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Reduced Basis Methods for Option Pricing

Acknowledgements

- joint work with / contributions from
  - Silke Glas, Antonia Mayerhofer, Andreas Rupp, Bernhard Wieland (all Ulm)
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The Heston Model

The Heston Model (European Option)

$$dS_t = \bar{\mu} S_t dt + \sqrt{\nu_t} S_t dz_1(t), \qquad d\nu_t = \kappa[\theta - \nu_t]dt + \sigma\sqrt{\nu_t} dz_2(t)$$

- ν_t : instantaneous variance — CIR (Cox-Ingersoll-Ross) process
- z_1, z_2: Wiener processes with correlation ρ
- $\bar{\mu}$: rate of return of the asset
- κ: revert rate of μ_t to θ
- θ: long variance
- σ: volatility of volatility
- parameters to be calibrated from market data

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Feynman-Kac theorem

$$\begin{aligned} \frac{\partial u}{\partial t} - \mathrm{div}(\underline{\underline{\alpha}}(t)\nabla u) + \underline{\beta}(t)\nabla u + \gamma(t)u &= 0 && \text{in } (0,T] \times D, \\ u &= 0 && \text{on } [0,T] \times \partial D \\ u(0) &= u_0 && \text{on } D \end{aligned}$$

with

$$\underline{\underline{\alpha}}(t) := \begin{pmatrix} \nu_t & \nu_t \sigma \rho \\ \nu_t \sigma \rho & \nu_t \sigma^2 \end{pmatrix}, \ \underline{\beta}(t) := -\begin{pmatrix} r(t) - \frac{1}{2}\nu_t - \frac{1}{2}\sigma\rho \\ \kappa\theta - \kappa\nu_t - \frac{1}{2}\sigma^2 \end{pmatrix}, \ \gamma(t) := r(t).$$

The Heston Model and RBM

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- calibration parameters: $\mu_1 := (r(t), \sigma, \varrho, \kappa, \theta)$ $(P = 5)$
- some may be stochastic,
  e.g. ν_t, $\sigma = \sigma(t, \omega)$, $\omega \in \Omega$, probability space $(\Omega, \mathcal{B}, P)$
- pricing parameter: $\mu_0 = u_0 \in L_2(D)$ (payoff: parameter function)

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$$\frac{\partial u}{\partial t} - \mathcal{A}(\mu_1, \omega; t)u = 0 \qquad \text{in } (0, T] \times D,$$

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Additional challenges

- several options/assets (WASC, CDOs):
  $\rightsquigarrow$ many coupled PDEs, high (space) dimension
- American options: variational inequalities (Haasdonk, Salomon, Wohlmuth; Glas, U.)
- stochastic coefficients
- jump models (Lévy): integral operators, PIDEs (Schwab et al., Kestler, ...)
- problems on infinite domains ($S \in [0, \infty)$) (Kestler, U.)
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- traders do not trust numerics ...

The Heston Model again: variable initial condition

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Parabolic PDEs / Space-Time variational formulation

- $V := H_0^1(D)$, $H := L_2(D)$, $V \hookrightarrow H \hookrightarrow V'$, $I := (0, T)$
- $$\langle \dot{u}(t), \phi \rangle_{V' \times V} + a(u(t), \phi) = \langle g(t), \phi \rangle_{V' \times V} \; \forall \phi \in V, t \in I (a.e.)$$
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- Integrate over time $\rightsquigarrow$ Trial space:
  - $\mathcal{Z} := L_2(I; V) := \{w : I \to V : \|w\|^2_{L_2(I;V)} := \int_I \|w(t)\|_V^2 \, dt < \infty\}$
    (Bochner space)
  - $\mathcal{X} := \{w \in \mathcal{Z} : \dot{w} \in \mathcal{Z}'\} = L_2(I; V) \cap H^1(I; V') \hookrightarrow C(\bar{I}; H)$
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$$\begin{aligned} b(u, v) &:= \int_I \langle \dot{u}(t), z(t) \rangle_{V' \times V} \, dt + \int_I a(u(t), z(t)) dt + (u(0), \zeta)_H \\ &= b_1(w, z) + (u(0), \zeta)_H \end{aligned}$$

$$f(v) := \int_I \langle g(t), z(t) \rangle_{V' \times V} \, dt + (\mu_0, \zeta)_H =: g_1(z) + (u_0, \zeta)_H$$

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Variational formulation (Petrov-Galerkin)

find $u \in \mathcal{X}$ s.t. $b(u, v) = f(v) \;\; \forall v \in \mathcal{Y}$.

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$$b(\mu_1; u, v) := \int_I \langle \dot{u}(t), z(t) \rangle_{V' \times V} \, dt + \int_I a(\mu_1; u(t), z(t)) dt + (w(0), \zeta)_H$$
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Why space-time?

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$$\beta := \inf_{w \in \mathcal{X}} \sup_{v \in \mathcal{Y}} \frac{b(\mu_1; w, v)}{\|w\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}}$$

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– (offline) dimension increased by one (cpu / memory)

Well-posedness / inf-sup-constant

- $C_e := \sup_{w \in \mathcal{X} \setminus \{0\}} \frac{\|w(0)\|_H}{\|w\|_{\mathcal{X}}} \leq \sqrt{3}, \qquad \varrho := \sup_{0 \neq \phi \in V} \frac{\|\phi\|_V}{\|\phi\|_H} \quad (\leq 1)$

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- $\beta_a^* := \inf\limits_{\mu_1\in\mathcal{D}_1} \inf\limits_{\phi\in V} \sup\limits_{\psi\in V} \dfrac{a(\mu_1;\psi,\phi)}{\|\phi\|_V \, \|\psi\|_V} > 0$

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- we look for: $\beta_b := \inf_{\mu_1\in\mathcal{D}_1} \inf_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}} \frac{b(\mu_1;w,v)}{\|w\|_\mathcal{X} \, \|v\|_\mathcal{Y}}$

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- inf-sup bounds:

$$\beta_{\text{coer}}^{\text{LB}}(\alpha,\lambda,M,C) := \frac{\min\{\min\{1,M^{-2}\}(\alpha-\lambda\varrho^2),1\}}{\sqrt{2\max\{1,(\beta_a^*)^{-1}\}+C^2}},$$

$$\beta_{\text{time}}^{\text{LB}}(\alpha,\lambda,M,C,T) := \frac{e^{-2\lambda T}}{\sqrt{\max\{2,1+2\lambda^2\varrho^4\}}}\ \beta_{\text{coer}}^{\text{LB}}(\alpha,0,M,C)$$

Proposition (Inf-sup bound(Schawb/Stevenson, U./Patera))

Let $a(\cdot;\cdot,\cdot)$ be bounded (M_a) and satisfy a Garding inequality (α_a, λ_a). Then,

$$\beta_b \geq \beta_b^{\text{LB}} := \max\{\beta_{coer}^{LB}(\alpha_a,\lambda_a,M_a,C_e),\beta_{time}^{LB}(\alpha_a,\lambda_a,M_a,C_e,T)\}.$$

Space-Time Discretization

- Note: trial and test spaces are *tensor products*:
  - $\mathcal{X} = H^1(I) \otimes V$ $\qquad\mathcal{Y} = \mathcal{Z} \times H := L_2(I; V) \times H = (L_2(I) \otimes V) \times H$

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- FE in time: $E_{\Delta t} = \{\sigma^1, \ldots, \sigma^K\} \subset H^1_{\{0\}}(I)$ (pw. linear)
  and $F_{\Delta t} \subset L_2(I)$ (pw. constant)
  w.r.t. $\mathcal{T}_{\text{time},\Delta t} := \{t^k = k\Delta t : 0 \leq k \leq K, \Delta t := \frac{T}{K}\}$

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  - $\mathcal{X} = H^1(I) \otimes V$ $\qquad \mathcal{Y} = \mathcal{Z} \times H := L_2(I; V) \times H = (L_2(I) \otimes V) \times H$
- FE in space: $V_h := \text{span}\{\phi_1, \ldots, \phi_{n_h}\}$ w.r.t. $\mathcal{T}_{\text{space},h}$
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Offline computations (Mayerhofer, U.)

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- stabilization or by stabilizer, double Greedy, ... (Andreev; Rozza et al, Dahmen, Welper, ...)

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- Recall: (5) won't be time-marching!
  - no sum up of time-discrete residuals
  - but: space-time

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1st step: Initial condiition

- construct $I_{N_0} \subset \mathcal{I}_M \subset H$ of (small) dimension N_0
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- also adaptive (Steih, U.)

Space-Time RBM 3/3

2nd step: Evolution with homogeneous initial conditions

- extend the 'space-only' function $u_N^{0,0}(\mu_0) \in I_{N_0} \subset H^1(\Omega)$
  to a space-time function $u_N^0(\mu_0) := \sigma^0 \otimes u_N^{0,0}(\mu_0) \in L_2(I; H^1(\Omega))$
  do this for $\mu_0^i \in S_{N_0}^0 := \{\mu_0^i : 1 \leq i \leq N_0\}$,

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  by snapshots $S_{N_1}^1 = \{\mu^j = (\mu_0^i, \mu_1^j) : 1 \leq j \leq N_1\} \subset S_{N_0}^0 \times \mathcal{D}_1 \subset \mathcal{D}$

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- compute $\breve{u}^j := \breve{u}_\eta(\mu^j) \in \breve{\mathcal{X}}_\eta$ (offline 2nd step — modified rhs)
  and set $\breve{X}_{N_1} := \text{span}\{\breve{u}^j : j = 1, \ldots, N_1\}$

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- for new $\mu = (\mu_0, \mu_1)$ define (stable) test space $Z_{N_1}(\mu_1) \in \mathcal{Z}_\delta$
  e.g. by supremizers

Space-Time RBM 3/3

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  where $\breve{u}_N(\mu) \in \breve{X}_{N_1}$ solves

$$b_1(\mu_1; \breve{u}_N(\mu), z_N) = \breve{f}(u_N^0(\mu_0), \mu_1; z_N) \qquad \forall z_N \in Z_{N_1}(\mu_1).$$

Space-Time RBM 3/3

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- NO time-marching!

Space-Time RBM 3/3

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$$b_1(\mu_1; \breve{u}_N(\mu), z_N) = \breve{f}(u_N^0(\mu_0), \mu_1; z_N) \qquad \forall z_N \in Z_{N_1}(\mu_1).$$

- NO time-marching!
- b_1 and $\breve{f}$ are tensor products!

Greedy for initial value

- determine by POD or adaptive approximation

Greedy for initial value

- determine by POD or adaptive approximation

  ... or
- $\Delta^0_{N_0}(\mu_0) := \|\mu_0 - \mu_0^N\|_H$

Greedy for initial value

1: Let $M^0_{\text{train}} \subset \mathcal{D}^0$ be the training set of initial values, $\text{tol}^0 > 0$ a tolerance.
2: Choose $\mu_0^1 \in M^0_{\text{train}}$, $S^0_1 := \{\mu_0^1\}$
3: **for** $N_0 = 1, \ldots, N_0^{\max}$ **do**
4: Compute $u_0^{N_0;0} = u_\delta^{0,0}(\mu_0^N) \in \mathcal{I}_M$ as in (3) % Offline 1st step
5: $\mu_0^{N_0+1} = \arg\max_{\mu_0 \in M^0_{\text{train}}} \Delta^0_{N_0}(\mu_0)$
6: **if** $\Delta^0_{N_0}(\mu_0^{N_0+1}) < \text{tol}^0$ **then** Stop **end if**
7: $S^0_{N_0+1} := S^0_{N_0} \cup \{\mu_0^{N_0+1}\}$
8: **end for**
9: $X^0_{N_0} := \text{span}\{u^{i;0} := \sigma^0 \otimes u_0^{i;0} : 1 \leq i \leq N_0\}$

Greedy for evolution

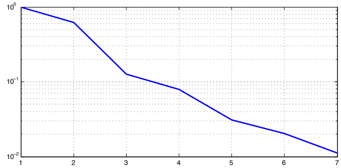
- $\Delta^1_{N_1}(\mu) := \beta_\delta^{-1} \| g_1(\mu_1) - b_1(\mu_1; u_N(\mu), \cdot) \|_{\mathcal{Y}'_\eta}$

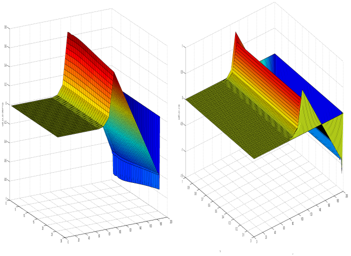
Greedy for evolution

Let $M_{\text{train}} \subset S^0_{N_0} \times M^1_{\text{train}}$ be the training set, $\text{tol}^1 > 0$ a tolerance.
Choose $\mu_1^{1,1} \in M^1_{\text{train}}$, $\mu^{1,1} := (\mu_0^1, \mu_1^{1,1})$, $S^1_1 := \{\mu^{1,1}\}$
Compute $u^{1,1;1} = \breve{u}_\delta(\mu^{1,1}) \in \breve{\mathcal{X}}_\delta$, $N_1 := 1$
for $i = 1, \ldots, N_0$ **do**
    for $j = 1, \ldots, N_1^{\max}$ **do**
        $\mu_1^j = \arg\max_{\mu_1 \in M^1_{\text{train}}} \Delta^1_{N_1}((\mu_0^i, \mu_1))$; $\mu^{i,j} := (\mu_0^i, \mu_1^j)$
        if $\Delta^1_{N_1}(\mu^{i,j}) < \text{tol}^1$ **then** $N_{i,1} := j$ **end for** j **end if**
        $N_1 := N_1 + 1$,
        Compute $u^{i,j;1} = \breve{u}_\delta(\mu^{i,j}) \in \breve{\mathcal{X}}_\delta$ % Offline 2nd step (e.g. C-N)
        $S^1_{N_1+1} := S^1_{N_1} \cup \{\mu^{i,j}\}$
    end for
end for

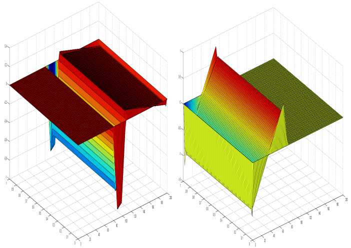
Numerical Results

- Heston model
- model payoff μ_0 by Bezier curves
- POD for initial value



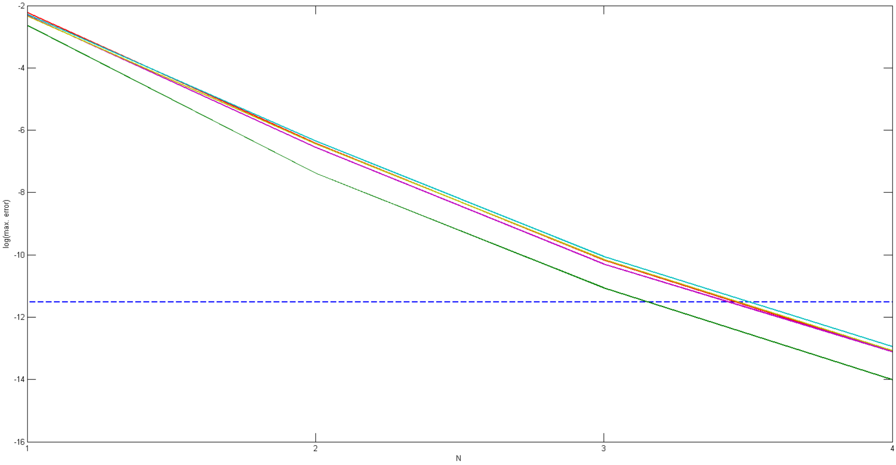


RB solution (internal / initial)



Extended initial values

Space-Time Errors



Erros vs. N_1 for different N_0 ($P = 1$ out of 5)
(for different parameter selections)

Recall: Additional challenges

- several options/assets (WASC, CDOs):
  $\rightsquigarrow$ many coupled PDEs, high (space) dimension
- American options: variational inequalities (Haasdonk, Salomon, Wohlmuth; Glas, U.)
- stochastic coefficients
- jump models (Lévy): integral operators, PIDEs (Schwab et al., Kestler, ...)
- problems on infinite domains ($S \in [0, \infty)$) (Kestler, U.)
- ...

CDO pricing model

CDO model: $\mathbf{N} = 2^n$ coupled PDEs: $j \in \{1, \dots \mathbf{N}\} = \mathcal{N}$

$$u_t^j(t,y) = -\frac{1}{2}\nabla \cdot (\mathbf{B}(t)\nabla u^j(t,y)) - \boldsymbol{\alpha}^T(t)\nabla u^j(t,y) + r(t,y)u^j(t,y) - \sum_{k \in \mathcal{N}\setminus\{j\}} q^{j,k}(t,y)(a^{j,k}(t,y) + u^k(t,y) - u^j(t,y)) - c^j(t,y), \quad (6a)$$

$$\mathbf{u}(t,y) = 0,\ t \in (0,T),\ \ y \in \partial\Omega, \quad (6b)$$

$$\mathbf{u}(T,y) = (u_T^0(y), \dots, u_T^{\mathbf{N}-1}(y))^T,\ y \in \Omega, \quad (6c)$$

- CDOs are one reason for the financial crisis
- coupling terms $q^{j,k}$ hardly known
- goal: find ways to control the market
  (sensitivities, restrictions to parameters, ...)

CDO Space-time variational formulation (Kiesel, Rupp, U.)

CDO model: $\mathbf{N} = 2^n$ coupled PDEs: $j \in \{1, \dots \mathbf{N}\} = \mathcal{N}$

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$$- \sum_{k \in \mathcal{N}\setminus\{j\}} q^{j,k}(t,y)(a^{j,k}(t,y) + u^k(t,y) - u^j(t,y)) - c^j(t,y),$$

$$\mathbf{u}(t,y) = 0,\ t \in (0,T),\ y \in \partial\Omega, \quad \mathbf{u}(T,y) = (u_T^0(y), \dots, u_T^{\mathbf{N}-1}(y))^T,\ y \in \Omega,$$

$$\begin{aligned}
\mathbf{X} &:= L_2(0,T;H_0^1(\Omega)^{\mathbf{N}}) \cap H^1(0,T;H^{-1}(\Omega)^{\mathbf{N}}) \\
\mathbf{Y} &:= L_2(0,T;H_0^1(\Omega)^{\mathbf{N}}) \times L_2(\Omega)^{\mathbf{N}}, \quad \mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \\
\mathbf{b}(\mu;\mathbf{u},\mathbf{v}) &:= \int_0^T [(\mathbf{u}_t(t), \mathbf{v}_1)_{0;\Omega} + \mathbf{a}(\mu;\mathbf{u}(t),\mathbf{v}_1)]\,dt + (\mathbf{u}(T), \mathbf{v}_2)_{0;\Omega} \\
\mathbf{f}(\mathbf{v}) &:= \int_0^T (\mathbf{f}(t), \mathbf{v}_1(t))_{0;\Omega} + (\mathbf{u}_T, \mathbf{v}_2)_{0;\Omega}
\end{aligned}$$

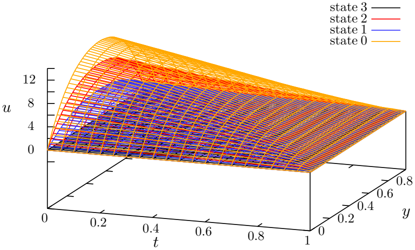
CDO space-time formulation

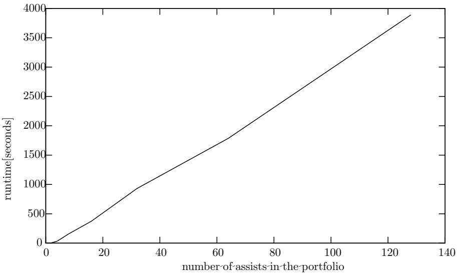
$$\mathbf{u} \in \mathbf{X}: \qquad \mathbf{b}(\mu;\mathbf{u},\mathbf{v}) = \mathbf{f}(\mathbf{v}) \quad \forall\, \mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{Y}. \tag{7}$$

HTucker simulation of CDOs

- use multiwavelets in space
  (Donovan, Geronimo, Hardin; Dijkema, Schwab, Stevenson)
- obtain equivalent ℓ_2-problem
  ($\rightarrow$ talk of W. Dahmen)
- can be written in tensor form
  (also space/time)

- use HTucker-format
  (Hackbusch, Kühn, Grasedyck, Kressner, ...)
  ($\rightarrow$ talk of R. Schneider)
- n: number of assets
- $\rightsquigarrow$ $\mathbf{N} = 2^n$ equations





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- ...

Parabolic Variational Inequality PVI(μ) (Glas, U.)

American / swing options $\rightsquigarrow$ obstacle problem: ($\rightarrow$ talks of K. Veroy, J. Salomon)

Parameterized Parabolic Variational Inequality:
For $\mu \in \mathcal{D}$, find $u(\mu; t) \in K(t)$, s.t. for all $v(t) \in K(t)$, $t \in (0, T) a.e.$

$$\langle u_t(\mu; t), v(t) - u(\mu; t)\rangle_{V' \times V} + a(\mu; u(\mu; t), v(t) - u(\mu; t)) \geqslant f(\mu; v(t) - u(\mu; t))$$

where

- $V \hookrightarrow H$ Hilbert Spaces
- $a(\mu; \cdot, \cdot) : \mathcal{D} \times V \times V \to \mathbb{R}$ (possibly non-coercive)
- $K(t) \subset V$ closed and convex set
- $f(\mu; \cdot) : V \to \mathbb{R}$

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Transfer into saddle point problem:

- W Hilbert space, $M \subset W$ convex cone
- $K(t) = \{v \in V | c(t; v, \eta) \leqslant g(\mu; \eta), \eta \in M\}$

For μ in $\mathcal{D}$, find $(u(\mu), \lambda(\mu)) \in V \times M$ such that for $t \in (0,T)$ a.e.

$$\begin{aligned} \langle u_t, v\rangle_{V' \times V} + a(\mu; u(\mu), v) + c(t; v, \lambda(\mu)) &= f(\mu; v), && v \in V \\ c(t; u(\mu), \eta - \lambda(\mu)) &\leqslant g(\mu; \eta - \lambda(\mu)), && \eta \in M. \end{aligned}$$

Space-Time Formulation of PVIs

$$\langle u_t(t), v(t)-u(t)\rangle + a(\mu; u(t), v(t)-u(t)) \geqslant f(\mu; v(t)-u(t)) \quad \forall v(t) \in V, t \in I \text{ a.e.}$$

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- $\mathcal{X} := \{w \in L_2(I; V) : \dot{w} \in L_2(I; V'), w(0) = 0\}$

$$\int_0^T \langle u_t, v-u\rangle dt + \int_0^T a(\mu; u, v-u) dt \geqslant \int_0^T f(\mu; v-u) dt \qquad \forall v \in \mathcal{X}$$

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$$\underbrace{\int_0^T \langle u_t, v-u\rangle dt + \int_0^T a(\mu; u, v-u) dt}_{b(\mu; u, v-u)} \geqslant \underbrace{\int_0^T f(\mu; v-u) dt}_{\tilde{f}(v-u;\mu)} \qquad \forall v \in \mathcal{X}$$

Petrov-Galerkin Problem

Space-time Saddle Point Problem:

For μ in $\mathcal{D}$, find $(u(\mu), \lambda(\mu)) \in \mathcal{X} \times \mathcal{M}$ ($\mathcal{M} \subseteq C(I; M)$) such that

$$\begin{aligned} b(\mu; u(\mu), v) + c(v, \lambda(\mu)) &= \tilde{f}(\mu; v), \qquad && v \in \mathcal{Y} := L_2(I; V) \\ c(u(\mu), \eta - \lambda(\mu)) &\leqslant g(\mu; \eta - \lambda(\mu)), \qquad && \eta \in \mathcal{M}. \end{aligned}$$

- Recall $\mathcal{X} \hookrightarrow C(I; H)$

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- Recall $\mathcal{X} \hookrightarrow C(I; H)$
- (Semi-)Norms:
  - $\|v\|_{\mathcal{Y}} := \|v\|^2_{L_2(I;V)}$
  - $\|v\|^2_{\mathcal{X}} := \|v\|^2_{L_2(I;V)} + \|v_t\|^2_{L_2(I;V')}$

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$$
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\end{aligned}
$$

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  - $\|v\|_{\mathcal{Y}} := \|v\|^2_{L_2(I;V)}$
  - $\|v\|^2_{\mathcal{X}} := \|v\|^2_{L_2(I;V)} + \|v_t\|^2_{L_2(I;V')}$
  - $|\!|\!|v|\!|\!|^2_{\mathcal{X}} := \|v\|^2_{L_2(I;V)} + \|v_t\|^2_{L_2(I;V')} + \|v(T)\|^2_H$ (U., Patera)

Petrov-Galerkin Problem

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  - $\|v\|^2_{\mathcal{X}} := \|v\|^2_{L_2(I;V)} + \|v_t\|^2_{L_2(I;V')}$
  - $|||v|||^2_{\mathcal{X}} := \|v\|^2_{L_2(I;V)} + \|v_t\|^2_{L_2(I;V')} + \|v(T)\|^2_H$ (U., Patera)
  - $[\![v]\!]^2_{\mathcal{X}} := \|v\|^2_{L_2(I;V)} + \|v(T)\|^2_H$ (weaker than $|||\cdot|||_{\mathcal{X}}$, $\|\cdot\|_{\mathcal{X}}$)
    $([\![v]\!]^2_{\mathcal{X}} := \|v\|^2_{L_2(I;V)} + \|v(T)\|^2_H < \|v\|^2_{L_2(I;V)} + \|v_t\|^2_{L_2(I;V')} + \|v(T)\|^2_H = |||v|||^2_{\mathcal{X}})$

Preliminaries

Properties/Assumptions:

(A1) Bilinear forms $b(\mu;\cdot,\cdot), c(\cdot,\cdot)$ bounded with constants γ_b, γ_c ✓

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(A2) The form $b(\mu; \cdot, \cdot)$ is *weakly coercive* with coercivity constant $\alpha_w > 0$, i.e.,

$$b(\mu; v, v) \geq \alpha_w \, [\![v]\!]^2_{\mathcal{X}}, \quad v \in \mathcal{X} \qquad ([\![v]\!]_{\mathcal{X}} < \|v\|_X)$$

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Proof: (in the coercive case)

$$\begin{aligned} b(\mu; v, v) &= \int_0^T \langle v_t, v\rangle dt + \int_0^T a(\mu; v, v) dt \\ &\geqslant \frac{1}{2}\|v(T)\|_H^2 + \int_0^T (\alpha_a \|v(t)\|_V^2 - \lambda_a \|v(t)\|_H^2) dt \\ &\geqslant \frac{1}{2}\|v(T)\|_H^2 + (\alpha_a - \lambda_a \varrho^2)\|v\|_Y^2 \\ &\geqslant \min\{1/2, \alpha_a - \lambda_a \varrho^2\} [\![v]\!]_{\mathcal{X}}^2 \end{aligned}$$

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(A3) The form $c(\cdot,\cdot)$ is inf-sup-stable on $\mathcal{Y} \times \mathcal{W}$, i.e. $\exists \beta_c > 0$:

$$\sup_{v \in \mathcal{Y}} \frac{c(v,q)}{\|v\|_{\mathcal{Y}} \|q\|_{\mathcal{W}}} \geqslant \beta_c \|q\|_{\mathcal{W}}, \quad \forall q \in \mathcal{W} (\subset L_2(I;W))$$

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$$b(\mu; v, w) \leqslant \gamma_s [\![v]\!]_{\mathcal{X}} |\!|\!| w |\!|\!|_X \text{for } v, w \in X \checkmark \text{(integration by parts)}$$

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- (A1-A4) $\rightsquigarrow$ well-posedness of the problem (Glas, U.; Lions/Stampacchia)

RBM: Error/Residual estimate 1/3

Residuals (space/time):

$$r_N(\mu; v) := b(\mu; u - u_N, v) + c(v, p - p_N), \qquad v \in \mathcal{Y},$$
$$s_N(\mu; q) := c(u_N, q) - g(\mu; q), \qquad q \in \mathcal{W},$$

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Projection: (from the stationary case; [HSW])

- $\pi : W \to M$ orthogonal with respect to $\langle \cdot, \cdot \rangle_\pi$ on W.
- Induced norm on W, $\|\eta\|_\pi := \sqrt{\langle \eta, \eta \rangle_\pi}$,
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- extend that to space/time: $\mathcal{W}$, $\mathcal{M}$

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Primal/Dual Error Relation

Properties (A1)-(A4) and $\inf_{q \in \mathcal{W}} \sup_{v \in \mathcal{X}} \frac{c(v, q)}{[\![v]\!]_{\mathcal{X}} \|q\|_{\mathcal{W}}} \geq \beta_c > 0$ (✓)

do not yield a primal/dual error relation like:

$$\|p - p_N\|_W \leq \frac{1}{\beta_1} (|\!|\!| r_N |\!|\!|_{X'} + \gamma_s [\![u - u_N]\!]_{\mathcal{X}}).$$

RBM: Error/Residual estimate 2/3

Assumption (D):

Assume the existence of an invertible mapping $D : \mathcal{M} \to \mathcal{X}$ such that

(1) $c(Dp, q) = \langle v, q \rangle_{\mathcal{W}}$, $p, q \in \mathcal{M}$

(2) $\exists C_D$, s.t. $|\!|\!| Dp |\!|\!|_{\mathcal{X}} \leqslant C_D \|p\|_{\mathcal{W}}$

- controls temporal movement/change of obstacle
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Primal/dual error relation

If (A1)-(A4), inf-sup and (D) hold, we have

$$\| p - p_N \|_{\mathcal{W}} \leqslant C_D (|\!|\!| r_N |\!|\!|_{X'} + \gamma_s [\![u - u_N]\!]_{\mathcal{X}}).$$

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- error w.r.t. weaker (semi-)norm $[\![\cdot]\!]_{\mathcal{X}}$, not $|||\cdot|||_{\mathcal{X}}$ or $\|\cdot\|_{\mathcal{X}}$

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Note:

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- (D) poses requirement on the movement of the obstacle (in time)
- choice of $c(\cdot,\cdot) \rightsquigarrow$ enforcement of obstacle (point wise, average, ...)

RBM: Error/Residual estimate 3/3

Error/residual estimate

Let (A1)-(A4), inf-sup, (D) hold. Then

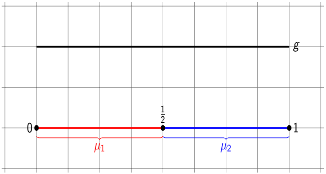
$$[\![u - u_N]\!]_{\mathcal{X}} := \Delta_u = c_1 + (c_1^2 + c_2)^{1/2}$$
$$\|p - p_N\|_{\mathcal{W}} := \Delta_p = C_D(|\!|\!| r_N |\!|\!|_{\mathcal{X}'} + \gamma_s \Delta_u)$$

$$c_1 := \frac{1}{2\alpha_w}([\![r_N]\!]_{\mathcal{X}'} + \gamma_s C_D \|\pi(\hat{s}_N)\|_{\mathcal{W}})$$
$$c_2 := \frac{1}{\alpha_w}(C_D |\!|\!| r_N |\!|\!|_{\mathcal{X}'} \|\pi(\hat{s}_N)\|_{\mathcal{W}} + \langle p_N, \pi(\hat{s}_N)\rangle_{\mathcal{W}})$$

Numerical example: 1-D Heat Conduction

Wire with two heat conductivities:

- $D := [0, 1]$, $D_1 := [0, \frac{1}{2})$, $D_2 := [\frac{1}{2}, 1]$
- $t \in [0, T]$
- $\mu := \mu_1 \chi_{[0,\frac{1}{2})} + \mu_2 \chi_{[\frac{1}{2},1]}$

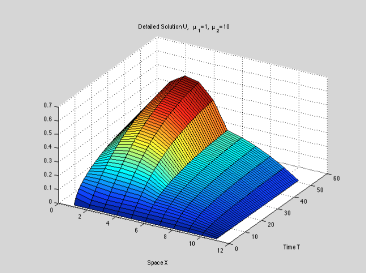


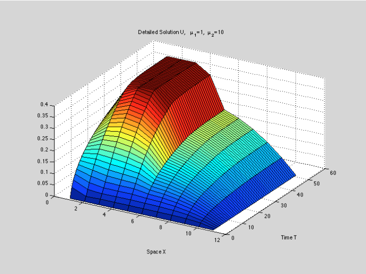
Strong Formulation:

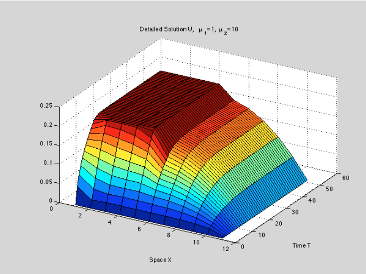
$$\begin{cases} u_t - \nabla(\mu \nabla u) \geqslant f, & x \in D,\ t \in [0, T] \\ \mu \dfrac{\partial u}{\partial n} = 1, & x \in \{0\},\ t \in [0, T] \\ u = 0, & x \in \{1\},\ t \in [0, T] \\ u(x, 0) = 0, & x \in D \end{cases}$$

Detailed Solution with obstacle

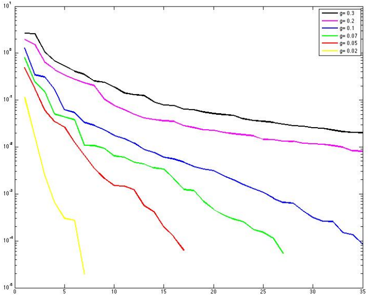
- $f = 1$
- Obstacle constant 0.6, 0.4, 0.2
- $D = [0, 1]$, #intervals $= 10$
- $T = 0.1$, #intervals $= 50$

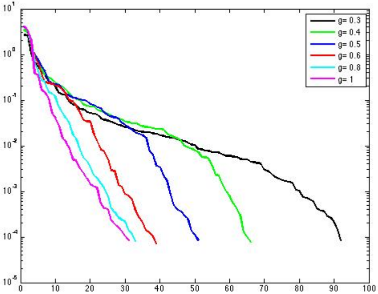






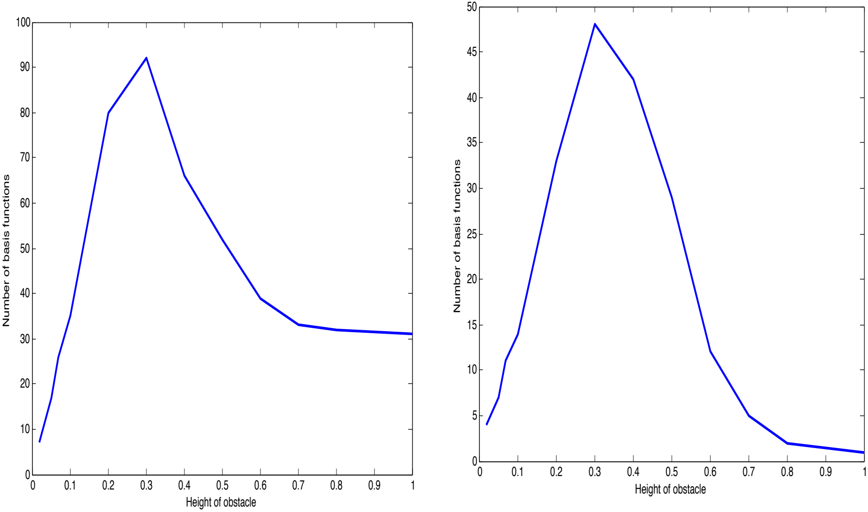
Greedy for primal basis





Error decay vs. obstacle

Greedy — # of basis functions vs. obstacle



N vs. obstacle

Recall: Additional challenges

- several options/assets (WASC, CDOs):
  $\rightsquigarrow$ many coupled PDEs, high (space) dimension
- American options: variational inequalities (Haasdonk, Salomon, Wohlmuth; Glas, U.)
- stochastic coefficients
- jump models (Lévy): integral operators, PIDEs (Schwab et al., Kestler, ...)
- problems on infinite domains ($S \in [0, \infty)$) (Kestler, U.)
- ...

PPDEs with stochastic parameters (U., Wieland)

(→ talk of G. Rozza)

- Deterministic parameter domain $\mathcal{D} \subset \mathbb{R}^P$, $\mu \in \mathcal{D}$ deterministic parameter
- Probability space $(\Omega, \mathcal{B}, P)$, $\omega \in \Omega$ probabilistic parameter
- $D \subset \mathbb{R}^d$ open, bounded (domain of PDE)
- Hilbert space $X \subset H^1(D)$ (boundary conditions), dimension $\mathcal{N}$ – truth

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Problem Formulation

For $(\mu, \omega) \in \mathcal{D} \times \Omega$ find $u = u(\mu, \omega) \in X$ s.t.

$$b(\mu, \omega; u, v) = f(\mu, \omega; v) \qquad \forall v \in X.$$

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Evaluate outputs of interest

$$\begin{aligned} s(\mu, \omega) &:= \ell\,(u(\mu, \omega);\ \mu), \\ \mathbb{E}(\mu) &:= \mathbb{E}\left[s(\mu, \cdot)\right], \\ \mathbb{V}(\mu) &:= \mathbb{E}\left[s^2(\mu, \cdot)\right] - \mathbb{E}\left[s(\mu, \cdot)\right]^2, \ldots \end{aligned}$$

Karhunen-Loève (KL) Expansion

Random variable:

$$\kappa(x;\mu,\omega) = \kappa_0(x;\mu) + \tilde{\kappa}(x;\mu,\omega), \quad \mathbb{E}[\tilde{\kappa}(x;\mu,\cdot)] = 0, \ \mathbb{E}[\kappa(x;\mu,\cdot)] = \kappa_0(x;\mu)$$

(empirical) covariance matrix $(x_i, x_j \in D)$

$$C = C(\mu) := \Big(Cov_\kappa(x_i, x_j)\Big)_{ij} = \Big(\mathbb{E}\big[\tilde{\kappa}(x_i;\mu,\cdot)\,\tilde{\kappa}(x_j;\mu,\cdot)\big]\Big)_{ij},$$

with eigenvalues $\lambda_k(\mu)$ and eigenfunctions $\kappa_k(x;\mu)$

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- λ_k often decreasing exponentially $(k \to \infty)$ $\rightsquigarrow$ truncate at $\bar{K} < \infty$
- ξ_k zero mean, unit variance, uncorrelated
- κ_k orthonormal

Further assumptions (for notational simplicity):

- f is deterministic and parameter independent,
- ℓ is deterministic and parameter independent,

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Variational Primal-Dual Problem

For $(\mu, \omega) \in \mathcal{D} \times \Omega$, find $u = u(\mu, \omega) \in X$ and $p = p(\mu, \omega) \in X$ s.t.

$$\begin{aligned} b(\mu, \omega; u, v) &= f(v) \qquad \forall v \in X, \\ b(\mu, \omega; v, p) &= -\ell(v) \qquad \forall v \in X. \end{aligned}$$

Further assumptions (for notational simplicity):

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Truncated Variational Primal-Dual Problem

For $(\mu, \omega) \in \mathcal{D} \times \Omega$, find $u^K = u^K(\mu, \omega) \in X$ and $p^K = p^K(\mu, \omega) \in X$ s.t.

$$\begin{aligned} b^K(\mu, \omega; u^K, v) &= f(v) \qquad \forall v \in X, \\ b^K(\mu, \omega; v, p^K) &= -\ell(v) \qquad \forall v \in X. \end{aligned}$$

KL Truncation

- Truncate KL series at some $K \ll \bar{K}$ (λ_k decrease fast)
- Truncated bilinear form $b^K(\mu, \omega; w, v)$

RB System

Reduced Basis System

- RB subspaces (Greedy) w.r.t. pairs (μ_i, ω_i)
  $X_N = \mathrm{span}\{u^K(\mu_i, \omega_i)\}_{i=1,\dots,N} = \mathrm{span}\{\zeta_i\}_{i=1,\dots,N} \subset X, \quad \tilde{X}_N \dots$
- evaluate and store parameter-independent terms

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RB Variational Problem

For $\mu \in \mathcal{D}$, $\omega \in \Omega$, find $u^{NK} \in X_N$ and $p^{NK} \in \tilde{X}_N$ s.t.

$$\begin{aligned} b^K(\mu, \omega; u^{NK}, v) &= f(v) \qquad \forall v \in X_N \\ b^K(\mu, \omega; v, p^{NK}) &= -\ell(v) \qquad \forall v \in \tilde{X}_N \end{aligned}$$

Complexity for each parameter pair (μ, ω):

- $\mathcal{O}(QKN^2)$ to assemble system
- $\mathcal{O}(N^3)$ to solve the system
- $\mathcal{O}(QKN^2)$ to evaluate output $s(\mu, \omega) = \ell(u^{NK}(\mu, \omega)) - r^K(\mu, \omega; p^{NK}(\mu, \omega))$

Primal and Dual Error Bounds

Proposition (Error bounds)

For the primal and dual problem, we have the error estimates

$$\begin{aligned} \|u - u^{NK}\|_X &\leq \Delta := \Delta_{RB} + \Delta_{KL} \\ \|p - p^{NK}\|_X &\leq \tilde{\Delta} := \Delta_{RB} + \tilde{\Delta}_{KL} \end{aligned}$$

Linear Output Error Bound

Proposition (RB output)

Using the correction term $r^K(p^{NK})$, the RB output is given by

$$s^{NK}(\mu, \omega) \quad := \quad \ell(u^{NK}) - r^K(p^{NK})$$

Proposition (Output error bound)

The output error bound is then given by

$$|s - s^{NK}| \quad \leq \quad \Delta_s \quad := \quad \alpha_{LB} \Delta \, \tilde{\Delta} \; + \; \delta_{KL}(p^{NK})$$

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- recall: $\Delta := \Delta_{RB} + \Delta_{KL}$
- Δ_{RB}, Δ_{KL} are multiplied $\Rightarrow$ only small N necessary
- δ_{KL} is more precise than Δ_{KL} and decreases fast in K

Quadratic Output

- Output of Interest: $\mathbb{V}(\mu) := \mathbb{E}\left[s^2(\mu,\cdot)\right] - \mathbb{E}\left[s(\mu,\cdot)\right]^2$

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Additional Dual Problems

For $(\mu,\omega) \in \mathcal{D} \times \Omega$, find $p_1,\ p_2 \in X$ s.t.

$$\text{(D-1)} \qquad b(\mu,\omega; v, p_1) = -2s^{NK}(\mu,\omega)\cdot \ell(v) \quad \forall v \in X$$
$$\text{(D-2)} \qquad b(\mu,\omega; v, p_2) = -2\,\mathbb{E}^{NK}(\mu)\cdot \ell(v) \quad \forall v \in X$$

Additional Dual RB Problems

For $(\mu,\omega) \in \mathcal{D} \times \Omega$, find $p_1^{NK} \in \tilde{X}_N^1$ and $p_2^{NK} \in \tilde{X}_N^2$ s.t.

$$\text{(RB-D-1)} \qquad b^K(\mu,\omega; v, p_1^{NK}) = -2s^{NK}(\mu,\omega)\cdot \ell(v) \quad \forall v \in \tilde{X}_N^1$$
$$\text{(RB-D-2)} \qquad b^K(\mu,\omega; v, p_2^{NK}) = -2\,\mathbb{E}^{NK}(\mu)\cdot \ell(v) \quad \forall v \in \tilde{X}_N^2$$

Variance Error

Analogously to Δ^{s^2}, we obtain

Squared expected value

$$|\mathbb{E}^2 - \mathbb{E}^{2,NK}| \quad \leq \quad \Delta^{\mathbb{E}^2} \quad := \quad \left(\Delta^{\mathbb{E}}\right)^2 + \mathbb{E}\Big[\alpha_{LB}\Delta\,\tilde{\Delta}^2\Big] \; + \; \mathbb{E}\left[\delta_{KL}(p_2^{NK})\right]$$

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Variance error bound

$$|\mathbb{V} - \mathbb{V}^{NK}| \leq \Delta^{\mathbb{V}} := \mathbb{E}\left[\Delta^{s^2}\right] + \Delta^{\mathbb{E}^2}$$

- Improved variance error bound for $\tilde{X}_N^1 = \tilde{X}_N^2$

Numerical example: Heat Transfer in Porous Media

Heat transfer in a wet sandstone with conductivity

$$\alpha(\mu, \omega; x) = (1 - \kappa(x; \omega))c_s + \kappa(x; \omega)\left(\mu c_w + (1 - \mu)c_a\right)$$

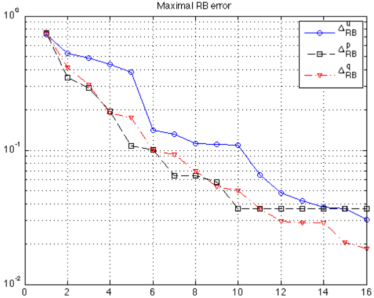
where

- c_s, c_w, c_a: conductivities of sandstone, water and air
- $\kappa(x; \omega) := \frac{\text{volume unit of pore space}}{\text{volume unit}} \in (0, 1)$
- $\mu \in \mathcal{D} = [0.01; 1]$: global saturation of water

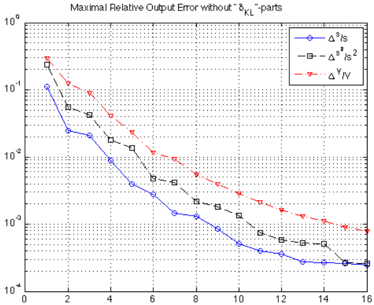
$$\begin{cases} -\nabla \cdot \Big(\alpha(\mu, \omega; x)\nabla u(\mu, \omega; x)\Big) & = & 0 & \forall x \in D := (0, 1)^2 \\ u(\mu, \omega; x) & = & 0 & \forall x \in \Gamma_D \\ \vec{n} \cdot \Big(\alpha(\mu, \omega; x)\nabla u(\mu, \omega; x)\Big) & = & 0 & \forall x \in \Gamma_N \\ \vec{n} \cdot \Big(\alpha(\mu, \omega; x)\nabla u(\mu, \omega; x)\Big) & = & g(\omega; x) & \forall x \in \Gamma_{out} \end{cases}$$

- Output: $s(\mu, \omega) := \int_{\Gamma_{OUT}} u(\mu, \omega; x) dx$

Convergence of Error Bounds



Maximal RB Error Bounds

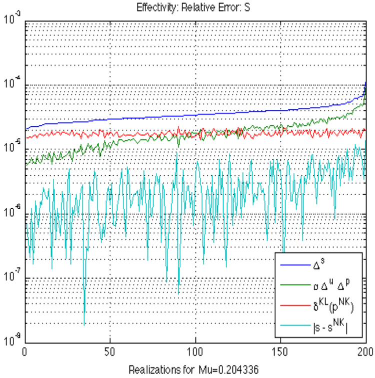


Maximal Relative Output Error Bounds

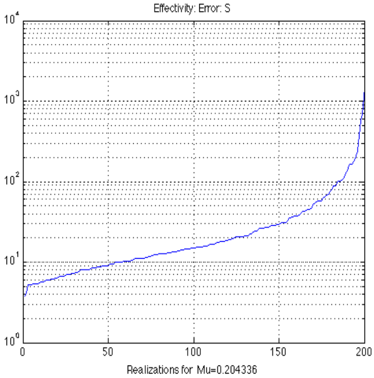
primal, dual, additional dual

linear, quadratic, variance

Variance Error Bounds



Error Contributions



Sorted Effectivity

(200 realizations; $\mu = 0.204336$)

linear, primal·dual, KL, true

Summary and Outlook

- initial value parameter functions in RB
  - space/time-variational formulation
  - separate space/time RB computation $\rightsquigarrow$ huge reduction
- CDO pricing with HTucker ($\mathbf{N} = 2^{128}$)
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  - tensor product structure
- parabolic variational inequalities
  - well-posedness in space/time
  - error/residual error estimator, also for non-coercive blf's
- PPDEs with stochastic coefficients (KL-expansion, quadratic outputs)

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Outlook:

- 'optimal' approximation of initial value (adaptive, dictionaries - K. Steih)
- optimize HTucker in space/time
- extensions (PIDEs, nonlinear, RB and adaptivity, ...)

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