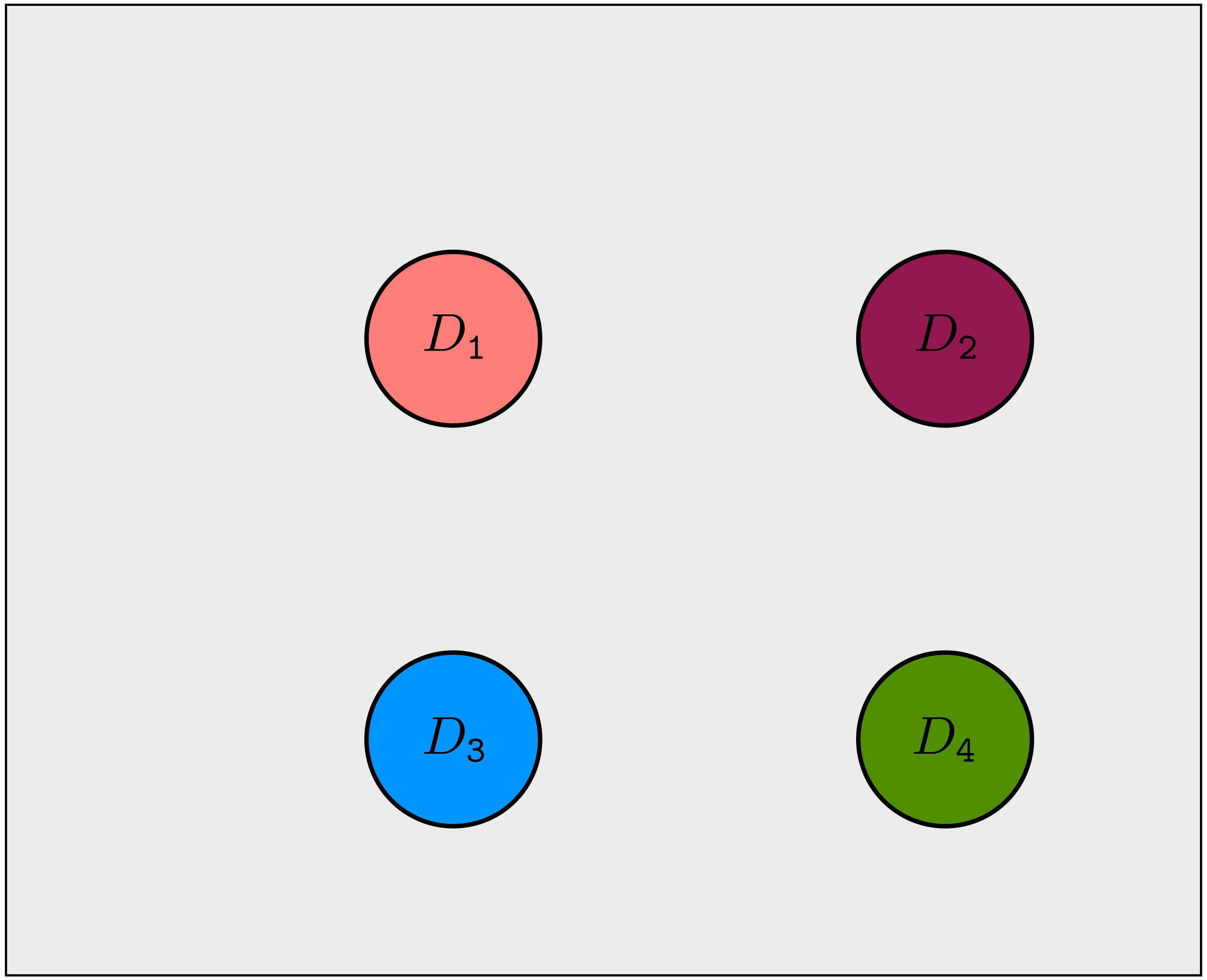
Easy implementation in parallel.
One LU-factorization per obstacle.

see book of [P.A. Martin]

Generalized Born Series - Idea



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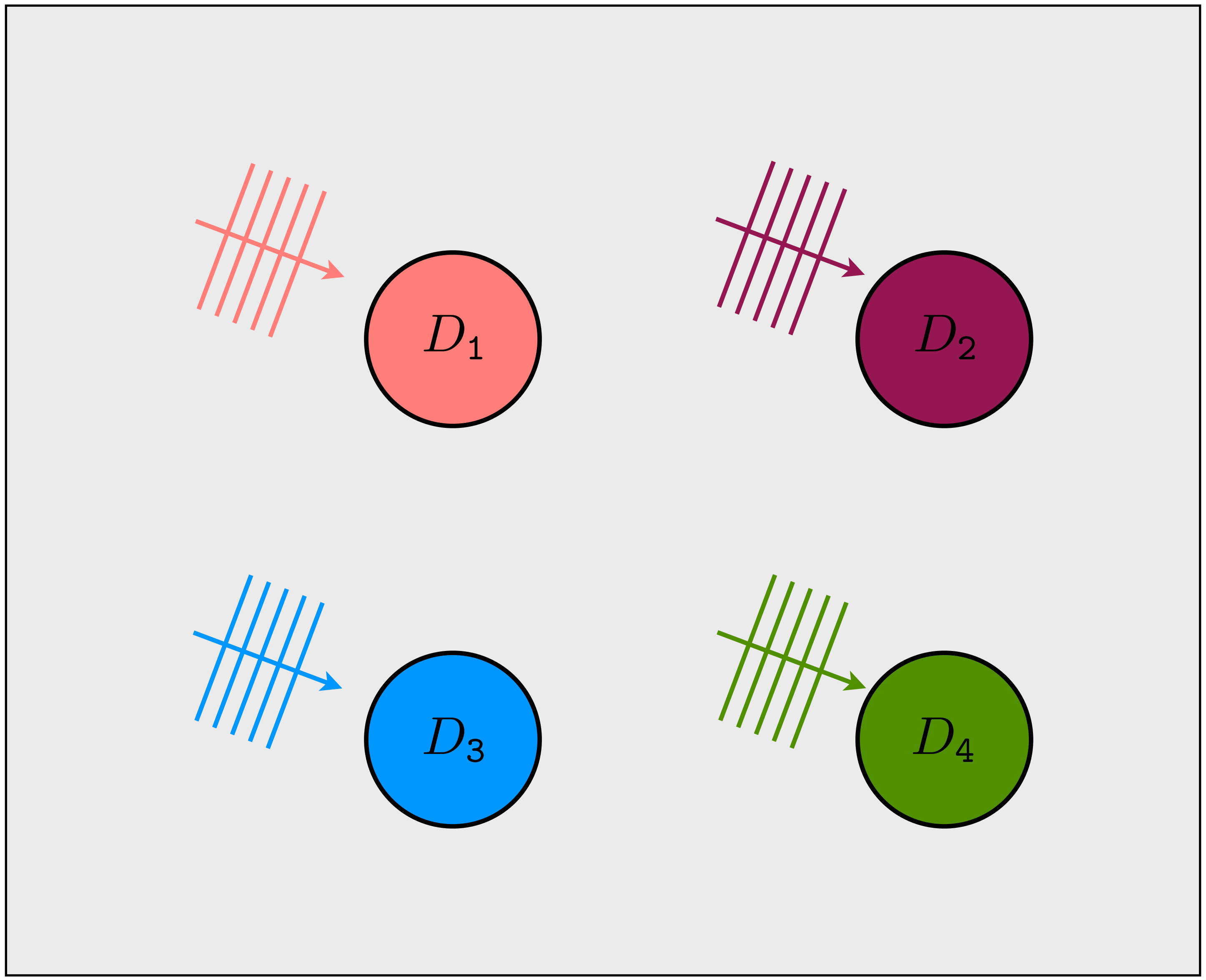
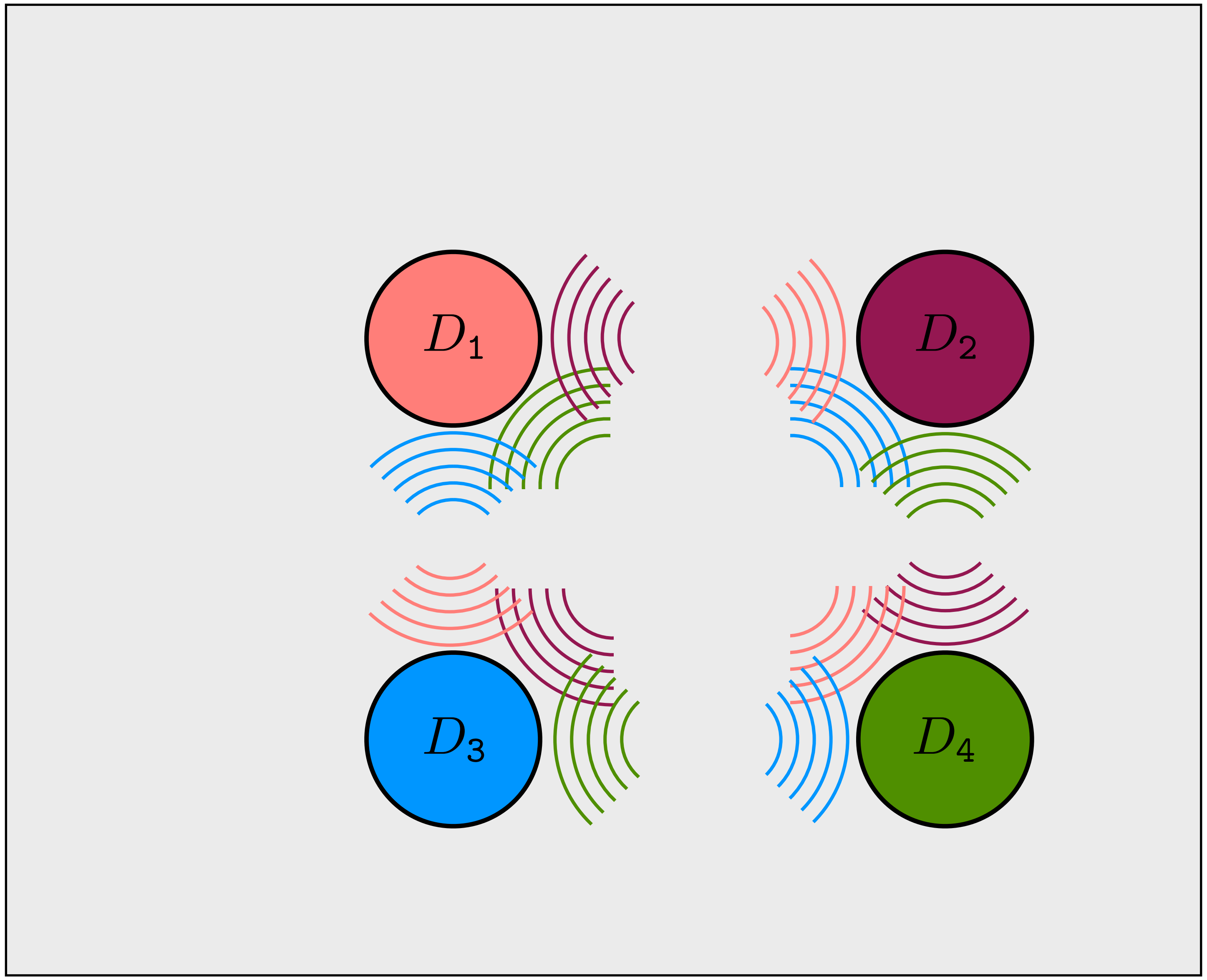


4 independent problems

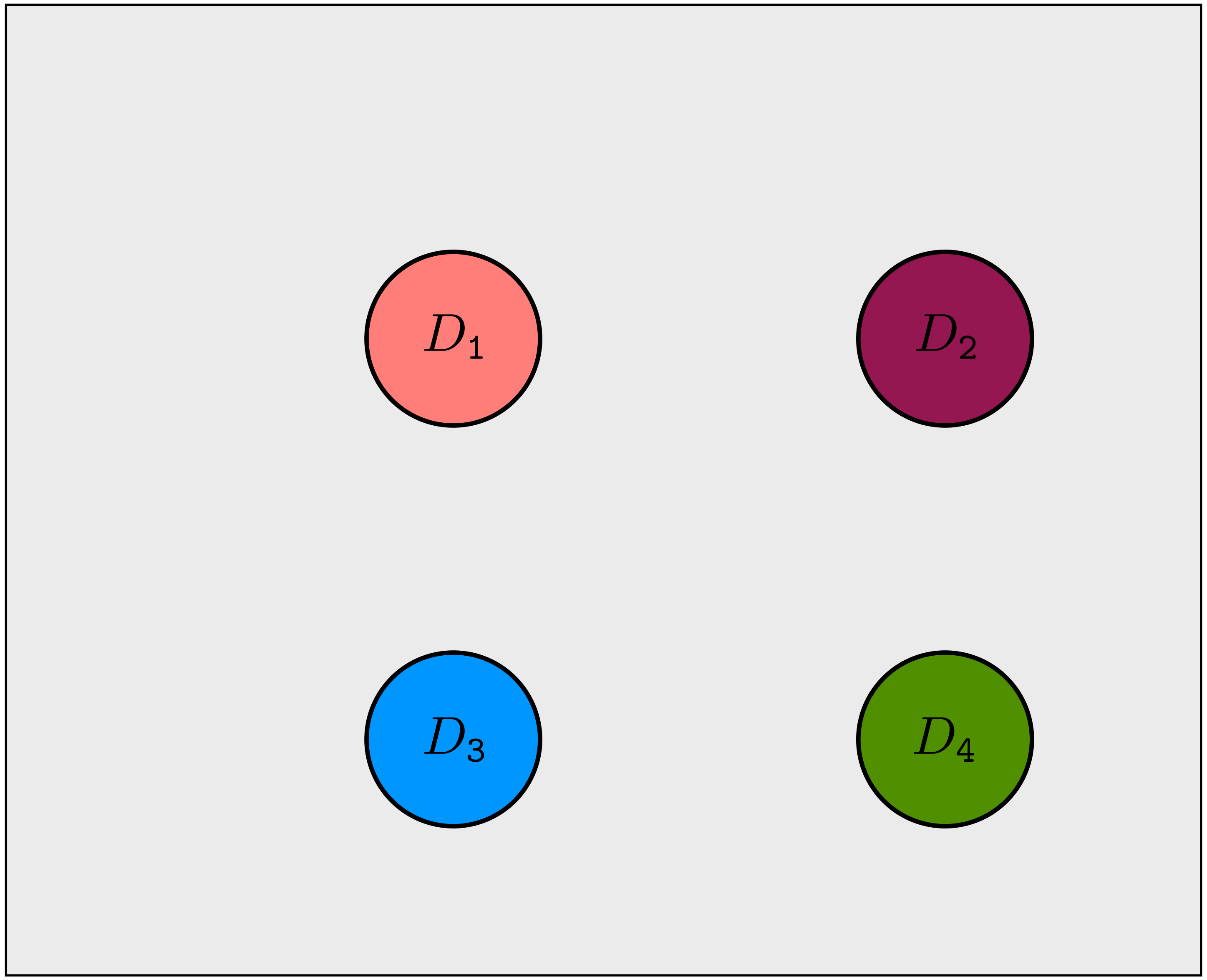
Generalized Born Series - Idea



4 independent problems

Interaction of reflected waves

Generalized Born Series - Idea

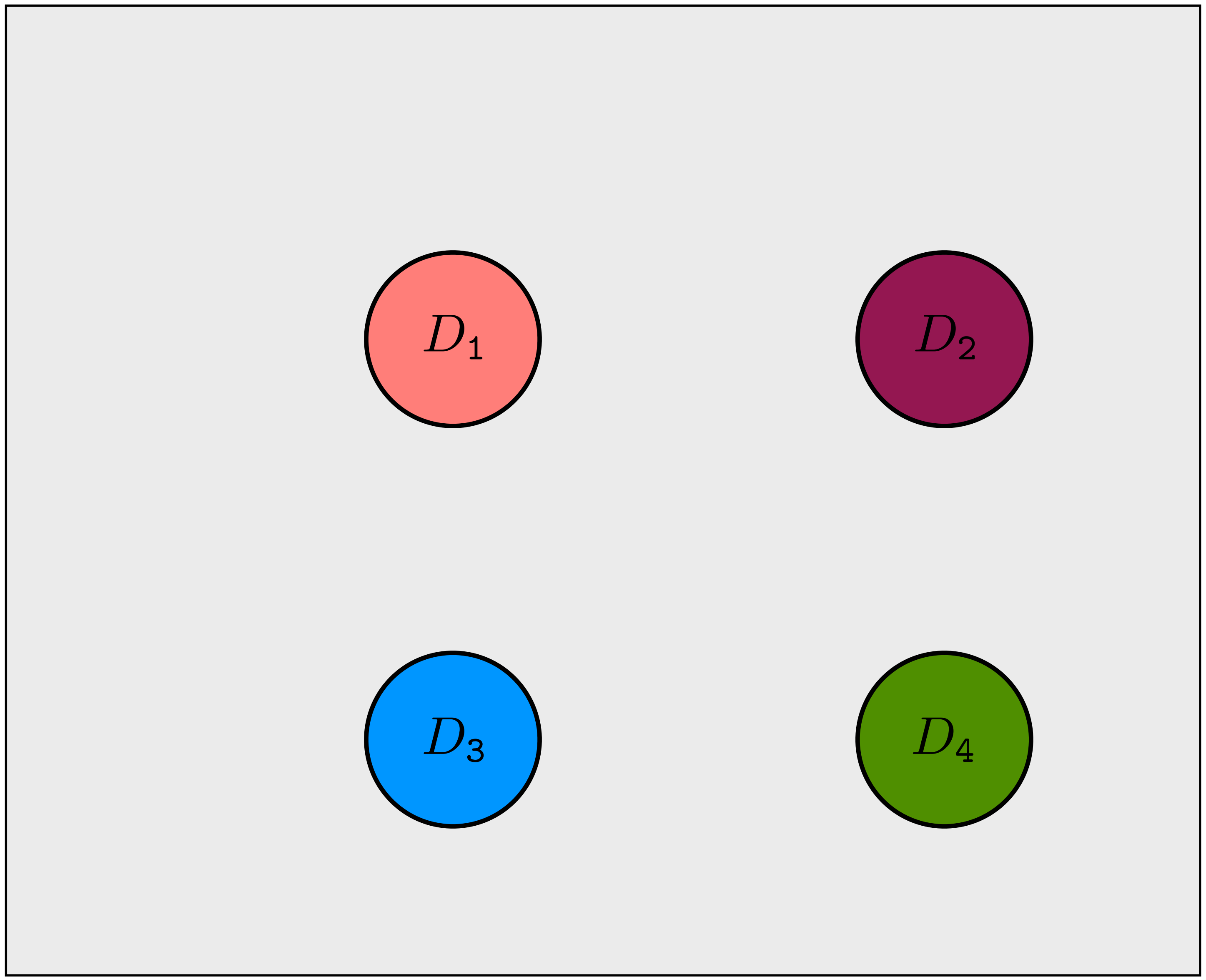


4 independent problems

Interaction of reflected waves

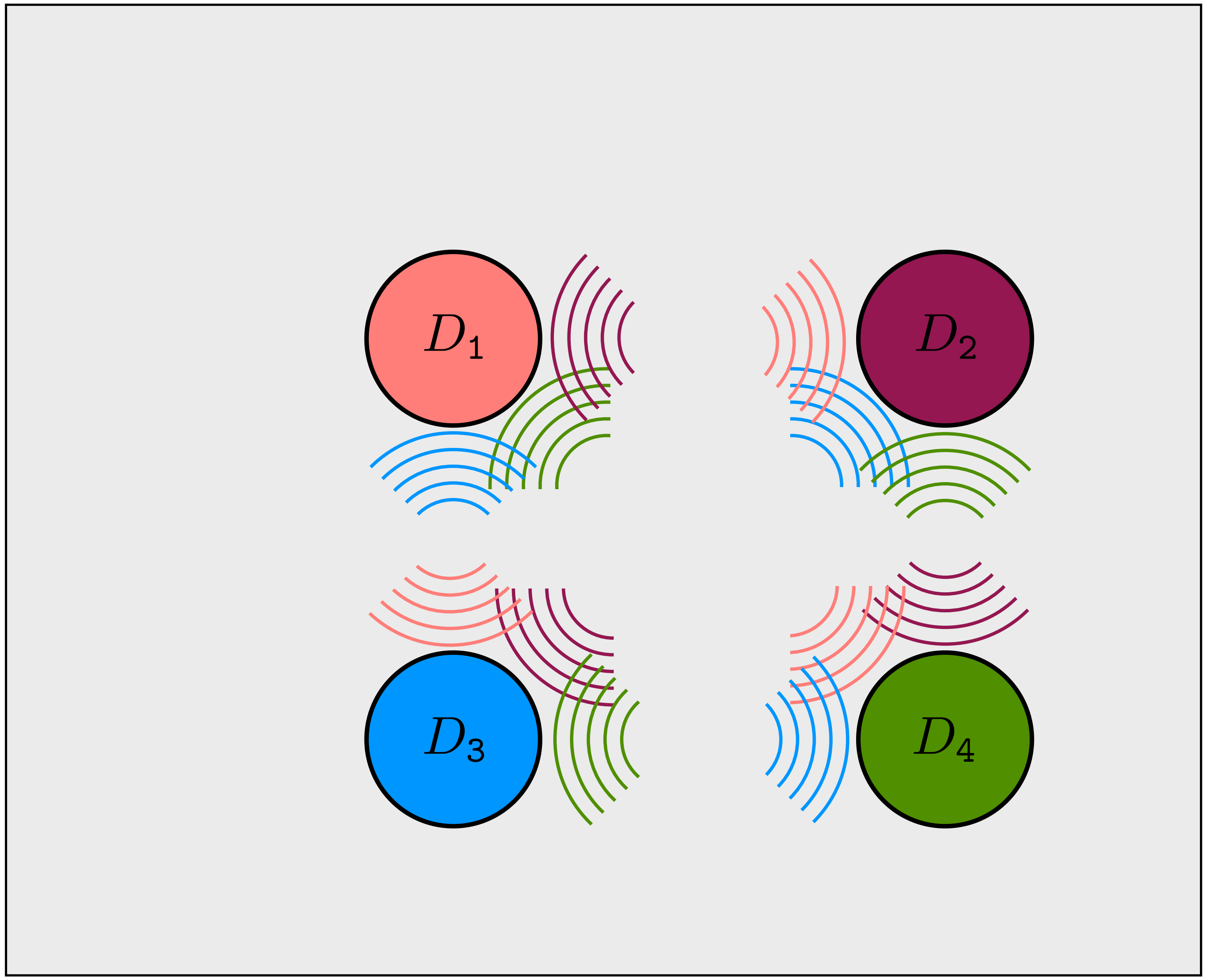
Updating

Generalized Born Series - Idea



4 independent problems

Generalized Born Series - Idea



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Interaction of reflected waves

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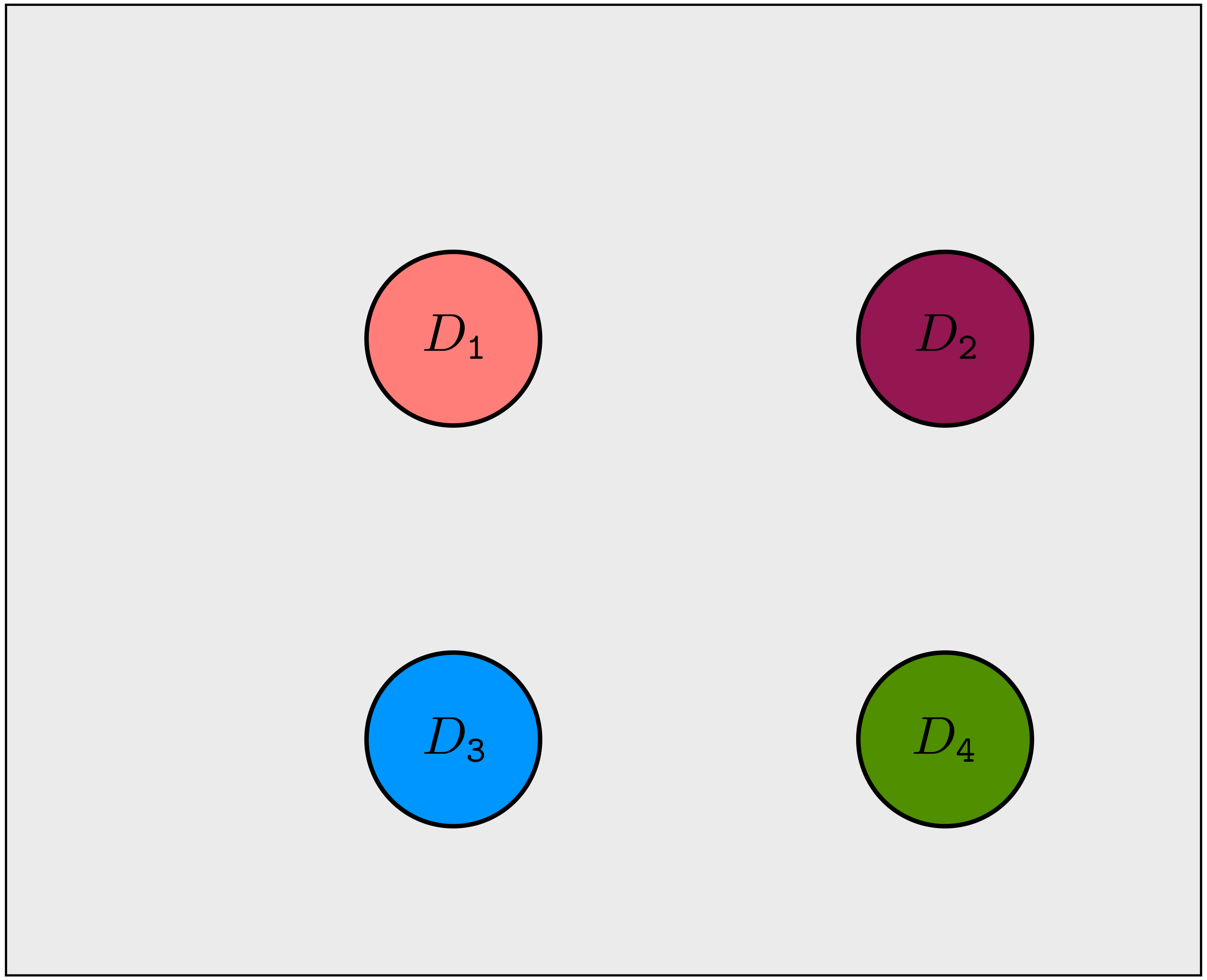

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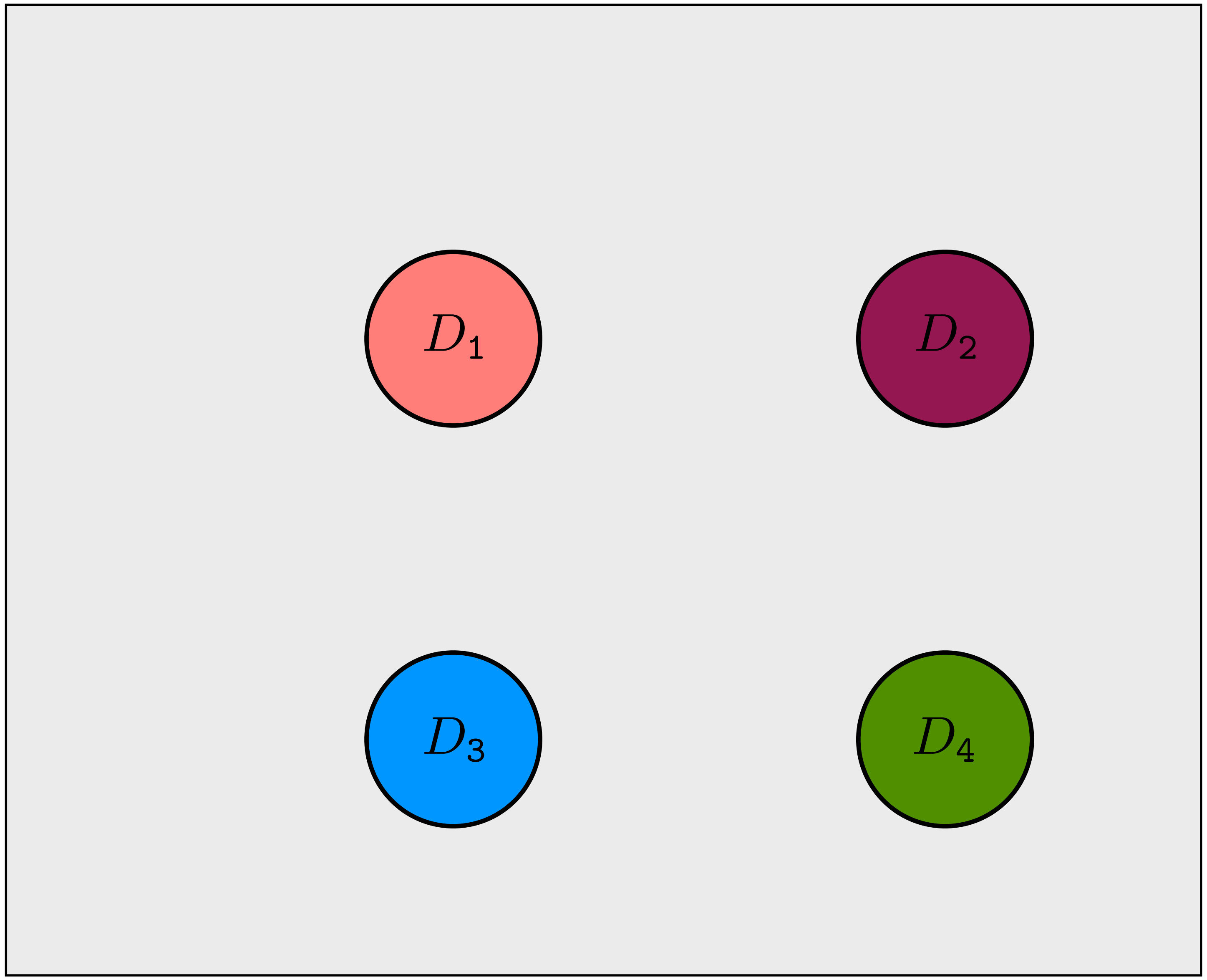

4 independent problems

Interaction of reflected waves

Updating

etc

Combination of model reduction and Generalized Born Series

General idea

Offline procedure:

1. Take a reference shape: Assemble a reduced basis $\mathbb{V}_N$ that represents accurately all solutions for $k \in [k^-, k^+]$, all possible angles and polarizations for the incident plane wave.
   - $\Rightarrow$ 5 parameters only.
   - $\Rightarrow$ The (certified) reduced basis space $\mathbb{V}_N$ can represent any solution on a single scatterer for any incident plane wave accurately.
   - $\Rightarrow$ Details of this step: first part of this talk.
2. Copy this reduced basis on all objects $D_{\mathtt{i}}$ and use it as approximation spaces.

Online procedure:

3. Solve the coupled problem iteratively ”à la Generalized Born series”, but with the reduced basis space as solution space on each obstacle.

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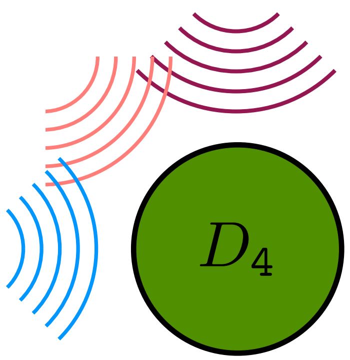
Idea: During each iteration, the reflected wave impinging on $D_{\mathtt{i}}$ can be approximated by a linear combination of plane waves. The reduced basis on $D_{\mathtt{i}}$ is trained to be accurate for such cases.



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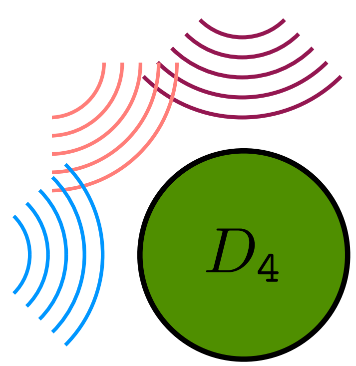
Limitations: Close objects! $\Longrightarrow$ Dipole-like interaction



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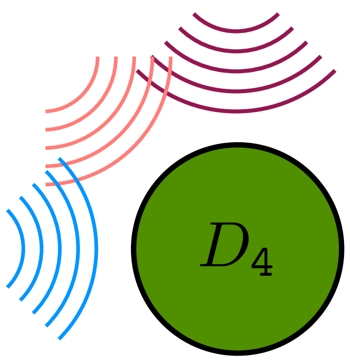
Remaining discussion:

1. Proper formulation of online part.
2. Efficient implementation (indep. of $\mathcal{N} = \dim(\mathbb{V}_h)$).
3. Numerical results.



Integration over reference shape

Goal: State sesquilinear form as integrals over the reference shapes (parameter indep.).

Here:

$$\mathrm{G}^{\mathtt{ij}}_{\boldsymbol{\mu}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}}) = \frac{e^{ik|T_{\mathtt{i}}(\hat{\boldsymbol{x}})-T_{\mathtt{j}}(\hat{\boldsymbol{y}})|}}{4\pi|T_{\mathtt{i}}(\hat{\boldsymbol{x}})-T_{\mathtt{j}}(\hat{\boldsymbol{y}})|},$$

$$|T_{\mathtt{i}}(\hat{\boldsymbol{x}})-T_{\mathtt{j}}(\hat{\boldsymbol{y}})| = \gamma_{\mathtt{i}}\left|\hat{\boldsymbol{x}} - \frac{\gamma_{\mathtt{j}}}{\gamma_{\mathtt{i}}}\mathrm{B}_{\mathtt{i}}^T\mathrm{B}_{\mathtt{j}}\hat{\boldsymbol{y}} + \frac{1}{\gamma_{\mathtt{i}}}\mathrm{B}_{\mathtt{i}}^T(\mathrm{b}_{\mathtt{i}}-\mathrm{b}_{\mathtt{j}})\right| = \gamma_{\mathtt{i}}|\hat{\boldsymbol{x}} - \gamma_{\mathtt{ij}}\mathrm{B}_{\mathtt{ij}}\hat{\boldsymbol{y}} + \mathrm{c}_{\mathtt{ij}}|$$

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Given the affine transformation $T_{\mathtt{i}}(\hat{\boldsymbol{x}}) = \gamma_{\mathtt{i}} \mathrm{B}_{\mathtt{i}} \hat{\boldsymbol{x}} + \mathrm{b}_{\mathtt{i}}$, write

$$
\begin{aligned}
& a^{\mathtt{ij}}[\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{\mu}] \\
&= ikZ \int_{\Gamma_{\mathtt{i}}(\boldsymbol{\mu})} \int_{\Gamma_{\mathtt{j}}(\boldsymbol{\mu})} \mathrm{G}_{k}(\boldsymbol{x}, \boldsymbol{y}) \left\{ \boldsymbol{u}(\boldsymbol{y}) \cdot \overline{\boldsymbol{v}(\boldsymbol{x})} - \tfrac{1}{k^2} \mathrm{div}_{\boldsymbol{y}} \boldsymbol{u}(\boldsymbol{y}) \overline{\mathrm{div}_{\boldsymbol{x}} \boldsymbol{v}(\boldsymbol{x})} \right\} d\boldsymbol{y}\, d\boldsymbol{x} \\
&= ikZ \gamma_{\mathtt{i}} \gamma_{\mathtt{j}} \int_{\hat{\Gamma}} \int_{\hat{\Gamma}} \mathrm{G}^{\mathtt{ij}}_{\boldsymbol{\mu}}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \left\{ \mathrm{B}_{\mathtt{j}} \hat{\boldsymbol{u}}(\hat{\boldsymbol{y}}) \cdot \mathrm{B}_{\mathtt{i}} \overline{\hat{\boldsymbol{v}}(\hat{\boldsymbol{x}})} - \tfrac{1}{k^2 \gamma_{\mathtt{i}} \gamma_{\mathtt{j}}} \mathrm{div}_{\hat{\boldsymbol{y}}} \hat{\boldsymbol{u}}(\hat{\boldsymbol{y}}) \overline{\mathrm{div}_{\hat{\boldsymbol{x}}} \hat{\boldsymbol{v}}(\hat{\boldsymbol{x}})} \right\} d\boldsymbol{y}\, d\boldsymbol{x} \\
&=: \hat{a}^{\mathtt{ij}}[\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}; \boldsymbol{\mu}]
\end{aligned}
$$

where $\hat{\boldsymbol{u}} = \hat{\mathcal{P}}(\boldsymbol{u})$ and $\hat{\boldsymbol{v}} = \hat{\mathcal{P}}(\boldsymbol{v})$ (Piola transformation).

Here:

$$
\mathrm{G}^{\mathtt{ij}}_{\boldsymbol{\mu}}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) = \frac{e^{ik|T_{\mathtt{i}}(\hat{\boldsymbol{x}}) - T_{\mathtt{j}}(\hat{\boldsymbol{y}})|}}{4\pi |T_{\mathtt{i}}(\hat{\boldsymbol{x}}) - T_{\mathtt{j}}(\hat{\boldsymbol{y}})|},
$$

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$$

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Formulation of Online part

Having previously assembled a reduced basis $\mathbb{V}_N$ for the reference shape, we restrict the solution space to $\mathbb{V}_N$.

$\rightarrow$ $\mathbb{V}_N$ represents accurately all solutions on the reference shape corresponding to all wave numbers $k \in [k^-, k^+]$ and incident plane wave of all angles and polarizations.

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Thus, assemble the matrices

$$\begin{aligned} \mathrm{A}^{\mathtt{ij}}_{\boldsymbol{\mu}} &= \hat{a}^{\mathtt{ij}}[\cdot,\cdot;\boldsymbol{\mu}]|_{\mathbb{V}_N\times\mathbb{V}_N}, \\ \mathrm{f}^{\mathtt{j}}_{\boldsymbol{\mu}} &= \hat{f}^{\mathtt{j}}[\cdot;\boldsymbol{\mu}]|_{\mathbb{V}_N}. \end{aligned}$$

Then, solve

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Remaining task: Affine decomposition of forms.

Efficient implementation - Empirical Interpolation Method (EIM)

Define the functions on which we apply EIM:

$$\mathcal{G}[\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}; \gamma_{\mathtt{i}} k, \gamma, \mathrm{B}, \mathrm{c}] = \frac{e^{i\gamma_{\mathtt{i}} k|\hat{\boldsymbol{x}} - \gamma \mathrm{B} \hat{\boldsymbol{y}} + \mathrm{c}|}}{4\pi|\hat{\boldsymbol{x}} - \gamma \mathrm{B} \hat{\boldsymbol{y}} + \mathrm{c}|}, \qquad \mathtt{i} \neq \mathtt{j}$$

$$\mathcal{G}_0[r; \gamma_{\mathtt{i}} k] = \frac{e^{i\gamma_{\mathtt{i}} kr}}{4\pi r}, \qquad r = |\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}|, \mathtt{i} = \mathtt{j},$$

with $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}} \in \hat{\Gamma}$ and $\gamma_{\mathtt{i}} k, \gamma \in \mathbb{R}$, $\mathrm{c} \in \mathbb{R}^3$, $\mathrm{B} \in SO(3)$.

Efficient implementation - Empirical Interpolation Method (EIM)

Define the functions on which we apply EIM:

$$\mathcal{G}[\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}; \gamma_{\mathtt{i}} k, \gamma, \mathrm{B}, \mathrm{c}] = \frac{e^{i\gamma_{\mathtt{i}} k|\hat{\boldsymbol{x}} - \gamma \mathrm{B} \hat{\boldsymbol{y}} + \mathrm{c}|}}{4\pi|\hat{\boldsymbol{x}} - \gamma \mathrm{B}\hat{\boldsymbol{y}} + \mathrm{c}|}, \qquad \mathtt{i} \neq \mathtt{j}$$

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Denoting $\boldsymbol{\mu} = (\gamma_{\mathtt{i}} k, \gamma, \mathrm{c}, \mathrm{B})$ resp. $\boldsymbol{\mu} = \gamma_{\mathtt{i}} k$, the EIM provides us $\{\boldsymbol{\mu}_m\}$, $\{\boldsymbol{\mu}_m^0\}$ such that

$$\mathcal{G}[\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}; \boldsymbol{\mu}] \approx \sum_{m=1}^{M} \alpha_m(\boldsymbol{\mu})\, \mathcal{G}[\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}; \boldsymbol{\mu}_m],$$

$$\mathcal{G}_0[r; \boldsymbol{\mu}] \approx \sum_{m=1}^{M} \alpha_m^0(\boldsymbol{\mu})\, \mathcal{G}_0[r; \boldsymbol{\mu}_m^0]$$

Efficient implementation - Empirical Interpolation Method (EIM)

Define the functions on which we apply EIM:

$$\mathcal{G}[\hat{\boldsymbol{x}},\hat{\boldsymbol{y}};\gamma_{\mathtt{i}}k,\gamma,\mathrm{B},\mathrm{c}] = \frac{e^{i\gamma_{\mathtt{i}}k|\hat{\boldsymbol{x}}-\gamma\mathrm{B}\hat{\boldsymbol{y}}+\mathrm{c}|}}{4\pi|\hat{\boldsymbol{x}}-\gamma\mathrm{B}\hat{\boldsymbol{y}}+\mathrm{c}|}, \qquad \mathtt{i}\neq\mathtt{j}$$

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$$\mathcal{G}_0[r;\boldsymbol{\mu}] \approx \sum_{m=1}^{M}\alpha^0_m(\boldsymbol{\mu})\,\mathcal{G}_0[r;\boldsymbol{\mu}^0_m]$$

if $\mathtt{i}\neq\mathtt{j}$:

$\Rightarrow$ 6-dimensional spatial space $\Omega=\hat{\Gamma}\times\hat{\Gamma}$

$\Rightarrow$ 8-dimensional paramater space $\mathbb{P}$.

Efficient implementation - Empirical Interpolation Method (EIM)

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$$\mathcal{G}_0[r;\boldsymbol{\mu}] \approx \sum_{m=1}^{M}\alpha_m^0(\boldsymbol{\mu})\,\mathcal{G}_0[r;\boldsymbol{\mu}_m^0]$$

if $\mathtt{i}\neq\mathtt{j}$:

$\Rightarrow$ 6-dimensional spatial space $\Omega=\hat{\Gamma}\times\hat{\Gamma}$

$\Rightarrow$ 8-dimensional paramater space $\mathbb{P}$.

Simplified: Only translations.

Efficient implementation - Affine decomposition

$\texttt{i} = \texttt{j}$:

$$\hat{a}^{\texttt{ii}}[\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}; \boldsymbol{\mu}] = ikZ\gamma_{\texttt{i}} \int_{\hat{\Gamma}} \int_{\hat{\Gamma}} \mathcal{G}_0[r; \gamma_{\texttt{i}} k]\, \hat{\boldsymbol{u}}(\hat{\boldsymbol{y}}) \cdot \overline{\hat{\boldsymbol{v}}(\hat{\boldsymbol{x}})}\, d\boldsymbol{y}\, d\boldsymbol{x} - \frac{iZ}{k} \int_{\hat{\Gamma}} \int_{\hat{\Gamma}} \mathcal{G}_0[r; \gamma_{\texttt{i}} k]\, \mathrm{div}_{\hat{\boldsymbol{y}}} \hat{\boldsymbol{u}}(\hat{\boldsymbol{y}}) \overline{\mathrm{div}_{\hat{\boldsymbol{x}}} \hat{\boldsymbol{v}}(\hat{\boldsymbol{x}})}\, d\boldsymbol{y}\, d\boldsymbol{x}$$

$$r = |\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}|$$

Efficient implementation - Affine decomposition

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$$r = |\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}|$$

Thus ...

$$\hat{a}^{\mathtt{ii}}[\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}; \boldsymbol{\mu}] \approx ik\gamma_{\mathtt{i}} Z \sum_{m=1}^{M} \alpha_m^0(\boldsymbol{\mu}) \int_{\hat{\Gamma}} \int_{\hat{\Gamma}} \mathcal{G}_0[r; \boldsymbol{\mu}_m^0]\, \hat{\boldsymbol{w}}(\hat{\boldsymbol{y}}) \cdot \overline{\hat{\boldsymbol{v}}(\hat{\boldsymbol{x}})}\, d\boldsymbol{y}\, d\boldsymbol{x} - \frac{iZ}{k\gamma_{\mathtt{i}}} \sum_{m=1}^{M} \alpha_m^0(\boldsymbol{\mu}) \int_{\hat{\Gamma}} \int_{\hat{\Gamma}} \mathcal{G}[r; \boldsymbol{\mu}_m^0]\, \mathrm{div}_{\hat{\boldsymbol{y}}} \hat{\boldsymbol{u}}(\hat{\boldsymbol{y}}) \overline{\mathrm{div}_{\hat{\boldsymbol{x}}} \hat{\boldsymbol{v}}(\hat{\boldsymbol{x}})}\, d\boldsymbol{y}\, d\boldsymbol{x}$$

Efficient implementation - Affine decomposition

$\mathtt{i} \neq \mathtt{j}$:

$$\hat{a}^{\mathtt{ij}}[\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}; \boldsymbol{\mu}] = ikZ\gamma_{\mathtt{i}}\gamma_{\mathtt{j}} \int_{\hat{\Gamma}}\int_{\hat{\Gamma}} \mathrm{G}^{\mathtt{ij}}_{\boldsymbol{\mu}}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \mathrm{B}_{\mathtt{j}} \hat{\boldsymbol{u}}(\hat{\boldsymbol{y}}) \cdot \mathrm{B}_{\mathtt{i}} \overline{\hat{\boldsymbol{v}}(\hat{\boldsymbol{x}})}\, d\boldsymbol{y}\, d\boldsymbol{x}$$
$$- \frac{iZ}{k} \int_{\hat{\Gamma}}\int_{\hat{\Gamma}} \mathrm{G}^{\mathtt{ij}}_{\boldsymbol{\mu}}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \mathrm{div}_{\hat{\boldsymbol{y}}} \hat{\boldsymbol{u}}(\hat{\boldsymbol{y}}) \overline{\mathrm{div}_{\hat{\boldsymbol{x}}} \hat{\boldsymbol{v}}(\hat{\boldsymbol{x}})}\, d\boldsymbol{y}\, d\boldsymbol{x}$$

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Note that:

$$\mathrm{B}_{\mathtt{j}} \hat{\boldsymbol{w}}(\hat{\boldsymbol{y}}) \cdot \mathrm{B}_{\mathtt{i}} \overline{\hat{\boldsymbol{v}}(\hat{\boldsymbol{x}})} = \sum_{l,n=1}^{3} \hat{\boldsymbol{w}}(\hat{\boldsymbol{y}})_l \, (\mathrm{B}_{\mathtt{j}}^T \mathrm{B}_{\mathtt{i}})_{ln} \, \overline{\hat{\boldsymbol{v}}(\hat{\boldsymbol{x}})}_n$$

$$\mathrm{G}^{\mathtt{ij}}_{\boldsymbol{\mu}}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) = \frac{1}{\gamma_{\mathtt{i}}} \mathcal{G}[\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}; \gamma_{\mathtt{i}} k, \gamma_{\mathtt{ij}}, \mathrm{B}_{\mathtt{ij}}, \mathrm{c}_{\mathtt{ij}}]$$

Efficient implementation - Affine decomposition

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Efficient implementation - Affine decomposition

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$$- \frac{iZ}{k\gamma_{\mathtt{j}}} \sum_{m=1}^{M} \alpha_m(\boldsymbol{\mu}) \int_{\hat{\Gamma}} \int_{\hat{\Gamma}} \mathcal{G}[\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}; \boldsymbol{\mu}_m] \mathrm{div}_{\hat{\boldsymbol{y}}} \hat{\boldsymbol{u}}(\hat{\boldsymbol{y}}) \overline{\mathrm{div}_{\hat{\boldsymbol{x}}} \hat{\boldsymbol{v}}(\hat{\boldsymbol{x}})} \, d\boldsymbol{y} \, d\boldsymbol{x}$$

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Efficient implementation - Affine decomposition

We are therefore able to write

$$\mathrm{A}_{\boldsymbol{\mu}}^{\mathtt{ij}} = \sum_{m=1}^{M} \Theta_m^{\mathtt{ij}}(\boldsymbol{\mu})\, \mathrm{A}_m, \qquad \text{and} \qquad \mathrm{f}_{\boldsymbol{\mu}}^{\mathtt{j}} = \sum_{m=1}^{M_{\mathtt{f}}} \Theta_{m,\mathtt{f}}^{\mathtt{j}}(\boldsymbol{\mu})\, \mathrm{f}_m$$

Note, given the interpolation points $\{\boldsymbol{\mu}_m\}_{m=1}^{M}$ from the EIM, and the reduced basis $\mathbb{V}_N$ on the reference shape, we can precompute $\{\mathrm{A}_m\}_{m=1}^{M}$ and $\{\mathrm{f}_m\}_{m=1}^{M_{\mathtt{f}}}$

Efficient implementation - Affine decomposition

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Note, given the interpolation points $\{\boldsymbol{\mu}_m\}_{m=1}^{M}$ from the EIM, and the reduced basis $\mathbb{V}_N$ on the reference shape, we can precompute $\{\mathrm{A}_m\}_{m=1}^{M}$ and $\{\mathrm{f}_m\}_{m=1}^{M_{\mathtt{f}}}$

Thus: For any new configuration ($\boldsymbol{\mu} \in \mathbb{P}$), which is described by

1. the wave number k,
2. the angle and polarization of incident plane wave, and
3. the geometrical configuration of each obstacle $1 \leq \mathtt{j} \leq \mathtt{J}$ which includes a rotation, stretch and a translation of the reference shape,

we can solve the coupled problem independently of $\mathcal{N} = \dim(\mathbb{V}_h)$.

Complexity

The computing time (on seq. computer) is dictated by

1. Assembling† matrices $\mathrm{A}_{\boldsymbol{\mu}}^{\mathtt{ij}}$: $\sim \mathtt{J}^2 N^2 M$.
2. Solving $\mathtt{J}$ N-dimensional dense systems (depending on solver).
3. Computing the RCS (no details, but indep. of $\mathcal{N}$).

† : Building the sum of matrices $\sum_{m=1}^{M} \Theta_m^{\mathtt{ij}}(\boldsymbol{\mu})\, \mathrm{A}_m^{\mathtt{ij}}$.
‡ : Assembling the matrices, i.e., numerical integration of weakly singular kernel etc.

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Without reduced basis approach, it would be

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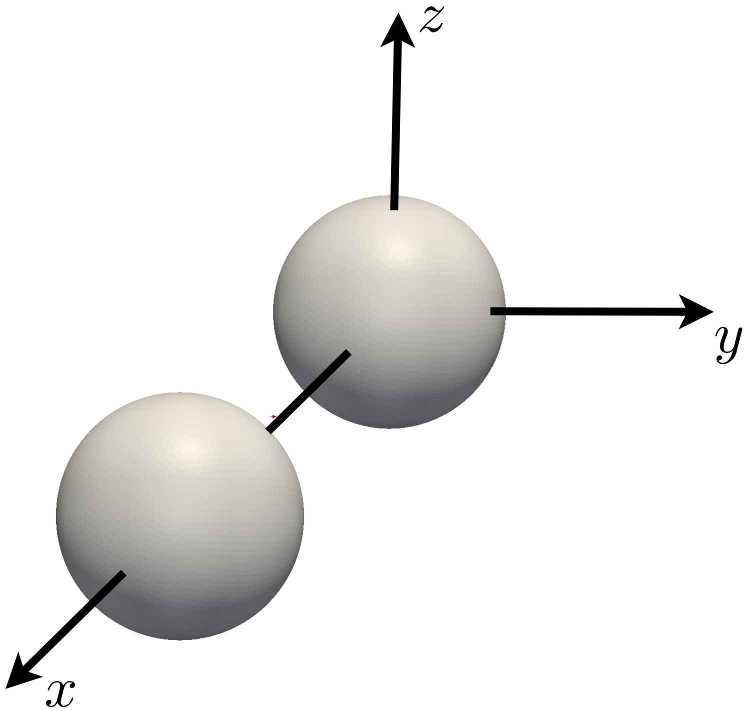
Comments: Remember that $N \ll \mathcal{N}$ (example of a sphere with $k \in [3,5]$: $N = 509$, $\mathcal{N} = 4320$):

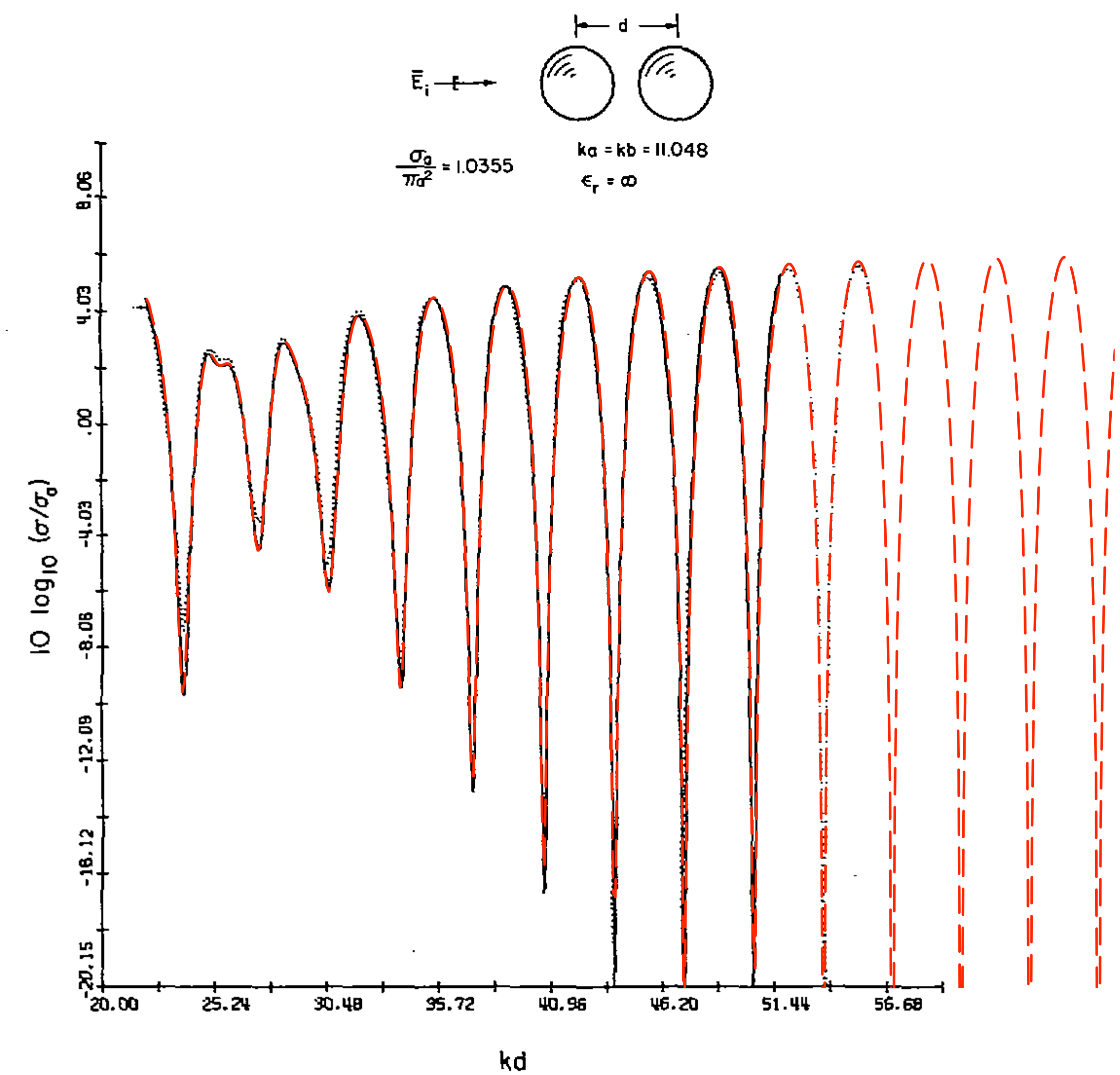
1. Speed up only if M is moderate: For shape modifications such as stretch and rotations M is not moderate.
   $\Rightarrow$ novel techniques exist and are under development such as *hp*-EIM etc ...
2. Speed up thanks to $N \ll \mathcal{N}$
3. No details, but similar comment as in 1.

Numerical results

2 Unit spheres - fixed wavenumber



Comparison with [Bruning and Lo]:

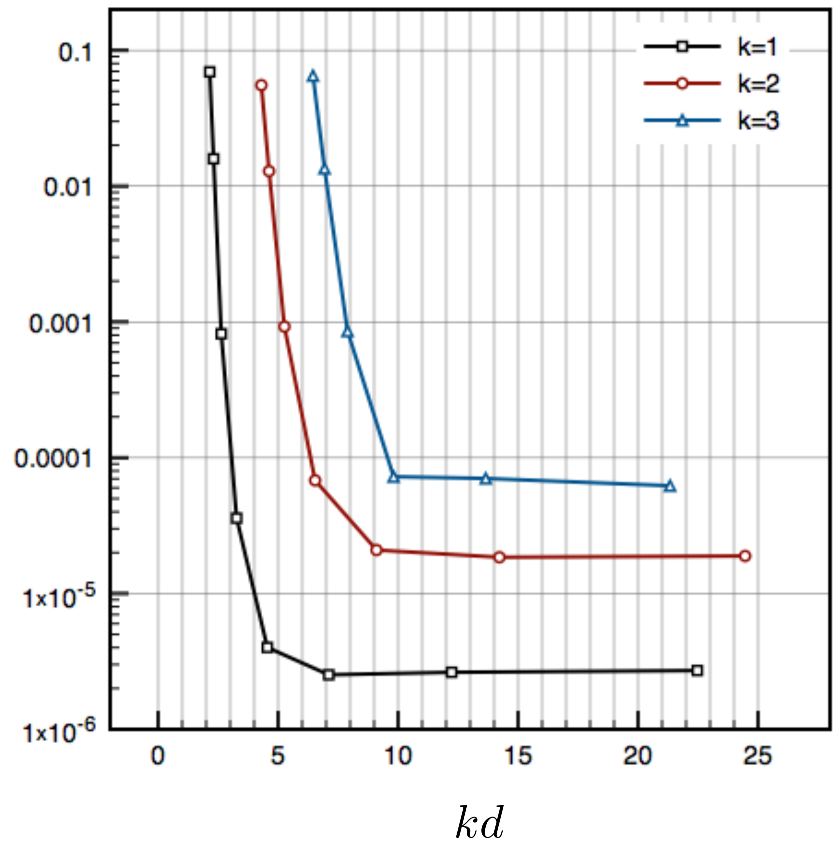


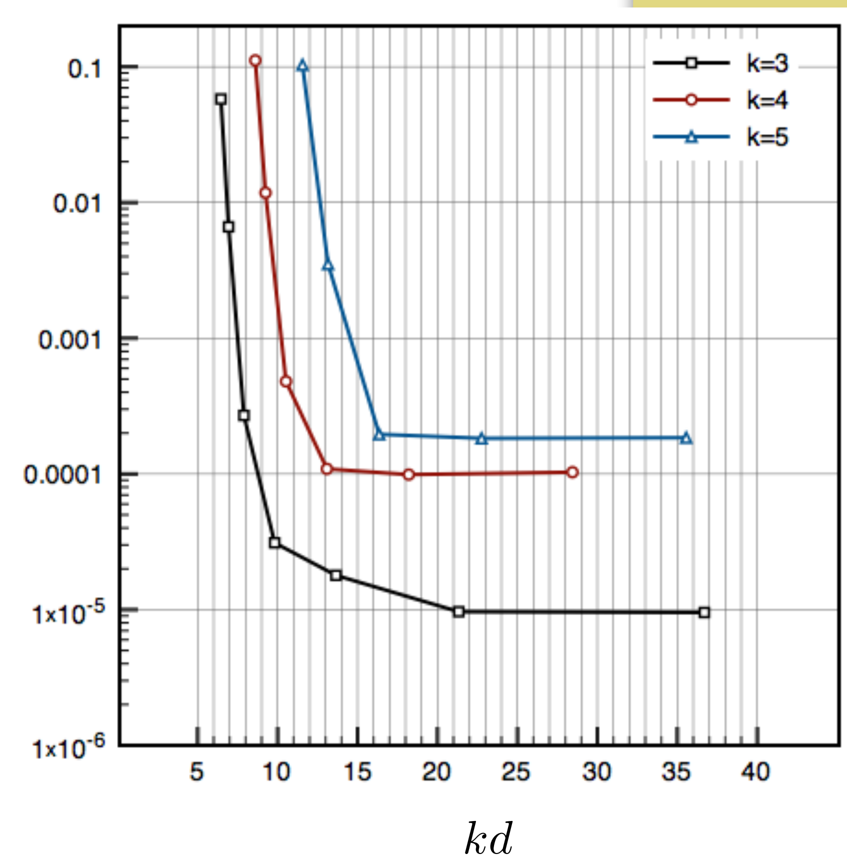
Endfire incidence and backscattering

2 Unit spheres - variable wavenumber

many sources of error

RCS: integral of current u_h



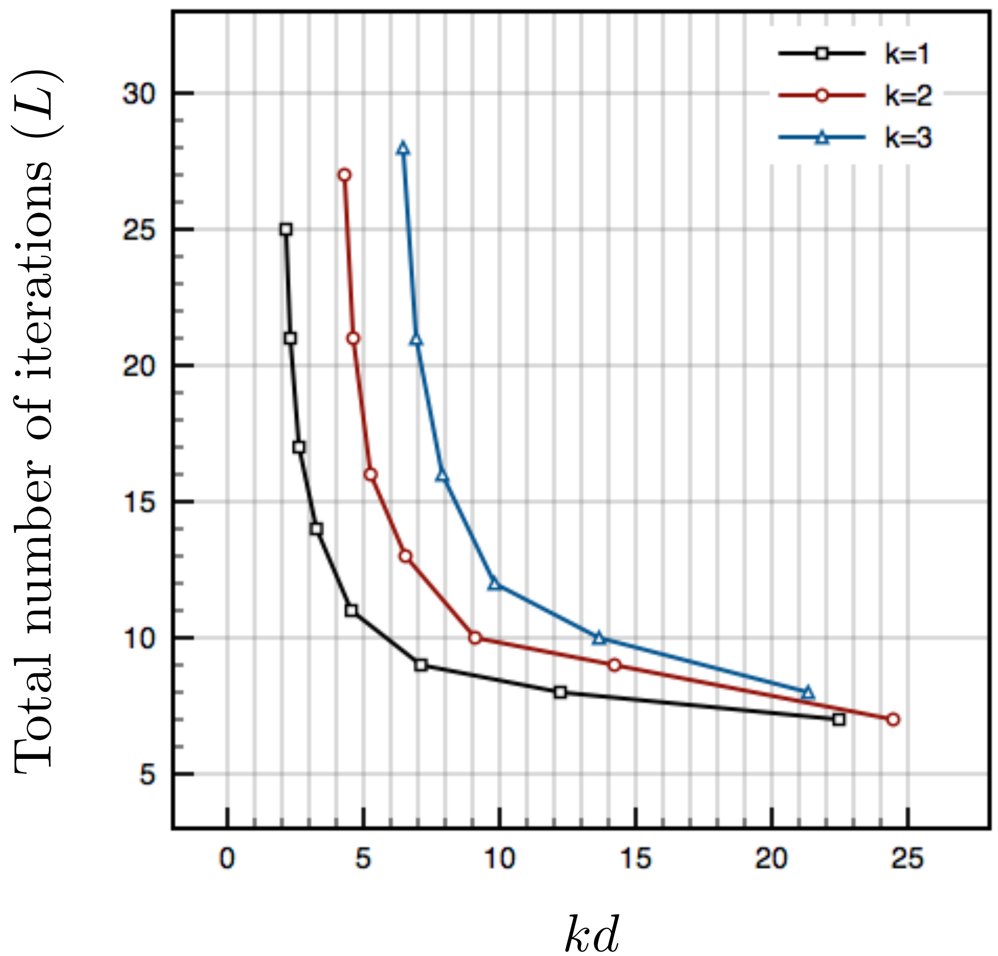


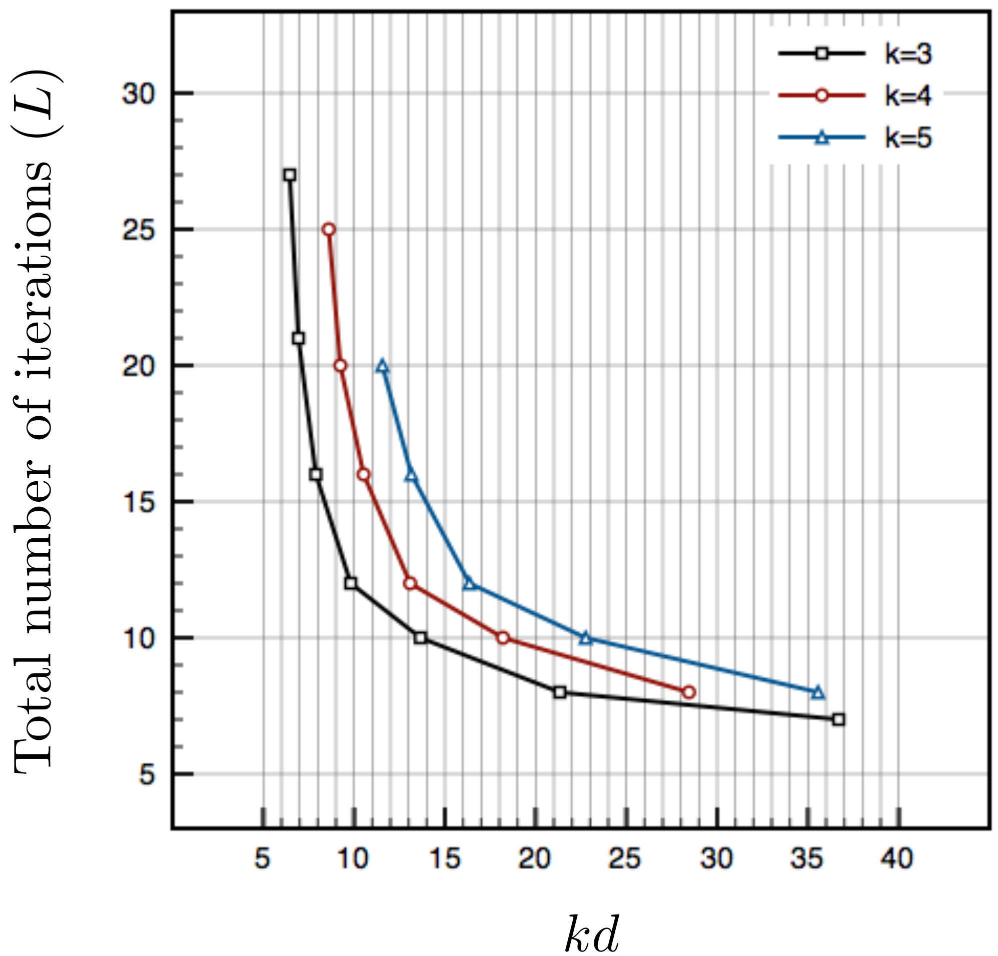
Error: Difference between BEM solution $\boldsymbol{u}_h(\boldsymbol{\mu})$ and RB approximation $\boldsymbol{u}_N(\boldsymbol{\mu})$ in $L^2(\Gamma)$-norm.

Sources of error:

1. Ability of reduced basis space $\mathbb{V}_N$ to represent the solution space. As closer the spheres get, as more the interaction is of dipole character. The reduced basis is however trained to respond for linear combinations of plane waves.
2. Accuracy of EIM and therefore the matrices $\mathrm{A}^{\mathtt{ij}}_{\boldsymbol{\mu}}$.
3. Truncation of generalized Born series.

2 Unit spheres - variable wavenumber



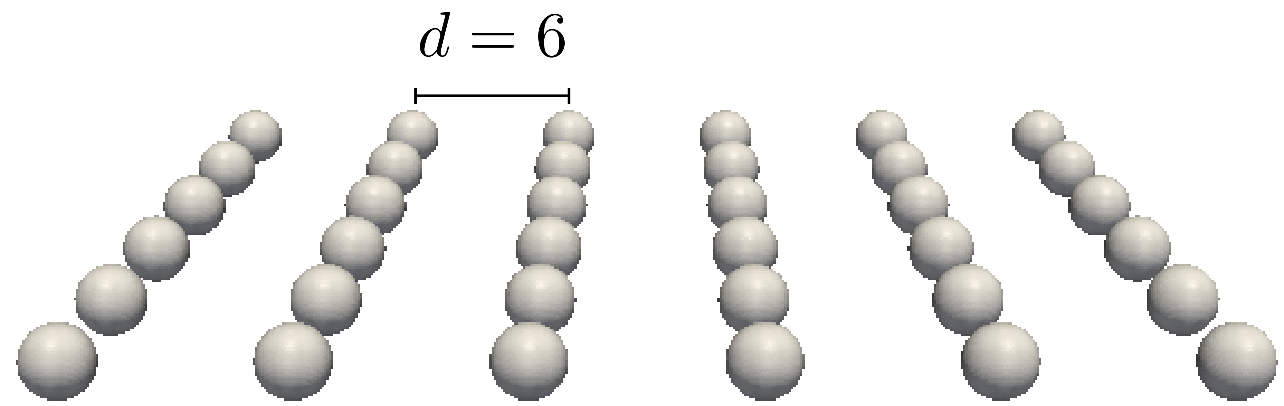


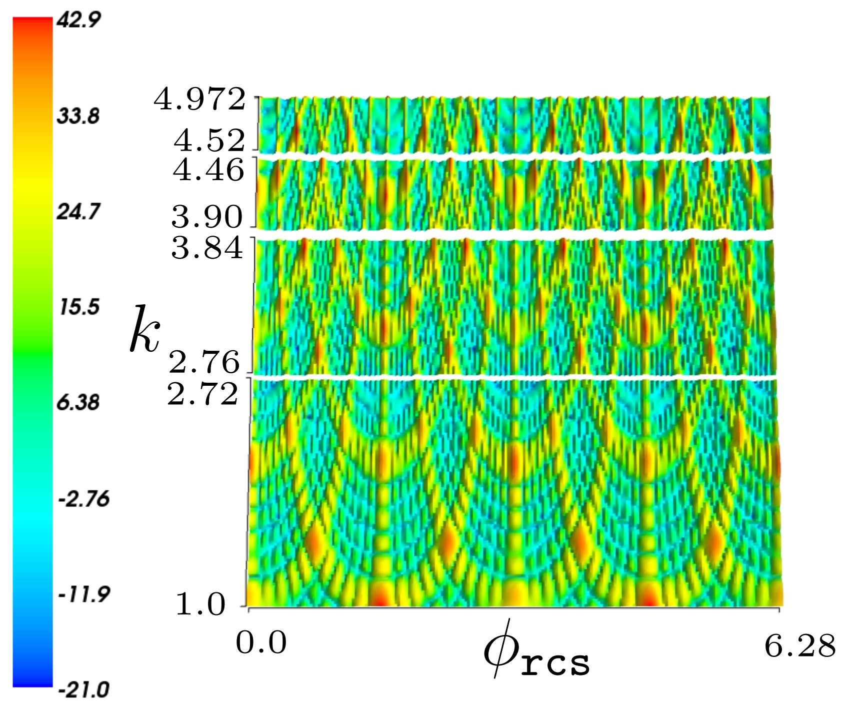
many sources of error

RCS: integral of current u_h

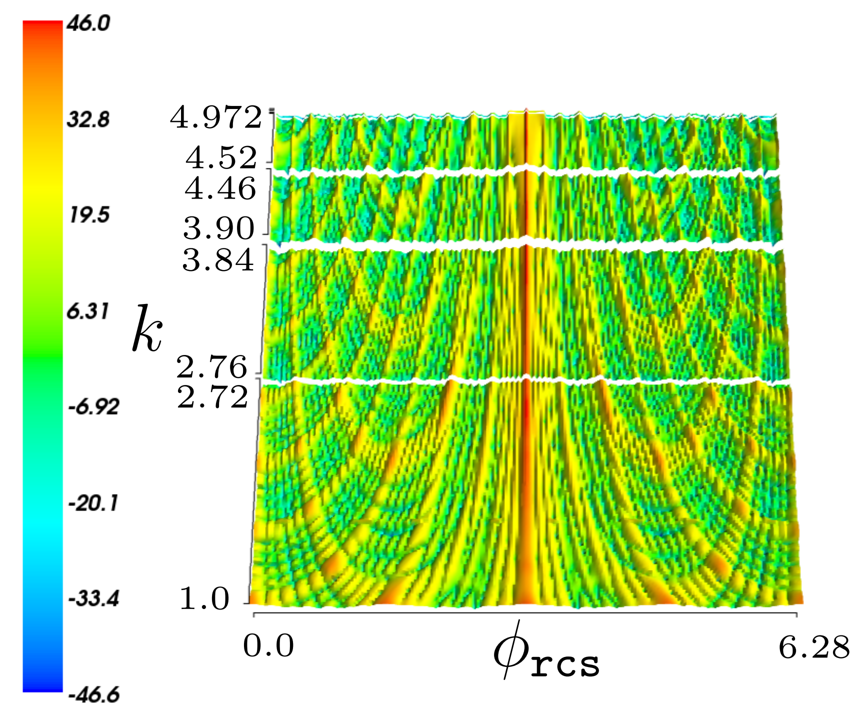
Lattice of spheres

Sender: $\theta = 0, \frac{\pi}{2}$, $\phi = 0$
Receiver: $\theta_{\mathtt{rcs}} = \frac{\pi}{2}$, $\phi_{\mathtt{rcs}} \in [0, 2\pi]$





$\theta = 0$



$\theta = \frac{\pi}{2}$

Lattice of unit spheres

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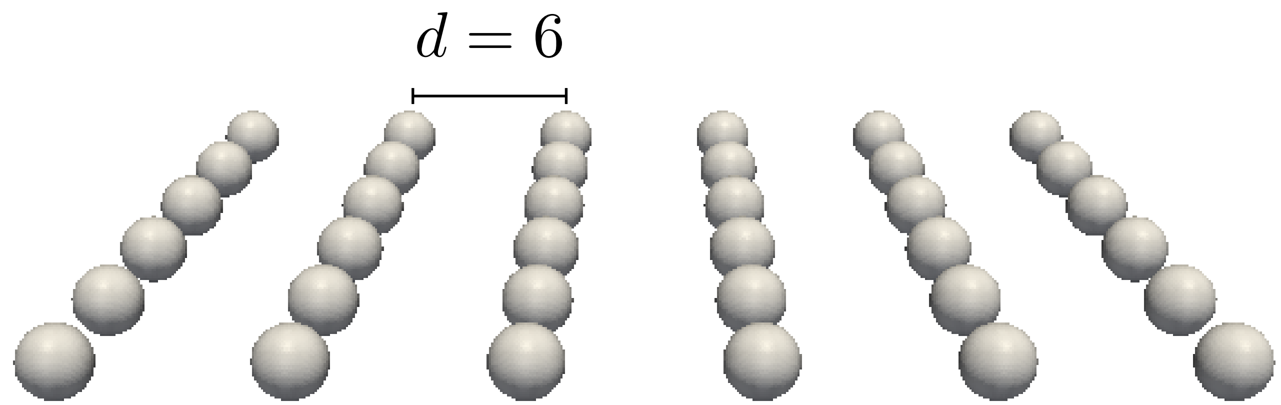
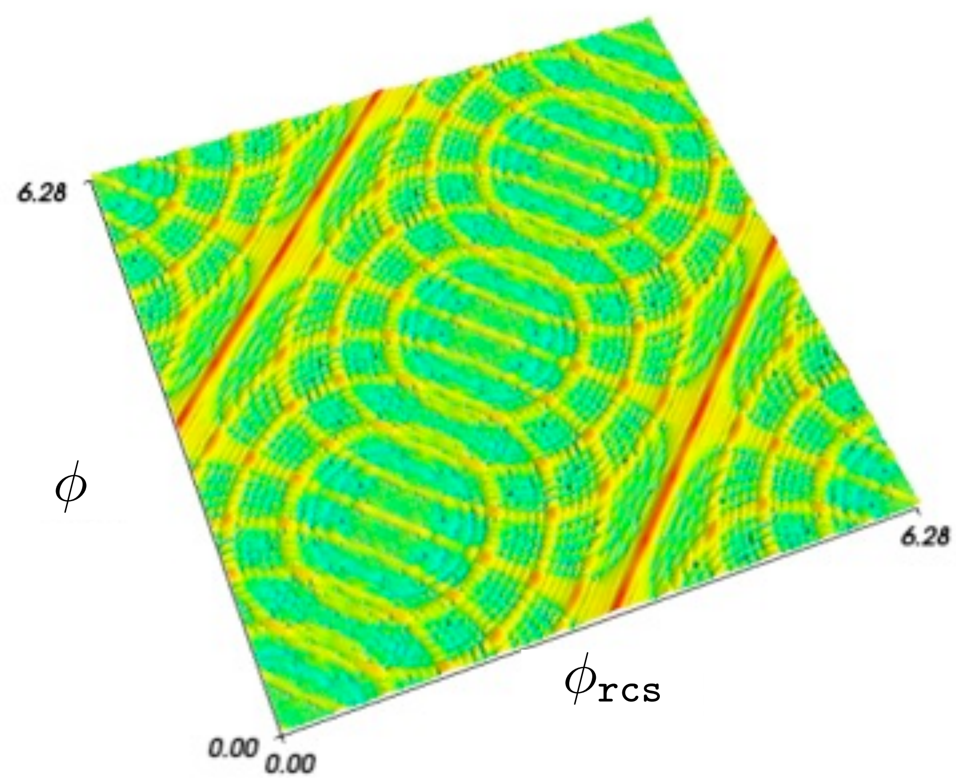
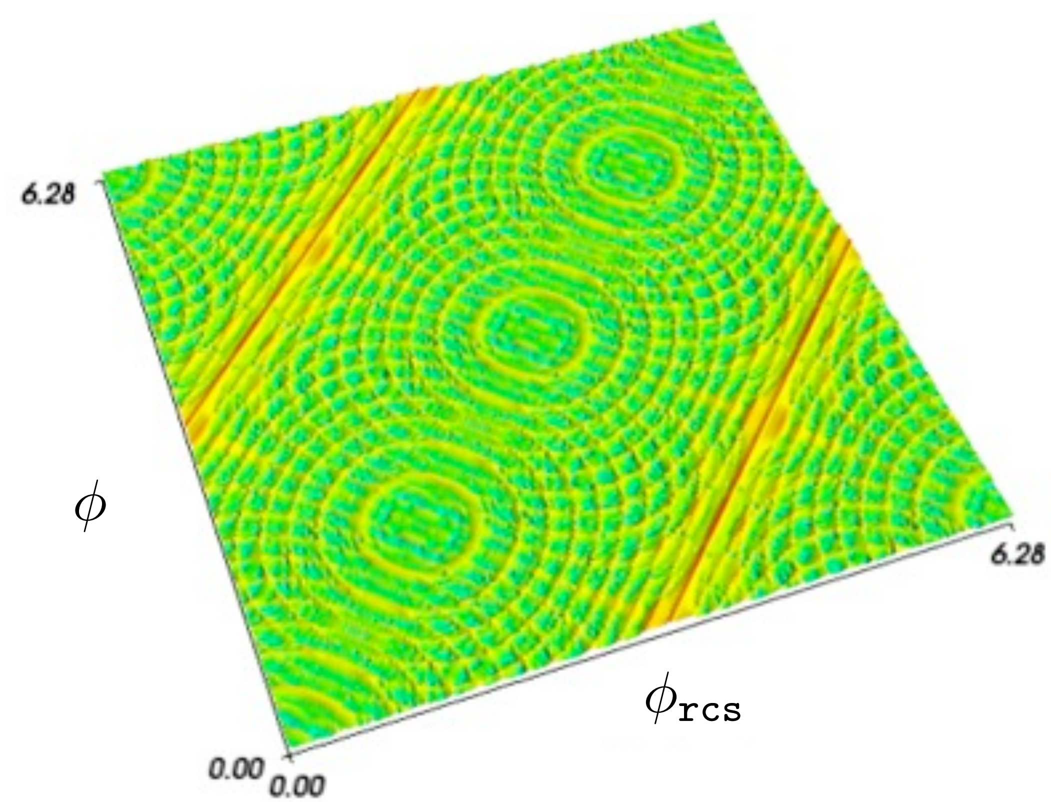




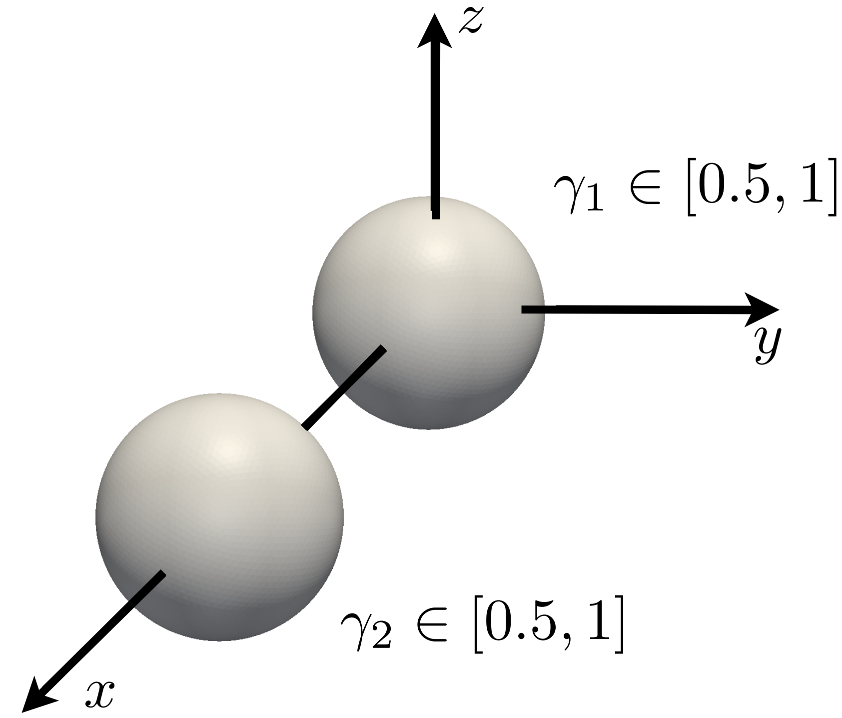
$k = 2$



$k = 5$

Stretch

Sender: $\theta = \phi = 0,\ k = 3,\ d = 4$



Stretch

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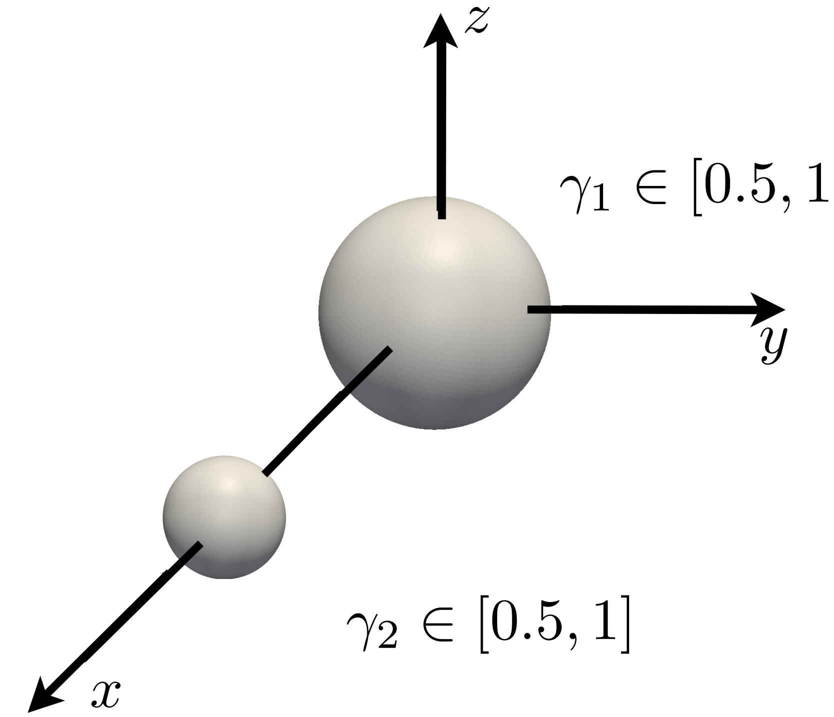
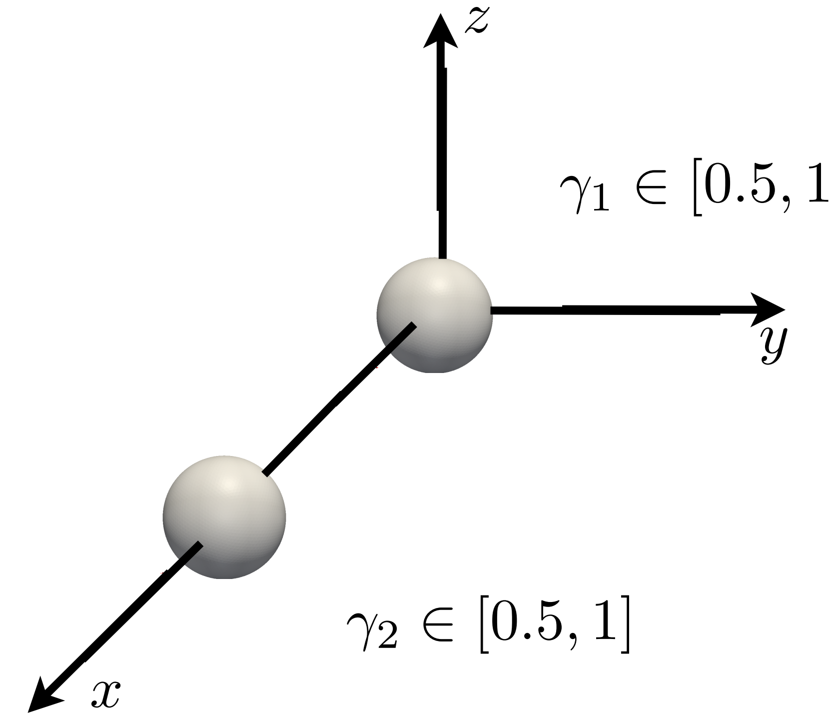


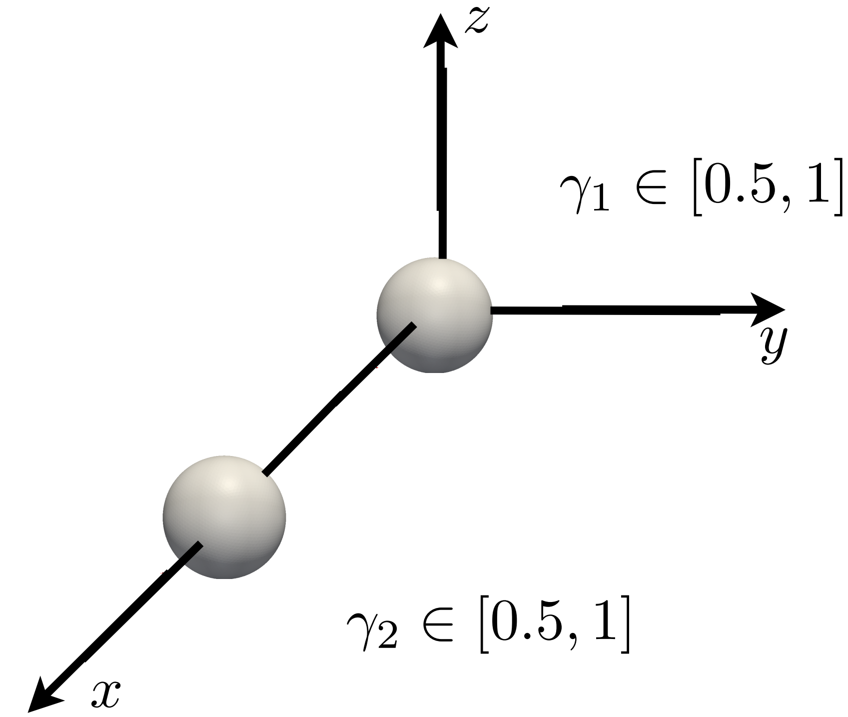
Stretch

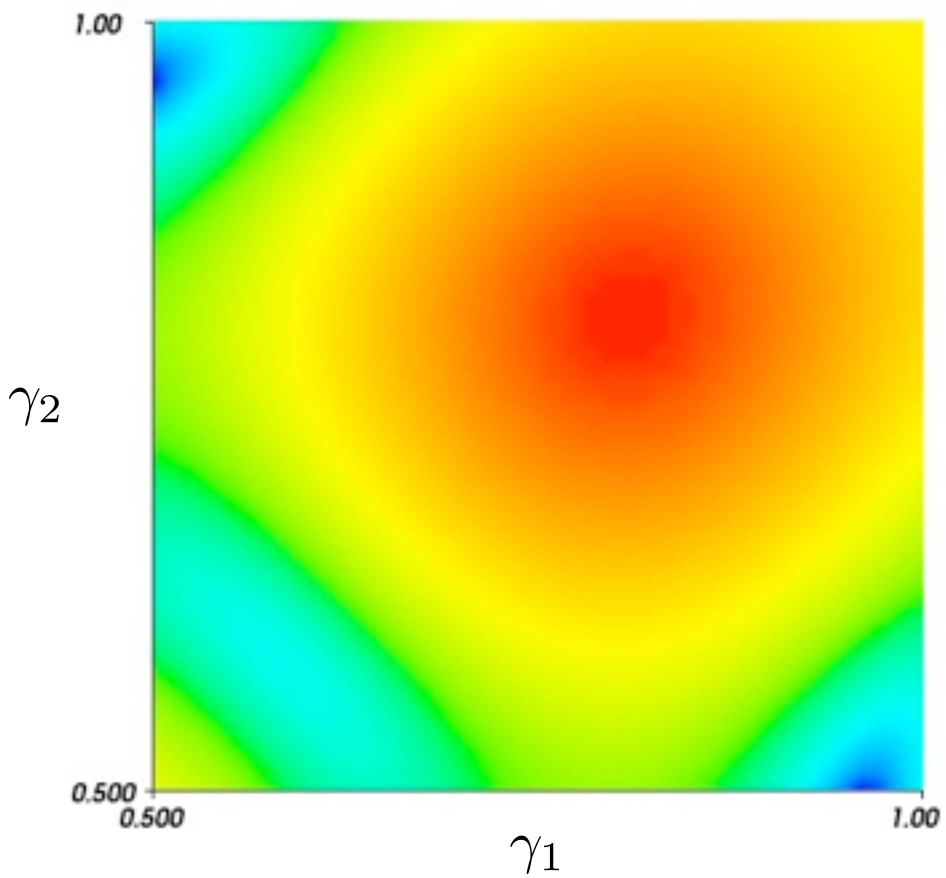
Sender: $\theta = \phi = 0,\ k = 3,\ d = 4$



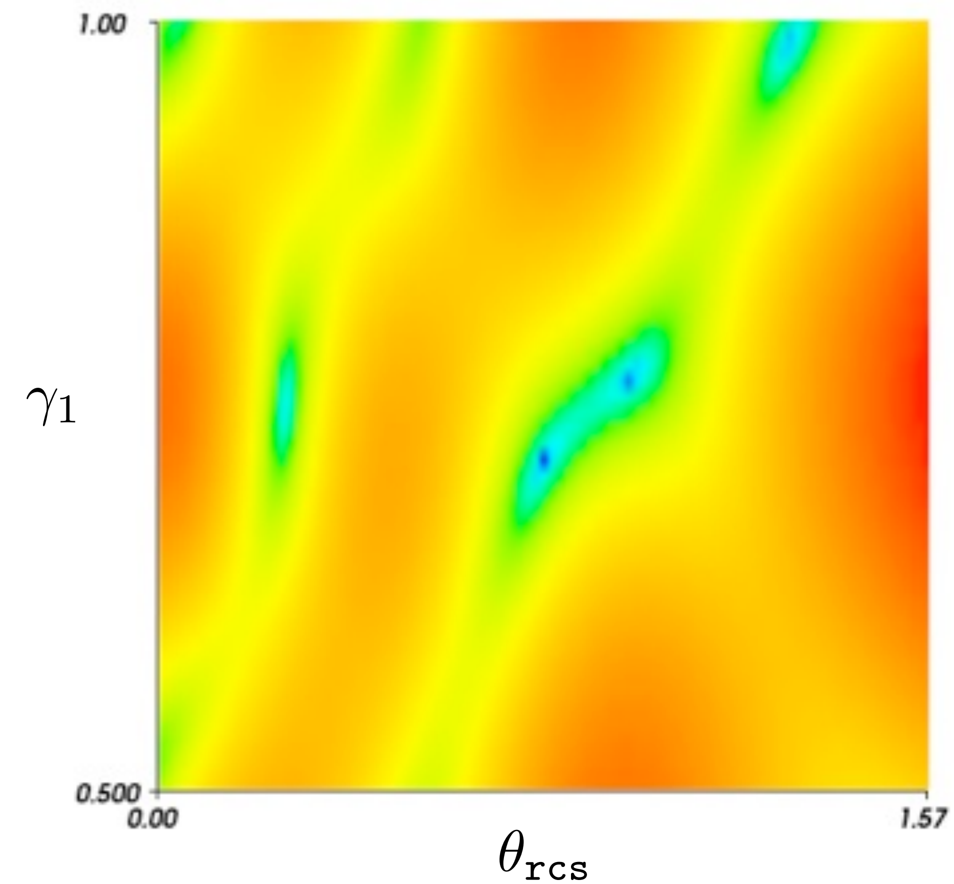
Stretch

Sender: $\theta = \phi = 0,\ k = 3,\ d = 4$





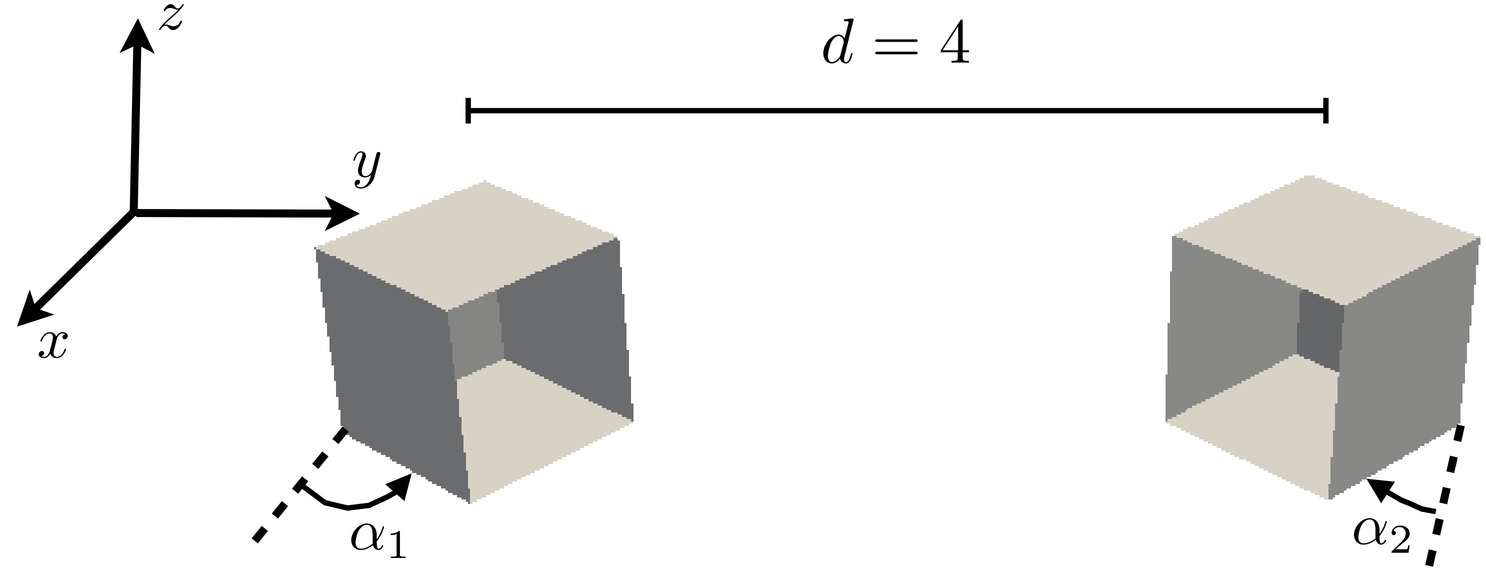
Receiver: $\theta_{\texttt{rcs}} = \phi_{\texttt{rcs}} = 0$



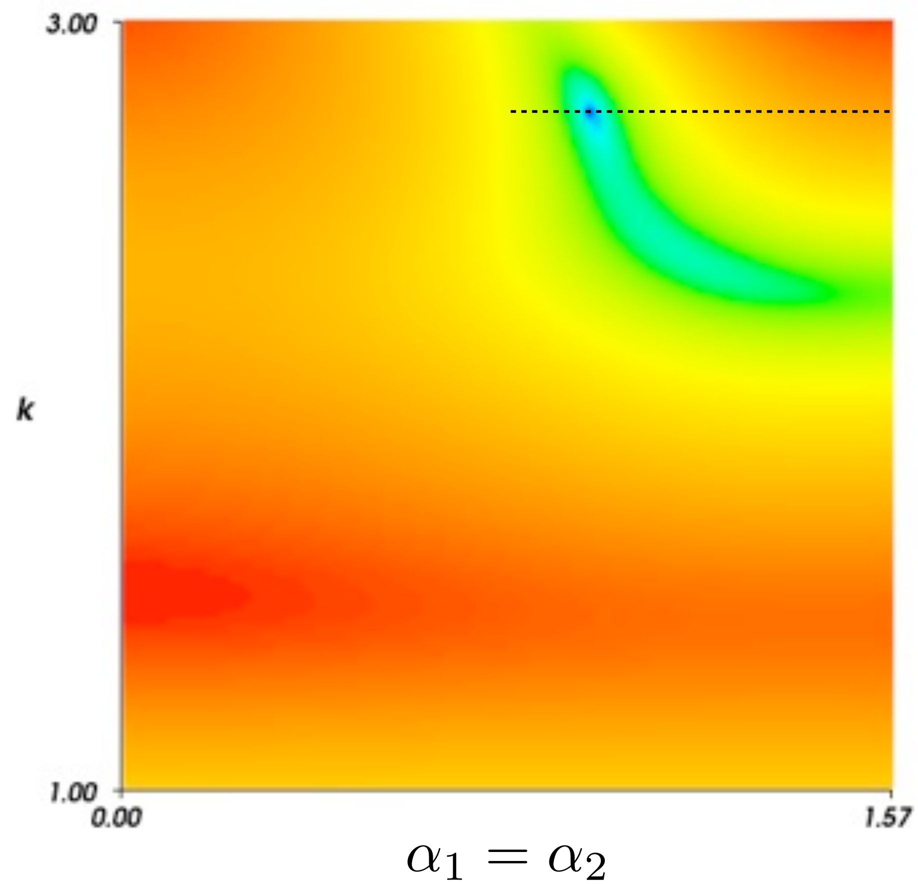
Receiver: $\theta_{\texttt{rcs}} \in [0, \frac{\pi}{2}],\ \phi_{\texttt{rcs}} = 0$

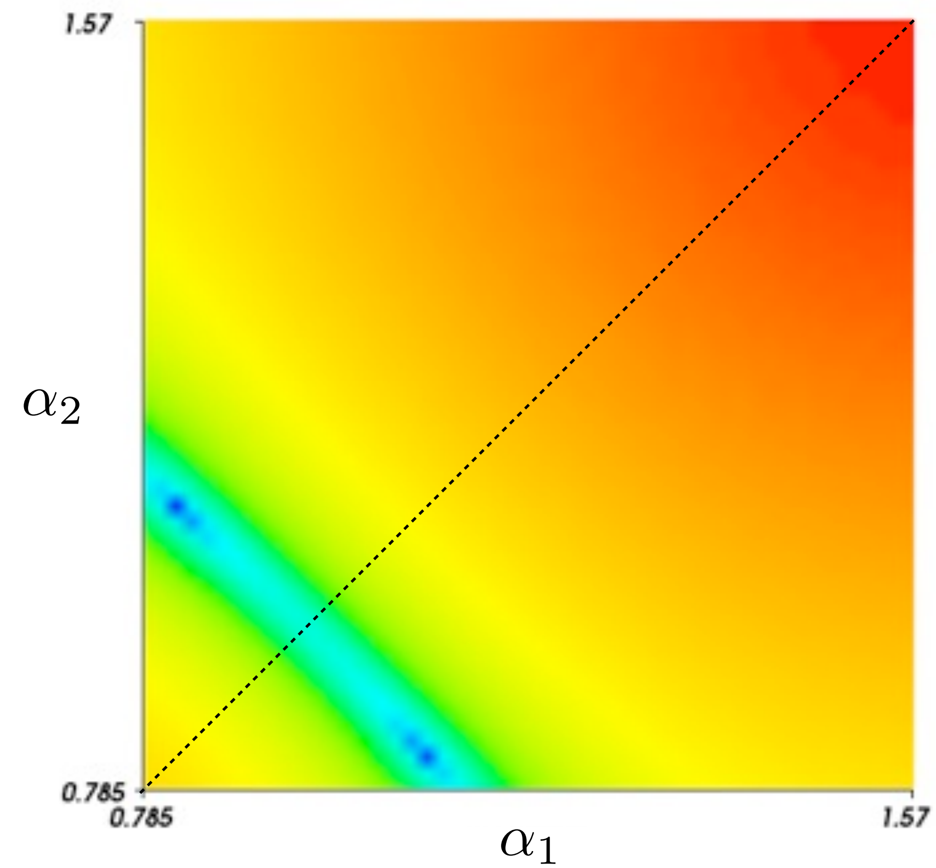
$\gamma_2 = 1.5 - \gamma$

Rotation



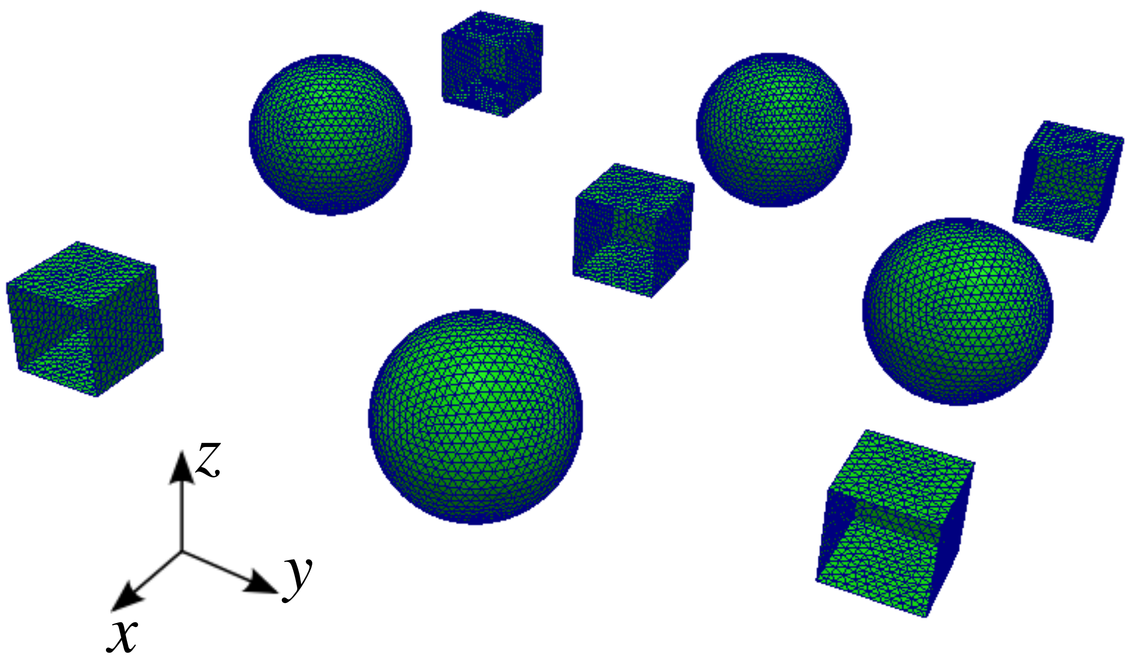
Sender: $\theta = \frac{\pi}{2}$, $\phi = 0$
Receiver: $\theta_{\texttt{rcs}} = \frac{\pi}{2}$, $\phi_{\texttt{rcs}} = 0$

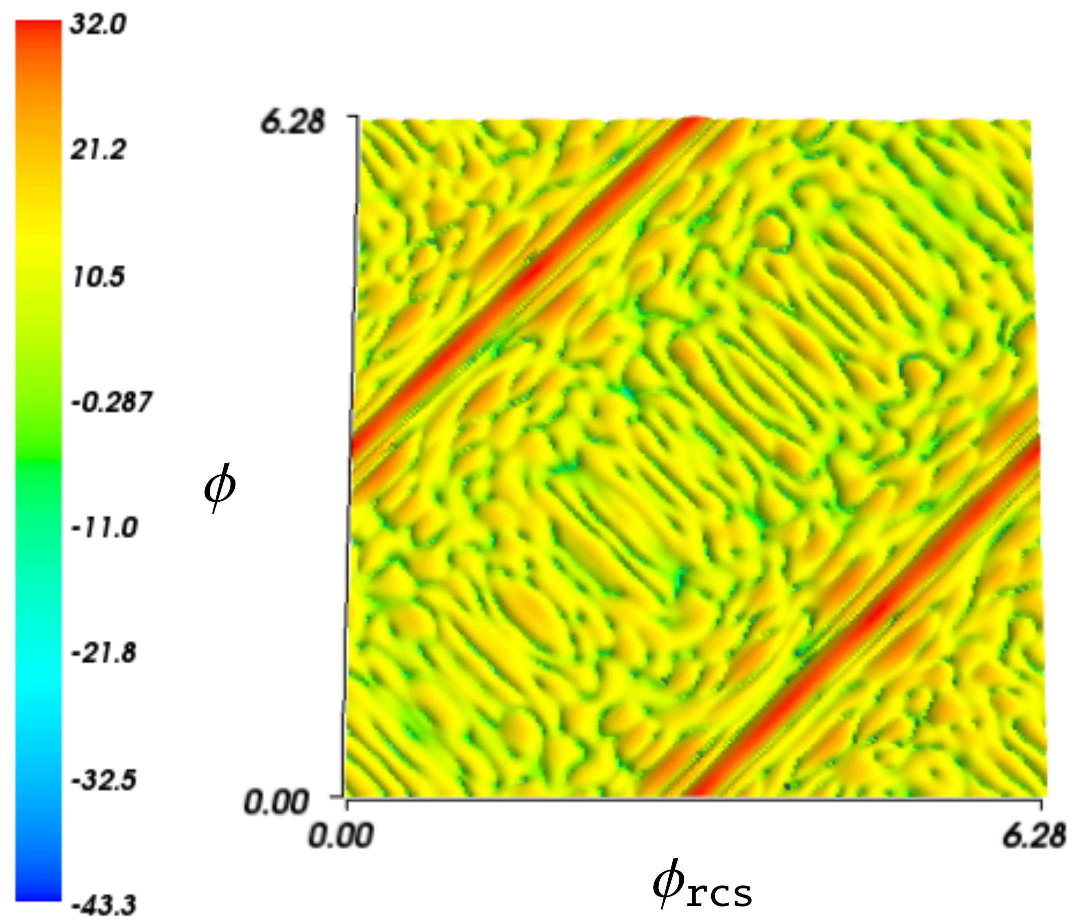




Different reference shapes

$k = 3$
Sender: $\theta = \frac{\pi}{2}$, $\phi \in [0, 2\pi]$
Receiver: $\theta_{\texttt{rcs}} = \frac{\pi}{2}$, $\phi_{\texttt{rcs}} \in [0, 2\pi]$





Conclusions

- RBM was applied to an integral equation $\Rightarrow$ EIM plays an important role.
- Previously, the RBM was designed to get significant speed-up for parametrized problems. Solving the problem always relied on an established solver (black-box). Here, we can solve configurations where the black-box solver would fail (memory, time).
  $\Rightarrow$ Similar in spirit to work of Patera, Eftang, etc ("lego") but no physical interface condition. Instead communication is through kernel function. In consequence, heavy use of EIM.
  $\Rightarrow$ Use of ROM to solve larger problems, i.e. design new solvers.
- IE are well suited for coupling several RB models.
- Translations only (no stretch, no rotation) simplifies and accelerates the approach.
- Generalization to CFIE straightforward.
- Bottleneck: large number of obstacles (scaling and convergence), low rank-structure of parametrized interaction