Correlation-based imaging in random media

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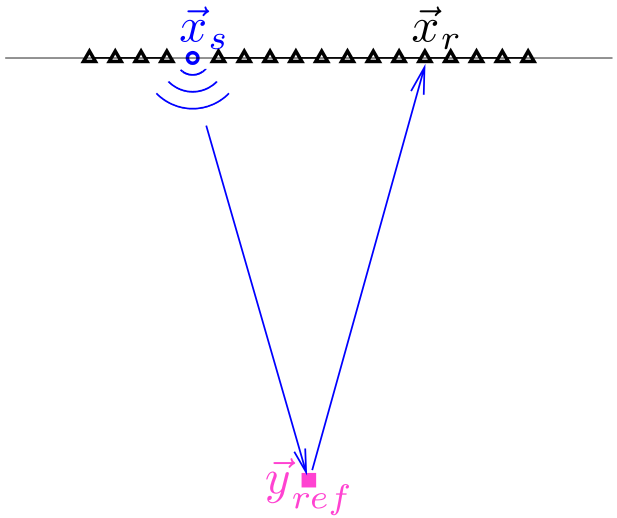
• Principle of sensor array imaging:
- probe an unknown medium with waves,
- record the waves transmitted through or reflected by the medium,
- process the recorded data to extract relevant information.

$\rightarrow$ What about imaging in the presence of measurement noise, medium noise, or source noise ?

Reflector imaging through a homogeneous medium



• Sensor array imaging of a reflector located at $\vec{\boldsymbol{y}}_{\text{ref}}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver. Measured data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \ldots, N_{\text{r}}, s = 1, \ldots, N_{\text{s}}\}$.

• Mathematical model:

$$\left(\frac{1}{c_0^2} + \frac{1}{c_{\text{ref}}^2} \mathbf{1}_{B_{\text{ref}}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}_{\text{ref}}) \right) \frac{\partial^2 u}{\partial t^2}(t, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_{\text{s}}) - \Delta_{\vec{\boldsymbol{x}}} u(t, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_{\text{s}}) = f(t)\delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_{\text{s}})$$

• Purpose of imaging: using the measured data, build an imaging function $\mathcal{I}(\vec{\boldsymbol{y}}^S)$ that would ideally look like $\frac{1}{c_{\text{ref}}^2} \mathbf{1}_{B_{\text{ref}}}(\vec{\boldsymbol{y}}^S - \vec{\boldsymbol{y}}_{\text{ref}})$, in order to extract the relevant information $(\vec{\boldsymbol{y}}_{\text{ref}}, B_{\text{ref}}, c_{\text{ref}})$ about the reflector.

- Classical imaging functions:

1) Least-Squares imaging: minimize the quadratic misfit between measured data and synthetic data obtained by solving the wave equation with a candidate $(\vec{\boldsymbol{y}}_{\rm test}, B_{\rm test}, c_{\rm test})$.

2) Reverse Time imaging: simplify Least-Squares imaging by "linearization" of the forward problem.

3) Kirchhoff Migration: simplify Reverse Time imaging by substituting travel time migration for full wave equation.

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3) Kirchhoff Migration: simplify Reverse Time imaging by substituting travel time migration for full wave equation.

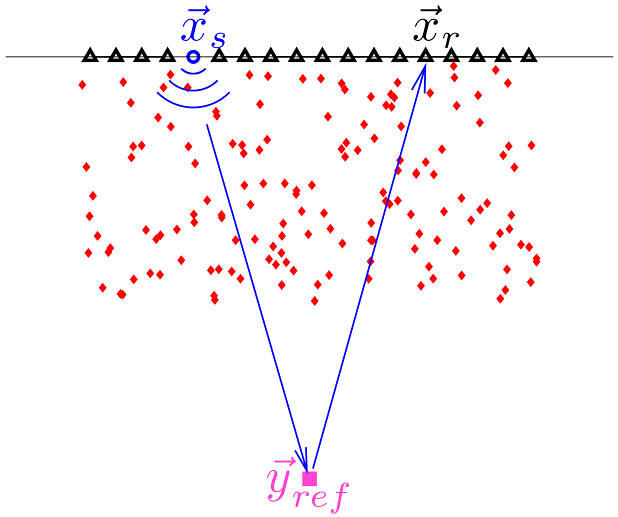
- Kirchhoff Migration function:

$$\mathcal{I}_{\text{KM}}(\vec{\boldsymbol{y}}^S) = \sum_{r=1}^{N_\text{r}} \sum_{s=1}^{N_\text{s}} u\Big(\frac{|\vec{\boldsymbol{x}}_s - \vec{\boldsymbol{y}}^S|}{c_0} + \frac{|\vec{\boldsymbol{y}}^S - \vec{\boldsymbol{x}}_r|}{c_0}, \vec{\boldsymbol{x}}_r ; \vec{\boldsymbol{x}}_s \Big)$$

It forms the image with the superposition of the backpropagated traces.
Here the travel time from $\vec{\boldsymbol{x}}$ to $\vec{\boldsymbol{y}}^S$ is $|\vec{\boldsymbol{y}}^S - \vec{\boldsymbol{x}}|/c_0$.
- Very robust with respect to (additive) measurement noise [1].
- Sensitive to medium noise: If the medium is scattering, then Kirchhoff Migration does not work.

[1] H. Ammari, J. Garnier, and K. Sølna, *Waves in Random and Complex Media* **22**, 40 (2012).

Imaging through a scattering medium



- Sensor array imaging of a reflector located at $\vec{\boldsymbol{y}}_{\rm ref}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver.

Data: $\{u(t,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s), r=1,\ldots,N_{\rm r}, s=1,\ldots,N_{\rm s}\}$.

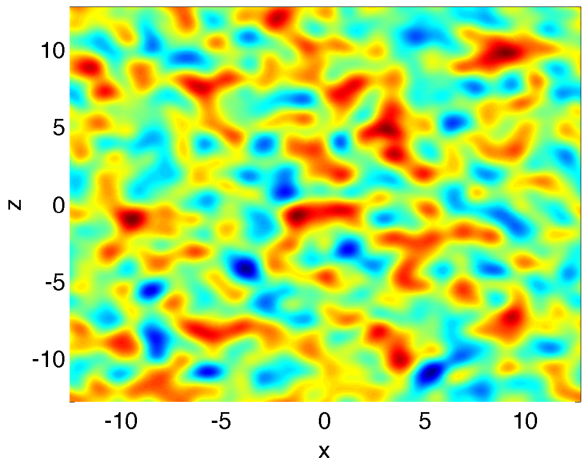
$$\Big(\frac{1}{c^2(\vec{\boldsymbol{x}})}+\frac{1}{c_{\rm ref}^2}\mathbf{1}_{B_{\rm ref}}(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}_{\rm ref})\Big)\frac{\partial^2 u}{\partial t^2}(t,\vec{\boldsymbol{x}};\vec{\boldsymbol{x}}_{\rm s})-\Delta_{\vec{\boldsymbol{x}}}u(t,\vec{\boldsymbol{x}};\vec{\boldsymbol{x}}_{\rm s})=f(t)\delta(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_{\rm s})$$

- Random medium model:

$$\frac{1}{c^2(\vec{\boldsymbol{x}})}=\frac{1}{c_0^2}\big(1+\mu(\vec{\boldsymbol{x}})\big)$$

c_0 is a reference speed,

$\mu(\vec{\boldsymbol{x}})$ is a zero-mean random process.



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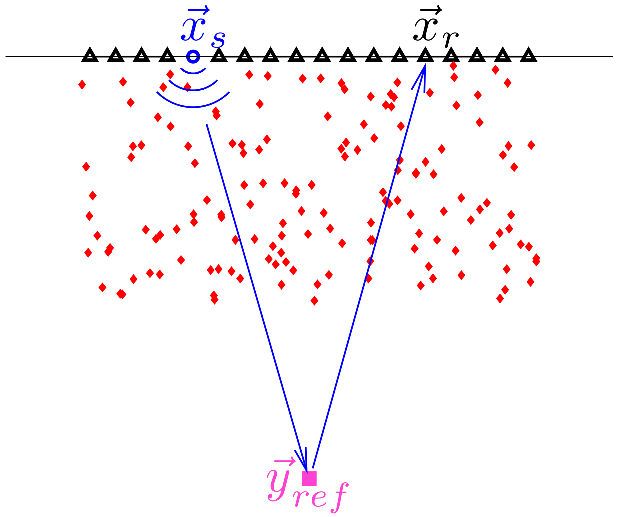

- Sensor array imaging of a reflector located at $\vec{\boldsymbol{y}}_{\rm ref}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver.

Data: $\{\hat{u}(\omega, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \dots, N_{\rm r}, s = 1, \dots, N_{\rm s}\}$.

$$\omega^2 \Big(\frac{1}{c^2(\vec{\boldsymbol{x}})} + \frac{1}{c_{\rm ref}^2} \mathbf{1}_{B_{\rm ref}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}_{\rm ref}) \Big) \hat{u}(\omega, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_{\rm s}) + \Delta_{\vec{\boldsymbol{x}}} \hat{u}(\omega, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_{\rm s}) = -\hat{f}(\omega) \delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_{\rm s})$$

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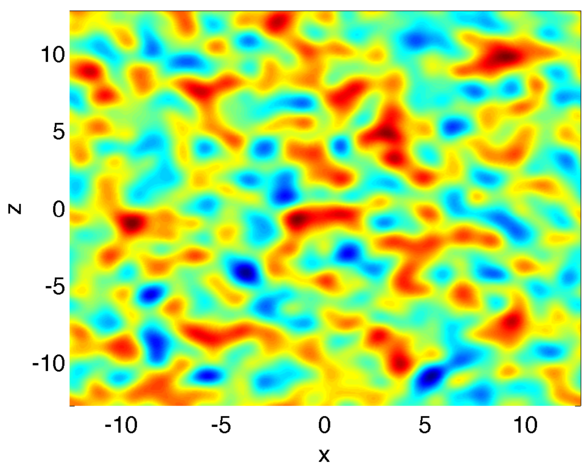

Strategy: Stochastic and multiscale analysis

• A stochastic and multiscale analysis is possible in different regimes of separation of scales (small wavelength, large propagation distance, small correlation length, ...).
$\hookrightarrow$ Analysis of the moments of $\hat{u}$.

• Compute the mean and variance of an imaging function $\mathcal{I}(\vec{\boldsymbol{y}}^S)$.
$\hookrightarrow$ resolution and stability analysis.
• Resolution analysis: What is the size of the smallest feature that can be distinguished ? Can be obtained by studying the mean imaging function $\mathbb{E}\big[\mathcal{I}(\vec{\boldsymbol{y}}^S)\big]$.
• Criterium for statistical stability:

$$\mathrm{SNR} := \frac{\mathbb{E}\big[\mathcal{I}(\vec{\boldsymbol{y}}^S)\big]}{\mathrm{Var}\big(\mathcal{I}(\vec{\boldsymbol{y}}^S)\big)^{1/2}} > 1$$

$\hookrightarrow$ design the imaging function to get good trade-off between stability and resolution.

Wave propagation in the random paraxial regime

- Consider the time-harmonic form of the scalar wave equation ($\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$)

$$(\partial_z^2 + \Delta_\perp)\hat{u} + \frac{\omega^2}{c_0^2}\big(1 + \mu(\boldsymbol{x}, z)\big)\hat{u} = 0.$$

Consider the paraxial regime "$\lambda \ll l_c \ll L$":

$$\omega \to \frac{\omega}{\varepsilon^4}, \qquad \mu(\boldsymbol{x}, z) \to \varepsilon^3 \mu\big(\frac{\boldsymbol{x}}{\varepsilon^2}, \frac{z}{\varepsilon^2}\big).$$

The function $\hat{\phi}^\varepsilon$ (slowly-varying envelope of a plane wave) defined by

$$\hat{u}^\varepsilon(\omega, \boldsymbol{x}, z) = e^{i\frac{\omega z}{\varepsilon^4 c_0}}\hat{\phi}^\varepsilon\big(\omega, \frac{\boldsymbol{x}}{\varepsilon^2}, z\big)$$

satisfies

$$\varepsilon^4 \partial_z^2 \hat{\phi}^\varepsilon + \left(2i\frac{\omega}{c_0}\partial_z \hat{\phi}^\varepsilon + \Delta_\perp \hat{\phi}^\varepsilon + \frac{\omega^2}{c_0^2}\frac{1}{\varepsilon}\mu\big(\boldsymbol{x}, \frac{z}{\varepsilon^2}\big)\hat{\phi}^\varepsilon\right) = 0.$$

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- In the regime $\varepsilon \ll 1$, the forward-scattering approximation in direction z is valid and $\hat{\phi} = \lim_{\varepsilon\to 0}\hat{\phi}^\varepsilon$ satisfies the Itô-Schrödinger equation [1]

$$2i\frac{\omega}{c_0}\partial_z\hat{\phi} + \Delta_\perp\hat{\phi} + \frac{\omega^2}{c_0^2}\dot{B}(\boldsymbol{x}, z)\hat{\phi} = 0$$

with $B(\boldsymbol{x}, z)$ Brownian field $\mathbb{E}[B(\boldsymbol{x}, z)B(\boldsymbol{x}', z')] = \gamma(\boldsymbol{x} - \boldsymbol{x}')\min(z, z')$,
$\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty}\mathbb{E}[\mu(\boldsymbol{0}, 0)\mu(\boldsymbol{x}, z)]dz$.

[1] J. Garnier and K. Sølna, *Ann. Appl. Probab.* **19**, 318 (2009).

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$$d\hat{\phi} = \frac{ic_0}{2\omega}\Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_0}\hat{\phi} \circ dB(\boldsymbol{x}, z)$$

with $B(\boldsymbol{x}, z)$ Brownian field $\mathbb{E}[B(\boldsymbol{x}, z)B(\boldsymbol{x}', z')] = \gamma(\boldsymbol{x} - \boldsymbol{x}') \min(z, z')$,
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Wave propagation in the random paraxial regime

- We introduce the fundamental solution $\hat{G}\big(\omega, (\boldsymbol{x}, z), (\boldsymbol{x}_0, z_0)\big)$:

$$d\hat{G} = \frac{ic_0}{2\omega}\Delta_\perp \hat{G} dz + \frac{i\omega}{2c_0}\hat{G} \circ dB(\boldsymbol{x}, z)$$

starting from $\hat{G}\big(\omega, (\boldsymbol{x}, z = z_0), (\boldsymbol{x}_0, z_0)\big) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$.

- In a homogeneous medium ($B \equiv 0$) the fundamental solution is

$$\hat{G}_0\big(\omega, (\boldsymbol{x}, z), (\boldsymbol{x}_0, z_0)\big) = \frac{\exp\Big(\frac{i\omega|\boldsymbol{x}-\boldsymbol{x}_0|^2}{2c_0|z-z_0|}\Big)}{2i\pi c_0\frac{|z-z_0|}{\omega}}.$$

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- In a random medium,

$$\mathbb{E}\big[\hat{G}\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\big] = \hat{G}_0\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\exp\Big(-\frac{\gamma(\boldsymbol{0})\omega^2|z-z_0|}{8c_0^2}\Big),$$

where $\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty}\mathbb{E}[\mu(\boldsymbol{0},0)\mu(\boldsymbol{x},z)]dz$.

- Strong damping of the coherent wave.

$\Longrightarrow$ Coherent imaging methods (such as Kirchhoff migration) fail.

Wave propagation in the random paraxial regime

- In a random medium,

$$
\begin{aligned}
&\mathbb{E}\Big[\hat{G}\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\overline{\hat{G}\big(\omega,(\boldsymbol{x}',z),(\boldsymbol{x}_0,z_0)\big)}\Big]\\
&=\hat{G}_0\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\overline{\hat{G}_0\big(\omega,(\boldsymbol{x}',z),(\boldsymbol{x}_0,z_0)\big)}\exp\Big(-\frac{\gamma_2(\boldsymbol{x}-\boldsymbol{x}')\omega^2|z-z_0|}{4c_0^2}\Big),
\end{aligned}
$$

where $\gamma_2(\boldsymbol{x})=\int_0^1\gamma(\mathbf{0})-\gamma(\boldsymbol{x}s)ds$ (note $\gamma_2(\mathbf{0})=0$).

- The fields at nearby points are correlated.
- Same results in frequency: The fields at nearby frequencies are correlated.

$\Longrightarrow$ One should migrate cross correlations for imaging.

Wave propagation in the random paraxial regime

- In a random medium,

$$\mathbb{E}\Big[\hat{G}\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\overline{\hat{G}\big(\omega,(\boldsymbol{x}',z),(\boldsymbol{x}_0,z_0)\big)}\Big]$$
$$= \hat{G}_0\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\overline{\hat{G}_0\big(\omega,(\boldsymbol{x}',z),(\boldsymbol{x}_0,z_0)\big)}\exp\Big(-\frac{\gamma_2(\boldsymbol{x}-\boldsymbol{x}')\omega^2|z-z_0|}{4c_0^2}\Big),$$

where $\gamma_2(\boldsymbol{x}) = \int_0^1 \gamma(\mathbf{0}) - \gamma(\boldsymbol{x}s)ds$ (note $\gamma_2(\mathbf{0}) = 0$).

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- In a random medium,

one can write a closed-form equation for the n-th order moment.
The fourth-order moments can be studied [1].

[1] J. Garnier and K. Sølna, *Comm. Part. Differ. Equat.* **39**, 626 (2014).

Wave propagation in the randomly layered regime

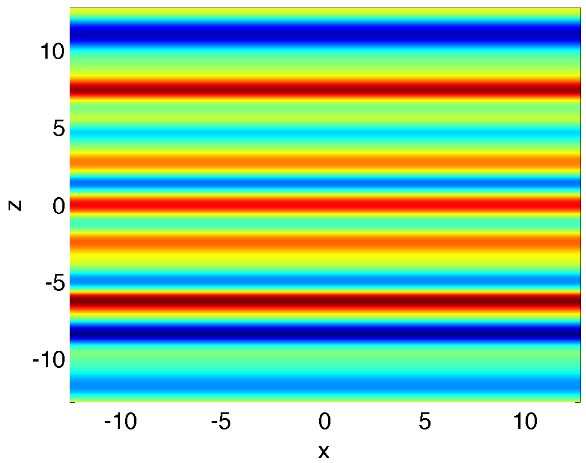
- Other regimes can be analyzed, for instance the randomly layered regime.

- Random medium model ($\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$):

$$\frac{1}{c^2(\vec{\boldsymbol{x}})} = \frac{1}{c_0^2}\big(1 + \mu(z)\big)$$

c_0 is a reference speed,

$\mu(z)$ is a zero-mean random process.



- Consider the time-harmonic form of the scalar wave equation ($\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$)

$$(\partial_z^2 + \Delta_\perp)\hat{u} + \frac{\omega^2}{c_0^2}\big(1 + \mu(z)\big)\hat{u} = 0$$

Consider the scaled regime "$l_c \ll \lambda \ll L$":

$$\omega \to \frac{\omega}{\varepsilon}, \qquad \mu(z) \to \mu\big(\frac{z}{\varepsilon^2}\big)$$

The moments of the solutions are known in the limit $\varepsilon \to 0$ [1].
They are characterized by transport equations.

[1] J.-P. Fouque, J. Garnier, G. Papanicolaou, and K. Sølna, *Wave propagation ...*, Springer, 2007.

• General results obtained by multiscale analysis: wave propagation can be described by a stochastic partial differential equation.

• General results obtained by stochastic analysis: the moments of the wave are solutions of transport equations.

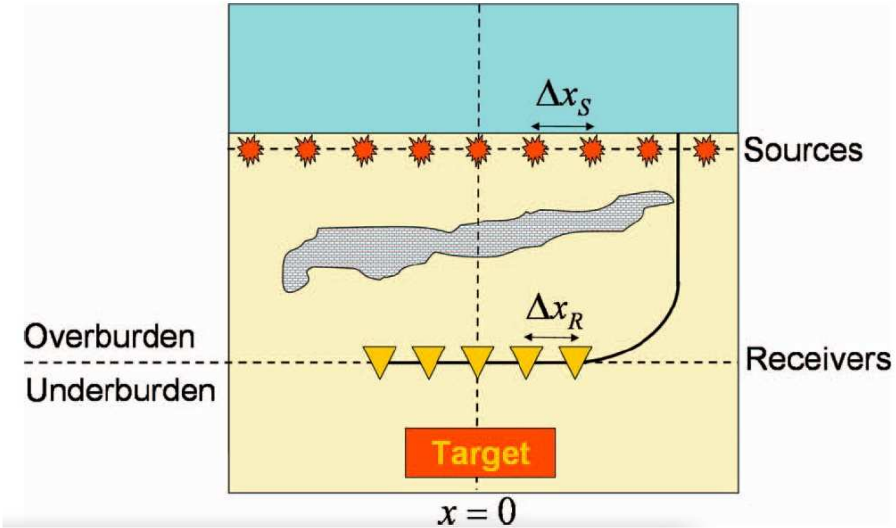
• The mean (coherent) wave is small.
$\Longrightarrow$ The Kirchhoff Migration function is unstable in randomly scattering media.

$$\frac{\mathbb{E}\big[\mathcal{I}_{\rm KM}(\vec{\boldsymbol{y}}^S)\big]}{\mathrm{Var}\big(\mathcal{I}_{\rm KM}(\vec{\boldsymbol{y}}^S)\big)^{1/2}} \ll 1$$

• The wave fluctuations at nearby points and nearby frequencies are correlated.
The wave correlations carry information about the medium.
$\Longrightarrow$ One should use local cross correlations for imaging.

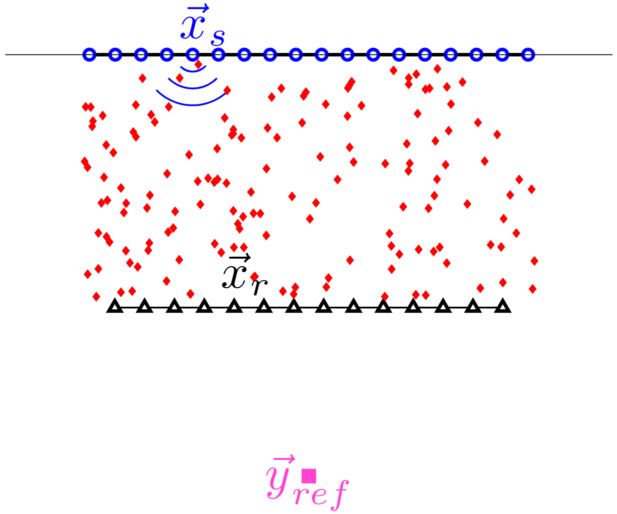
• Results obtained with media with rapid decorrelations. Not so many results with media with long-range correlations.

Imaging below an "overburden"



From van der Neut and Bakulin (2009)

Imaging below an overburden

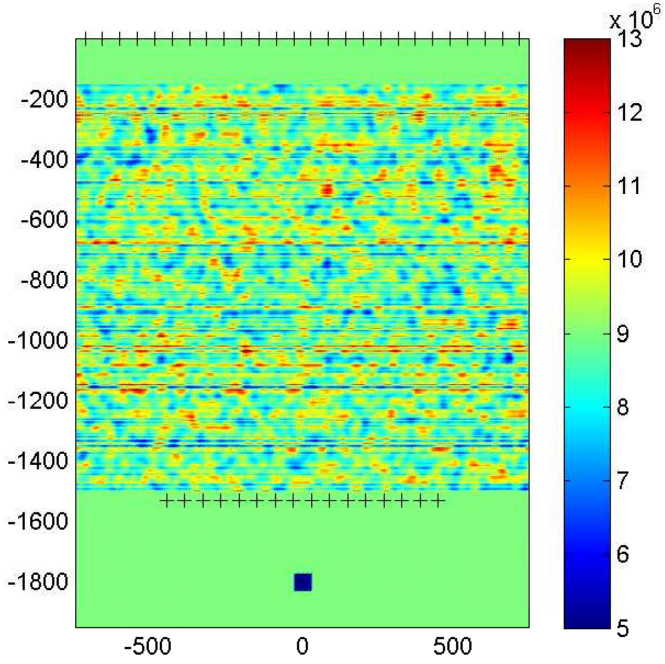


Array imaging of a reflector at $\vec{\boldsymbol{y}}_{\rm ref}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver located below the scattering medium. Data: $\{u(t,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s), r=1,\ldots,N_{\rm r}, s=1,\ldots,N_{\rm s}\}$.

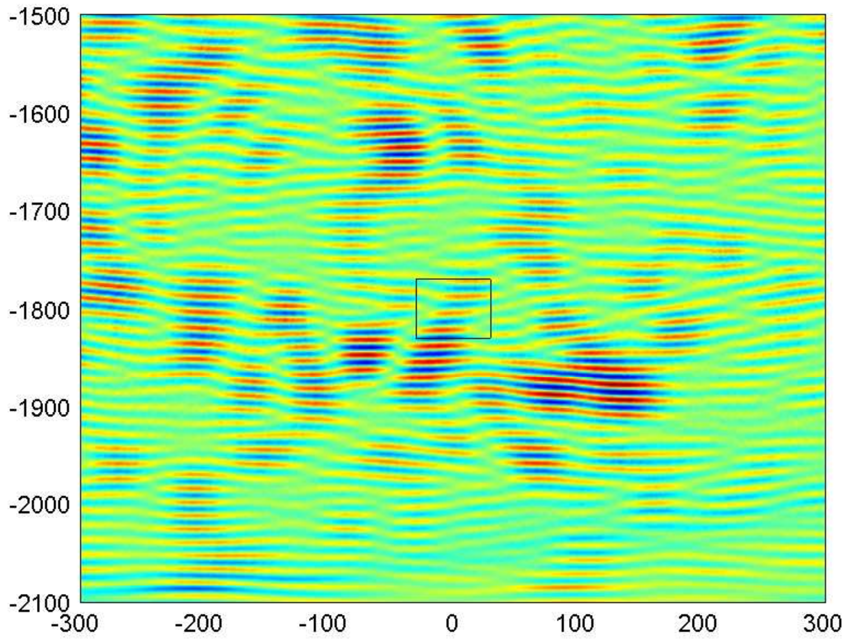
If the overburden is scattering, then Kirchhoff Migration does not work:

$$\mathcal{I}_{\rm KM}(\vec{\boldsymbol{y}}^S) = \sum_{r=1}^{N_{\rm r}} \sum_{s=1}^{N_{\rm s}} u\Big(\frac{|\vec{\boldsymbol{x}}_s - \vec{\boldsymbol{y}}^S|}{c_0} + \frac{|\vec{\boldsymbol{y}}^S - \vec{\boldsymbol{x}}_r|}{c_0}, \vec{\boldsymbol{x}}_r ; \vec{\boldsymbol{x}}_s \Big)$$

Numerical simulations



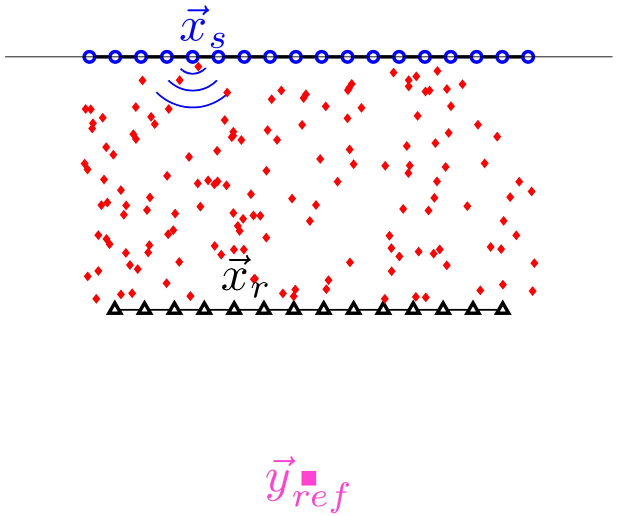
Computational setup



Kirchhoff Migration

(simulations carried out by Chrysoula Tsogka, University of Crete)

Imaging below an overburden



$\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver. Data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \dots, N_\mathrm{r}, s = 1, \dots, N_\mathrm{s}\}$.

Image with migration of the cross correlation matrix:

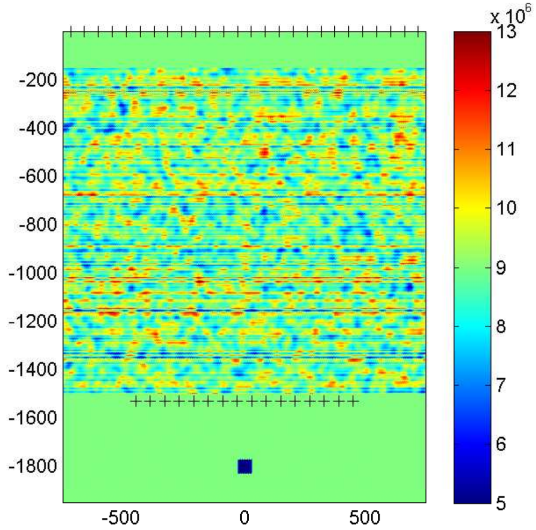
$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \sum_{r,r'=1}^{N_\mathrm{r}} \mathcal{C}\Big(\frac{|\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{y}}^S|}{c_0} + \frac{|\vec{\boldsymbol{y}}^S - \vec{\boldsymbol{x}}_{r'}|}{c_0}, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_{r'} \Big),$$

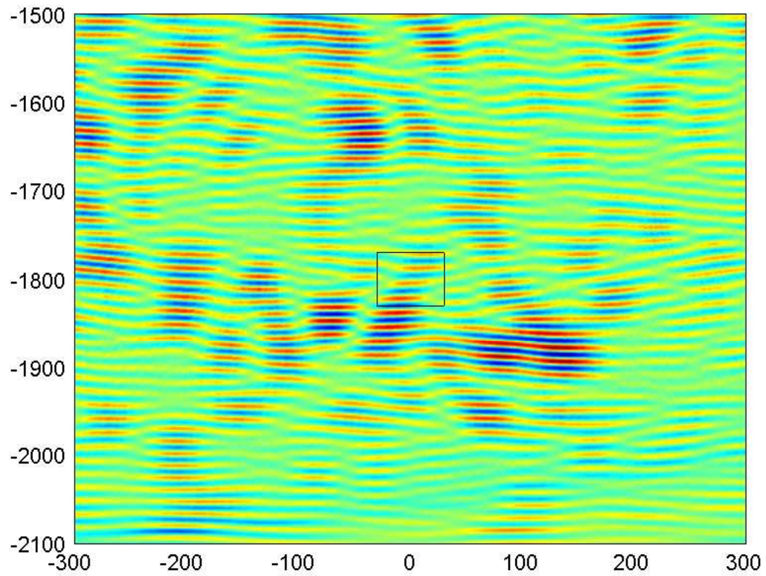
with

$$\mathcal{C}(\tau, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_{r'}) = \sum_{s=1}^{N_\mathrm{s}} \int u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s) u(t+\tau, \vec{\boldsymbol{x}}_{r'}; \vec{\boldsymbol{x}}_s) dt \ , \qquad r, r' = 1, \dots, N_\mathrm{r}$$

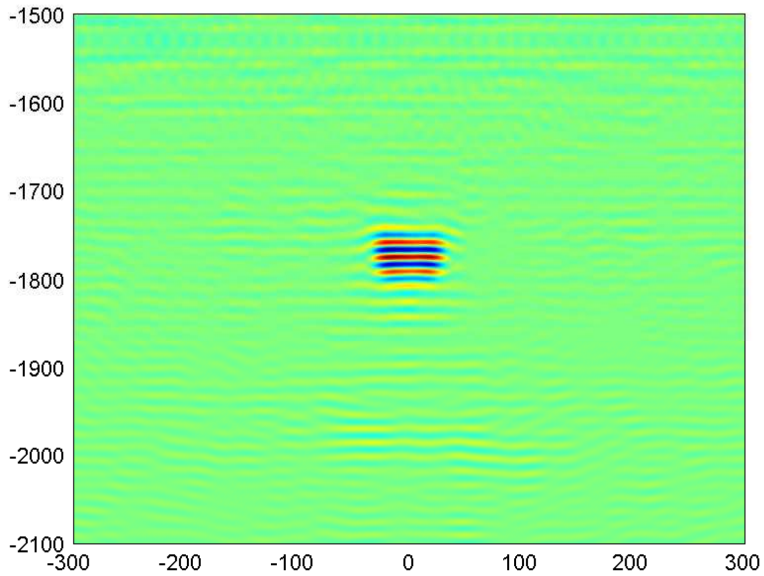
The choice of the cross correlations to be migrated (which pairs of receivers, which pairs of sources) is important !

Numerical simulations





Kirchhoff Migration



Cross Correlation Migration

Further results

• Use of ambient noise sources.

One can apply correlation-based imaging techniques to signals emitted by ambient noise sources.

$\hookrightarrow$ Useful for applications in seismology: travel time tomography, volcano monitoring, oil reservoir monitoring.

• Use of higher-order correlations.

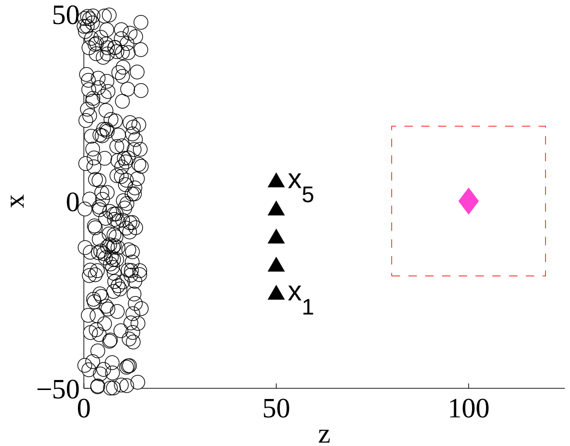
One can apply imaging techniques based on special fourth-order cross correlations.

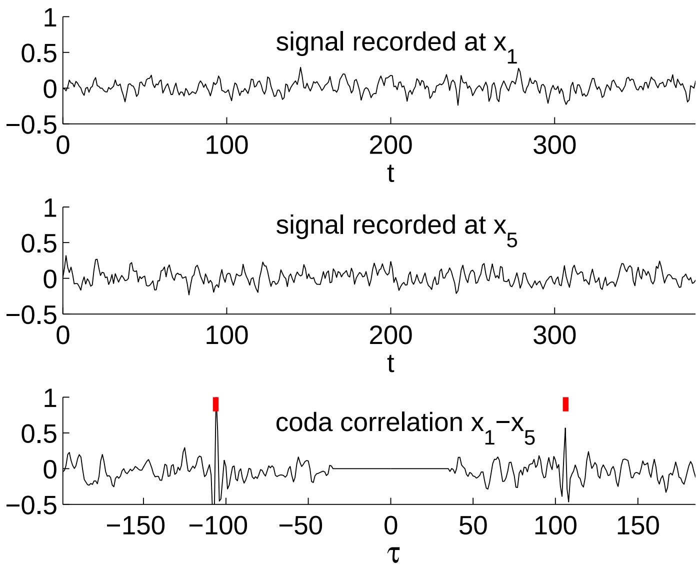
Passive sensor imaging of a reflector

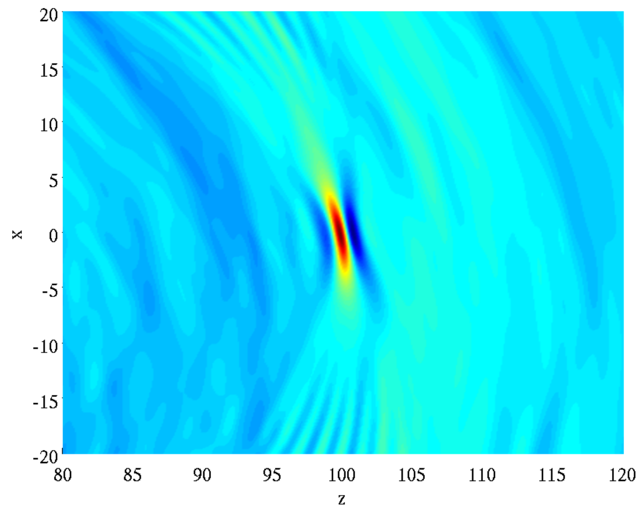
- Ambient noise sources ($\circ$) emit stationary random signals.
- The signals $(u(t,\vec{\boldsymbol{x}}_r))_{r=1,\dots,N_\mathrm{r}}$ are recorded by the receivers $(\vec{\boldsymbol{x}}_r)_{r=1,\dots,N_\mathrm{r}}$ ($\blacktriangle$).
- The cross correlation matrix is computed and migrated:

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \sum_{r,r'=1}^{N_\mathrm{r}} \mathcal{C}_T\Big(\frac{|\vec{\boldsymbol{x}}_{r'}-\vec{\boldsymbol{y}}^S|}{c_0}+\frac{|\vec{\boldsymbol{x}}_{r}-\vec{\boldsymbol{y}}^S|}{c_0},\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_{r'}\Big)$$

$$\text{with}\quad \mathcal{C}_T(\tau,\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_{r'}) = \frac{1}{T}\int_0^T u(t+\tau,\vec{\boldsymbol{x}}_{r'})u(t,\vec{\boldsymbol{x}}_r)dt$$







Provided the ambient noise illumination is long (in time) and diversified (in angle and frequency): good stability [1].

[1] J. Garnier and G. Papanicolaou, *SIAM J. Imaging Sciences* **2**, 396 (2009).

Conclusions

- Multiscale and stochastic analysis are useful to understand the structure of the data in sensor array imaging.
- In scattering media one should migrate *well chosen* cross correlations of data, not data themselves.
- Method can be applied with ambient noise sources instead of controlled sources.