Online-Efficient RB Methods for Contact and Other Problems in Nonlinear Solid Mechanics

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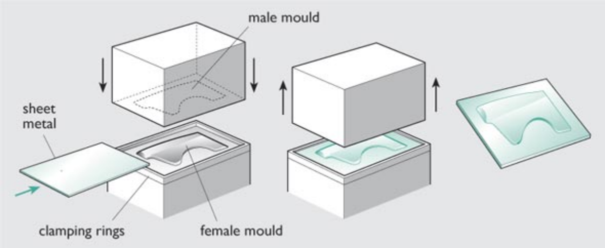
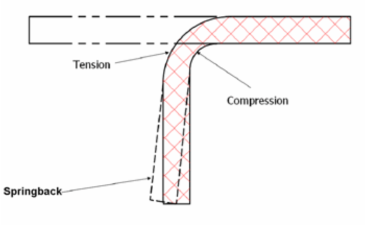
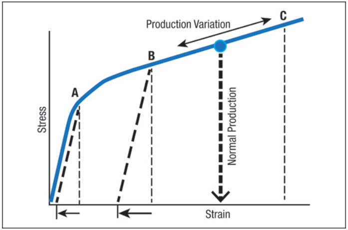
Joint work with Z. Zhang and E. Bader

14–18 April 2014

Motivating Example: Manufacturing







Common Thread

Contact

Friction

Elastoplasticity

Common Thread

Variational Inequalities (VIs)

Contact

Friction

Elastoplasticity

Common Thread

	Variational Inequalities (VIs)
Contact	Elliptic VI of the 1st kind (EVI-1)
Friction	
Elastoplasticity	

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Common Thread

	Variational Inequalities (VIs)
Contact	Elliptic VI of the 1st kind (EVI-1)
Friction	Elliptic VI of the 2nd kind (EVI-2)
Elastoplasticity	

Common Thread

	Variational Inequalities (VIs)
Contact	Elliptic VI of the 1st kind (EVI-1)
Friction	Elliptic VI of the 2nd kind (EVI-2)
Elastoplasticity	Parabolic VI (with EVI-1,2)

RB for Parametrized VIs

Haasdonk, Salomon & Wohlmuth (SIAM J Num Anal, 2012)

- Reduced Basis Method (RBM) for EVI-1

Haasdonk, Salomon & Wohlmuth (Num Math & Adv App, 2011)

- RBM for PVI-1

Glas & Urban (preprint, 2013)

- RBM for PVI-1 through space-time formulation

RB for Parametrized VIs

[HSW12]

- RB approximation and error estimation for **EVI-1**
- **Partial** offline/online computational decomposition
- Online cost to evaluate error estimates depends on $\mathcal{N}_{\mathrm{FE}}$
- Numerical results for **one-dimensional** obstacle problem

Difficulties

- High online cost for more complex 2- or 3-D problems
- Applicable only to EVI-1 (or PVI-1)

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The Plan

EVIs of the 1st kind

- Simple Obstacle Problem
- General Formulation
- Reduced Basis Method [HSW12]

Proposed Methods

- Method D
- Method R

Summary & Perspectives

Obstacle Problem

Region of no contact

$$\begin{aligned} -\nabla^2 u - f &= 0 \\ u &< g \end{aligned}$$

Region of contact

$$\begin{aligned} -\nabla^2 u - f &\geq 0 \\ u &= g \end{aligned}$$

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Obstacle Problem

Admissible Displacements

$$K = \{\ v \text{ sufficiently smooth} \mid v \leq g \text{ in } \Omega\ \}$$

Constrained Minimization Statement

$$u = \arg \inf_{v \in K} \int_\Omega \left(\frac{1}{2} \nabla v \cdot \nabla v - f v \right) dx$$

Weak Form

$$\int_\Omega \nabla u \cdot \nabla (v - u)\, dx \geq \int_\Omega f (v - u)\, dx, \qquad \forall\, v \in K$$

Obstacle Problem

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Elliptic Variational Inequality - 1st kind

Admissible Set

$$\boldsymbol{K} \text{ a convex subset of } \boldsymbol{V}$$

Constrained Minimization Statement

$$u = \arg \inf_{v \in K} \frac{1}{2} a(v, v) - f(v)$$

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$$a(u, v - u) \geq f(v - u) \qquad \forall\, v \in K$$

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Elliptic Variational Inequality - 1st kind

Admissible Set

$$K = \{\ v \in V \mid b(v,\eta) \le g(\eta),\ \forall\ \eta \in M\ \}$$

Saddle Point Inequality

$$\begin{aligned} a(u,v) + b(v,\lambda) &= f(v) && \forall\ v \in V \\ b(u,\eta-\lambda) &\le g(\eta-\lambda) && \forall\ \eta \in M \end{aligned}$$

where $u \in V$, $\lambda \in M$.

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Elliptic Variational Inequality - 1st kind

Admissible Set

$$\boldsymbol{K} = \{ \boldsymbol{v} \in \boldsymbol{V} \mid \boldsymbol{b}(\boldsymbol{v}, \boldsymbol{\eta}) \leq \boldsymbol{g}(\boldsymbol{\eta}), \ \forall \ \boldsymbol{\eta} \in \boldsymbol{M} \ \}$$

Saddle Point Inequality

$$\begin{aligned} \boldsymbol{a}(\boldsymbol{u}, \boldsymbol{v}) + \boldsymbol{b}(\boldsymbol{v}, \boldsymbol{\lambda}) &= \boldsymbol{f}(\boldsymbol{v}) && \forall \ \boldsymbol{v} \in \boldsymbol{V} \\ \boldsymbol{b}(\boldsymbol{u}, \boldsymbol{\eta} - \boldsymbol{\lambda}) &\leq \boldsymbol{g}(\boldsymbol{\eta} - \boldsymbol{\lambda}) && \forall \ \boldsymbol{\eta} \in \boldsymbol{M} \end{aligned}$$

where $\boldsymbol{u} \in \boldsymbol{V}$, $\boldsymbol{\lambda} \in \boldsymbol{M}$.

Elliptic Variational Inequality - 1st kind

KKT Conditions

The solution $(\boldsymbol{u}, \boldsymbol{\lambda}) \in \boldsymbol{V} \times \boldsymbol{M}$ satisfies

$\boldsymbol{Au} + \boldsymbol{B}^T\boldsymbol{\lambda}$	$=$	$\boldsymbol{f}$	**STATIONARITY**
$\boldsymbol{g} - \boldsymbol{Bu}$	$\geq$	$\boldsymbol{0}$	**PRIMAL FEASIBILITY**
$\boldsymbol{\lambda}$	$\geq$	$\boldsymbol{0}$	**DUAL FEASIBILITY**
$\boldsymbol{\lambda}^T(\boldsymbol{g} - \boldsymbol{Bu})$	$=$	$\boldsymbol{0}$	**COMPLEMENTARITY**

Elliptic Variational Inequality - 1st kind

Parametrized KKT Conditions

The solution $(\boldsymbol{u}, \boldsymbol{\lambda}) \in \boldsymbol{V} \times \boldsymbol{M}$ satisfies

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\begin{aligned}
\boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}^T\boldsymbol{\lambda} &= \boldsymbol{f} && \textbf{STATIONARITY} \\
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\boldsymbol{\lambda} &\geq \boldsymbol{0} && \textbf{DUAL FEASIBILITY} \\
\boldsymbol{\lambda}^T(\boldsymbol{g} - \boldsymbol{B}\boldsymbol{u}) &= \boldsymbol{0} && \textbf{COMPLEMENTARITY}
\end{aligned}
$$

Elliptic Variational Inequality - 1st kind

Parametrized KKT Conditions

The solution $(u(\mu), \lambda(\mu)) \in V \times M$ satisfies

$$\begin{aligned} A(\mu)u(\mu) + B^T(\mu)\lambda(\mu) &= f(\mu) && \textbf{STATIONARITY} \\ g(\mu) - B(\mu)u(\mu) &\geq 0 && \textbf{PRIMAL FEASIBILITY} \\ \lambda(\mu) &\geq 0 && \textbf{DUAL FEASIBILITY} \\ \lambda^T(g(\mu) - B(\mu)u(\mu)) &= 0 && \textbf{COMPLEMENTARITY} \end{aligned}$$

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\boldsymbol{\lambda}^{\boldsymbol{T}}(\boldsymbol{g} - \boldsymbol{B}\boldsymbol{u}) &= \boldsymbol{0} && \textbf{COMPLEMENTARITY}
\end{aligned}
$$

Reduced Basis Method for EVI-1 [HSW12]

Following **[HSW12]**, we introduce $\boldsymbol{1 \leq i \leq N}$

$$\boldsymbol{W_N} \quad = \quad \textbf{span}\{\ \boldsymbol{\lambda}(\boldsymbol{\mu_i})\ \}$$

$\boldsymbol{\lambda}$-SNAPSHOTS

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$$\boldsymbol{W_N} = \mathrm{span}\{\ \boldsymbol{\lambda}(\boldsymbol{\mu_i})\ \}$$

$\boldsymbol{\lambda}$-SNAPSHOTS

$$\boldsymbol{V_N} = \mathrm{span}\{\ \boldsymbol{u}(\boldsymbol{\mu_i}),\ \boldsymbol{T}\boldsymbol{\lambda}(\boldsymbol{\mu_i})\ \} = \mathrm{span}\{\ \boldsymbol{\varphi_j},\ \mathbf{1} \leq \boldsymbol{j} \leq \boldsymbol{N_u}\ \}$$

$\boldsymbol{u}$-SNAPSHOTS

$+$ SUPREMIZERS

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$$\boldsymbol{W_N} = \mathbf{span}\{\ \boldsymbol{\lambda}(\boldsymbol{\mu_i})\ \}$$ $\boldsymbol{\lambda}$-SNAPSHOTS

$$\boldsymbol{V_N} = \mathbf{span}\{\ \boldsymbol{u}(\boldsymbol{\mu_i}),\ \boldsymbol{T}\boldsymbol{\lambda}(\boldsymbol{\mu_i})\ \}$$ $\boldsymbol{u}$-SNAPSHOTS + SUPREMIZERS

$$= \mathbf{span}\{\ \boldsymbol{\varphi_j},\ \mathbf{1} \leq \boldsymbol{j} \leq \boldsymbol{N_u}\ \}$$

$$\boldsymbol{M_N} = \mathbf{span}_+\{\ \boldsymbol{\lambda}(\boldsymbol{\mu_i})\ \}$$ CONVEX CONE

$$= \left\{\ \sum_{i=1}^{N} \boldsymbol{\alpha_i}\boldsymbol{\lambda}(\boldsymbol{\mu_i}) \,|\, \boldsymbol{\alpha_i} \geq \mathbf{0}\ \right\}$$

Reduced Basis Method for EVI-1 [HSW12]

We then define our RB approximations as

$$
\begin{aligned}
u_N(\mu) &= \sum_{i=1}^{N_u} \underline{u}_{Ni}(\mu)\,\varphi_i && \in V_N \\
\lambda_N(\mu) &= \sum_{i=1}^{N_\lambda} \underline{\lambda}_{Ni}(\mu)\,\lambda(\mu_i) && \in M_N
\end{aligned}
$$

where $u_N \in V_N$ and $\lambda_N \in M_N$ satisfy

$$
\begin{aligned}
a(u_N, v) + b(v, \lambda_N) &= f(v) && \forall\, v \in V_N \\
b(u_N, \eta - \lambda_N) &\le g(\eta - \lambda_N) && \forall\, \eta \in M_N
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Reduced Basis Method for EVI-1 [HSW12]

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Reduced Basis Method for EVI-1 [HSW12]

The coefficients $\underline{\boldsymbol{u}}_{\boldsymbol{N}}(\boldsymbol{\mu}) \in \mathbb{R}^{\boldsymbol{N_u}}$ and $\underline{\boldsymbol{\lambda}}_{\boldsymbol{N}}(\boldsymbol{\mu}) \in \mathbb{R}^{\boldsymbol{N_\lambda}}$ satisfy

$$
\begin{aligned}
\boldsymbol{A_N}\underline{\boldsymbol{u}}_{\boldsymbol{N}} + {\boldsymbol{B_N}}^{\boldsymbol{T}}\underline{\boldsymbol{\lambda}}_{\boldsymbol{N}} &= \boldsymbol{f_N} \\
\boldsymbol{g_N} - \boldsymbol{B_N}\underline{\boldsymbol{u}}_{\boldsymbol{N}} &\geq \boldsymbol{0} \\
\underline{\boldsymbol{\lambda}}_{\boldsymbol{N}} &\geq \boldsymbol{0} \\
\underline{\boldsymbol{\lambda}}_{\boldsymbol{N}}^{\boldsymbol{T}}(\boldsymbol{g_N} - \boldsymbol{B_N}\underline{\boldsymbol{u}}_{\boldsymbol{N}}) &= \boldsymbol{0}
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How can we quantify the error $\|u - u_N\|_V$?

Reduced Basis Method for EVI-1 [HSW12]

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How can we quantify the error $\|\boldsymbol{u} - \boldsymbol{u_N}\|_{\boldsymbol{V}}$?

Reduced Basis Method for EVI-1 [HSW12]

Substituting u_N and λ_N into the original problem

$$
\begin{aligned}
r_{\mathbf{E}} &= f - A\, u_N - B^T \lambda_N && \textbf{EQUALITY RESIDUAL} \\
r_{\mathbf{I}} &= B\, u_N - g && \textbf{“INEQUALITY RESIDUAL”}
\end{aligned}
$$

Following [HSW12], error is indicated by

$$
\begin{aligned}
r_{\mathbf{E}} &\neq 0 \\
[r_{\mathbf{I}}]_+ &= [B\, u_N - g]_+ && \textbf{component-wise positive part}
\end{aligned}
$$

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Reduced Basis Method for EVI-1 [HSW12]

Substituting $\boldsymbol{u_N}$ and $\boldsymbol{\lambda_N}$ into the original problem

$$\boldsymbol{r_{\mathbf{E}}} = \boldsymbol{f} - \boldsymbol{A}\,\boldsymbol{u_N} - \boldsymbol{B}^{\boldsymbol{T}}\boldsymbol{\lambda_N}$$

EQUALITY RESIDUAL

$$\boldsymbol{r_{\mathbf{I}}} = \boldsymbol{B}\,\boldsymbol{u_N} - \boldsymbol{g}$$

"INEQUALITY RESIDUAL"

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component-wise positive part

Reduced Basis Method for EVI-1 [HSW12]

The RB approximation errors can be bounded by **[PROP 4.2]**

$$\|u - u_N\|_V \leq \Delta_u := c_1 + \sqrt{c_1^2 + c_2}$$

$$\|\lambda - \lambda_N\|_W \leq \Delta_\lambda := \frac{1}{\beta}\left(\|r_E\|_{V'} + \gamma_a \Delta_u\right)$$

Here, the constants are given by

$$c_1 := \frac{1}{2\alpha}\left(\|r_E\|_{V'} + \frac{\gamma_a}{\beta}\delta_1\right) \qquad c_2 := \frac{1}{\alpha}\left(\frac{\|r_E\|_{V'}}{\beta}\delta_1 + \delta_2\right)$$

$$\delta_1 := \|\pi(\hat{e}_I)\|_W \qquad \delta_2 := \langle \lambda_N, \pi(\hat{e}_I)\rangle_W$$

where $\pi : W \to M$ is a (generally nonlinear) projection operator.

Reduced Basis Method for EVI-1 [HSW12]

For the case $\boldsymbol{W} = \boldsymbol{V}'$, **[HSW12]** proposes

$$\underline{\boldsymbol{\pi}}(\underline{\boldsymbol{\eta}}) = (\underline{\boldsymbol{M}}^{\boldsymbol{W}})^{-1}[\underline{\boldsymbol{M}}^{\boldsymbol{W}}\underline{\boldsymbol{\eta}}]_+$$

so that

$$\begin{aligned} \boldsymbol{\delta_1} &= [\boldsymbol{B}\boldsymbol{u_N} - \boldsymbol{g}]_+^{\boldsymbol{T}}\, \boldsymbol{M}^{\boldsymbol{V}}\, [\boldsymbol{B}\boldsymbol{u_N} - \boldsymbol{g}]_+ \\ \boldsymbol{\delta_2} &= \boldsymbol{\lambda}_{\boldsymbol{N}}^{\boldsymbol{T}}\, [\boldsymbol{B}\boldsymbol{u_N} - \boldsymbol{g}]_+ \end{aligned}$$

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This requires $O(\mathcal{N}_{\mathbf{FE}})$ operations **online**.

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The Plan

EVIs of the 1st kind

- Simple Obstacle Problem
- General Formulation
- Reduced Basis Method [HSW12]

Proposed Methods

- Method D
- Method R

Summary & Perspectives

Method D

Observation

Recall from **[HSW12]**

$$r_{\mathrm{I}} = B\,u_N - g \qquad\qquad [r_{\mathrm{I}}]_+ = [B\,u_N - g]_+$$

"INEQUALITY RESIDUAL" **ERROR INDICATOR**

Note that $-r_{\mathrm{I}}$ is in fact an approximation to the slack variable

$$s := g - B\,u \geq 0$$

Method D

Observation

Recall from **[HSW12]**

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"INEQUALITY RESIDUAL" **ERROR INDICATOR**

Note that $-r_{\mathbf{I}}$ is in fact an approximation to the **slack variable**

$$s := g - B\,u \geq 0$$

Method D

Assuming that B is parameter-independent and that B^{-1} exists,

$$u = B^{-1}(g - s)$$

Method D

Assuming that $\boldsymbol{B}$ is parameter-independent and that $\boldsymbol{B}^{-1}$ exists,

$$\boldsymbol{u} = \boldsymbol{B}^{-1}(\boldsymbol{g} - \boldsymbol{s})$$

We can introduce, in addition to our primal problem,

$$\begin{array}{rcl} \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}^T\boldsymbol{\lambda} & = & \boldsymbol{f} \\ \boldsymbol{g} - \boldsymbol{B}\boldsymbol{u} & \geq & \boldsymbol{0} \\ \boldsymbol{\lambda} & \geq & \boldsymbol{0} \\ \boldsymbol{\lambda}^T(\boldsymbol{g} - \boldsymbol{B}\boldsymbol{u}) & = & \boldsymbol{0} \end{array}$$

Method D

Assuming that B is parameter-independent and that B^{-1} exists,

$$u = B^{-1}(g - s)$$

We can introduce, in addition to our primal problem, a **dual** problem

$$\begin{aligned} Au + B^T\lambda &= f \\ g - Bu &\geq 0 \\ \lambda &\geq 0 \\ \lambda^T(g - Bu) &= 0 \end{aligned}$$

$$\begin{aligned} \tilde{A}s - \lambda &= \tilde{f} \\ s &\geq 0 \\ \lambda &\geq 0 \\ \lambda^T s &= 0 \end{aligned}$$

where $\tilde{A} := B^{-T}AB^{-1}$ and $\tilde{f} := B^{-T}(AB^{-1}g - f)$.

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D is for Dual

Approximation

In addition to the primal RB spaces, we introduce $\quad 1 \leq i \leq N_\lambda$

$$W'_N \quad = \quad \text{span}\{\ s(\mu_i)\ \} \qquad\qquad s\text{-SNAPSHOTS}$$

and compute our RB approximation for s

$$s_N(\mu) \quad = \quad \sum_{i=1}^{N_u} \underline{s}_{Ni}(\mu)\, s(\mu_i) \quad \in W'_N$$

by solving . . .

D is for Dual

Approximation

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D is for Dual

...for the coefficients $\underline{\boldsymbol{s}}_{\boldsymbol{N}}(\boldsymbol{\mu}) \in \mathbb{R}^{\boldsymbol{N_s}}$ and $\underline{\boldsymbol{\lambda}}^{\boldsymbol{s}}_{\boldsymbol{N}}(\boldsymbol{\mu}) \in \mathbb{R}^{\boldsymbol{N_\lambda}}$

$$
\begin{aligned}
\tilde{\boldsymbol{A}}_{\boldsymbol{N}} \underline{\boldsymbol{s}}_{\boldsymbol{N}} - \underline{\boldsymbol{\lambda}}^{\boldsymbol{s}}_{\boldsymbol{N}} &= \tilde{\boldsymbol{f}}_{\boldsymbol{N}} \\
\underline{\boldsymbol{s}}_{\boldsymbol{N}} &\geq \boldsymbol{0} \\
\underline{\boldsymbol{\lambda}}^{\boldsymbol{s}}_{\boldsymbol{N}} &\geq \boldsymbol{0} \\
(\underline{\boldsymbol{\lambda}}^{\boldsymbol{s}}_{\boldsymbol{N}})^{\boldsymbol{T}} \underline{\boldsymbol{s}}_{\boldsymbol{N}} &= \boldsymbol{0}
\end{aligned}
$$

We now define an intermediate approximation to u

$$u^{s_N} := B^{-1}(g - s_N)$$

and make the following observation ...

D is for Dual

... for the coefficients $\underline{\boldsymbol{s}}_{\boldsymbol{N}}(\boldsymbol{\mu}) \in \mathbb{R}^{N_s}$ and $\underline{\boldsymbol{\lambda}}^s_{\boldsymbol{N}}(\boldsymbol{\mu}) \in \mathbb{R}^{N_\lambda}$

$$
\begin{aligned}
\tilde{A}_N \underline{s}_N - \underline{\lambda}^s_N &= \tilde{f}_N \\
\underline{s}_N &\geq 0 \\
\underline{\lambda}^s_N &\geq 0 \\
(\underline{\lambda}^s_N)^T \underline{s}_N &= 0
\end{aligned}
$$

We now define an intermediate approximation to $\boldsymbol{u}$

$$
u^{s_N} := B^{-1}(g - s_N)
$$

and make the following observation ...

D is for Dual

Note that the condition

$$\boldsymbol{g_N} - \boldsymbol{B_N}\underline{\boldsymbol{u}}_{\boldsymbol{N}} \geq \boldsymbol{0}$$

was insufficient to ensure that

$$\boldsymbol{g} - \boldsymbol{B}\boldsymbol{u_N} \geq \boldsymbol{0}$$

but that

$$\underline{s}_N \geq 0$$

suffices to ensure that

$$s_N = g - Bu^{s_N} \geq 0$$

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D is for Dual

However, $\boldsymbol{u}^{s_N}$ is expensive to compute, so we introduce

$$\boldsymbol{W}_N^s = \mathbf{span}\{\boldsymbol{W}^{-1}\boldsymbol{B}\,\boldsymbol{u}(\boldsymbol{\mu}_i)\} = \mathbf{span}\{\boldsymbol{W}^{-1}\boldsymbol{B}\,\boldsymbol{\varphi}_i\}$$

and compute our final RB approximation $\boldsymbol{u}_N^{s_N}$ from

$$\langle \boldsymbol{B}\boldsymbol{u}_N^{s_N}, \boldsymbol{\eta}\rangle_{\boldsymbol{W}',\boldsymbol{W}} = \langle \boldsymbol{g} - \boldsymbol{s}_N, \boldsymbol{\eta}\rangle_{\boldsymbol{W}',\boldsymbol{W}}, \qquad \forall \boldsymbol{\eta} \in \boldsymbol{W}_N^s.$$

We then decompose the error into two parts

$$\|u - u_N^{s_N}\|_V \leq \|u - u^{s_N}\|_V + \|u^{s_N} - u_N^{s_N}\|_V$$

and show that . . .

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and show that . . .

D is for Dual

…the errors in are bounded by

$$\begin{aligned}
\|u - u^{s_N}\|_V &\leq \Delta_u^1 := c_1 + \sqrt{c_1^2 + c_2} \\
\|u^{s_N} - u_N^{s_N}\|_V &\leq \Delta_u^2 := \frac{\|r_2\|_{W'}}{\beta} \\
\|\lambda^u - \lambda_N^u\|_W &\leq \Delta_\lambda := \frac{1}{\beta}\left(\|r_1\|_{V'} + \gamma_a\, \Delta_u^1\right)
\end{aligned}$$

Here,

$$c_1 := \frac{1}{2\alpha}\|r_1\|_{V'} \qquad c_2 := \frac{1}{\alpha}\lambda_N^T s_N$$

$$r_1 := f - AB^{-1}(g - s_N) + B^T \lambda_N$$

$$r_2 := g - s_N - B\, u_N^{s_N}$$

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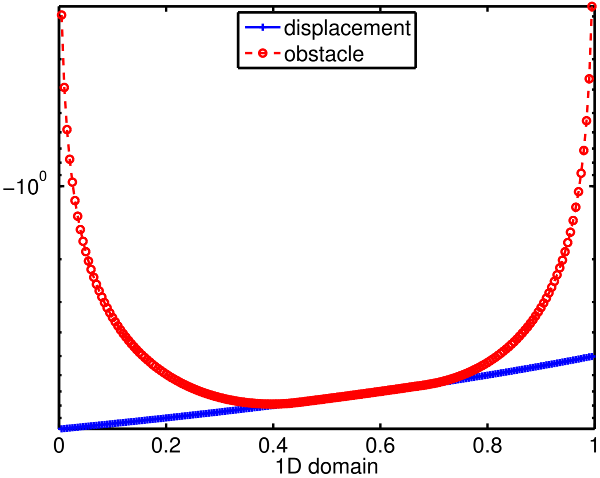
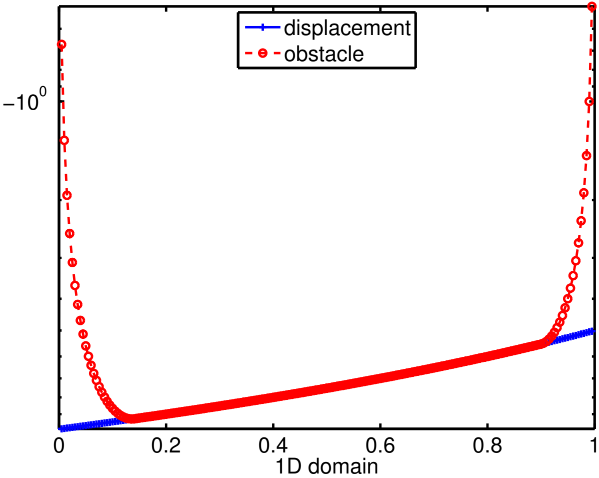
Numerical Results - 1D



Numerical Results - 1D



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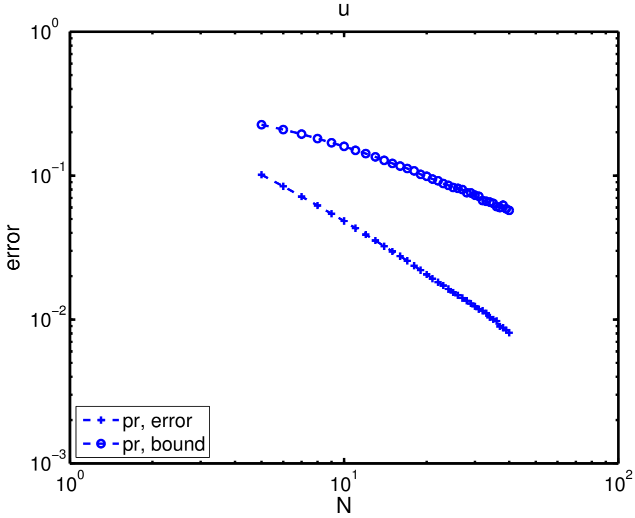

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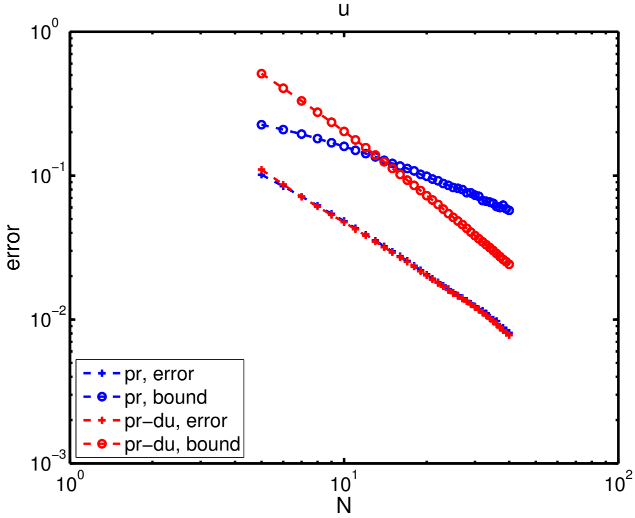

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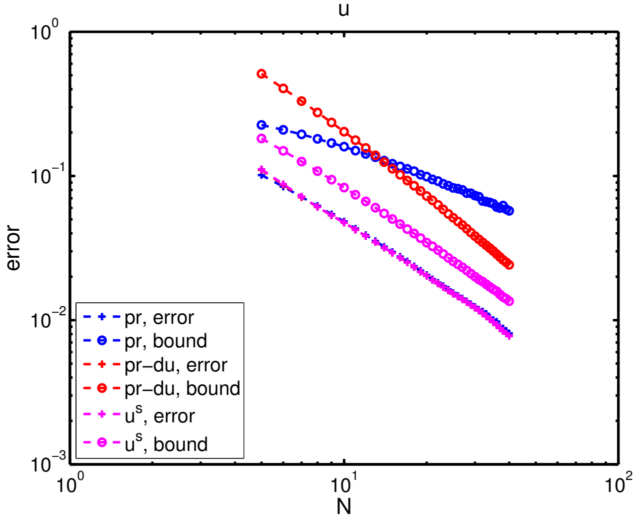
Numerical Results - 1D



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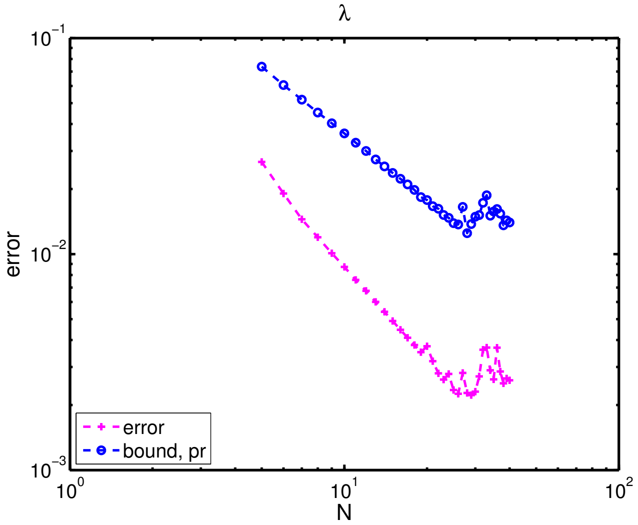

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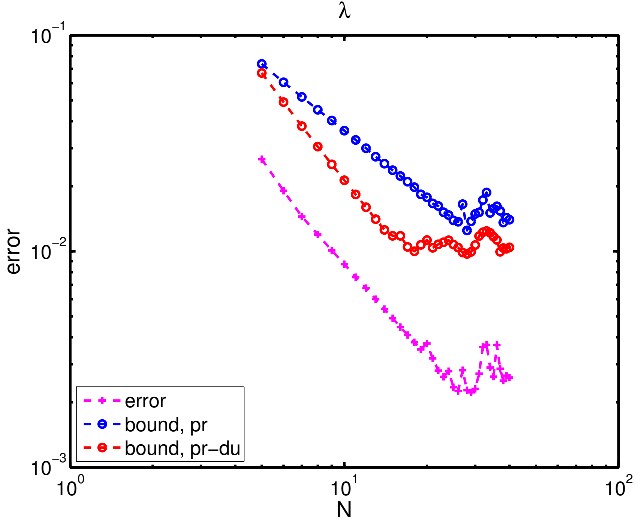

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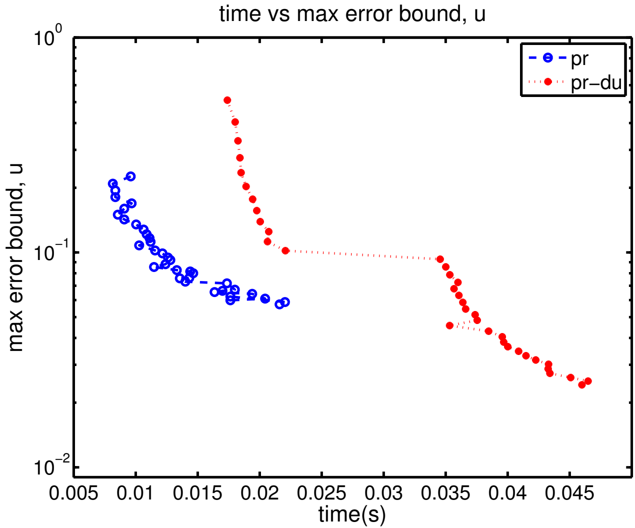
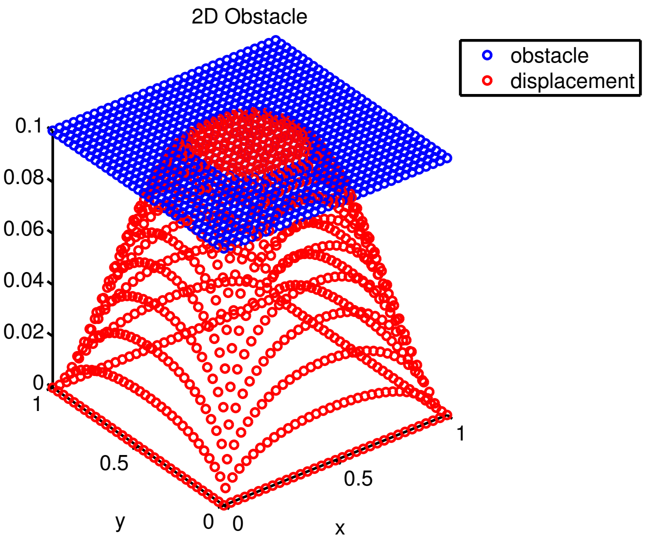


Numerical Results - 2D



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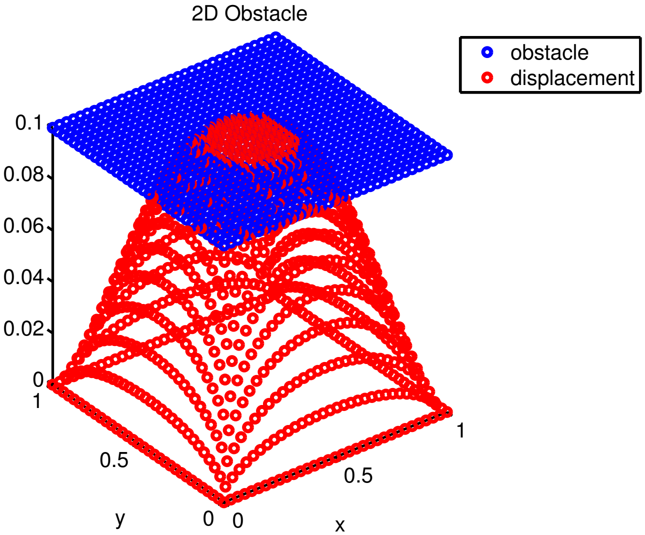

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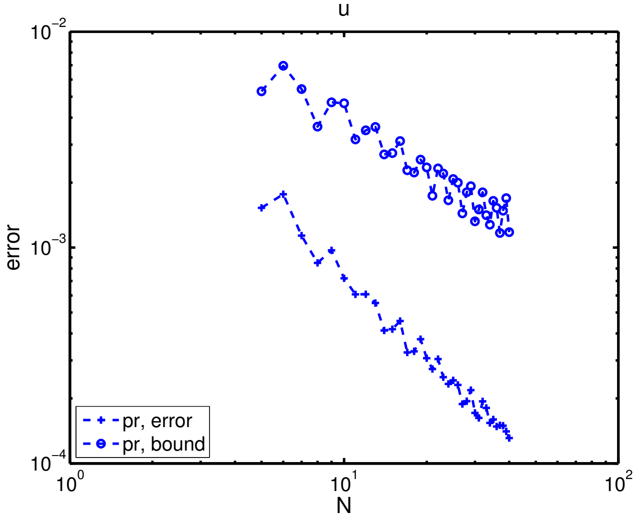

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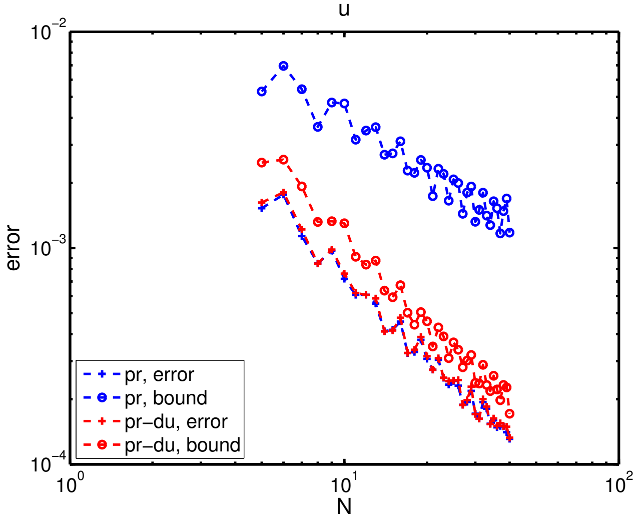

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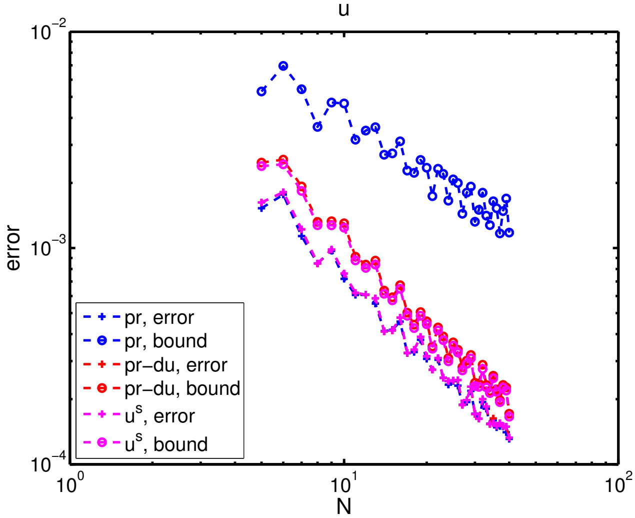

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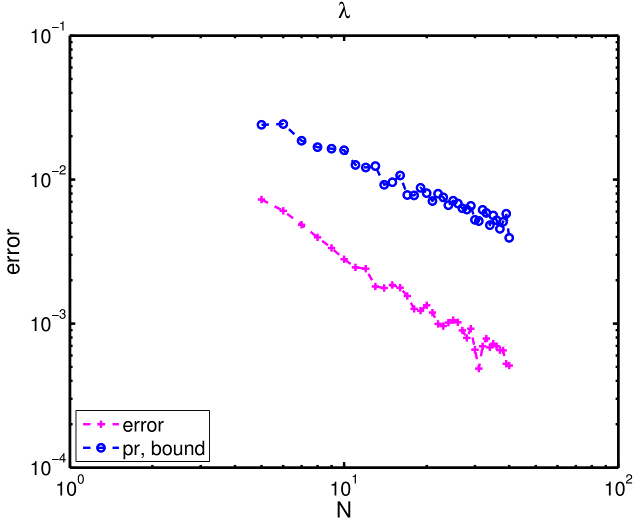

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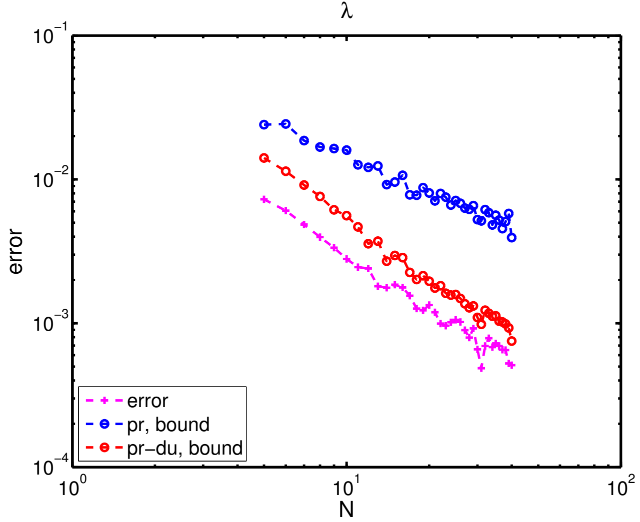

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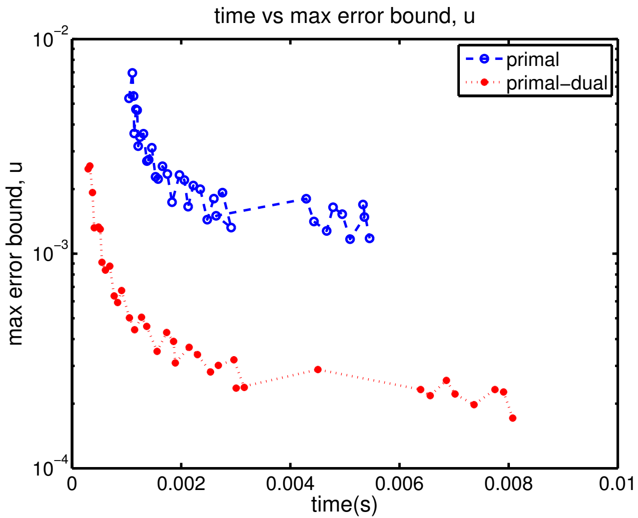

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Mid-Talk Summary

We developed an **online-efficient certified** reduced basis method for elliptic variational inequalities of the first kind.

We introduce a **dual problem** to enable computation of **sharp** and **inexpensive** *a posteriori* error bounds.

The **online** computational cost depends on N, Q, but **not** on $\mathcal{N}_{\mathbf{FE}}$.

However, the method is not applicable to

- problems where B is μ-dependent
- EVIs of the **second** kind

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Coulomb Friction

Equilibrium $$-\sigma_{ij,j} = 0$$

Constitutive Law $$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

Strain-Displacement $$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

Boundary Conditions

DISPLACEMENT $u_i = 0$ on Γ_u

APPLIED TRACTION $\sigma_n = g_i$ on Γ_g

CONTACT $\sigma_n < 0$ on Γ_C

FRICTION: If $|\sigma_t| < \nu_F|\sigma_n|$ then $u_t = 0$

If $|\sigma_t| = \nu_F|\sigma_n|$ then $u_t = -\lambda\sigma_t$ for some $\lambda > 0$

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Coulomb Friction

Variational Formulation

The displacement $\boldsymbol{u} \in \boldsymbol{K}$ satisfies

$$\boldsymbol{a}(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + \boldsymbol{j}(\boldsymbol{u}, \boldsymbol{v}) - \boldsymbol{j}(\boldsymbol{u}, \boldsymbol{u}) \geq \boldsymbol{f}(\boldsymbol{v} - \boldsymbol{u}) \qquad \forall \boldsymbol{v} \in \boldsymbol{K}$$

where

$$\boldsymbol{j}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\boldsymbol{\Omega}} \boldsymbol{\nu_F} |\boldsymbol{\sigma_n}(\boldsymbol{u})| |\boldsymbol{v_t}|$$

See, e.g., **[Han & Reddy, 1999]**

Variational Inequalities

First Kind

$$u = \arg \inf_{v \in K} \frac{1}{2} a(v,v) - f(v)$$

where K is a convex subset of V.

Second Kind

$$u = \arg \inf_{v \in V} \frac{1}{2} a(v,v) + j(v) - f(v)$$

where the functional j is nondifferentiable.

Method R

We transform the **constrained** minimization problem (EVI-1)
into an **unconstrained** minimization problem.

Start with an **interior point** and replace the constraint
with a **barrier function.**

The barrier causes the objective function to increase without bound
as u approaches the constraint.

See, e.g., [Weiser, SIAM J Optim, 2005]
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R is for Regularize

Obstacle Problem

Let the admissible set be given by

$$K = \{\, v \in V \mid v \leq g \text{ in } \Omega \,\}$$

We introduce u_ν

$$u_\nu = \arg\inf_{v \in V} \frac{1}{2} a(v,v) - f(v) - \nu \int_\Omega \log\left(g - v\right)\, d\Omega$$

$$\Rightarrow \quad a(u,v) - f(v) + \nu \int_\Omega \frac{v}{g-u}\, d\Omega \;=\; 0, \quad \forall\, v \in V$$

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R is for Regularize

For problems of the form

$$a(u,v) - f(v) + \langle h(u), v\rangle_{V',V} \;=\; 0, \quad \forall\, v \in V$$

where $h(\cdot;\mu)$ is nonlinear, we can approximate h
using the **Empirical Interpolation Method:**

$$h(u(x;\mu);\mu) \approx h^u_M(x;\mu) = \sum_{m=1}^{M} q_m(x)\varphi^u_{Mm}(\mu)$$

where

$$\sum_{m=1}^{M} q_m(x_i)\varphi^u_{Mm}(\mu) = h(u(x_i;\mu);\mu), \quad 1 \le i \le M,$$

x_i are interpolation pts, and q_m are chosen by a greedy procedure.

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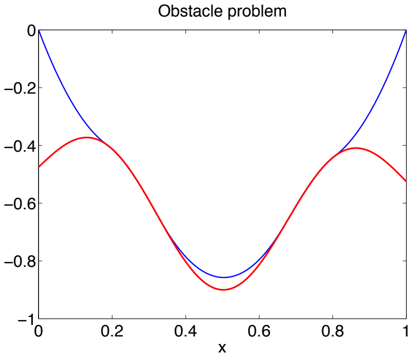
B is for Barrier

The **Empirical Interpolation Method** provides

- affine approximations to non-affine and/or nonlinear forms
- efficient *a posteriori* error estimators (in some cases, bounds)

See, e.g., **[Barrault, Maday, Nguyen, & Patera, CR Math, 2004]**, **[Grepl, Maday, Nguyen, & Patera, M2AN, 2007].**

RB Method for Problems in Solid Mechanics



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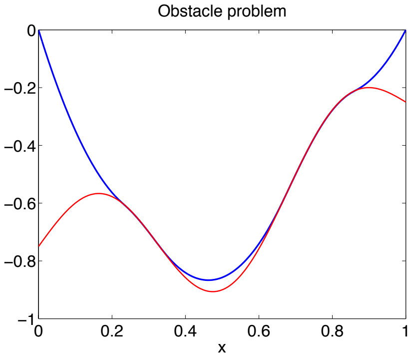

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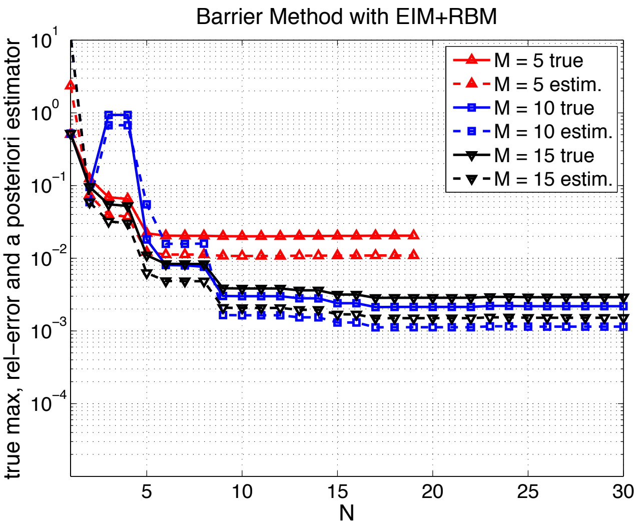

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Summary & Perspectives

- We proposed two **online-efficient** RB approaches for VIs:
  - **Primal-Dual Approach**
  - **Regularization Approach**

  motivated by problems in nonlinear solid mechanics.

- We intend to explore:
  - extension to Parabolic VIs
  - combination with work on finite deformation [with L. Zanon]
  - connection to optimal control problems with control and/or state constraints [with M. Grepl & M. Kaercher]

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