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SPECTRAL CORRECTNESS OF THE SIMPLICIAL DISCONTINUOUS GALERKIN APPROXIMATION OF THE FIRST-ORDER FORM OF MAXWELL'S EQUATIONS WITH DISCONTINUOUS COEFFICIENTS*

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Abstract. The paper analyzes the discontinuous Galerkin approximation of Maxwell's equations written in first-order form and with nonhomogeneous magnetic permeability and electric permittivity. Although the Sobolev smoothness index of the solution may be smaller than $\frac{1}{2}$, it is shown that the approximation converges strongly and is therefore spectrally correct. The convergence proof uses the notion of involution and is based on a deflated inf-sup condition and a duality argument. One essential idea is that the smoothness index of the dual solution is always larger than $\frac{1}{2}$ irrespective of the regularity of the material properties.

Key words. Maxwell's equations, involution, finite elements, discontinuous Galerkin, duality argument, spectral correctness

MSC codes. 65M60, 65M12, 65N30, 35L02, 35L05, 35Q61

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1. Introduction.

The main motivation of the present work is the construction of space and time approximation techniques for nonlinear conservation equations, such as the Euler–Maxwell equations, or in ideal magnetohydrodynamics. Our objective is to construct approximation methods that are invariant-domain and involution preserving. For instance, for the Euler–Maxwell equations, the invariant domain concerns pointwise values of the density and the energy, whereas the involutions mean that the magnetic and electric fields remain in the image of the curl operator (in the absence of free charges), as stated by Gauss's laws. Preserving the involutions is essential to establishing compactness of the solution operator and ensures that the approximate solution behaves properly over long times (see, e.g., [25], [15]).

In the paper, we focus on the discontinuous Galerkin (dG) approximation of Maxwell's equations written as first-order conservation equations. Our main result establishes that the dG approximation is involution preserving and spectrally correct. We do so by proving the strong convergence in the $\boldsymbol{L}^2$-norm of the discrete solution operator. We emphasize that our goal is not to approximate the Maxwell eigenvalue problem per se, and that, for that specific purpose, it may be more convenient to work with the second-order formulation of Maxwell's equations. However, working with the second-order formulation is not an option in the context of magnetohydrodynamics. This is why we insist on working with the first-order formulation.

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The dG approximation of Maxwell's equations written in first-order form and with constant properties has been shown to be spectrally correct in [31]. The main limitation of [31], however, is that it relies on the solution to the boundary-value problem having a Sobolev smoothness index larger than $\frac{1}{2}$ for square integrable right-hand sides. The argument presented in [31] fails when the magnetic permeability and electric permittivity are discontinuous as the smoothness index of the solution to the boundary-value problem is lower than $\frac{1}{2}$ in this case. The result presented in the paper significantly improves that reported in [31] since it is based on two novel arguments. The first one consists of establishing stability by proving a deflated inf-sup condition (see Lemma 5.1). The second key idea is based on the observation that, irrespective of the smoothness of the magnetic permeability and electric permittivity, the solution to the dual problem lives in a Sobolev space with a smoothness regularity index that is always larger than $\frac{1}{2}$ (see Lemma 5.3). Strong convergence of the dG approximation is then proved by using a duality argument à la Aubin–Nitsche, following the seminal work in [40]. In the context of dG methods, a duality argument inspired by [40] is used in Chaumont-Frelet [16] for the Helmholtz problem and in Chaumont-Frelet and Ern [17] for Maxwell's equations in second-order form in the frequency domain.

The eigenvalue problem for Maxwell's equations with discontinuous properties written in second-order form has been investigated in [13], and the dG approximation has been shown therein to be spectrally correct. The method has been numerically tested in [14]. Conforming methods using edge elements have been investigated in [22] and in [29] for solving the boundary-value problem with low regularity. A stabilized continuous finite element method has been studied in [10] and has been shown to be spectrally correct when the Sobolev smoothness index is smaller than $\frac{1}{2}$. Very few papers have addressed the dG approximation of Maxwell's equations in first-order form. Numerical experiments performed in [33, section 5] and [34, pp. 513–514] have revealed that the dG approximation performs well in the time domain and that the approximation of the eigenvalue problem seems to be spurious-free even when the magnetic permeability and electric permittivity are discontinuous. Similar observations are made in [1, section V] and [23, section 5]. We provide here a mathematical proof of these observations.

The paper is organized as follows. We introduce the eigenvalue problem and recall some theoretical results from the literature in section 2. The discrete dG setting is described in section 3. We give in Lemma 3.1 and (3.14) a precise characterization of the discrete involutions satisfied by the discrete eigenvalue problem. Section 4 contains two preliminary results: discrete Poincaré–Steklov inequalities for involution-preserving discrete fields (Lemma 4.1) and a consistency bound for the discrete curl operators introduced in the dG setting (Lemma 4.2). The error analysis is performed in section 5. The two main steps are the deflated inf-sup condition established in Lemma 5.1 and the duality argument developed in section 5.2. The main convergence result of the paper is stated in Theorem 5.9. Finally, some standard results on Helmholtz decompositions are collected in Appendix A. We point out that the present analysis can be extended to establish that the dG approximation of the grad-div problem written in first-order form converges strongly, which, in turn, implies that the dG approximation of the linear wave equation written in first-order form is involution preserving and spectrally correct; the details are omitted for brevity.

2. Continuous setting. In this section, we present the functional setting to formulate the exact eigenvalue problem and the associated boundary-value problem. Most of the results listed here can be found in the literature; see e.g., [2], [5], [12], [24], [26], [32], [35].

2.1. Domain, model parameters, involutions. We consider a material with magnetic permeability, μ, and electric permittivity, ϵ, occupying the domain $D\subset\mathbb{R}^d$, $d=3$, which is assumed to be an open, bounded, connected polyhedron with Lipschitz boundary. The boundary of D is denoted ∂D and its unit outward normal $\boldsymbol{n}_D$. The domain D can have a general topology. In particular, D can be multiply connected, and ∂D can have several connected components. The material properties μ and ϵ can be heterogeneous, and, in particular, they can take discontinuous values. To fix the ideas, we assume that there is a partition of D into a finite number of disjoint Lipschitz polyhedra such that μ and ϵ are piecewise smooth on this partition. The above assumptions imply that D can be exactly covered with affine simplicial meshes, and each covering can be made compatible with the partition of D on which the scalar fields μ and ϵ are piecewise smooth.

The magnetic permeability and electric permittivity of vacuum are denoted μ_0,ϵ_0, respectively. To simplify the presentation, we assume throughout the paper that the ratios $\mu_0^{-1}\mu$ and $\epsilon_0^{-1}\epsilon$ are of order unity, so that these ratios can be hidden in the generic constants used in the error analysis. To be dimensionally consistent, we introduce a length scale, ℓ_D, associated with D; it can be, for instance, the diameter of D. Recalling that $\mathfrak{c}:=(\mu_0\epsilon_0)^{-\frac{1}{2}}$ is the speed of light, we introduce the following quantity which scales as the reciprocal of a time scale:

$$\omega:=\mathfrak{c}\ell_D^{-1}. \tag{2.1}$$

In what follows, for positive real numbers A,B, we abbreviate as $A\lesssim B$ the inequality $A\le CB$, where C is a generic constant whose value can change at each occurrence as long as it is independent of the mesh size (whenever relevant), the parameters μ_0, ϵ_0, ℓ_D, ω, and any fields involved in the inequality. The value of C can depend on the ratios $\operatorname{ess\,inf}\frac{\mu}{\mu_0}$, $\operatorname{ess\,sup}\frac{\mu}{\mu_0}$, $\operatorname{ess\,inf}\frac{\varepsilon}{\varepsilon_0}$, and $\operatorname{ess\,sup}\frac{\varepsilon}{\varepsilon_0}$ and on the shape-regularity of the mesh and the polynomial degree used in the dG approximation (whenever relevant).

Let us consider Maxwell's eigenvalue problem: Find $\lambda\in\mathbb{C}\setminus\{0\}$ and a nonzero pair of fields $\boldsymbol{H},\boldsymbol{E}:D\to\mathbb{C}^d$ such that

$$-\nabla\times\boldsymbol{E}=\frac{\omega}{\lambda}\mu\boldsymbol{H},\qquad \nabla_0\times\boldsymbol{H}=\frac{\omega}{\lambda}\epsilon\boldsymbol{E},\qquad \boldsymbol{H}\times\boldsymbol{n}|_{\partial D}=\boldsymbol{0}. \tag{2.2}$$

Here, $\nabla_0\times$ denotes the curl operator acting on vector fields that have a zero tangential trace at the boundary. An important observation from (2.2) is that $\mu\boldsymbol{H}$ is in the image of the curl operator, and $\epsilon\boldsymbol{E}$ is in the image of the curl operator acting on fields with zero tangential boundary condition. Following the terminology used in [8], [25], we henceforth refer to these properties as involutions.

2.2. Functional spaces. We now formalize the notion of involution by introducing a proper functional framework. We use standard notation for Lebesgue and Sobolev spaces. We use boldface fonts for $\mathbb{C}^d$-valued fields and functional spaces composed of such fields. The space $\boldsymbol{L}^2(D)$ is composed of Lebesgue integrable fields that are square integrable, and its canonical inner product is denoted $(\cdot,\cdot)_{\boldsymbol{L}^2(D)}$. As the solution to Maxwell's problem is a pair $(\boldsymbol{H},\boldsymbol{E})$, we introduce the product space $L^{\mathrm{c}}:=\boldsymbol{L}^2(D)\times\boldsymbol{L}^2(D)$. It is convenient to consider the norm

$$\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\mathrm{c}}}:=\big\{\|\mu^{\frac{1}{2}}\boldsymbol{f}\|^2_{\boldsymbol{L}^2(D)}+\|\epsilon^{\frac{1}{2}}\boldsymbol{g}\|^2_{\boldsymbol{L}^2(D)}\big\}^{\frac{1}{2}}. \tag{2.3}$$

Next, we define the Hilbert space

$$\boldsymbol{H}(\mathbf{curl};D):=\{\boldsymbol{h}\in\boldsymbol{L}^2(D)\mid\nabla\times\boldsymbol{h}\in\boldsymbol{L}^2(D)\}, \tag{2.4}$$

equipped with the natural graph norm $\|\boldsymbol{h}\|^2_{\boldsymbol{H}(\mathbf{curl};D)} := \|\boldsymbol{h}\|^2_{\boldsymbol{L}^2(D)} + \ell_D^2\|\nabla\times\boldsymbol{h}\|^2_{\boldsymbol{L}^2(D)}$. We also consider the closed subspace

$$\boldsymbol{H}_0(\mathbf{curl};D) := \{\boldsymbol{h}\in\boldsymbol{H}(\mathbf{curl};D) \mid \gamma^{\mathrm{c}}_{\partial D}(\boldsymbol{h}) = \mathbf{0}\}, \tag{2.5}$$

where $\gamma^{\mathrm{c}}_{\partial D} : \boldsymbol{H}(\mathbf{curl};D) \to \boldsymbol{H}^{-\frac{1}{2}}(\partial D)$ is the extension by density of the tangent trace operator such that $\gamma^{\mathrm{c}}_{\partial D}(\boldsymbol{h}) = \boldsymbol{h}|_{\partial D}\times\boldsymbol{n}_D$ for every smooth field $\boldsymbol{h}\in\boldsymbol{H}^r(D)$, $r>\frac{1}{2}$, with $\boldsymbol{n}_D$ the unit outward normal to D. We consider the operators $\nabla\times : \boldsymbol{H}(\mathbf{curl};D) \ni \boldsymbol{e} \longmapsto \nabla\times\boldsymbol{e} \in \boldsymbol{L}^2(D)$ and $\nabla_0\times : \boldsymbol{H}_0(\mathbf{curl};D) \ni \boldsymbol{h} \longmapsto \nabla\times\boldsymbol{h}\in\boldsymbol{L}^2(D)$. These operators are adjoint to each other since we have

$$(\nabla_0\times\boldsymbol{h},\boldsymbol{e})_{\boldsymbol{L}^2(D)} = (\boldsymbol{h},\nabla\times\boldsymbol{e})_{\boldsymbol{L}^2(D)} \quad \forall(\boldsymbol{h},\boldsymbol{e})\in\boldsymbol{H}_0(\mathbf{curl};D)\times\boldsymbol{H}(\mathbf{curl};D). \tag{2.6}$$

Since the operators $\nabla_0\times$ and $\nabla\times$ are closed, the involution properties for (2.2) are equivalent to asserting that

$$\mu\boldsymbol{H}\in\mathrm{im}(\nabla\times) = (\ker\nabla_0\times)^\perp, \qquad \epsilon\boldsymbol{E}\in\mathrm{im}(\nabla_0\times) = (\ker\nabla\times)^\perp, \tag{2.7}$$

where the symbol $^\perp$ denotes the orthogonality in $\boldsymbol{L}^2(D)$. Thus, after setting

$$\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D) := \ker\nabla\times = \{\boldsymbol{e}\in\boldsymbol{H}(\mathbf{curl};D)\mid\nabla\times\boldsymbol{e}=\mathbf{0}\}, \tag{2.8a}$$
$$\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D) := \ker\nabla_0\times = \{\boldsymbol{h}\in\boldsymbol{H}_0(\mathbf{curl};D)\mid\nabla_0\times\boldsymbol{h}=\mathbf{0}\}, \tag{2.8b}$$

it is natural to introduce the following closed subspaces:

$$\boldsymbol{X}^{\mathrm{c}}_{\mu,0} := \{\boldsymbol{h}\in\boldsymbol{H}_0(\mathbf{curl};D)\mid\mu\boldsymbol{h}\in\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^\perp\}, \tag{2.9a}$$
$$\boldsymbol{X}^{\mathrm{c}}_{\epsilon} := \{\boldsymbol{e}\in\boldsymbol{H}(\mathbf{curl};D)\mid\epsilon\boldsymbol{e}\in\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D)^\perp\}, \tag{2.9b}$$
$$\boldsymbol{X}^{\mathrm{c}}_{0} := \boldsymbol{H}_0(\mathbf{curl};D)\cap\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^\perp, \tag{2.9c}$$
$$\boldsymbol{X}^{\mathrm{c}} := \boldsymbol{H}(\mathbf{curl};D)\cap\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D)^\perp. \tag{2.9d}$$

Introducing the $\boldsymbol{L}^2$-orthogonal projections

$$\boldsymbol{\Pi}^{\mathrm{c}} : \boldsymbol{L}^2(D)\to\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D), \tag{2.10a}$$
$$\boldsymbol{\Pi}^{\mathrm{c}}_0 : \boldsymbol{L}^2(D)\to\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D), \tag{2.10b}$$

we can rewrite

$$\boldsymbol{X}^{\mathrm{c}}_{\mu,0} = \{\boldsymbol{h}\in\boldsymbol{H}_0(\mathbf{curl};D)\mid\boldsymbol{\Pi}^{\mathrm{c}}_0(\mu\boldsymbol{h})=\mathbf{0}\}, \tag{2.11a}$$
$$\boldsymbol{X}^{\mathrm{c}}_{\epsilon} = \{\boldsymbol{e}\in\boldsymbol{H}(\mathbf{curl};D)\mid\boldsymbol{\Pi}^{\mathrm{c}}(\epsilon\boldsymbol{e})=\mathbf{0}\}, \tag{2.11b}$$
$$\boldsymbol{X}^{\mathrm{c}}_{0} = \{\boldsymbol{\eta}\in\boldsymbol{H}_0(\mathbf{curl};D)\mid\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{\eta})=\mathbf{0}\}, \tag{2.11c}$$
$$\boldsymbol{X}^{\mathrm{c}} = \{\boldsymbol{\varepsilon}\in\boldsymbol{H}(\mathbf{curl};D)\mid\boldsymbol{\Pi}^{\mathrm{c}}(\boldsymbol{\varepsilon})=\mathbf{0}\}, \tag{2.11d}$$

and the involution properties (2.7) are equivalent to asserting that

$$\boldsymbol{\Pi}^{\mathrm{c}}_0(\mu\boldsymbol{H})=\mathbf{0}, \qquad \boldsymbol{\Pi}^{\mathrm{c}}(\epsilon\boldsymbol{E})=\mathbf{0}. \tag{2.12}$$

Remark 2.1 (topology of D). Referring to Appendix A.1 for the definition of the partition $\{\Gamma_i\}_{i\in\{1:I\}}$ and the cuts $\{\Sigma_j\}_{j\in\{1:J\}}$, we see that the definition (A.1) and the characterization (A.3) give $\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^\perp = \boldsymbol{H}^\Gamma(\mathrm{div}=0;D)$ and $\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D)^\perp = \boldsymbol{H}^\Sigma_0(\mathrm{div}=0;D)$, so that

$$\boldsymbol{X}^{\mathrm{c}}_{\mu,0} = \Big\{\boldsymbol{\eta}\in \boldsymbol{H}_0(\mathbf{curl};D)\mid \nabla\cdot(\mu\boldsymbol{\eta})=0,\ \int_{\Gamma_i}\mu\boldsymbol{\eta}\cdot\boldsymbol{n}\,\mathrm{d}s=0\ \forall i\in\{1{:}I\}\Big\}, \tag{2.13}$$

$$\boldsymbol{X}^{\mathrm{c}}_{\epsilon} = \Big\{\boldsymbol{\varepsilon}\in \boldsymbol{H}(\mathbf{curl};D)\mid \nabla\cdot(\epsilon\boldsymbol{\varepsilon})=0, \epsilon\boldsymbol{\varepsilon}\cdot\boldsymbol{n}_{|\partial D}=0, \int_{\Sigma_j}\epsilon\boldsymbol{\varepsilon}\cdot\boldsymbol{n}\,\mathrm{d}s=0\ \forall j\in\{1{:}J\}\Big\}, \tag{2.14}$$

$$\boldsymbol{X}^{\mathrm{c}}_{0} = \Big\{\boldsymbol{\eta}\in \boldsymbol{H}_0(\mathbf{curl};D)\cap\boldsymbol{H}(\mathrm{div}=0;D)\mid \int_{\Gamma_i}\boldsymbol{\eta}\cdot\boldsymbol{n}\,\mathrm{d}s=0\ \forall i\in\{1{:}I\}\Big\}, \tag{2.15}$$

$$\boldsymbol{X}^{\mathrm{c}} = \Big\{\boldsymbol{\varepsilon}\in \boldsymbol{H}(\mathbf{curl};D)\cap\boldsymbol{H}_0(\mathrm{div}=0;D)\mid \int_{\Sigma_j}\boldsymbol{\varepsilon}\cdot\boldsymbol{n}\,\mathrm{d}s=0\ \forall j\in\{1{:}J\}\Big\}, \tag{2.16}$$

where, for all $i\in\{1{:}I\}$ and all $j\in\{1{:}J\}$, the integrals over the $(d-1)$-manifolds Γ_i and Σ_j should be understood as duality products with the indicator functions of Γ_i and Σ_j, respectively. When the topology of D is trivial, we have $\boldsymbol{X}^{\mathrm{c}}_0=\boldsymbol{H}_0(\mathbf{curl};D)\cap \boldsymbol{H}(\mathrm{div}=0;D)$ and $\boldsymbol{X}^{\mathrm{c}}=\boldsymbol{H}(\mathbf{curl};D)\cap\boldsymbol{H}_0(\mathrm{div}=0;D)$. In this case, the involutions (2.7) are equivalent to Gauss's laws (in the absence of free charges), i.e., $\nabla\cdot(\varepsilon\boldsymbol{E})=0$ and $\nabla\cdot(\mu\boldsymbol{H})=0$. This is no longer the case when the topology of D is nontrivial.

2.3. Preliminary results. We now recall standard results about the operators $\nabla\times$ and $\nabla_0\times$ which we are going to invoke later in the paper.

LEMMA 2.2 (isomorphisms). *The following operators are isomorphisms:*

$$\nabla\times:\boldsymbol{X}^{\mathrm{c}}\to\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^{\perp},\qquad \nabla_0\times:\boldsymbol{X}^{\mathrm{c}}_0\to\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D)^{\perp}, \tag{2.17a}$$

$$\nabla\times:\boldsymbol{X}^{\mathrm{c}}_{\epsilon}\to\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^{\perp},\qquad \nabla_0\times:\boldsymbol{X}^{\mathrm{c}}_{\mu,0}\to\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D)^{\perp}. \tag{2.17b}$$

Proof. See the appendix. ∎

The operators $\nabla\times$ and $\nabla_0\times$ can be extended by duality to $\nabla\times:\boldsymbol{L}^2(D)\to(\boldsymbol{X}^{\mathrm{c}}_0)'$ and $\nabla_0\times:\boldsymbol{L}^2(D)\to(\boldsymbol{X}^{\mathrm{c}})'$. The following result is a straightforward consequence of Lemma 2.2; see also [12, sections 4-5].

COROLLARY 2.3 (weak Poincaré–Steklov (in)equalities). *The following equalities hold:*

$$\|\boldsymbol{e}\|_{\boldsymbol{L}^2(D)}=\ell_D\|\nabla\times\boldsymbol{e}\|_{(\boldsymbol{X}^{\mathrm{c}}_0)'}\qquad \forall\boldsymbol{e}\in\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D)^{\perp}, \tag{2.18a}$$

$$\|\boldsymbol{h}\|_{\boldsymbol{L}^2(D)}=\ell_D\|\nabla_0\times\boldsymbol{h}\|_{(\boldsymbol{X}^{\mathrm{c}})'}\qquad \forall\boldsymbol{h}\in\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^{\perp}, \tag{2.18b}$$

with $\|\nabla\times\boldsymbol{e}\|_{(\boldsymbol{X}^{\mathrm{c}}_0)'}:=\sup_{\boldsymbol{\eta}\in\boldsymbol{X}^{\mathrm{c}}_0}\frac{|(\boldsymbol{e},\nabla_0\times\boldsymbol{\eta})_{\boldsymbol{L}^2(D)}|}{\ell_D\|\nabla_0\times\boldsymbol{\eta}\|_{\boldsymbol{L}^2(D)}}$, $\|\nabla_0\times\boldsymbol{h}\|_{(\boldsymbol{X}^{\mathrm{c}})'}:=\sup_{\boldsymbol{\varepsilon}\in\boldsymbol{X}^{\mathrm{c}}}\frac{|(\boldsymbol{h},\nabla\times\boldsymbol{\varepsilon})_{\boldsymbol{L}^2(D)}|}{\ell_D\|\nabla\times\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2(D)}}$.

The next result is a consequence of the elliptic regularity theory and is proved in [9], [36]. The difference in regularity shift between the case of constant coefficients ($s'\in(\frac{1}{2},1]$) and the case of heterogeneous coefficients ($s\in(0,\frac{1}{2}]$) plays an important role in the paper. For the case of constant (or smooth) coefficients, we refer the reader to Costabel [24, Thm. 2] (see also Birman and Solomyak [5, Thm. 3.1]) and to Amrouche et al. [2, Prop. 3.7] whenever D is a Lipschitz polyhedron. The reader is also referred to Ciarlet [21, Thm. 16] for particular situations with heterogeneous properties for which smoothness is established with $s>\frac{1}{2}$.

LEMMA 2.4 (regularity shift). (i) *There is* $s'\in(\frac{1}{2},1]$ *such that, for all* $(\boldsymbol{\eta},\boldsymbol{\varepsilon})\in\boldsymbol{X}^{\mathrm{c}}_0\times\boldsymbol{X}^{\mathrm{c}}$,

$$|\boldsymbol{\eta}|_{\boldsymbol{H}^{s'}(D)}\lesssim\ell_D^{1-s'}\|\nabla_0\times\boldsymbol{\eta}\|_{\boldsymbol{L}^2(D)},\qquad |\boldsymbol{\varepsilon}|_{\boldsymbol{H}^{s'}(D)}\lesssim\ell_D^{1-s'}\|\nabla\times\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2(D)}. \tag{2.19}$$

(ii) *There is* $s\in(0,\frac{1}{2}]$ *such that, for all* $(\boldsymbol{h},\boldsymbol{e})\in\boldsymbol{X}^{\mathrm{c}}_{\mu,0}\times\boldsymbol{X}^{\mathrm{c}}_{\epsilon}$,

$$|\boldsymbol{h}|_{\boldsymbol{H}^{s}(D)}\lesssim\ell_D^{1-s}\|\nabla_0\times\boldsymbol{h}\|_{\boldsymbol{L}^2(D)},\qquad |\boldsymbol{e}|_{\boldsymbol{H}^{s}(D)}\lesssim\ell_D^{1-s}\|\nabla\times\boldsymbol{e}\|_{\boldsymbol{L}^2(D)}. \tag{2.20}$$

Remark 2.5 (Poincaré–Steklov inequalities). Notice in passing that as a consequence of (2.19), the compactness of the embedding $\boldsymbol{H}^{s'}(D) \subset \boldsymbol{L}^2(D)$, the injectivity of $\nabla_0\times$ and $\nabla\times$ on $\boldsymbol{X}_0^{\rm c}$ and $\boldsymbol{X}^{\rm c}$, respectively, and the Petree–Tartar lemma (see, e.g., [30, Lem. A.20]), the following Poincaré–Steklov inequalities hold for all $\boldsymbol{\eta} \in \boldsymbol{X}_0^{\rm c}$ and all $\boldsymbol{\varepsilon} \in \boldsymbol{X}^{\rm c}$:

$$\|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(D)} \lesssim \ell_D \|\nabla_0 \times \boldsymbol{\eta}\|_{\boldsymbol{L}^2(D)}, \qquad \|\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2(D)} \lesssim \ell_D \|\nabla \times \boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2(D)}. \tag{2.21}$$

Similar inequalities hold on $\boldsymbol{X}_{\mu,0}^{\rm c}$ and $\boldsymbol{X}_{\epsilon}^{\rm c}$, respectively.

2.4. Eigenvalue and boundary-value problems. Using the notation introduced above, the eigenvalue problem (2.2) consists of seeking $\lambda \in \mathbb{C}\backslash\{0\}$ and a nonzero pair $(\boldsymbol{H}, \boldsymbol{E}) \in \boldsymbol{X}_{\mu,0}^{\rm c} \times \boldsymbol{X}_{\epsilon}^{\rm c}$ so that

$$-\nabla \times \boldsymbol{E} = \frac{\omega}{\lambda} \mu \boldsymbol{H}, \qquad \nabla_0 \times \boldsymbol{H} = \frac{\omega}{\lambda} \epsilon \boldsymbol{E}. \tag{2.22}$$

We now define the boundary-value operator $T : L^{\rm c} \to L^{\rm c}$ associated with (2.22). For all $(\boldsymbol{f}, \boldsymbol{g}) \in L^{\rm c}$, one cannot simply require that the pair $(\boldsymbol{H}, \boldsymbol{E}) := T(\boldsymbol{f}, \boldsymbol{g})$ solve $-\nabla \times \boldsymbol{E} = \omega \mu \boldsymbol{f}$ and $\nabla_0 \times \boldsymbol{H} = \omega \epsilon \boldsymbol{g}$ (the factor ω is only introduced for dimensional consistency). Indeed, this problem only makes sense if $\mu \boldsymbol{f} \in \operatorname{im}(\nabla\times) = \ker(\boldsymbol{\Pi}_0^{\rm c})$ and $\epsilon \boldsymbol{g} \in \operatorname{im}(\nabla_0\times) = \ker(\boldsymbol{\Pi}^{\rm c})$. To circumvent this difficulty, we define $T(\boldsymbol{f}, \boldsymbol{g})$ as the unique pair $(\boldsymbol{H}, \boldsymbol{E}) \in \boldsymbol{X}_{\mu,0}^{\rm c} \times \boldsymbol{X}_{\epsilon}^{\rm c}$ so that

$$-\nabla \times \boldsymbol{E} = \omega (\boldsymbol{I} - \boldsymbol{\Pi}_0^{\rm c})(\mu \boldsymbol{f}), \tag{2.23a}$$

$$\nabla_0 \times \boldsymbol{H} = \omega (\boldsymbol{I} - \boldsymbol{\Pi}^{\rm c})(\epsilon \boldsymbol{g}). \tag{2.23b}$$

The existence and uniqueness of the solution to (2.23) is a consequence of (2.17b) and the definition of the projectors $\boldsymbol{\Pi}_0^{\rm c}$ and $\boldsymbol{\Pi}^{\rm c}$.

LEMMA 2.6 (a priori estimates and compactness). (i) *The solution* $(\boldsymbol{H}, \boldsymbol{E}) \in \boldsymbol{X}_{\mu,0}^{\rm c} \times \boldsymbol{X}_{\epsilon}^{\rm c}$ *of* (2.23) *satisfies the a priori estimate*

$$\ell_D \big(\mu_0^{\frac12} \|\nabla_0 \times \boldsymbol{H}\|_{\boldsymbol{L}^2(D)} + \epsilon_0^{\frac12} \|\nabla \times \boldsymbol{E}\|_{\boldsymbol{L}^2(D)}\big) \lesssim \|(\boldsymbol{f}, \boldsymbol{g})\|_{L^{\rm c}}. \tag{2.24a}$$

(ii) *There is* $s \in (0, \frac12]$ *such that*

$$|\boldsymbol{H}|_{\boldsymbol{H}^s(D)} \lesssim \ell_D^{1-s} \|\nabla_0 \times \boldsymbol{H}\|_{\boldsymbol{L}^2(D)}, \quad |\boldsymbol{E}|_{\boldsymbol{H}^s(D)} \lesssim \ell_D^{1-s} \|\nabla \times \boldsymbol{E}\|_{\boldsymbol{L}^2(D)}, \tag{2.24b}$$

and the operator $T : L^{\rm c} \to L^{\rm c}$ *is compact.*

Proof. (i) Equations (2.23) give $\|\nabla \times \boldsymbol{E}\|_{\boldsymbol{L}^2(D)} \le \omega \|\mu \boldsymbol{f}\|_{\boldsymbol{L}^2(D)} \lesssim \omega \mu_0^{\frac12} \|\mu^{\frac12} \boldsymbol{f}\|_{\boldsymbol{L}^2(D)}$ and, similarly, we obtain $\|\nabla_0 \times \boldsymbol{H}\|_{\boldsymbol{L}^2(D)} \lesssim \omega \varepsilon_0^{\frac12} \|\varepsilon^{\frac12} \boldsymbol{g}\|_{\boldsymbol{L}^2(D)}$. The estimate (2.24a) is obtained by observing that $\omega \mu_0^{\frac12} \epsilon_0^{\frac12} = \ell_D^{-1}$.

(ii) The estimates (2.24b) come from (2.20) and imply the compactness of T owing to the Rellich–Kondrachov embedding theorem. ∎

Notice that for all $\lambda \in \mathbb{C}\backslash\{0\}$, the nonzero pair $(\boldsymbol{H}, \boldsymbol{E}) \in \boldsymbol{X}_{\mu,0}^{\rm c} \times \boldsymbol{X}_{\epsilon}^{\rm c}$ solves the eigenvalue problem (2.22) if and only if $\big(\lambda, (\boldsymbol{H}, \boldsymbol{E})\big)$ is an eigenpair of T.

Remark 2.7 (equivalent definition). An equivalent definition of the operator $T : L^{\rm c} \to L^{\rm c}$ is that, for all $(\boldsymbol{f}, \boldsymbol{g}) \in L^{\rm c}$, $T(\boldsymbol{f}, \boldsymbol{g}) := (\boldsymbol{H}, \boldsymbol{E})$ is the unique pair in $\boldsymbol{H}_0(\mathbf{curl}; D) \times \boldsymbol{H}(\mathbf{curl}; D)$ solving the boundary-value problem

$$\omega\mathbf{\Pi}_0^{\mathrm{c}}(\mu\boldsymbol{H}) - \nabla\times\boldsymbol{E} = \omega(\boldsymbol{I} - \mathbf{\Pi}_0^{\mathrm{c}})(\mu\boldsymbol{f}), \tag{2.25a}$$

$$\omega\mathbf{\Pi}^{\mathrm{c}}(\epsilon\boldsymbol{E}) + \nabla_0\times\boldsymbol{H} = \omega(\boldsymbol{I} - \mathbf{\Pi}^{\mathrm{c}})(\epsilon\boldsymbol{g}). \tag{2.25b}$$

Indeed, since $\mathbf{\Pi}_0^{\mathrm{c}}((\boldsymbol{I} - \mathbf{\Pi}_0^{\mathrm{c}})(\mu\boldsymbol{f})) = \mathbf{\Pi}^{\mathrm{c}}((\boldsymbol{I} - \mathbf{\Pi}^{\mathrm{c}})(\epsilon\boldsymbol{g})) = \boldsymbol{0}$ and

$$\mathbf{\Pi}_0^{\mathrm{c}}(\nabla\times\boldsymbol{E}) = \mathbf{\Pi}^{\mathrm{c}}(\nabla_0\times\boldsymbol{H}) = \boldsymbol{0},$$

the pair $(\boldsymbol{H},\boldsymbol{E}) \in \boldsymbol{H}_0(\mathbf{curl};D)\times\boldsymbol{H}(\mathbf{curl};D)$ solves (2.25) if and only if the pair $(\boldsymbol{H},\boldsymbol{E})$ sits in $\boldsymbol{X}_{\mu,0}^{\mathrm{c}}\times\boldsymbol{X}_{\epsilon}^{\mathrm{c}}$ and solves (2.23).

Remark 2.8 (shifting or not shifting). One traditional way of dealing with the boundary-value problem consists of avoiding the use of the projections $\mathbf{\Pi}_0^{\mathrm{c}}$ and $\mathbf{\Pi}^{\mathrm{c}}$ by adding a zero-order term to the operator. Specifically, one considers a shifted operator $S : L^{\mathrm{c}} \to L^{\mathrm{c}}$ defined as follows: For all $(\boldsymbol{f},\boldsymbol{g}) \in L^{\mathrm{c}}$, $S(\boldsymbol{f},\boldsymbol{g}) := (\boldsymbol{H},\boldsymbol{E})$ is the pair in $\boldsymbol{H}_0(\mathbf{curl};D)\times\boldsymbol{H}(\mathbf{curl};D) \subset L^{\mathrm{c}}$ such that

$$\omega\mu\boldsymbol{H} - \nabla\times\boldsymbol{E} = \omega\mu\boldsymbol{f}, \tag{2.26a}$$

$$\omega\epsilon\boldsymbol{E} + \nabla_0\times\boldsymbol{H} = \omega\epsilon\boldsymbol{g}. \tag{2.26b}$$

This problem is well-posed and has a unique solution. One observes that $(\lambda,(\boldsymbol{H},\boldsymbol{E}))$ is an eigenpair of the operator T iff $(\frac{\lambda}{1+\lambda},(\boldsymbol{H},\boldsymbol{E}))$ is an eigenpair of the operator S. Hence, the shifting does not change the eigenstructure of the problem, and one can address the spectral correctness of the approximation of (2.22) by proving the spectral correctness of the approximation to the shifted problem. The slight difficulty is that the operator S is not compact, so that one cannot invoke the now standard Bramble–Babuška–Osborn theory. One can instead invoke the theory developed in [27]; see, for instance, Property 1 and Theorem 2 therein. For instance, this is the approach followed in [13] for the dG approximation of the Maxwell operator in second-order form. The route we follow in the paper is different: instead of shifting the problem, we formulate the boundary-value problem by projecting the data on the involution-preserving space $\boldsymbol{X}_{\mu,0}^{\mathrm{c}}\times\boldsymbol{X}_{\epsilon}^{\mathrm{c}}$. We shall proceed similarly at the discrete level and project the data on the discrete counterpart of $\boldsymbol{X}_{\mu,0}^{\mathrm{c}}\times\boldsymbol{X}_{\epsilon}^{\mathrm{c}}$. The key advantage of this approach is that we can then apply the usual theory for the spectral approximation of compact operators.

3. Discrete eigenvalue problem.

In this section, we introduce the discrete setting to formulate the discrete eigenvalue problem.

3.1. dG setting.

Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular family of affine simplicial meshes such that each mesh covers D exactly and is compatible with the partition of D associated with the discontinuities of the material properties. For simplicity, we assume in the paper that the meshes are quasi-uniform. The symbol K denotes a generic mesh cell, h_K its diameter, and $\boldsymbol{n}_K$ its outward unit normal. We define $\tilde{h}$ as the piecewise constant function on $\mathcal{T}_h$ such that $\tilde{h}|_K = h_K$ for all $K \in \mathcal{T}_h$; we set $h := \|\tilde{h}\|_{L^\infty(D)}$. The set of mesh faces, $\mathcal{F}_h$, is split into the subset of mesh interfaces, say $\mathcal{F}_h^\circ$, and the subset of mesh boundary faces, say $\mathcal{F}_h^\partial$. Each interface is shared by two distinct mesh cells which we denote by K_l, K_r. Each boundary face is shared by one mesh cell, K_l, and the boundary, ∂D. For every mesh face $F \in \mathcal{F}_h$, h_F denotes the diameter of F. Every mesh interface $F \in \mathcal{F}_h^\circ$ is oriented by the unit normal, $\boldsymbol{n}_F$, pointing from K_l to K_r. Every boundary face $F \in \mathcal{F}_h^\partial$ is oriented by the unit normal $\boldsymbol{n}_F := \boldsymbol{n}_D$.

Let $k \geq 0$ be the polynomial degree. We insist that $k = 0$ is a legitimate choice for the theory developed in the paper. Let $\mathbb{P}_{k,d}$ be the vector space over $\mathbb{C}$ composed of d-variate polynomials of total degree at most k, and set $\mathbb{P}_{k,d} := [\mathbb{P}_{k,d}]^d$. Consider the vector-valued broken polynomial spaces

$$\boldsymbol{P}_k^{\rm b}(\mathcal{T}_h) := \{\boldsymbol{w}_h \in \boldsymbol{L}^2(D) \mid \boldsymbol{w}_{h|K} \in \mathbb{P}_{k,d}\, \forall K \in \mathcal{T}_h\}. \tag{3.1}$$

We define the $\boldsymbol{L}^2$-orthogonal projection onto the broken polynomial space $\boldsymbol{P}_k^{\rm b}(\mathcal{T}_h)$,

$$\boldsymbol{\Pi}_h^{\rm b} : \boldsymbol{L}^2(D) \to \boldsymbol{P}_k^{\rm b}(\mathcal{T}_h). \tag{3.2}$$

For all $K \in \mathcal{T}_h$, all $F \in \mathcal{F}_h$ with $F \subset \partial K$, and all $\boldsymbol{w}_h \in \boldsymbol{P}_k^{\rm b}(\mathcal{T}_h)$, we define the local trace operators such that $\gamma_{K,F}^{\rm g}(\boldsymbol{w}_h)(\boldsymbol{x}) := \boldsymbol{w}_h|_K(\boldsymbol{x})$, $\gamma_{K,F}^{\rm c}(\boldsymbol{w}_h)(\boldsymbol{x}) := \boldsymbol{w}_h|_K(\boldsymbol{x}) \times \boldsymbol{n}_F$ for a.e. $\boldsymbol{x} \in F$. Then, for all $F \in \mathcal{F}_h^\circ$ and ${\rm x} \in \{{\rm g}, {\rm c}\}$, we define the jump and average operators such that

$$[\![\boldsymbol{w}_h]\!]_F^{\rm x} := \gamma_{K_l,F}^{\rm x}(\boldsymbol{w}_h) - \gamma_{K_r,F}^{\rm x}(\boldsymbol{w}_h), \qquad \{\!\{\boldsymbol{w}_h\}\!\}_F^{\rm x} := \frac{1}{2}\big(\gamma_{K_l,F}^{\rm x}(\boldsymbol{w}_h) + \gamma_{K_r,F}^{\rm x}(\boldsymbol{w}_h)\big). \tag{3.3}$$

To allow for more compact expressions, we also set $[\![\boldsymbol{w}_h]\!]_F^{\rm x} := \{\!\{\boldsymbol{w}_h\}\!\}_F^{\rm x} := \gamma_{K_l,F}^{\rm x}(\boldsymbol{w}_h)$ for all $F \in \mathcal{F}_h^\partial$. We define the jump sesquilinear forms such that for all $\boldsymbol{H}_h, \boldsymbol{h}_h, \boldsymbol{E}_h, \boldsymbol{e}_h \in \boldsymbol{P}_k^{\rm b}(\mathcal{T}_h)$,

$$s_h^{\rm c}(\boldsymbol{H}_h, \boldsymbol{h}_h) := \sum_{F \in \mathcal{F}_h} ([\![\boldsymbol{H}_h]\!]_F^{\rm c}, [\![\boldsymbol{h}_h]\!]_F^{\rm c})_{\boldsymbol{L}^2(F)}, \qquad s_h^{{\rm c},\circ}(\boldsymbol{E}_h, \boldsymbol{e}_h) := \sum_{F \in \mathcal{F}_h^\circ} ([\![\boldsymbol{E}_h]\!]_F^{\rm c}, [\![\boldsymbol{e}_h]\!]_F^{\rm c})_{\boldsymbol{L}^2(F)}.$$

These sesquilinear forms induce the following seminorms which we henceforth call jump seminorms:

$$|\boldsymbol{h}_h|_{\rm J}^{\rm c} := s_h^{\rm c}(\boldsymbol{h}_h, \boldsymbol{h}_h)^{\frac{1}{2}}, \qquad |\boldsymbol{e}_h|_{\rm J}^{{\rm c},\circ} := s_h^{{\rm c},\circ}(\boldsymbol{e}_h, \boldsymbol{e}_h)^{\frac{1}{2}}. \tag{3.4}$$

The discrete curl operators

$$\boldsymbol{C}_{h0} : \boldsymbol{P}_k^{\rm b}(\mathcal{T}_h) \to \boldsymbol{P}_k^{\rm b}(\mathcal{T}_h), \qquad \boldsymbol{C}_h : \boldsymbol{P}_k^{\rm b}(\mathcal{T}_h) \to \boldsymbol{P}_k^{\rm b}(\mathcal{T}_h) \tag{3.5}$$

are defined as follows: For all $(\boldsymbol{h}_h, \boldsymbol{e}_h) \in \boldsymbol{P}_k^{\rm b}(\mathcal{T}_h) \times \boldsymbol{P}_k^{\rm b}(\mathcal{T}_h)$, $\boldsymbol{C}_{h0}(\boldsymbol{h}_h)$ and $\boldsymbol{C}_h(\boldsymbol{e}_h)$ are the unique members of $\boldsymbol{P}_k^{\rm b}(\mathcal{T}_h)$ such that the following identities hold true for all $\boldsymbol{e}_h', \boldsymbol{h}_h' \in \boldsymbol{P}_k^{\rm b}(\mathcal{T}_h)$:

$$(\boldsymbol{C}_{h0}(\boldsymbol{h}_h), \boldsymbol{e}_h')_{\boldsymbol{L}^2(D)} := (\nabla_h \times \boldsymbol{h}_h, \boldsymbol{e}_h')_{\boldsymbol{L}^2(D)} + \sum_{F \in \mathcal{F}_h} ([\![\boldsymbol{h}_h]\!]_F^{\rm c}, \{\!\{\boldsymbol{e}_h'\}\!\}_F^{\rm g})_{\boldsymbol{L}^2(F)}, \tag{3.6a}$$

$$(\boldsymbol{C}_h(\boldsymbol{e}_h), \boldsymbol{h}_h')_{\boldsymbol{L}^2(D)} := (\nabla_h \times \boldsymbol{e}_h, \boldsymbol{h}_h')_{\boldsymbol{L}^2(D)} + \sum_{F \in \mathcal{F}_h^\circ} ([\![\boldsymbol{e}_h]\!]_F^{\rm c}, \{\!\{\boldsymbol{h}_h'\}\!\}_F^{\rm g})_{\boldsymbol{L}^2(F)}, \tag{3.6b}$$

where $\nabla_h \times$ denotes the broken curl operator (evaluated cellwise). The following integration-by-parts formula holds (compare with (2.6)):

$$(\boldsymbol{C}_{h0}(\boldsymbol{h}_h), \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)} = (\boldsymbol{h}_h, \boldsymbol{C}_h(\boldsymbol{e}_h))_{\boldsymbol{L}^2(D)} \quad \forall (\boldsymbol{h}_h, \boldsymbol{e}_h) \in \boldsymbol{P}_k^{\rm b}(\mathcal{T}_h) \times \boldsymbol{P}_k^{\rm b}(\mathcal{T}_h). \tag{3.7}$$

3.2. Discrete eigenvalue problem and discrete involution. We define the discrete space $L_h^{\rm c} := \boldsymbol{P}_k^{\rm b}(\mathcal{T}_h) \times \boldsymbol{P}_k^{\rm b}(\mathcal{T}_h)$. Notice that $L_h^{\rm c} \subset L^{\rm c}$. We define the sesquilinear form $a_h : L_h^{\rm c} \times L_h^{\rm c} \to \mathbb{C}$ such that

$$\begin{aligned} a_h\big((\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)\big) := &- (\boldsymbol{C}_h(\boldsymbol{E}_h), \boldsymbol{h}_h)_{\boldsymbol{L}^2(D)} + (\boldsymbol{C}_{h0}(\boldsymbol{H}_h), \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)} \\ &+ \kappa_H s_h^{\rm c}(\boldsymbol{H}_h, \boldsymbol{h}_h) + \kappa_E s_h^{{\rm c},\circ}(\boldsymbol{E}_h, \boldsymbol{e}_h), \end{aligned} \tag{3.8}$$

with $\kappa_H := \mu_0\omega\ell_D = (\mu_0/\epsilon_0)^{\frac{1}{2}}$ and $\kappa_E := \epsilon_0\omega\ell_D = (\epsilon_0/\mu_0)^{\frac{1}{2}}$; notice that $\kappa_H\kappa_E = 1$. The discrete eigenvalue problem consists of finding $\lambda_h \in \mathbb{C}\setminus\{0\}$ and a nonzero pair $(\boldsymbol{H}_h, \boldsymbol{E}_h) \in L_h^{\mathrm{c}}$ such that the following holds for all $(\boldsymbol{h}_h, \boldsymbol{e}_h) \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)\times\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$:

$$a_h\big((\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)\big) = \frac{\omega}{\lambda_h}\big((\mu\boldsymbol{H}_h, \epsilon\boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)\big)_{\boldsymbol{L}^2(D)\times\boldsymbol{L}^2(D)}. \tag{3.9}$$

Introducing the operator $A_h : L_h^{\mathrm{c}} \to L_h^{\mathrm{c}}$ such that, for all $(\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h) \in L_h^{\mathrm{c}}$,

$$\big(A_h(\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)\big)_{\boldsymbol{L}^2(D)\times\boldsymbol{L}^2(D)} := a_h\big((\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)\big),$$

the discrete eigenvalue problem (3.9) can be rewritten as

$$A_h(\boldsymbol{H}_h, \boldsymbol{E}_h) = \frac{\omega}{\lambda_h}(\mu\boldsymbol{H}_h, \epsilon\boldsymbol{E}_h). \tag{3.10}$$

Hence, the discrete involution property consists of requesting that $(\mu\boldsymbol{H}_h, \epsilon\boldsymbol{E}_h)$ be in the range of A_h, i.e., a member of $\ker(A_h^{\mathsf{T}})^{\perp}$ (notice that $A_h^{\mathsf{T}} : L_h^{\mathrm{c}} \to L_h^{\mathrm{c}}$, and recall that the symbols $^{\mathsf{T}}$ and $^{\perp}$ refer to the orthogonality in $\boldsymbol{L}^2(D)$). Thus, our first step is to identify the kernel of A_h^{T}.

Lemma 3.1 (discrete involution). *The following holds:*

$$\ker(A_h^{\mathsf{T}}) = \big(\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \cap \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0}; D)\big) \times \big(\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathbf{curl}=\mathbf{0}; D)\big). \tag{3.11}$$

Proof. (1) Let $(\boldsymbol{h}_h, \boldsymbol{e}_h) \in \ker(A_h^{\mathsf{T}})$, i.e., we have $a_h\big((\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)\big) = 0$ for all $(\boldsymbol{H}_h, \boldsymbol{E}_h) \in L_h^{\mathrm{c}}$. This implies $a_h\big((\boldsymbol{h}_h, \boldsymbol{e}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)\big) = 0$, and we infer from (3.8) and (3.7) that $|\boldsymbol{h}_h|_{\mathrm{J}}^{\mathrm{c}} = |\boldsymbol{e}_h|_{\mathrm{J}}^{\mathrm{c},\circ} = 0$. Hence, $\boldsymbol{h}_h \in \boldsymbol{H}_0(\mathbf{curl}; D)$ and $\boldsymbol{e}_h \in \boldsymbol{H}(\mathbf{curl}; D)$. This, in turn, implies that it is legitimate to consider the test functions $\boldsymbol{H}_h = \boldsymbol{C}_h(\boldsymbol{e}_h) = \nabla\times\boldsymbol{e}_h$ and $\boldsymbol{E}_h = \boldsymbol{C}_{h0}(\boldsymbol{h}_h) = \nabla_0\times\boldsymbol{h}_h$, which gives $\nabla_0\times\boldsymbol{h}_h = \nabla\times\boldsymbol{e}_h = \mathbf{0}$. In conclusion, $\ker(A_h^{\mathsf{T}}) \subset \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0}; D) \times \boldsymbol{H}(\mathbf{curl}=\mathbf{0}; D)$.

(2) The proof of $\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0}; D) \times \boldsymbol{H}(\mathbf{curl}=\mathbf{0}; D) \subset \ker(A_h^{\mathsf{T}})$ is similar. ∎

Our next important result is that $\ker(A_h^{\mathsf{T}})$ can be characterized by means of curl-free Nédélec finite element subspaces. Let $\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)$ be the $\boldsymbol{H}(\mathbf{curl}; D)$-conforming finite element space composed of the piecewise Nédélec polynomials of the first family and of degree $k \geq 0$, and let $\boldsymbol{P}_{k0}^{\mathrm{c}}(\mathcal{T}_h)$ be its subspace composed of discrete fields with zero tangential trace on ∂D. We set

$$\boldsymbol{P}_k^{\mathrm{c}}(\mathbf{curl}=\mathbf{0}; \mathcal{T}_h) := \{\boldsymbol{e}_h \in \boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h) \mid \nabla\times\boldsymbol{e}_h = \mathbf{0}\}, \tag{3.12a}$$
$$\boldsymbol{P}_{k0}^{\mathrm{c}}(\mathbf{curl}=\mathbf{0}; \mathcal{T}_h) := \{\boldsymbol{h}_h \in \boldsymbol{P}_{k0}^{\mathrm{c}}(\mathcal{T}_h) \mid \nabla_0\times\boldsymbol{h}_h = \mathbf{0}\}. \tag{3.12b}$$

Notice that $\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)$ is not a subspace of $\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$ (actually $\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h) \subsetneq \boldsymbol{P}_{k+1}^{\mathrm{b}}(\mathcal{T}_h)$), but we are going to make use of the following well-known results (see, e.g., [30, Chaps. 15 and 18]):

$$\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathbf{curl}=\mathbf{0}; D) = \boldsymbol{P}_k^{\mathrm{c}}(\mathbf{curl}=\mathbf{0}; \mathcal{T}_h), \tag{3.13a}$$
$$\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \cap \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0}; D) = \boldsymbol{P}_{k0}^{\mathrm{c}}(\mathbf{curl}=\mathbf{0}; \mathcal{T}_h). \tag{3.13b}$$

Therefore, (3.11) can be reformulated as follows:

$$\ker(A_h^{\mathsf{T}}) = \boldsymbol{P}_{k0}^{\mathrm{c}}(\mathbf{curl}=\mathbf{0}; \mathcal{T}_h) \times \boldsymbol{P}_k^{\mathrm{c}}(\mathbf{curl}=\mathbf{0}; \mathcal{T}_h). \tag{3.14}$$

Since the discrete involutions are equivalent to being $\boldsymbol{L}^2$-orthogonal to $\ker(A_h^{\mathsf{T}})$, it is natural to define the $\boldsymbol{L}^2$-orthogonal projections,

$$\boldsymbol{\Pi}_h^{\mathrm{c}} : \boldsymbol{L}^2(D) \to \boldsymbol{P}_k^{\mathrm{c}}(\mathbf{curl}=\mathbf{0};\mathcal{T}_h), \tag{3.15a}$$

$$\boldsymbol{\Pi}_{h0}^{\mathrm{c}} : \boldsymbol{L}^2(D) \to \boldsymbol{P}_{k0}^{\mathrm{c}}(\mathbf{curl}=\mathbf{0};\mathcal{T}_h), \tag{3.15b}$$

and to introduce the following discrete subspaces:

$$\boldsymbol{X}_{\mu,h0}^{\mathrm{c}} := \{\boldsymbol{h}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \mid \boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\mu \boldsymbol{h}_h) = \mathbf{0}\}, \tag{3.16a}$$

$$\boldsymbol{X}_{\epsilon,h}^{\mathrm{c}} := \{\boldsymbol{e}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \mid \boldsymbol{\Pi}_{h}^{\mathrm{c}}(\epsilon \boldsymbol{e}_h) = \mathbf{0}\}. \tag{3.16b}$$

In conclusion, the discrete involutions associated with the discrete eigenvalue problem (3.10) can be rewritten as

$$\boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\mu \boldsymbol{H}_h) = \boldsymbol{\Pi}_{h}^{\mathrm{c}}(\epsilon \boldsymbol{E}_h) = \mathbf{0}. \tag{3.17}$$

Remark 3.2 (simplices). The identities (3.13) only hold true for simplicial finite elements. The result is false on quadrangles and hexahedra. Actually, the dG approximation of Maxwell's equations on quadrangles and hexahedra is not spectrally correct. The reader is referred to [13, Rem. 7.14] and [7, Rem. 3], where similar observations are made in the context of the approximation of Maxwell's equations written in second-order form.

3.3. Discrete boundary-value problem. We now define $T_h : L^{\mathrm{c}} \to L_h^{\mathrm{c}} \subset L^{\mathrm{c}}$, the discrete counterpart of the operator $T : L^{\mathrm{c}} \to L^{\mathrm{c}}$ defined in section 2.4. For all $(\boldsymbol{f},\boldsymbol{g}) \in L^{\mathrm{c}}$, we let $T_h(\boldsymbol{f},\boldsymbol{g}) := (\boldsymbol{H}_h,\boldsymbol{E}_h)$ be the unique pair in $\boldsymbol{X}_{\mu,h0}^{\mathrm{c}} \times \boldsymbol{X}_{\epsilon,h}^{\mathrm{c}}$ so that the following equality holds for all $(\boldsymbol{h}_h,\boldsymbol{e}_h) \in \boldsymbol{X}_{\mu,h0}^{\mathrm{c}} \times \boldsymbol{X}_{\epsilon,h}^{\mathrm{c}}$:

$$a_h\big((\boldsymbol{H}_h,\boldsymbol{E}_h),(\boldsymbol{h}_h,\boldsymbol{e}_h)\big) := \omega\big((\boldsymbol{I}-\boldsymbol{\Pi}_{h0}^{\mathrm{c}})(\mu\boldsymbol{f}),\boldsymbol{h}_h\big)_{\boldsymbol{L}^2(D)} + \omega\big((\boldsymbol{I}-\boldsymbol{\Pi}_{h}^{\mathrm{c}})(\epsilon\boldsymbol{g}),\boldsymbol{e}_h\big)_{\boldsymbol{L}^2(D)}. \tag{3.18}$$

Notice in passing that equality actually holds in (3.18) for all $(\boldsymbol{h}_h,\boldsymbol{e}_h) \in L_h^{\mathrm{c}}$ because $\boldsymbol{X}_{\mu,h0}^{\mathrm{c}} \times \boldsymbol{X}_{\epsilon,h}^{\mathrm{c}} = (\boldsymbol{I}-\boldsymbol{\Pi}_{h0}^{\mathrm{c}})(\mu L_h^{\mathrm{c}}) \times (\boldsymbol{I}-\boldsymbol{\Pi}_{h}^{\mathrm{c}})(\epsilon L_h^{\mathrm{c}})$ and $\ker(A_h^{\mathsf{T}}) = \boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\mu L_h^{\mathrm{c}}) \times \boldsymbol{\Pi}_{h}^{\mathrm{c}}(\epsilon L_h^{\mathrm{c}})$. The following result is a consequence of Lemma 5.1(ii) below.

LEMMA 3.3 (discrete well-posedness). *The problem* (3.18) *has a unique solution.*

As (3.18) also holds for all $(\boldsymbol{h}_h,\boldsymbol{e}_h) \in L_h^{\mathrm{c}}$, we infer that for all $\lambda_h \neq 0$ and all $(\boldsymbol{H}_h,\boldsymbol{E}_h) \in L_h^{\mathrm{c}}$, $\big(\lambda_h,(\boldsymbol{H}_h,\boldsymbol{E}_h)\big)$ is an eigenpair of T_h if and only if $\big(\frac{1}{\lambda_h},(\boldsymbol{H}_h,\boldsymbol{E}_h)\big)$ is an eigenpair of A_h (see (3.10)). Therefore, proving that the spectrum of T_h is pollution-free is equivalent to proving that the spectrum of A_h is pollution-free. We do so in section 5 by establishing that $\lim_{h\to 0}\|T-T_h\|_{\mathcal{L}(L^{\mathrm{c}};L^{\mathrm{c}})} = 0$.

Remark 3.4 (equivalent definition). In the same spirit as in Remark 2.7 and [31], we can devise an equivalent definition of the discrete operator $T_h : L^{\mathrm{c}} \to L_h^{\mathrm{c}}$. We introduce the augmented sesquilinear form $a_h^{\pi} : L_h^{\mathrm{c}} \times L_h^{\mathrm{c}} \to \mathbb{C}$ such that

$$\begin{aligned} a_h^{\pi}\big((\boldsymbol{H}_h,\boldsymbol{E}_h),(\boldsymbol{h}_h,\boldsymbol{e}_h)\big) := {}& \omega(\boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\mu\boldsymbol{H}_h),\boldsymbol{h}_h)_{\boldsymbol{L}^2(D)} + \omega(\boldsymbol{\Pi}_{h}^{\mathrm{c}}(\epsilon\boldsymbol{E}_h),\boldsymbol{e}_h)_{\boldsymbol{L}^2(D)} \\ & + a_h\big((\boldsymbol{H}_h,\boldsymbol{E}_h),(\boldsymbol{h}_h,\boldsymbol{e}_h)\big). \end{aligned} \tag{3.19}$$

Then, for all $(\boldsymbol{f},\boldsymbol{g}) \in L^{\mathrm{c}}$, $T_h(\boldsymbol{f},\boldsymbol{g}) := (\boldsymbol{H}_h,\boldsymbol{E}_h)$ is defined as the unique pair in L_h^{c} such that, for all $(\boldsymbol{h}_h,\boldsymbol{e}_h) \in L_h^{\mathrm{c}}$,

$$a_h^{\pi}\big((\boldsymbol{H}_h,\boldsymbol{E}_h),(\boldsymbol{h}_h,\boldsymbol{e}_h)\big) := \omega\big((\boldsymbol{I}-\boldsymbol{\Pi}_{h0}^{\mathrm{c}})(\mu\boldsymbol{f}),\boldsymbol{h}_h\big)_{\boldsymbol{L}^2(D)} + \omega\big((\boldsymbol{I}-\boldsymbol{\Pi}_{h}^{\mathrm{c}})(\epsilon\boldsymbol{g}),\boldsymbol{e}_h\big)_{\boldsymbol{L}^2(D)}. \tag{3.20}$$

One readily verifies that the pair $(\boldsymbol{H}_h, \boldsymbol{E}_h) \in L_h^{\rm c}$ solves (3.20) if and only if $(\boldsymbol{H}_h, \boldsymbol{E}_h)$ sits in $\boldsymbol{X}^{\rm c}_{\mu,h0} \times \boldsymbol{X}^{\rm c}_{\epsilon,h}$ and solves (3.18).

4. Preliminary results. In this section, we establish two preliminary results: two discrete Poincaré–Steklov inequalities in the involution-preserving subspaces defined by (3.16), and a consistency bound on the discrete curl operators.

4.1. Quasi-interpolation operators. In what follows, we invoke two types of quasi-interpolation operators. The first type, considered in the proofs of Lemma 4.1 and Lemma 5.11, are the averaging operators $\mathcal{I}_{h0}^{\rm c,av} : \boldsymbol{P}_k^{\rm b}(\mathcal{T}_h) \to \boldsymbol{P}_{k0}^{\rm c}(\mathcal{T}_h)$ and $\mathcal{I}_h^{\rm c,av} : \boldsymbol{P}_k^{\rm b}(\mathcal{T}_h) \to \boldsymbol{P}_k^{\rm c}(\mathcal{T}_h)$ analyzed in [28, section 6]. The key approximation property of these operators is that, for all $\boldsymbol{v}_h \in \boldsymbol{P}_k^{\rm b}(\mathcal{T}_h)$,

$$\|\boldsymbol{v}_h - \mathcal{I}_{h0}^{\rm c,av}(\boldsymbol{v}_h)\|_{\boldsymbol{L}^2(D)} \lesssim h^{\frac{1}{2}} |\boldsymbol{v}_h|_{\rm J}^{\rm c}, \qquad \|\boldsymbol{v}_h - \mathcal{I}_{h}^{\rm c,av}(\boldsymbol{v}_h)\|_{\boldsymbol{L}^2(D)} \lesssim h^{\frac{1}{2}} |\boldsymbol{v}_h|_{\rm J}^{\rm c,\circ}. \tag{4.1}$$

The second type, invoked in the proofs of Lemma 4.1, Lemma 5.1, and Lemma 5.11, are the commuting approximation operators $\mathcal{J}_{h0}^{\rm c} : \boldsymbol{L}^2(D) \to \boldsymbol{P}_{k0}^{\rm c}(\mathcal{T}_h)$ $\mathcal{J}_{h}^{\rm c} : \boldsymbol{L}^2(D) \to \boldsymbol{P}_{k}^{\rm c}(\mathcal{T}_h)$ devised in [30, section 23.3] following the seminal ideas in [3], [19], [20], [41]. We also invoke the operators $\mathcal{J}_{h0}^{\rm d} : \boldsymbol{L}^2(D) \to \boldsymbol{P}_{k0}^{\rm d}(\mathcal{T}_h)$ and $\mathcal{J}_{h}^{\rm d} : \boldsymbol{L}^2(D) \to \boldsymbol{P}_{k}^{\rm d}(\mathcal{T}_h)$, which commute with $\mathcal{J}_{h0}^{\rm c}$ and $\mathcal{J}_{h}^{\rm c}$, where $\boldsymbol{P}_{k0}^{\rm d}(\mathcal{T}_h)$ and $\boldsymbol{P}_{k}^{\rm d}(\mathcal{T}_h)$ are the Raviart–Thomas spaces of degree k associated with the mesh $\mathcal{T}_h$ with and without boundary conditions, respectively. The properties of these operators invoked in the paper are the following: (i) they are projections; (ii) they map curl-free fields to curl-free fields; and (iii) they are stable in $\boldsymbol{L}^2$ and enjoy optimal approximation properties in the $\boldsymbol{L}^2$-norm for fields in $\boldsymbol{H}^r(D)$, $r \in (0,1]$. As these operators are not locally defined, their approximation properties involve the global mesh-size. This is not an issue here since we have assumed the meshes to be quasi-uniform. We notice that an interesting extension of our work is to lift this restriction by considering the operators devised in Chaumont-Frelet and Vohralík [18].

4.2. Discrete Poincaré–Steklov inequalities. We prove in this section discrete versions of the Poincaré–Steklov (in)equalities stated in Corollary 2.3. Recall the discrete involution-preserving subspaces $\boldsymbol{X}^{\rm c}_{\mu,h0}$ and $\boldsymbol{X}^{\rm c}_{\epsilon,h}$ defined in (3.16).

LEMMA 4.1 (discrete Poincaré–Steklov inequalities). *The following bounds hold:*

$$\|\boldsymbol{h}_h\|_{\boldsymbol{L}^2(D)} \lesssim \ell_D \|\nabla_0 \times \boldsymbol{h}_h\|_{(\boldsymbol{X}^{\rm c})'} + h^{\frac{1}{2}} |\boldsymbol{h}_h|_{\rm J}^{\rm c} \qquad \forall \boldsymbol{h}_h \in \boldsymbol{X}^{\rm c}_{\mu,h0}, \tag{4.2a}$$

$$\|\boldsymbol{e}_h\|_{\boldsymbol{L}^2(D)} \lesssim \ell_D \|\nabla \times \boldsymbol{e}_h\|_{(\boldsymbol{X}_0^{\rm c})'} + h^{\frac{1}{2}} |\boldsymbol{e}_h|_{\rm J}^{\rm c,\circ} \qquad \forall \boldsymbol{e}_h \in \boldsymbol{X}^{\rm c}_{\epsilon,h}. \tag{4.2b}$$

Proof. We only prove (4.2a) since the proof of (4.2b) is similar. We revisit the proof of Lemma 3.2 in [31] to account for the presence of discontinuous material properties. Let $\boldsymbol{h}_h \in \boldsymbol{X}^{\rm c}_{\mu,h0}$. We define

$$\boldsymbol{h}_h^{\rm c} := \mathcal{I}_{h0}^{\rm c,av}(\boldsymbol{h}_h), \qquad \boldsymbol{\xi} := \boldsymbol{h}_h^{\rm c} - \boldsymbol{\Pi}_0^{\rm c}(\boldsymbol{h}_h^{\rm c}).$$

As $\boldsymbol{\xi} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}$, the weak Poincaré–Steklov (in)equality (2.18b) gives

$$\|\boldsymbol{\xi}\|_{\boldsymbol{L}^2(D)} = \ell_D \|\nabla_0 \times \boldsymbol{\xi}\|_{(\boldsymbol{X}^{\rm c})'} = \ell_D \|\nabla_0 \times \boldsymbol{h}_h^{\rm c}\|_{(\boldsymbol{X}^{\rm c})'}. \tag{4.3}$$

Since $\boldsymbol{h}_h^{\rm c} \in \boldsymbol{P}_{k0}^{\rm c}(\mathcal{T}_h)$, we have $\mathcal{J}_{h0}^{\rm c}(\boldsymbol{h}_h^{\rm c}) = \boldsymbol{h}_h^{\rm c}$, whence

$$\boldsymbol{h}_h^{\rm c} - \mathcal{J}_{h0}^{\rm c}(\boldsymbol{\xi}) = \mathcal{J}_{h0}^{\rm c}(\boldsymbol{h}_h^{\rm c} - \boldsymbol{\xi}) = \mathcal{J}_{h0}^{\rm c}(\boldsymbol{\Pi}_0^{\rm c}(\boldsymbol{h}_h^{\rm c})).$$

The commuting property of $\mathcal{J}^{\mathrm{c}}_{h0}$ implies that

$$\nabla_0\times(\boldsymbol{h}^{\mathrm{c}}_h-\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\xi}))=\nabla_0\times(\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{h}^{\mathrm{c}}_h)))=\mathcal{J}^{\mathrm{d}}_{h0}(\nabla_0\times(\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{h}^{\mathrm{c}}_h)))=\mathcal{J}^{\mathrm{d}}_{h0}(\boldsymbol{0})=\boldsymbol{0}.$$

Hence, $\boldsymbol{h}^{\mathrm{c}}_h-\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\xi})\in\boldsymbol{P}^{\mathrm{c}}_{k0}(\mathcal{T}_h)\cap\boldsymbol{H}_0(\mathbf{curl}=\boldsymbol{0};D)=\boldsymbol{P}^{\mathrm{c}}_{k0}(\mathbf{curl}=\boldsymbol{0};\mathcal{T}_h)$. As $\boldsymbol{h}_h\in\boldsymbol{X}^{\mathrm{c}}_{\mu,h0}$, we have $\mu\boldsymbol{h}_h\in\boldsymbol{P}^{\mathrm{c}}_{k0}(\mathbf{curl}=\boldsymbol{0};\mathcal{T}_h)^{\perp}$, so that

$$\begin{aligned}\|\mu^{\frac12}\boldsymbol{h}_h\|^2_{\boldsymbol{L}^2(D)}&=(\mu\boldsymbol{h}_h,\boldsymbol{h}_h-\boldsymbol{h}^{\mathrm{c}}_h)_{\boldsymbol{L}^2(D)}+(\mu\boldsymbol{h}_h,\boldsymbol{h}^{\mathrm{c}}_h-\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\xi}))_{\boldsymbol{L}^2(D)}+(\mu\boldsymbol{h}_h,\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\xi}))_{\boldsymbol{L}^2(D)}\\&=(\mu\boldsymbol{h}_h,\boldsymbol{h}_h-\boldsymbol{h}^{\mathrm{c}}_h)_{\boldsymbol{L}^2(D)}+(\mu\boldsymbol{h}_h,\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\xi}))_{\boldsymbol{L}^2(D)}.\end{aligned}$$

Invoking the Cauchy–Schwarz inequality, the $\boldsymbol{L}^2$-stability of $\mathcal{J}^{\mathrm{c}}_{h0}$, and (4.3), we obtain

$$\begin{aligned}\mu_0^{-\frac12}\|\mu^{\frac12}\boldsymbol{h}_h\|_{\boldsymbol{L}^2(D)}&\lesssim\|\boldsymbol{h}_h-\boldsymbol{h}^{\mathrm{c}}_h\|_{\boldsymbol{L}^2(D)}+\|\boldsymbol{\xi}\|_{\boldsymbol{L}^2(D)}\\&=\|\boldsymbol{h}_h-\boldsymbol{h}^{\mathrm{c}}_h\|_{\boldsymbol{L}^2(D)}+\ell_D\|\nabla_0\times\boldsymbol{h}^{\mathrm{c}}_h\|_{(\boldsymbol{X}^{\mathrm{c}})'}.\end{aligned}$$

Using the triangle inequality, the inequality $\ell_D\|\nabla_0\times\boldsymbol{\phi}\|_{(\boldsymbol{X}^{\mathrm{c}})'}\le\|\boldsymbol{\phi}\|_{\boldsymbol{L}^2(D)}$ which holds for all $\boldsymbol{\phi}\in\boldsymbol{L}^2(D)$, and the approximation properties of $\mathcal{I}^{\mathrm{c,av}}_{h0}$, we infer that

$$\begin{aligned}\mu_0^{-\frac12}\|\mu^{\frac12}\boldsymbol{h}_h\|_{\boldsymbol{L}^2(D)}&\lesssim\|\boldsymbol{h}_h-\boldsymbol{h}^{\mathrm{c}}_h\|_{\boldsymbol{L}^2(D)}+\ell_D\|\nabla_0\times\boldsymbol{h}_h\|_{(\boldsymbol{X}^{\mathrm{c}})'}\\&\lesssim h^{\frac12}|\boldsymbol{h}_h|^{\mathrm{c}}_{\mathrm{J}}+\ell_D\|\nabla_0\times\boldsymbol{h}_h\|_{(\boldsymbol{X}^{\mathrm{c}})'}.\end{aligned}$$

This proves (4.2a). ∎

4.3. Consistency bound on discrete curl operators. Since $\boldsymbol{P}^{\mathrm{b}}_k(\mathcal{T}_h)$ is not a subset of $\boldsymbol{H}(\mathbf{curl};D)$, it is useful to define the spaces

$$\boldsymbol{V}^{H}_{\sharp}:=\boldsymbol{H}_0(\mathbf{curl};D)+\boldsymbol{P}^{\mathrm{b}}_k(\mathcal{T}_h),\qquad\boldsymbol{V}^{E}_{\sharp}:=\boldsymbol{H}(\mathbf{curl};D)+\boldsymbol{P}^{\mathrm{b}}_k(\mathcal{T}_h).\tag{4.4}$$

It is then convenient to extend the jump sesquilinear forms and the discrete curl operators defined above to the spaces $\boldsymbol{V}^{H}_{\sharp}$ and $\boldsymbol{V}^{E}_{\sharp}$. For this purpose, we notice that although the sums in (4.4) are not direct, every field $\boldsymbol{h}_h$ in $\boldsymbol{H}_0(\mathbf{curl};D)\cap\boldsymbol{P}^{\mathrm{b}}_k(\mathcal{T}_h)$ satisfies $[\![\boldsymbol{h}_h]\!]^{\mathrm{c}}_F=\boldsymbol{0}$ for all $F\in\mathcal{F}_h$, and $\boldsymbol{C}_{h0}(\boldsymbol{h}_h)=\nabla_0\times\boldsymbol{h}_h$ because $\nabla_0\times\big(\boldsymbol{H}_0(\mathbf{curl};D)\cap\boldsymbol{P}^{\mathrm{b}}_k(\mathcal{T}_h)\big)\subset\boldsymbol{P}^{\mathrm{b}}_k(\mathcal{T}_h)$. Similarly, every field $\boldsymbol{e}_h$ in $\boldsymbol{H}(\mathbf{curl};D)\cap\boldsymbol{P}^{\mathrm{b}}_k(\mathcal{T}_h)$ satisfies $[\![\boldsymbol{e}_h]\!]^{\mathrm{c}}_F=\boldsymbol{0}$ for all $F\in\mathcal{F}^{\circ}_h$, and $\boldsymbol{C}_h(\boldsymbol{e}_h)=\nabla\times\boldsymbol{e}_h$. It is therefore legitimate to set, for all $\boldsymbol{h}=\tilde{\boldsymbol{h}}+\boldsymbol{h}_h\in\boldsymbol{V}^{H}_{\sharp}$ with $\tilde{\boldsymbol{h}}\in\boldsymbol{H}_0(\mathbf{curl};D)$ and $\boldsymbol{h}_h\in\boldsymbol{P}^{\mathrm{b}}_k(\mathcal{T}_h)$, and for all $\boldsymbol{e}=\tilde{\boldsymbol{e}}+\boldsymbol{e}_h\in\boldsymbol{V}^{E}_{\sharp}$ with $\tilde{\boldsymbol{e}}\in\boldsymbol{H}(\mathbf{curl};D)$ and $\boldsymbol{e}_h\in\boldsymbol{P}^{\mathrm{b}}_k(\mathcal{T}_h)$,

$$\boldsymbol{C}_{h0}(\boldsymbol{h}):=\nabla_0\times\tilde{\boldsymbol{h}}+\boldsymbol{C}_{h0}(\boldsymbol{h}_h),\qquad[\![\boldsymbol{h}]\!]^{\mathrm{c}}_F:=[\![\boldsymbol{h}_h]\!]^{\mathrm{c}}_F\qquad\forall F\in\mathcal{F}_h,\tag{4.5a}$$

$$\boldsymbol{C}_{h}(\boldsymbol{e}):=\nabla\times\tilde{\boldsymbol{e}}+\boldsymbol{C}_{h}(\boldsymbol{e}_h),\qquad[\![\boldsymbol{e}]\!]^{\mathrm{c}}_F:=[\![\boldsymbol{e}_h]\!]^{\mathrm{c}}_F\qquad\forall F\in\mathcal{F}^{\circ}_h.\tag{4.5b}$$

For simplicity, we use the same symbols for the extended operators; in particular, we now have $\boldsymbol{C}_{h0}:\boldsymbol{V}^{H}_{\sharp}\to\boldsymbol{L}^2(D)$ and $\boldsymbol{C}_h:\boldsymbol{V}^{E}_{\sharp}\to\boldsymbol{L}^2(D)$. The following commutators are useful for estimating the defects introduced by the above extensions:

$$\delta_h(\boldsymbol{h}_h,\boldsymbol{e}):=(\boldsymbol{h}_h,\nabla\times\boldsymbol{e})_{\boldsymbol{L}^2(D)}-(\boldsymbol{C}_{h0}(\boldsymbol{h}_h),\boldsymbol{e})_{\boldsymbol{L}^2(D)},\tag{4.6a}$$

$$\delta^{\circ}_h(\boldsymbol{e}_h,\boldsymbol{h}):=(\boldsymbol{e}_h,\nabla_0\times\boldsymbol{h})_{\boldsymbol{L}^2(D)}-(\boldsymbol{C}_{h}(\boldsymbol{e}_h),\boldsymbol{h})_{\boldsymbol{L}^2(D)}\tag{4.6b}$$

for all $\boldsymbol{h}_h,\boldsymbol{e}_h\in\boldsymbol{P}^{\mathrm{b}}_k(\mathcal{T}_h)$, all $\boldsymbol{e}\in\boldsymbol{H}(\mathbf{curl};D)$, and all $\boldsymbol{h}\in\boldsymbol{H}_0(\mathbf{curl};D)$.

LEMMA 4.2 (Consistency bound on discrete curl operators). *The following inequalities hold for all* $(\boldsymbol{h}_h,\boldsymbol{e}_h)\in\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)\times\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$ *and all* $(\boldsymbol{\eta},\boldsymbol{\varepsilon})\in\boldsymbol{X}_0^{\mathrm{c}}\times\boldsymbol{X}^{\mathrm{c}}$*:*

$$|\delta_h(\boldsymbol{h}_h,\boldsymbol{\varepsilon})|\lesssim(h/\ell_D)^{s'-\frac12}\ell_D^{\frac12}|\boldsymbol{h}_h|_{\mathrm{J}}^{\mathrm{c}}\|\nabla\times\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2(D)},\tag{4.7a}$$

$$|\delta_h^{\circ}(\boldsymbol{e}_h,\boldsymbol{\eta})|\lesssim(h/\ell_D)^{s'-\frac12}\ell_D^{\frac12}|\boldsymbol{e}_h|_{\mathrm{J}}^{\mathrm{c},\circ}\|\nabla_0\times\boldsymbol{\eta}\|_{\boldsymbol{L}^2(D)},\tag{4.7b}$$

where $s'\in(\frac12,1]$ *is the regularity pickup introduced in* (2.19).

Proof. We only prove (4.7a) since the proof of the other estimate is similar. Let $\boldsymbol{h}_h\in\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$ and $\boldsymbol{\varepsilon}\in\boldsymbol{X}^{\mathrm{c}}$. Using the definition of the commutator (4.6a) and integrating by parts, we obtain

$$\begin{aligned}\delta_h(\boldsymbol{h}_h,\boldsymbol{\varepsilon})&=(\boldsymbol{h}_h,\nabla\times\boldsymbol{\varepsilon})_{\boldsymbol{L}^2(D)}-(\boldsymbol{C}_{h0}(\boldsymbol{h}_h),\boldsymbol{\varepsilon})_{\boldsymbol{L}^2(D)}\\&=(\nabla_h\times\boldsymbol{h}_h,\boldsymbol{\varepsilon})_{\boldsymbol{L}^2(D)}+\sum_{F\in\mathcal{F}_h}([\![\boldsymbol{h}_h]\!]_F^{\mathrm{c}},\{\!\{\boldsymbol{\varepsilon}\}\!\}_F^{\mathrm{g}})_{\boldsymbol{L}^2(F)}-(\boldsymbol{C}_{h0}(\boldsymbol{h}_h),\boldsymbol{\varepsilon})_{\boldsymbol{L}^2(D)}.\end{aligned}$$

Notice that the summation over the mesh faces is meaningful since the regularity estimate (2.19) implies that $\boldsymbol{\varepsilon}\in\boldsymbol{H}^{s'}(D)$ with $s'>\frac12$. Moreover, we have $(\boldsymbol{C}_{h0}(\boldsymbol{h}_h),\boldsymbol{\varepsilon})_{\boldsymbol{L}^2(D)}=(\boldsymbol{C}_{h0}(\boldsymbol{h}_h),\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{\varepsilon}))_{\boldsymbol{L}^2(D)}$. Thus, using the definition (3.6a) of the operator $\boldsymbol{C}_{h0}$, we infer that

$$\begin{aligned}\delta_h(\boldsymbol{h}_h,\boldsymbol{\varepsilon})&=(\nabla_h\times\boldsymbol{h}_h,\boldsymbol{\varepsilon})_{\boldsymbol{L}^2(D)}+\sum_{F\in\mathcal{F}_h}([\![\boldsymbol{h}_h]\!]_F^{\mathrm{c}},\{\!\{\boldsymbol{\varepsilon}\}\!\}_F^{\mathrm{g}})_{\boldsymbol{L}^2(F)}-(\boldsymbol{C}_{h0}(\boldsymbol{h}_h),\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{\varepsilon}))_{\boldsymbol{L}^2(D)}\\&=(\nabla_h\times\boldsymbol{h}_h,\boldsymbol{\varepsilon}-\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{\varepsilon}))_{\boldsymbol{L}^2(D)}+\sum_{F\in\mathcal{F}_h}([\![\boldsymbol{h}_h]\!]_F^{\mathrm{c}},\{\!\{\boldsymbol{\varepsilon}-\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{\varepsilon})\}\!\}_F^{\mathrm{g}})_{\boldsymbol{L}^2(F)}\\&=\sum_{F\in\mathcal{F}_h}([\![\boldsymbol{h}_h]\!]_F^{\mathrm{c}},\{\!\{\boldsymbol{\varepsilon}-\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{\varepsilon})\}\!\}_F^{\mathrm{g}})_{\boldsymbol{L}^2(F)}.\end{aligned}$$

Using the Cauchy–Schwarz inequality, together with the boundedness of the trace operator $\boldsymbol{H}^{s'}(\widehat{K})\ni\widehat{\boldsymbol{v}}\mapsto\widehat{\boldsymbol{v}}|_{\widehat{F}}\in\boldsymbol{L}^2(\widehat{F})$ from the reference element $\widehat{K}$ to any reference face $\widehat{F}$ of $\widehat{K}$ (recall that $s'>\frac12$), and the shape-regularity of the mesh sequence, we infer that $|([\![\boldsymbol{h}_h]\!]_F^{\mathrm{c}},\{\!\{\boldsymbol{\varepsilon}-\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{\varepsilon})\}\!\}_F^{\mathrm{g}})_{\boldsymbol{L}^2(F)}|\lesssim\|[\![\boldsymbol{h}_h]\!]_F^{\mathrm{c}}\|_{\boldsymbol{L}^2(F)}h_F^{-\frac12}(\|\boldsymbol{\varepsilon}-\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{\varepsilon})\|_{\boldsymbol{L}^2(K_l\cup K_r)}+h_K^{s'}|\boldsymbol{\varepsilon}-\boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{\varepsilon})|_{\boldsymbol{H}^{s'}(K_l\cup K_r)})$. Hence, $|\delta_h(\boldsymbol{h}_h,\boldsymbol{\varepsilon})|\lesssim h^{s'-\frac12}|\boldsymbol{h}_h|_{\mathrm{J}}^{\mathrm{c}}|\boldsymbol{\varepsilon}|_{\boldsymbol{H}^{s'}(D)}$. The assertion follows by invoking the regularity estimate (2.19). ∎

5. Strong convergence.

The goal of this section is to establish our main result, namely that $\lim_{h\to0}\|T-T_h\|_{\mathcal{L}(L^{\mathrm{c}};L^{\mathrm{c}})}=0$. The proof consists of two main steps. First, we establish a deflated inf-sup condition restricted to the involution-preserving subspaces $\boldsymbol{X}_{\mu,h0}^{\mathrm{c}}\times\boldsymbol{X}_{\epsilon,h}^{\mathrm{c}}$. Then we prove convergence by using a duality argument à la Aubin–Nitsche. The essential part of the argument is that the smoothness index of the dual solution is always larger than $\frac12$ even if the coefficients are heterogeneous. Using the duality argument seems to be necessary, as relying on the deflated inf-sup condition is not enough to prove convergence when the smoothness index of the solution is less than $\frac12$.

5.1. Deflated inf-sup condition.

We equip the space L_h^{c} with the mesh-dependent norm

$$\begin{aligned}\|(\boldsymbol{h}_h,\boldsymbol{e}_h)\|_{\flat,h}&:=\omega^{\frac12}\|(\boldsymbol{h}_h,\boldsymbol{e}_h)\|_{L^{\mathrm{c}}}\\&\quad+\kappa_H^{\frac12}\{\|\tilde{h}^{\frac12}\boldsymbol{C}_{h0}(\boldsymbol{h}_h)\|_{\boldsymbol{L}^2(D)}+|\boldsymbol{h}_h|_{\mathrm{J}}^{\mathrm{c}}\}+\kappa_E^{\frac12}\{\|\tilde{h}^{\frac12}\boldsymbol{C}_h(\boldsymbol{e}_h)\|_{\boldsymbol{L}^2(D)}+|\boldsymbol{e}_h|_{\mathrm{J}}^{\mathrm{c},\circ}\}.\end{aligned}\tag{5.1}$$

Recall also the norm defined in (2.3), i.e., $\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\mathrm{c}}}^2:=\|\mu^{\frac12}\boldsymbol{f}\|_{\boldsymbol{L}^2(D)}^2+\|\epsilon^{\frac12}\boldsymbol{g}\|_{\boldsymbol{L}^2(D)}^2$.

LEMMA 5.1 (deflated inf-sup condition).
(i) *The following holds for every discrete pair* $(\boldsymbol{H}_h,\boldsymbol{E}_h)$ *in* $\boldsymbol{X}^{\mathrm{c}}_{\mu,h0}\times\boldsymbol{X}^{\mathrm{c}}_{\epsilon,h}$*:*

$$\omega^{\frac12}\|(\boldsymbol{H}_h,\boldsymbol{E}_h)\|_{\flat,h}\lesssim \sup_{(\boldsymbol{h}_h,\boldsymbol{e}_h)\in L^{\mathrm{c}}_h}\frac{|a_h\big((\boldsymbol{H}_h,\boldsymbol{E}_h),(\boldsymbol{h}_h,\boldsymbol{e}_h)\big)|}{\|(\boldsymbol{h}_h,\boldsymbol{e}_h)\|_{L^{\mathrm{c}}}}. \tag{5.2}$$

(ii) *The discrete boundary-value problem* (3.18) *is well-posed, and its solution satisfies the a priori estimate* $\|(\boldsymbol{H}_h,\boldsymbol{E}_h)\|_{\flat,h}\lesssim\omega^{\frac12}\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\mathrm{c}}}$.

Proof. Let $\mathbb{S}$ denote the supremum on the right-hand side of (5.2).
(1) Taking $(\boldsymbol{h}_h,\boldsymbol{e}_h):=(\boldsymbol{H}_h,\boldsymbol{E}_h)$ and using integration by parts (see (3.7)) gives

$$\kappa_H(|\boldsymbol{H}_h|^{\mathrm{c}}_{\mathrm{J}})^2+\kappa_E(|\boldsymbol{E}_h|^{\mathrm{c},\circ}_{\mathrm{J}})^2=a_h\big((\boldsymbol{H}_h,\boldsymbol{E}_h),(\boldsymbol{H}_h,\boldsymbol{E}_h)\big)\le\mathbb{S}\|(\boldsymbol{H}_h,\boldsymbol{E}_h)\|_{L^{\mathrm{c}}}. \tag{5.3}$$

(2) Taking $(\boldsymbol{h}_h,\boldsymbol{e}_h):=(-\kappa_E\tilde{h}\boldsymbol{C}_h(\boldsymbol{E}_h),\boldsymbol{0})$ and using that $\kappa_H\kappa_E=1$ gives

$$\begin{aligned}\kappa_E\|\tilde{h}^{\frac12}\boldsymbol{C}_h(\boldsymbol{E}_h)\|^2_{\boldsymbol{L}^2(D)}&=a_h((\boldsymbol{H}_h,\boldsymbol{E}_h),(\boldsymbol{h}_h,\boldsymbol{e}_h))-\kappa_H s^{\mathrm{c}}_h(\boldsymbol{H}_h,\boldsymbol{h}_h)\\&\le\mathbb{S}\|\mu^{\frac12}\boldsymbol{h}_h\|_{\boldsymbol{L}^2(D)}+s^{\mathrm{c}}_h(\boldsymbol{H}_h,\tilde{h}\boldsymbol{C}_h(\boldsymbol{E}_h)).\end{aligned}$$

We bound the first term on the right-hand side as follows:

$$\|\mu^{\frac12}\boldsymbol{h}_h\|_{\boldsymbol{L}^2(D)}\lesssim(\mu_0\ell_D\kappa_E)^{\frac12}\kappa_E^{\frac12}\|\tilde{h}^{\frac12}\boldsymbol{C}_h(\boldsymbol{E}_h)\|_{\boldsymbol{L}^2(D)}=\omega^{-\frac12}\kappa_E^{\frac12}\|\tilde{h}^{\frac12}\boldsymbol{C}_h(\boldsymbol{E}_h)\|_{\boldsymbol{L}^2(D)},$$

where we used that $h\le\ell_D$ and $\ell_D\mu_0\kappa_E=\omega^{-1}$. Invoking the shape-regularity of the mesh sequence and an inverse trace inequality gives $\|[\![\tilde{h}\boldsymbol{C}_h(\boldsymbol{E}_h)]\!]^{\mathrm{c}}_F\|_{\boldsymbol{L}^2(F)}\lesssim h_F^{\frac12}\|[\![\tilde{h}^{\frac12}\boldsymbol{C}_h(\boldsymbol{E}_h)]\!]^{\mathrm{c}}_F\|_{\boldsymbol{L}^2(F)}\lesssim\|\tilde{h}^{\frac12}\boldsymbol{C}_h(\boldsymbol{E}_h)\|_{\boldsymbol{L}^2(K_l\cup K_r)}$. Hence, we can estimate the second term as follows:

$$\big|s^{\mathrm{c}}_h(\boldsymbol{H}_h,\tilde{h}\boldsymbol{C}_h(\boldsymbol{E}_h))\big|\le|\boldsymbol{H}_h|^{\mathrm{c}}_{\mathrm{J}}|\tilde{h}\boldsymbol{C}_h(\boldsymbol{E}_h)|^{\mathrm{c}}_{\mathrm{J}}\lesssim|\boldsymbol{H}_h|^{\mathrm{c}}_{\mathrm{J}}\|\tilde{h}^{\frac12}\boldsymbol{C}_h(\boldsymbol{E}_h)\|_{\boldsymbol{L}^2(D)}.$$

Using again that $\kappa_H\kappa_E=1$, this gives

$$\kappa_E\|\tilde{h}^{\frac12}\boldsymbol{C}_h(\boldsymbol{E}_h)\|^2_{\boldsymbol{L}^2(D)}\lesssim\big(\omega^{-\frac12}\mathbb{S}+\kappa_H^{\frac12}|\boldsymbol{H}_h|^{\mathrm{c}}_{\mathrm{J}}\big)\kappa_E^{\frac12}\|\tilde{h}^{\frac12}\boldsymbol{C}_h(\boldsymbol{E}_h)\|_{\boldsymbol{L}^2(D)}.$$

Therefore, we conclude that

$$\kappa_E^{\frac12}\|\tilde{h}^{\frac12}\boldsymbol{C}_h(\boldsymbol{E}_h)\|_{\boldsymbol{L}^2(D)}\lesssim\omega^{-\frac12}\mathbb{S}+\kappa_H^{\frac12}|\boldsymbol{H}_h|^{\mathrm{c}}_{\mathrm{J}}.$$

Similarly, testing (3.18) with $(\boldsymbol{h}_h,\boldsymbol{e}_h):=(\boldsymbol{0},\kappa_H\tilde{h}\boldsymbol{C}_{h0}(\boldsymbol{H}_h))$ gives

$$\kappa_H^{\frac12}\|\tilde{h}^{\frac12}\boldsymbol{C}_{h0}(\boldsymbol{H}_h)\|_{\boldsymbol{L}^2(D)}\lesssim\omega^{-\frac12}\mathbb{S}+\kappa_E^{\frac12}|\boldsymbol{E}_h|^{\mathrm{c},\circ}_{\mathrm{J}}.$$

Gathering the above two bounds and invoking (5.3) gives

$$\kappa_H^{\frac12}\|\tilde{h}^{\frac12}\boldsymbol{C}_{h0}(\boldsymbol{H}_h)\|_{\boldsymbol{L}^2(D)}+\kappa_E^{\frac12}\|\tilde{h}^{\frac12}\boldsymbol{C}_h(\boldsymbol{E}_h)\|_{\boldsymbol{L}^2(D)}\lesssim\omega^{-\frac12}\mathbb{S}+\mathbb{S}^{\frac12}\|(\boldsymbol{H}_h,\boldsymbol{E}_h)\|^{\frac12}_{L^{\mathrm{c}}}. \tag{5.4}$$

(3) We now estimate $\|\nabla_0\times\boldsymbol{H}_h\|_{(\boldsymbol{X}^{\mathrm{c}})'}$ and $\|\nabla\times\boldsymbol{E}_h\|_{(\boldsymbol{X}^{\mathrm{c}}_0)'}$. Let $\boldsymbol{\varepsilon}\in\boldsymbol{X}^{\mathrm{c}}$. Recalling the definition of the commutator δ_h in (4.6a) and setting $\boldsymbol{\varepsilon}_h:=\mathcal{J}^{\mathrm{c}}_h(\boldsymbol{\varepsilon})$, we have

$$\begin{aligned}(\boldsymbol{H}_h,\nabla\times\boldsymbol{\varepsilon})_{\boldsymbol{L}^2(D)}&=(\boldsymbol{C}_{h0}(\boldsymbol{H}_h),\boldsymbol{\varepsilon})_{\boldsymbol{L}^2(D)}+\delta_h(\boldsymbol{H}_h,\boldsymbol{\varepsilon})\\&=(\boldsymbol{C}_{h0}(\boldsymbol{H}_h),\boldsymbol{\varepsilon}_h)_{\boldsymbol{L}^2(D)}+(\boldsymbol{C}_{h0}(\boldsymbol{H}_h),\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_h)_{\boldsymbol{L}^2(D)}+\delta_h(\boldsymbol{H}_h,\boldsymbol{\varepsilon})\\&=a_h((\boldsymbol{H}_h,\boldsymbol{E}_h),(\boldsymbol{0},\boldsymbol{\varepsilon}_h))+(\boldsymbol{C}_{h0}(\boldsymbol{H}_h),\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_h)_{\boldsymbol{L}^2(D)}+\delta_h(\boldsymbol{H}_h,\boldsymbol{\varepsilon}),\end{aligned}$$

where the last equality follows from the fact that $\boldsymbol{\varepsilon}_h$ has zero tangential jumps. Using the approximation properties of $\mathcal{J}_h^{\mathrm{c}}$, the commutator estimate (4.7a), and the estimate $\|\boldsymbol{\varepsilon}_h\|_{\boldsymbol{L}^2(D)} \lesssim \|\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2(D)} + h^{s'}|\boldsymbol{\varepsilon}|_{\boldsymbol{H}^{s'}(D)} \lesssim \ell_D\|\nabla\times\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2(D)}$ with $s' \in (\frac{1}{2}, 1]$ (which follows from the regularity property (2.19) since $\boldsymbol{\varepsilon} \in \boldsymbol{X}^{\mathrm{c}}$, and the Poincaré–Steklov inequality $\|\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2(D)} \lesssim \ell_D\|\nabla\times\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2(D)}$ stated in (2.21)), we obtain

$$
\begin{aligned}
|(\boldsymbol{H}_h, \nabla\times\boldsymbol{\varepsilon})_{\boldsymbol{L}^2(D)}| &\lesssim \mathbb{S}\,\epsilon_0^{\frac{1}{2}}\|\boldsymbol{\varepsilon}_h\|_{\boldsymbol{L}^2(D)} + h^{s'-\frac{1}{2}}\|\tilde{h}^{\frac{1}{2}}\boldsymbol{C}_{h0}(\boldsymbol{H}_h)\|_{\boldsymbol{L}^2(D)}|\boldsymbol{\varepsilon}|_{\boldsymbol{H}^{s'}(D)} \\
&\quad + (h/\ell_D)^{s'-\frac{1}{2}}\ell_D^{\frac{1}{2}}|\boldsymbol{H}_h|_{\mathrm{J}}^{\mathrm{c}}\|\nabla\times\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2(D)} \\
&\lesssim \big\{\epsilon_0^{\frac{1}{2}}\mathbb{S} + (h/\ell_D)^{s'-\frac{1}{2}}\ell_D^{-\frac{1}{2}}(\|\tilde{h}^{\frac{1}{2}}\boldsymbol{C}_{h0}(\boldsymbol{H}_h)\|_{\boldsymbol{L}^2(D)} + |\boldsymbol{H}_h|_{\mathrm{J}}^{\mathrm{c}})\big\} \\
&\quad \times \ell_D\|\nabla\times\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2(D)}.
\end{aligned}
$$

Hence, recalling the definition of $\|\nabla_0\times\boldsymbol{H}_h\|_{(\boldsymbol{X}^{\mathrm{c}})'}$ introduced in Corollary 2.3, we have

$$
\|\nabla_0\times\boldsymbol{H}_h\|_{(\boldsymbol{X}^{\mathrm{c}})'} \lesssim \epsilon_0^{\frac{1}{2}}\mathbb{S} + (h/\ell_D)^{s'-\frac{1}{2}}\ell_D^{-\frac{1}{2}}(\|\tilde{h}^{\frac{1}{2}}\boldsymbol{C}_{h0}(\boldsymbol{H}_h)\|_{\boldsymbol{L}^2(D)} + |\boldsymbol{H}_h|_{\mathrm{J}}^{\mathrm{c}}).
$$

We proceed similarly to establish the following estimate where $s' \in (\frac{1}{2}, 1]$:

$$
\|\nabla\times\boldsymbol{E}_h\|_{(\boldsymbol{X}_0^{\mathrm{c}})'} \lesssim \mu_0^{\frac{1}{2}}\mathbb{S} + (h/\ell_D)^{s'-\frac{1}{2}}\ell_D^{-\frac{1}{2}}(\|\tilde{h}^{\frac{1}{2}}\boldsymbol{C}_h(\boldsymbol{E}_h)\|_{\boldsymbol{L}^2(D)} + |\boldsymbol{E}_h|_{\mathrm{J}}^{\mathrm{c},\circ}).
$$

(4) Since $\boldsymbol{H}_h \in \boldsymbol{X}_{\mu,h0}^{\mathrm{c}}$, we use the discrete Poincaré–Steklov inequality (4.2a) in Lemma 4.1 and the above estimate on $\|\nabla_0\times\boldsymbol{H}_h\|_{(\boldsymbol{X}^{\mathrm{c}})'}$ to infer that

$$
\begin{aligned}
\|\mu^{\frac{1}{2}}\boldsymbol{H}_h\|_{\boldsymbol{L}^2(D)} &\lesssim \mu_0^{\frac{1}{2}}\|\boldsymbol{H}_h\|_{\boldsymbol{L}^2(D)} \\
&\lesssim \mu_0^{\frac{1}{2}}(\ell_D\|\nabla_0\times\boldsymbol{H}_h\|_{(\boldsymbol{X}^{\mathrm{c}})'} + h^{\frac{1}{2}}|\boldsymbol{H}_h|_{\mathrm{J}}^{\mathrm{c}}) \\
&\le \mu_0^{\frac{1}{2}}(\ell_D\|\nabla_0\times\boldsymbol{H}_h\|_{(\boldsymbol{X}^{\mathrm{c}})'} + (h/\ell_D)^{s'-\frac{1}{2}}\ell_D^{\frac{1}{2}}|\boldsymbol{H}_h|_{\mathrm{J}}^{\mathrm{c}}) \\
&\lesssim \omega^{-1}\big\{\mathbb{S} + (h/\ell_D)^{s'-\frac{1}{2}}\omega^{\frac{1}{2}}\kappa_H^{\frac{1}{2}}(\|\tilde{h}^{\frac{1}{2}}\boldsymbol{C}_{h0}(\boldsymbol{H}_h)\|_{\boldsymbol{L}^2(D)} + |\boldsymbol{H}_h|_{\mathrm{J}}^{\mathrm{c}})\big\},
\end{aligned}
$$

where we used $s' \in (\frac{1}{2}, 1]$ and $h^{\frac{1}{2}} \le (h/\ell_D)^{s'-\frac{1}{2}}\ell_D^{\frac{1}{2}}$ on the third line, and $\ell_D(\mu_0\epsilon_0)^{\frac{1}{2}} = \omega^{-1}$ and $\mu_0\ell_D\kappa_H^{-1} = \omega^{-1}$ on the fourth line. We proceed similarly to establish that

$$
\|\epsilon^{\frac{1}{2}}\boldsymbol{E}_h\|_{\boldsymbol{L}^2(D)} \lesssim \omega^{-1}\big\{\mathbb{S} + (h/\ell_D)^{s'-\frac{1}{2}}\omega^{\frac{1}{2}}\kappa_E^{\frac{1}{2}}(\|\tilde{h}^{\frac{1}{2}}\boldsymbol{C}_h(\boldsymbol{E}_h)\|_{\boldsymbol{L}^2(D)} + |\boldsymbol{E}_h|_{\mathrm{J}}^{\mathrm{c},\circ})\big\}.
$$

Gathering the above two bounds, invoking (5.3) and (5.4), and using $h \le \ell_D$ gives

$$
\|(\boldsymbol{H}_h, \boldsymbol{E}_h)\|_{L^{\mathrm{c}}} \lesssim \omega^{-1}\mathbb{S} + \omega^{-\frac{1}{2}}\mathbb{S}^{\frac{1}{2}}\|(\boldsymbol{H}_h, \boldsymbol{E}_h)\|_{L^{\mathrm{c}}}^{\frac{1}{2}}.
$$

Invoking Young's inequality, we conclude that $\|(\boldsymbol{H}_h, \boldsymbol{E}_h)\|_{L^{\mathrm{c}}} \lesssim \omega^{-1}\mathbb{S}$. Substituting this bound into (5.3) and (5.4) and putting everything together proves (5.2).

(5) The deflated inf-sup condition (5.2) readily implies that the boundary-value problem (3.18) admits a unique solution (recall that (3.18) actually holds for all $(\boldsymbol{h}_h, \boldsymbol{e}_h) \in L_h^{\mathrm{c}}$). Moreover, the a priori estimate on the solution follows from the Cauchy–Schwarz inequality and (5.2). ∎

Remark 5.2 (standard inf-sup condition). Using Lemma 3.1 and the deflated inf-sup condition (5.2), one readily shows that, for all $(\boldsymbol{H}_h, \boldsymbol{E}_h) \in \boldsymbol{X}_{\mu,h0}^{\mathrm{c}}\times\boldsymbol{X}_{\epsilon,h}^{\mathrm{c}}$,

$$
\omega^{\frac{1}{2}}\|(\boldsymbol{H}_h, \boldsymbol{E}_h)\|_{\flat,h} \lesssim \sup_{(\boldsymbol{h}_h,\boldsymbol{e}_h)\in\boldsymbol{X}_{\mu,h0}^{\mathrm{c}}\times\boldsymbol{X}_{\epsilon,h}^{\mathrm{c}}} \frac{|a_h\big((\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)\big)|}{\|(\boldsymbol{h}_h, \boldsymbol{e}_h)\|_{L^{\mathrm{c}}}},
$$

i.e., the supremum can be restricted to $\boldsymbol{X}_{\mu,h0}^{\mathrm{c}}\times\boldsymbol{X}_{\epsilon,h}^{\mathrm{c}}$. Letting $J_h : \boldsymbol{X}_{\mu,h0}^{\mathrm{c}}\times\boldsymbol{X}_{\epsilon,h}^{\mathrm{c}} \to L_h^{\mathrm{c}}$ be the canonical injection, the above inf-sup condition means that the restricted operator $J_h^{\mathsf{T}}A_hJ_h$ is an isomorphism over $\boldsymbol{X}_{\mu,h0}^{\mathrm{c}}\times\boldsymbol{X}_{\epsilon,h}^{\mathrm{c}}$.

5.2. Duality argument. Let $(\boldsymbol{H},\boldsymbol{E})$ be the solution to the continuous problem (2.25), and let $(\boldsymbol{H}_h,\boldsymbol{E}_h)$ be the solution to the discrete problem (3.18). We define the errors

$$\boldsymbol{\delta h} := \boldsymbol{H} - \boldsymbol{H}_h, \qquad \boldsymbol{\delta e} := \boldsymbol{E} - \boldsymbol{E}_h. \tag{5.5}$$

We consider the following dual problem: Find $(\boldsymbol{\eta},\boldsymbol{\varepsilon})$ in $\boldsymbol{H}_0(\mathbf{curl};D)\times\boldsymbol{H}(\mathbf{curl};D)$ such that

$$\omega\mu\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta}) + \nabla\times\boldsymbol{\varepsilon} = \omega(\boldsymbol{I}-\boldsymbol{\Pi}_0^{\mathrm{c}})(\mu\boldsymbol{\delta h}), \tag{5.6a}$$
$$\omega\epsilon\boldsymbol{\Pi}^{\mathrm{c}}(\boldsymbol{\varepsilon}) - \nabla_0\times\boldsymbol{\eta} = \omega(\boldsymbol{I}-\boldsymbol{\Pi}^{\mathrm{c}})(\epsilon\boldsymbol{\delta e}). \tag{5.6b}$$

The following regularity estimate on the dual solution plays a key role in the analysis.

Lemma 5.3 (well-posedness and stability).

(i) *The dual problem* (5.6) *has a unique solution* $(\boldsymbol{\eta},\boldsymbol{\varepsilon})$, *and this solution sits in* $\boldsymbol{X}_0^{\mathrm{c}}\times\boldsymbol{X}^{\mathrm{c}}$.

(ii) *The solution* $(\boldsymbol{\eta},\boldsymbol{\varepsilon})$ *satisfies the following a priori estimates with* $s'\in(\frac12,1]$*:*

$$\ell_D\big(\mu_0^{\frac12}\|\nabla_0\times\boldsymbol{\eta}\|_{\boldsymbol{L}^2(D)} + \epsilon_0^{\frac12}\|\nabla\times\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2(D)}\big) \lesssim \|(\boldsymbol{\delta h},\boldsymbol{\delta e})\|_{L^{\mathrm{c}}}, \tag{5.7a}$$
$$|\boldsymbol{\eta}|_{\boldsymbol{H}^{s'}(D)} \lesssim \ell_D^{1-s'}\|\nabla_0\times\boldsymbol{\eta}\|_{\boldsymbol{L}^2(D)}, \qquad |\boldsymbol{\varepsilon}|_{\boldsymbol{H}^{s'}(D)} \lesssim \ell_D^{1-s'}\|\nabla\times\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2(D)}. \tag{5.7b}$$

Proof. (i) Testing (5.6a) with $\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta})$ gives

$$\omega(\mu\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta}),\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta}))_{\boldsymbol{L}^2(D)} + (\nabla\times\boldsymbol{\varepsilon},\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta}))_{\boldsymbol{L}^2(D)} = \omega((\boldsymbol{I}-\boldsymbol{\Pi}_0^{\mathrm{c}})(\mu\boldsymbol{\delta h}),\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta}))_{\boldsymbol{L}^2(D)} = 0,$$

where we used the $\boldsymbol{L}^2$-orthogonality of $\boldsymbol{\Pi}_0^{\mathrm{c}}$. Since $(\nabla\times\boldsymbol{\varepsilon},\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta}))_{\boldsymbol{L}^2(D)}=0$ because $\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta})\in\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)$, we conclude that $\|\mu^{\frac12}\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta})\|_{\boldsymbol{L}^2(D)}=0$. Similarly, we obtain $\|\epsilon^{\frac12}\boldsymbol{\Pi}^{\mathrm{c}}(\boldsymbol{\varepsilon})\|_{\boldsymbol{L}^2(D)}=0$ by testing (5.6b) with $\boldsymbol{\Pi}^{\mathrm{c}}(\boldsymbol{\varepsilon})$. Altogether, this proves that

$$\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta}) = \boldsymbol{\Pi}^{\mathrm{c}}(\boldsymbol{\varepsilon}) = \mathbf{0}. \tag{5.8}$$

The existence and uniqueness of a solution to (5.6) is then a direct consequence of (2.17a) in Lemma 2.2. Moreover, (5.8) means that $(\boldsymbol{\eta},\boldsymbol{\varepsilon})\in\boldsymbol{X}_0^{\mathrm{c}}\times\boldsymbol{X}^{\mathrm{c}}$.

(ii) The a priori estimate (5.7a) follows from (5.6), (5.8), and $\ell_D\omega(\mu_0\epsilon_0)^{\frac12}=1$. The a priori estimate (5.7b) is a consequence of the regularity estimate (2.19). ∎

Lemma 5.4 ($\boldsymbol{L}^2$-error representation). *The following holds:*

$$\omega\|(\boldsymbol{\delta h},\boldsymbol{\delta e})\|_{L^{\mathrm{c}}}^2 = \theta_{\mathrm{app}} + \theta_{\mathrm{gal}} + \theta_{\mathrm{crl}} + \theta_{\mathrm{div}}, \tag{5.9}$$

with the approximation error, the Galerkin orthogonality error, the curl commuting error, and the divergence conformity error defined as follows:

$$\theta_{\mathrm{app}} := a_h((\boldsymbol{\delta h},\boldsymbol{\delta e}),(\boldsymbol{\eta}-\boldsymbol{\eta}_h,\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_h)), \tag{5.10a}$$
$$\theta_{\mathrm{gal}} := a_h((\boldsymbol{\delta h},\boldsymbol{\delta e}),(\boldsymbol{\eta}_h,\boldsymbol{\varepsilon}_h)), \tag{5.10b}$$
$$\theta_{\mathrm{crl}} := \delta_h(\boldsymbol{\delta h},\boldsymbol{\varepsilon}) - \delta_h^{\circ}(\boldsymbol{\delta e},\boldsymbol{\eta}), \tag{5.10c}$$
$$\theta_{\mathrm{div}} := \omega\big\{(\boldsymbol{\delta h},\boldsymbol{\Pi}_0^{\mathrm{c}}(\mu\boldsymbol{\delta h}))_{\boldsymbol{L}^2(D)} + (\boldsymbol{\delta e},\boldsymbol{\Pi}^{\mathrm{c}}(\epsilon\boldsymbol{\delta e}))_{\boldsymbol{L}^2(D)}\big\}, \tag{5.10d}$$

with $\boldsymbol{\eta}_h := \boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{\eta})$, $\boldsymbol{\varepsilon}_h := \boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{\varepsilon})$, $\boldsymbol{\Pi}_h^{\mathrm{b}}$ *defined in* (3.2), *and the commutators* δ_h *and* δ_h° *defined in* (4.6).

Proof. We have $s_h^{\mathrm{c}}(\cdot,\boldsymbol{\eta}) = s_h^{\mathrm{c},\circ}(\cdot,\boldsymbol{\varepsilon}) = 0$. Recalling the definition of the sesquilinear form a_h (see (3.8)) and the definition of the commutators (see (4.6)) and using that $\nabla\times\boldsymbol{\varepsilon} = \omega(\boldsymbol{I}-\boldsymbol{\Pi}_0^{\mathrm{c}})(\mu\boldsymbol{\delta h})$ and $-\nabla_0\times\boldsymbol{\eta} = \omega(\boldsymbol{I}-\boldsymbol{\Pi}^{\mathrm{c}})(\epsilon\boldsymbol{\delta e})$, we infer that

$$\begin{aligned}
a_h((\boldsymbol{\delta h},\boldsymbol{\delta e}),(\boldsymbol{\eta},\boldsymbol{\varepsilon})) &= -(\boldsymbol{C}_h(\boldsymbol{\delta e}),\boldsymbol{\eta})_{\boldsymbol{L}^2(D)} + (\boldsymbol{C}_{h0}(\boldsymbol{\delta h}),\boldsymbol{\varepsilon})_{\boldsymbol{L}^2(D)} \\
&= -(\boldsymbol{\delta e},\nabla_0\times\boldsymbol{\eta})_{\boldsymbol{L}^2(D)} + \delta_h^{\circ}(\boldsymbol{\delta e},\boldsymbol{\eta}) + (\boldsymbol{\delta h},\nabla\times\boldsymbol{\varepsilon})_{\boldsymbol{L}^2(D)} - \delta_h(\boldsymbol{\delta h},\boldsymbol{\varepsilon}) \\
&= \omega\big\{(\boldsymbol{\delta h},(\boldsymbol{I}-\boldsymbol{\Pi}_0^{\mathrm{c}})(\mu\boldsymbol{\delta h}))_{\boldsymbol{L}^2(D)} + (\boldsymbol{\delta e},(\boldsymbol{I}-\boldsymbol{\Pi}^{\mathrm{c}})(\epsilon\boldsymbol{\delta e}))_{\boldsymbol{L}^2(D)}\big\} - \theta_{\mathrm{crl}} \\
&= \omega\big\{\|\mu^{\frac12}\boldsymbol{\delta h}\|^2_{\boldsymbol{L}^2(D)} + \|\epsilon^{\frac12}\boldsymbol{\delta e}\|^2_{\boldsymbol{L}^2(D)}\big\} - \theta_{\mathrm{div}} - \theta_{\mathrm{crl}} \\
&= \omega\|(\boldsymbol{\delta h},\boldsymbol{\delta e})\|^2_{L^{\mathrm{c}}} - \theta_{\mathrm{div}} - \theta_{\mathrm{crl}}.
\end{aligned}$$

Since $a_h((\boldsymbol{\delta h},\boldsymbol{\delta e}),(\boldsymbol{\eta},\boldsymbol{\varepsilon})) = \theta_{\mathrm{app}} + \theta_{\mathrm{gal}}$, reorganizing the terms proves (5.9). ∎

We shall use the following a priori estimates which follow from Lemma 2.6 and Lemma 5.1(ii), respectively:

$$\ell_D\big(\mu_0^{\frac12}\|\nabla_0\times\boldsymbol{H}\|_{\boldsymbol{L}^2(D)} + \epsilon_0^{\frac12}\|\nabla\times\boldsymbol{E}\|_{\boldsymbol{L}^2(D)}\big) \lesssim \|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\mathrm{c}}}, \tag{5.11a}$$

$$\|(\boldsymbol{H}_h,\boldsymbol{E}_h)\|_{\flat,h} \lesssim \omega^{\frac12}\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\mathrm{c}}}. \tag{5.11b}$$

To simplify the notation, we also set

$$c_{\mathrm{rot}}(\boldsymbol{\eta},\boldsymbol{\varepsilon}) := \ell_D\big(\mu_0^{\frac12}\|\nabla_0\times\boldsymbol{\eta}\|_{\boldsymbol{L}^2(D)} + \epsilon_0^{\frac12}\|\nabla\times\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2(D)}\big) \tag{5.12}$$

and observe that $c_{\mathrm{rot}}(\boldsymbol{\eta},\boldsymbol{\varepsilon}) \lesssim \|(\boldsymbol{\delta h},\boldsymbol{\delta e})\|_{L^{\mathrm{c}}}$ owing to (5.7a).

LEMMA 5.5 (bound on approximation error). *The following holds:*

$$|\theta_{\mathrm{app}}| \lesssim (h/\ell_D)^{s'-\frac12}\omega\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\mathrm{c}}}c_{\mathrm{rot}}(\boldsymbol{\eta},\boldsymbol{\varepsilon}). \tag{5.13}$$

Proof. We have

$$\begin{aligned}
\theta_{\mathrm{app}} =& -(\boldsymbol{C}_h(\boldsymbol{\delta e}),\boldsymbol{\eta}-\boldsymbol{\eta}_h)_{\boldsymbol{L}^2(D)} + (\boldsymbol{C}_{h0}(\boldsymbol{\delta h}),\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_h)_{\boldsymbol{L}^2(D)} \\
&+ \kappa_H s_h^{\mathrm{c}}(\boldsymbol{\delta h},\boldsymbol{\eta}-\boldsymbol{\eta}_h) + \kappa_E s_h^{\mathrm{c},\circ}(\boldsymbol{\delta e},\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_h) \\
=& -(\nabla\times\boldsymbol{E},\boldsymbol{\eta}-\boldsymbol{\eta}_h)_{\boldsymbol{L}^2(D)} + (\nabla_0\times\boldsymbol{H},\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_h)_{\boldsymbol{L}^2(D)} \\
&- \kappa_H s_h^{\mathrm{c}}(\boldsymbol{H}_h,\boldsymbol{\eta}-\boldsymbol{\eta}_h) - \kappa_E s_h^{\mathrm{c},\circ}(\boldsymbol{E}_h,\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_h),
\end{aligned}$$

since $(\boldsymbol{C}_h(\boldsymbol{E}_h),\boldsymbol{\eta}-\boldsymbol{\eta}_h)_{\boldsymbol{L}^2(D)} = (\boldsymbol{C}_{h0}(\boldsymbol{H}_h),\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_h)_{\boldsymbol{L}^2(D)} = 0$ by definition of the $\boldsymbol{L}^2$-orthogonal projection $\boldsymbol{\Pi}_h^{\mathrm{b}}$ and since $s_h^{\mathrm{c}}(\boldsymbol{H},\cdot) = s_h^{\mathrm{c},\circ}(\boldsymbol{E},\cdot) = 0$. Invoking the Cauchy–Schwarz inequality and the approximation properties of $\boldsymbol{\Pi}_h^{\mathrm{b}}$, we obtain

$$\begin{aligned}
&\big|-(\nabla\times\boldsymbol{E},\boldsymbol{\eta}-\boldsymbol{\eta}_h)_{\boldsymbol{L}^2(D)} + (\nabla_0\times\boldsymbol{H},\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_h)_{\boldsymbol{L}^2(D)}\big| \\
&\quad\lesssim \|\nabla\times\boldsymbol{E}\|_{\boldsymbol{L}^2(D)}h^{s'}|\boldsymbol{\eta}|_{H^{s'}(D)} + \|\nabla_0\times\boldsymbol{H}\|_{\boldsymbol{L}^2(D)}h^{s'}|\boldsymbol{\varepsilon}|_{H^{s'}(D)}.
\end{aligned}$$

Invoking the a priori estimate (5.11a) on $(\boldsymbol{H},\boldsymbol{E})$ and the regularity estimate (5.7b) on $(\boldsymbol{\eta},\boldsymbol{\varepsilon})$ gives

$$\big|-(\nabla\times\boldsymbol{E},\boldsymbol{\eta}-\boldsymbol{\eta}_h)_{\boldsymbol{L}^2(D)} + (\nabla_0\times\boldsymbol{H},\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_h)_{\boldsymbol{L}^2(D)}\big| \lesssim (h/\ell_D)^{s'}\omega\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\mathrm{c}}}c_{\mathrm{rot}}(\boldsymbol{\eta},\boldsymbol{\varepsilon}).$$

Moreover, recalling that $s' \in (\frac12,1]$ and proceeding as at the end of the proof of Lemma 4.2, we obtain $|\boldsymbol{\eta}-\boldsymbol{\eta}_h|_{\mathrm{J}}^{\mathrm{c}} \lesssim h^{s'-\frac12}|\boldsymbol{\eta}|_{\boldsymbol{H}^{s'}(D)} \lesssim (h/\ell_D)^{s'-\frac12}\ell_D^{\frac12}\|\nabla_0\times\boldsymbol{\eta}\|_{\boldsymbol{L}^2(D)}$ and $|\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_h|_{\mathrm{J}}^{\mathrm{c},\circ} \lesssim (h/\ell_D)^{s'-\frac12}\ell_D^{\frac12}\|\nabla\times\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2(D)}$. Then, the a priori estimate (5.11b) gives

$$
\begin{aligned}
&\big|\kappa_H s_h^{\mathrm{c}}(\boldsymbol{H}_h, \boldsymbol{\eta}-\boldsymbol{\eta}_h) + \kappa_E s_h^{\mathrm{c},\circ}(\boldsymbol{E}_h, \boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_h)\big| \\
&\quad \lesssim \kappa_H^{\frac12}|\boldsymbol{H}_h|_{\mathrm{J}}^{\mathrm{c}}\kappa_H^{\frac12}|\boldsymbol{\eta}-\boldsymbol{\eta}_h|_{\mathrm{J}}^{\mathrm{c}} + \kappa_E^{\frac12}|\boldsymbol{E}_h|_{\mathrm{J}}^{\mathrm{c},\circ}\kappa_E^{\frac12}|\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_h|_{\mathrm{J}}^{\mathrm{c},\circ} \\
&\quad \lesssim \omega^{\frac12}\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\mathrm{c}}}(h/\ell_D)^{s'-\frac12}\left(\kappa_H^{\frac12}\ell_D^{\frac12}\|\nabla_0\times\boldsymbol{\eta}\|_{\boldsymbol{L}^2(D)} + \kappa_E^{\frac12}\ell_D^{\frac12}\|\nabla\times\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2(D)}\right) \\
&\quad = (h/\ell_D)^{s'-\frac12}\omega\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\mathrm{c}}}c_{\mathrm{rot}}(\boldsymbol{\eta},\boldsymbol{\varepsilon}),
\end{aligned}
$$

since $\kappa_H^{\frac12}\ell_D^{\frac12} = \omega^{\frac12}\ell_D\mu_0^{\frac12}$ and $\kappa_E^{\frac12}\ell_D^{\frac12} = \omega^{\frac12}\ell_D\epsilon_0^{\frac12}$. Combining the above two estimates and using $h \le \ell_D$ proves the assertion. ∎

LEMMA 5.6 (bound on Galerkin orthogonality error). *The following holds:*

$$
|\theta_{\mathrm{gal}}| \lesssim (h/\ell_D)^{s'}\omega\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\mathrm{c}}}c_{rot}(\boldsymbol{\eta},\boldsymbol{\varepsilon}). \tag{5.14}
$$

Proof. Recalling the definition (5.10b) of θ_{gal} and the definition (5.5) of $\boldsymbol{\delta h}$ and $\boldsymbol{\delta e}$ and using that a_h is linear with respect to its first argument, we have

$$
\theta_{\mathrm{gal}} = a_h((\boldsymbol{H},\boldsymbol{E}),(\boldsymbol{\eta}_h,\boldsymbol{\varepsilon}_h)) - a_h((\boldsymbol{H}_h,\boldsymbol{E}_h),(\boldsymbol{\eta}_h,\boldsymbol{\varepsilon}_h)).
$$

Since $s_h^{\mathrm{c}}(\boldsymbol{H},\cdot) = s_h^{\mathrm{c},\circ}(\boldsymbol{E},\cdot) = 0$ and $\boldsymbol{C}_{h0}(\boldsymbol{H}) = \nabla_0\times\boldsymbol{H}$, $\boldsymbol{C}_h(\boldsymbol{E}) = \nabla\times\boldsymbol{E}$, we have

$$
\begin{aligned}
a_h((\boldsymbol{H},\boldsymbol{E}),(\boldsymbol{\eta}_h,\boldsymbol{\varepsilon}_h)) &= -(\nabla\times\boldsymbol{E},\boldsymbol{\eta}_h)_{\boldsymbol{L}^2(D)} + (\nabla_0\times\boldsymbol{H},\boldsymbol{\varepsilon}_h)_{\boldsymbol{L}^2(D)} \\
&= \omega\big\{((\boldsymbol{I}-\boldsymbol{\Pi}_0^{\mathrm{c}})(\mu\boldsymbol{f}),\boldsymbol{\eta}_h)_{\boldsymbol{L}^2(D)} + ((\boldsymbol{I}-\boldsymbol{\Pi}^{\mathrm{c}})(\epsilon\boldsymbol{g}),\boldsymbol{\varepsilon}_h)_{\boldsymbol{L}^2(D)}\big\}.
\end{aligned}
$$

Moreover, by definition of the discrete solution $(\boldsymbol{H}_h,\boldsymbol{E}_h)$, we have

$$
a_h((\boldsymbol{H}_h,\boldsymbol{E}_h),(\boldsymbol{\eta}_h,\boldsymbol{\varepsilon}_h)) = \omega\big\{((\boldsymbol{I}-\boldsymbol{\Pi}_{h0}^{\mathrm{c}})(\mu\boldsymbol{f}),\boldsymbol{\eta}_h)_{\boldsymbol{L}^2(D)} + ((\boldsymbol{I}-\boldsymbol{\Pi}_h^{\mathrm{c}})(\epsilon\boldsymbol{g}),\boldsymbol{\varepsilon}_h)_{\boldsymbol{L}^2(D)}\big\}.
$$

Hence, we have

$$
\theta_{\mathrm{gal}} = \omega\big\{((\boldsymbol{\Pi}_0^{\mathrm{c}}-\boldsymbol{\Pi}_{h0}^{\mathrm{c}})(\mu\boldsymbol{f}),\boldsymbol{\eta}_h)_{\boldsymbol{L}^2(D)} + ((\boldsymbol{\Pi}^{\mathrm{c}}-\boldsymbol{\Pi}_h^{\mathrm{c}})(\epsilon\boldsymbol{g}),\boldsymbol{\varepsilon}_h)_{\boldsymbol{L}^2(D)}\big\}.
$$

We have $\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{\eta}) = \boldsymbol{0}$, and, owing to Lemma 5.10 (see section 5.3 below), we also have $\boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\boldsymbol{\eta}) = \boldsymbol{0}$. Using the Cauchy–Schwarz inequality and the approximation properties of $\boldsymbol{\Pi}_h^{\mathrm{b}}$ gives

$$
\begin{aligned}
\omega|((\boldsymbol{\Pi}_{h0}^{\mathrm{c}}-\boldsymbol{\Pi}_0^{\mathrm{c}})(\mu\boldsymbol{f}),\boldsymbol{\eta}_h)_{\boldsymbol{L}^2(D)}| &= \omega|((\boldsymbol{\Pi}_{h0}^{\mathrm{c}}-\boldsymbol{\Pi}_0^{\mathrm{c}})(\mu\boldsymbol{f}),\boldsymbol{\eta}_h-\boldsymbol{\eta})_{\boldsymbol{L}^2(D)}| \\
&\lesssim \omega\mu_0^{\frac12}\|\mu^{\frac12}\boldsymbol{f}\|_{\boldsymbol{L}^2(D)}h^{s'}|\boldsymbol{\eta}|_{\boldsymbol{H}^{s'}(D)} \\
&\lesssim \omega\mu_0^{\frac12}\|\mu^{\frac12}\boldsymbol{f}\|_{\boldsymbol{L}^2(D)}(h/\ell_D)^{s'}\ell_D\|\nabla_0\times\boldsymbol{\eta}\|_{\boldsymbol{L}^2(D)} \\
&\lesssim (h/\ell_D)^{s'}\omega\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\mathrm{c}}}c_{\mathrm{rot}}(\boldsymbol{\eta},\boldsymbol{\varepsilon}).
\end{aligned}
$$

A similar bound holds for $\omega|((\boldsymbol{\Pi}_h^{\mathrm{c}}-\boldsymbol{\Pi}^{\mathrm{c}})(\epsilon\boldsymbol{g}),\boldsymbol{\varepsilon}_h)_{\boldsymbol{L}^2(D)}|$. This completes the proof. ∎

LEMMA 5.7 (bound on curl commuting error). *The following holds:*

$$
|\theta_{crl}| \lesssim (h/\ell_D)^{s'-\frac12}\omega\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\mathrm{c}}}c_{rot}(\boldsymbol{\eta},\boldsymbol{\varepsilon}). \tag{5.15}
$$

Proof. We observe that

$$
\theta_{\mathrm{crl}} = -\delta_h(\boldsymbol{H}_h,\boldsymbol{\varepsilon}) + \delta_h^{\circ}(\boldsymbol{E}_h,\boldsymbol{\eta}),
$$

since $\delta_h(\boldsymbol{H},\boldsymbol{\varepsilon}) = \delta_h^{\circ}(\boldsymbol{E},\boldsymbol{\eta}) = 0$. Invoking Lemma 4.2 then gives

$$
|\theta_{\mathrm{crl}}| \lesssim (h/\ell_D)^{s'-\frac12}\ell_D^{\frac12}\big\{|\boldsymbol{H}_h|_{\mathrm{J}}^{\mathrm{c}}\|\nabla\times\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2(D)} + |\boldsymbol{E}_h|_{\mathrm{J}}^{\mathrm{c},\circ}\|\nabla_0\times\boldsymbol{\eta}\|_{\boldsymbol{L}^2(D)}\big\}.
$$

Using that $\ell_D\kappa_H\epsilon_0=\omega^{-1}$, we obtain

$$\begin{aligned}\ell_D^{\frac12}|\boldsymbol{H}_h|_{\rm J}^{\rm c}\|\nabla\times\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2(D)}&=\kappa_H^{\frac12}|\boldsymbol{H}_h|_{\rm J}^{\rm c}(\ell_D\kappa_H\epsilon_0)^{-\frac12}\ell_D\epsilon_0^{\frac12}\|\nabla\times\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2(D)}\\&\lesssim\omega^{\frac12}\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\rm c}}\omega^{\frac12}c_{\rm rot}(\boldsymbol{\eta},\boldsymbol{\varepsilon})\\&=\omega\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\rm c}}c_{\rm rot}(\boldsymbol{\eta},\boldsymbol{\varepsilon}).\end{aligned}$$

A similar bound holds true for $\ell_D^{\frac12}|\boldsymbol{E}_h|_{\rm J}^{{\rm c},\circ}\|\nabla_0\times\boldsymbol{\eta}\|_{\boldsymbol{L}^2(D)}$. This completes the proof. ∎

LEMMA 5.8 (bound on divergence conformity error). *Let $s\in(0,\frac12]$ be the smoothness index of the solution $(\boldsymbol{H},\boldsymbol{E})\in\boldsymbol{X}_{\mu,0}^{\rm c}\times\boldsymbol{X}_{\epsilon}^{\rm c}$ to the problem* (2.23) *(see Lemma* 2.6*). Let $s'\in(\frac12,1]$ be the smoothness index of the dual solution $(\boldsymbol{\eta},\boldsymbol{\varepsilon})\in\boldsymbol{H}_0(\mathbf{curl};D)\times\boldsymbol{H}(\mathbf{curl};D)$ to the problem* (5.6) *(see Lemma* 5.3*). Then the following estimate holds with $\sigma:=\min(s,s'-\frac12)>0$:*

$$|\theta_{\rm div}|\lesssim(h/\ell_D)^\sigma\omega\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\rm c}}\|(\boldsymbol{\delta h},\boldsymbol{\delta e})\|_{L^{\rm c}}.\tag{5.16}$$

Proof. The proof crucially relies on Lemma 5.11 recorded below in section 5.3. Since $\boldsymbol{\Pi}_{h0}^{\rm c}(\mu\boldsymbol{\delta h})=\mathbf{0}$, we can apply the estimate (5.18a) with $\boldsymbol{B}:=\mu\boldsymbol{\delta h}$, which gives

$$\begin{aligned}\big|(\boldsymbol{\delta h},\boldsymbol{\Pi}_0^{\rm c}(\mu\boldsymbol{\delta h}))_{\boldsymbol{L}^2(D)}\big|\lesssim\Big\{&(h/\ell_D)^s\ell_D^s|\boldsymbol{H}|_{\boldsymbol{H}^s(D)}+(h/\ell_D)^{s'}\ell_D\|\nabla_0\times\boldsymbol{H}\|_{\boldsymbol{L}^2(D)}\\&+(h/\ell_D)^{s'-\frac12}\ell_D^{\frac12}(\|\tilde h^{\frac12}\boldsymbol{C}_{h0}(\boldsymbol{H}_h)\|_{\boldsymbol{L}^2(D)}+|\boldsymbol{H}_h|_{\rm J}^{\rm c})\Big\}\\&\times\|\boldsymbol{\Pi}_0^{\rm c}(\mu\boldsymbol{\delta h})\|_{\boldsymbol{L}^2(D)}.\end{aligned}$$

Since $\ell_D^s|\boldsymbol{H}|_{\boldsymbol{H}^s(D)}\lesssim\ell_D\|\nabla_0\times\boldsymbol{H}\|_{\boldsymbol{L}^2(D)}\le\mu_0^{-\frac12}\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\rm c}}$ owing to the continuous a priori estimate (5.11a) and also invoking the discrete a priori estimate (5.11b), we obtain, since $h\le\ell_D$ and $s\le s'$,

$$\begin{aligned}\big|(\boldsymbol{\delta h},\boldsymbol{\Pi}_0^{\rm c}(\mu\boldsymbol{\delta h}))_{\boldsymbol{L}^2(D)}\big|&\lesssim(h/\ell_D)^\sigma\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\rm c}}\mu_0^{-\frac12}\|\boldsymbol{\Pi}_0^{\rm c}(\mu\boldsymbol{\delta h})\|_{\boldsymbol{L}^2(D)}\\&\lesssim(h/\ell_D)^\sigma\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\rm c}}\|\mu^{\frac12}\boldsymbol{\delta h}\|_{\boldsymbol{L}^2(D)}.\end{aligned}$$

Proceeding similarly, one proves that

$$\big|(\boldsymbol{\delta e},\boldsymbol{\Pi}^{\rm c}(\epsilon\boldsymbol{\delta e}))_{\boldsymbol{L}^2(D)}\big|\lesssim(h/\ell_D)^\sigma\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\rm c}}\|\epsilon^{\frac12}\boldsymbol{\delta e}\|_{\boldsymbol{L}^2(D)}.$$

Putting the above two bounds together proves the claim. ∎

We are now ready to state the main result of the paper which, owing to standard spectral approximation results (see, e.g., [11, Lem. 2.2], [39, Thms. 3 and 4], [6, Prop. 7.4]), proves the spectral correctness of the dG approximation. Recall that $\sigma:=\min(s,s'-\frac12)>0$.

THEOREM 5.9 (convergence). *We have $\|T-T_h\|_{\mathcal{L}(L^{\rm c};L^{\rm c})}\lesssim(h/\ell_D)^\sigma$.*

Proof. Combining the $\boldsymbol{L}^2$-error representation formula (5.9) together with the bounds from Lemmas 5.5–5.8, and since $c_{\rm rot}(\boldsymbol{\eta},\boldsymbol{\varepsilon})\lesssim\|(\boldsymbol{\delta h},\boldsymbol{\delta e}\|_{L^{\rm c}}$ owing to (5.7a), we infer that

$$\omega\|(\boldsymbol{\delta h},\boldsymbol{\delta e})\|_{L^{\rm c}}^2\lesssim(h/\ell_D)^\sigma\omega\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\rm c}}\|(\boldsymbol{\delta h},\boldsymbol{\delta e})\|_{L^{\rm c}},$$

where we used that $\sigma\le s'-\frac12<s'$ and $h\le\ell_D$. This implies that $\|(\boldsymbol{\delta h},\boldsymbol{\delta e})\|_{L^{\rm c}}\lesssim(h/\ell_D)^\sigma\|(\boldsymbol{f},\boldsymbol{g})\|_{L^{\rm c}}$, whence the claim. ∎

5.3. Technical lemmas. We start by recording the following result whose proof is omitted for brevity.

LEMMA 5.10 (discrete projections). *The following holds:*

$$\mathbf{\Pi}^{\mathrm{c}}_{h0} \circ \mathbf{\Pi}^{\mathrm{c}}_{0} = \mathbf{\Pi}^{\mathrm{c}}_{h0}, \qquad \mathbf{\Pi}^{\mathrm{c}}_{h} \circ \mathbf{\Pi}^{\mathrm{c}} = \mathbf{\Pi}^{\mathrm{c}}_{h}. \tag{5.17}$$

Next we establish a result that plays a central role in the analysis to estimate the divergence conformity error.

LEMMA 5.11 (divergence conformity). *Let $s \in (0, \frac{1}{2}]$. For all $\boldsymbol{H} \in \boldsymbol{H}_0(\mathbf{curl}; D) \cap \boldsymbol{H}^s(D)$, all $\boldsymbol{H}_h \in \boldsymbol{P}^{\mathrm{b}}_k(\mathcal{T}_h)$, and all $\boldsymbol{B} \in \boldsymbol{L}^2(D)$ satisfying $\mathbf{\Pi}^{\mathrm{c}}_{h0}(\boldsymbol{B}) = \mathbf{0}$, we have*

$$\begin{aligned}\left|(\boldsymbol{H} - \boldsymbol{H}_h, \mathbf{\Pi}^{\mathrm{c}}_0(\boldsymbol{B}))_{\boldsymbol{L}^2(D)}\right| \lesssim \Big\{ &(h/\ell_D)^s \ell_D^s |\boldsymbol{H}|_{\boldsymbol{H}^s(D)} + (h/\ell_D)^{s'} \ell_D \|\nabla_0 \times \boldsymbol{H}\|_{\boldsymbol{L}^2(D)} \\ &+ (h/\ell_D)^{s'-\frac{1}{2}} \ell_D^{\frac{1}{2}} (\|\tilde{h}^{\frac{1}{2}} \boldsymbol{C}_{h0}(\boldsymbol{H}_h)\|_{\boldsymbol{L}^2(D)} + |\boldsymbol{H}_h|^{\mathrm{c}}_{\mathrm{J}}) \Big\} \\ &\times \|\mathbf{\Pi}^{\mathrm{c}}_0(\boldsymbol{B})\|_{\boldsymbol{L}^2(D)},\end{aligned} \tag{5.18a}$$

and for all $\boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl}; D) \cap \boldsymbol{H}^s(D)$, all $\boldsymbol{E}_h \in \boldsymbol{P}^{\mathrm{b}}_k(\mathcal{T}_h)$, and all $\boldsymbol{D} \in \boldsymbol{L}^2(D)$ satisfying $\mathbf{\Pi}^{\mathrm{c}}_h(\boldsymbol{D}) = \mathbf{0}$, we have

$$\begin{aligned}\left|(\boldsymbol{E} - \boldsymbol{E}_h, \mathbf{\Pi}^{\mathrm{c}}(\boldsymbol{D}))_{\boldsymbol{L}^2(D)}\right| \lesssim \Big\{ &(h/\ell_D)^s \ell_D^s |\boldsymbol{E}|_{\boldsymbol{H}^s(D)} + (h/\ell_D)^{s'} \ell_D \|\nabla \times \boldsymbol{E}\|_{\boldsymbol{L}^2(D)} \\ &+ (h/\ell_D)^{s'-\frac{1}{2}} \ell_D^{\frac{1}{2}} (\|\tilde{h}^{\frac{1}{2}} \boldsymbol{C}_{h}(\boldsymbol{E}_h)\|_{\boldsymbol{L}^2(D)} + |\boldsymbol{E}_h|^{\mathrm{c},\circ}_{\mathrm{J}}) \Big\} \\ &\times \|\mathbf{\Pi}^{\mathrm{c}}(\boldsymbol{D})\|_{\boldsymbol{L}^2(D)},\end{aligned} \tag{5.18b}$$

where $s' \in (\frac{1}{2}, 1]$ results from the regularity estimate (2.19).

Proof. We only prove the estimate (5.18a), since the proof of (5.18b) is similar.

(1) Using the commuting property of $\mathcal{J}^{\mathrm{c}}_{h0}$, we obtain $\nabla \times \mathcal{J}^{\mathrm{c}}_{h0}(\mathbf{\Pi}^{\mathrm{c}}_0(\boldsymbol{H} - \boldsymbol{H}_h)) = \mathbf{0}$. This implies that $\mathcal{J}^{\mathrm{c}}_{h0}(\mathbf{\Pi}^{\mathrm{c}}_0(\boldsymbol{H} - \boldsymbol{H}_h))$ is a member of $\boldsymbol{P}^{\mathrm{c}}_{k0}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$. Moreover, invoking Lemma 5.10, we have $\mathbf{\Pi}^{\mathrm{c}}_{h0}(\mathbf{\Pi}^{\mathrm{c}}_0(\boldsymbol{B})) = \mathbf{\Pi}^{\mathrm{c}}_{h0}(\boldsymbol{B}) = \mathbf{0}$ by assumption. Hence, $\mathbf{\Pi}^{\mathrm{c}}_0(\boldsymbol{B}) \in \boldsymbol{P}^{\mathrm{c}}_{k0}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^{\perp}$. This, in turn, implies the following identity:

$$(\boldsymbol{H} - \boldsymbol{H}_h, \mathbf{\Pi}^{\mathrm{c}}_0(\boldsymbol{B}))_{\boldsymbol{L}^2(D)} = (\boldsymbol{H} - \boldsymbol{H}_h - \mathcal{J}^{\mathrm{c}}_{h0}(\mathbf{\Pi}^{\mathrm{c}}_0(\boldsymbol{H} - \boldsymbol{H}_h)), \mathbf{\Pi}^{\mathrm{c}}_0(\boldsymbol{B}))_{\boldsymbol{L}^2(D)}.$$

Using the $\boldsymbol{L}^2$-orthogonality of $\mathbf{\Pi}^{\mathrm{c}}_0$ followed by the Cauchy–Schwarz inequality gives

$$\left|(\boldsymbol{H} - \boldsymbol{H}_h, \mathbf{\Pi}^{\mathrm{c}}_0(\boldsymbol{B}))_{\boldsymbol{L}^2(D)}\right| \le \|(\boldsymbol{I} - \mathcal{J}^{\mathrm{c}}_{h0})(\mathbf{\Pi}^{\mathrm{c}}_0(\boldsymbol{H} - \boldsymbol{H}_h))\|_{\boldsymbol{L}^2(D)} \|\mathbf{\Pi}^{\mathrm{c}}_0(\boldsymbol{B})\|_{\boldsymbol{L}^2(D)}.$$

It remains to bound $\|(\boldsymbol{I} - \mathcal{J}^{\mathrm{c}}_{h0})(\mathbf{\Pi}^{\mathrm{c}}_0(\boldsymbol{H} - \boldsymbol{H}_h))\|_{\boldsymbol{L}^2(D)}$.

(2) Let us set $\boldsymbol{H}^{\mathrm{c}}_h := \mathcal{I}^{\mathrm{c,av}}_{h0}(\boldsymbol{H}_h)$ and $\boldsymbol{w} := \boldsymbol{H} - \boldsymbol{H}^{\mathrm{c}}_h - \mathbf{\Pi}^{\mathrm{c}}_0(\boldsymbol{H} - \boldsymbol{H}^{\mathrm{c}}_h)$. Then $\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}; D)$ with $\nabla_0 \times \boldsymbol{w} = \nabla_0 \times (\boldsymbol{H} - \boldsymbol{H}^{\mathrm{c}}_h)$, and $\mathbf{\Pi}^{\mathrm{c}}_0(\boldsymbol{w}) = \mathbf{0}$. Hence, $\boldsymbol{w}$ is a member of $\boldsymbol{X}^{\mathrm{c}}_0$. The regularity estimate (2.19) followed by the triangle inequality gives

$$|\boldsymbol{w}|_{\boldsymbol{H}^{s'}} \lesssim \ell_D^{1-s'} \|\nabla_0 \times (\boldsymbol{H} - \boldsymbol{H}^{\mathrm{c}}_h)\|_{\boldsymbol{L}^2(D)} \le \ell_D^{1-s'} (\|\nabla_0 \times \boldsymbol{H}\|_{\boldsymbol{L}^2(D)} + \|\nabla_0 \times \boldsymbol{H}^{\mathrm{c}}_h\|_{\boldsymbol{L}^2(D)}).$$

Invoking the triangle inequality, the approximation properties of $\mathcal{J}^{\mathrm{c}}_{h0}$, and the fact that $(\boldsymbol{I} - \mathcal{J}^{\mathrm{c}}_{h0})(\boldsymbol{H}^{\mathrm{c}}_h) = \mathbf{0}$ (since $\boldsymbol{H}^{\mathrm{c}}_h \in \boldsymbol{P}^{\mathrm{c}}_{k0}(\mathcal{T}_h)$), this implies that

$$\begin{aligned}&\|(\boldsymbol{I} - \mathcal{J}^{\mathrm{c}}_{h0})(\mathbf{\Pi}^{\mathrm{c}}_0(\boldsymbol{H} - \boldsymbol{H}^{\mathrm{c}}_h))\|_{\boldsymbol{L}^2(D)} \\ &\qquad \lesssim \|(\boldsymbol{I} - \mathcal{J}^{\mathrm{c}}_{h0})(\boldsymbol{w})\|_{\boldsymbol{L}^2(D)} + \|(\boldsymbol{I} - \mathcal{J}^{\mathrm{c}}_{h0})(\boldsymbol{H} - \boldsymbol{H}^{\mathrm{c}}_h)\|_{\boldsymbol{L}^2(D)} \\ &\qquad \lesssim h^{s'} |\boldsymbol{w}|_{\boldsymbol{H}^{s'}(D)} + \|(\boldsymbol{I} - \mathcal{J}^{\mathrm{c}}_{h0})(\boldsymbol{H})\|_{\boldsymbol{L}^2(D)} \\ &\qquad \lesssim (h/\ell_D)^{s'} \ell_D (\|\nabla_0 \times \boldsymbol{H}\|_{\boldsymbol{L}^2(D)} + \|\nabla_0 \times \boldsymbol{H}^{\mathrm{c}}_h\|_{\boldsymbol{L}^2(D)}) + (h/\ell_D)^s \ell_D^s |\boldsymbol{H}|_{\boldsymbol{H}^s(D)}.\end{aligned}$$

Moreover, invoking the triangle inequality, the shape-regularity of the mesh sequence, and the approximation properties of $\mathcal{I}_{h0}^{\mathrm{c,av}}$, we infer that

$$\begin{aligned}\|\nabla_0{\times}\boldsymbol{H}_h^{\mathrm{c}}\|_{\boldsymbol{L}^2(D)} &\lesssim \|\boldsymbol{C}_{h0}(\boldsymbol{H}_h^{\mathrm{c}}-\boldsymbol{H}_h)\|_{\boldsymbol{L}^2(D)} + \|\boldsymbol{C}_{h0}(\boldsymbol{H}_h)\|_{\boldsymbol{L}^2(D)} \\ &\lesssim h^{-\frac12}(\|\tilde{h}^{\frac12}\boldsymbol{C}_{h0}(\boldsymbol{H}_h)\|_{\boldsymbol{L}^2(D)} + |\boldsymbol{H}_h|_{\mathrm{J}}^{\mathrm{c}}).\end{aligned}$$

We conclude that

$$\begin{aligned}\|(\boldsymbol{I}-\mathcal{J}_{h0}^{\mathrm{c}})(\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{H}-\boldsymbol{H}_h^{\mathrm{c}}))\|_{\boldsymbol{L}^2(D)} &\lesssim (h/\ell_D)^{s'}\ell_D\|\nabla_0{\times}\boldsymbol{H}\|_{\boldsymbol{L}^2(D)} + (h/\ell_D)^{s}\ell_D^{s}|\boldsymbol{H}|_{\boldsymbol{H}^s(D)} \\ &\quad + (h/\ell_D)^{s'-\frac12}\ell_D^{\frac12}(\|\tilde{h}^{\frac12}\boldsymbol{C}_{h0}(\boldsymbol{H}_h)\|_{\boldsymbol{L}^2(D)} + |\boldsymbol{H}_h|_{\mathrm{J}}^{\mathrm{c}}).\end{aligned}$$

Finally, using the triangle inequality, the $\boldsymbol{L}^2$-stability of $\mathcal{J}_{h0}^{\mathrm{c}}$ and that of $\boldsymbol{\Pi}_0^{\mathrm{c}}$, and the approximation properties of $\mathcal{I}_{h0}^{\mathrm{c,av}}$, we obtain

$$\begin{aligned}\|(\boldsymbol{I}-\mathcal{J}_{h0}^{\mathrm{c}})(\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{H}-\boldsymbol{H}_h))\|_{\boldsymbol{L}^2(D)} &\lesssim \|(\boldsymbol{I}-\mathcal{J}_{h0}^{\mathrm{c}})(\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{H}-\boldsymbol{H}_h^{\mathrm{c}}))\|_{\boldsymbol{L}^2(D)} \\ &\quad + \|(\boldsymbol{I}-\mathcal{J}_{h0}^{\mathrm{c}})(\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{H}_h^{\mathrm{c}}-\boldsymbol{H}_h))\|_{\boldsymbol{L}^2(D)} \\ &\lesssim \|(\boldsymbol{I}-\mathcal{J}_{h0}^{\mathrm{c}})(\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{H}-\boldsymbol{H}_h^{\mathrm{c}}))\|_{\boldsymbol{L}^2(D)} + h^{\frac12}|\boldsymbol{H}_h|_{\mathrm{J}}^{\mathrm{c}}.\end{aligned}$$

The assertion follows readily. □

Appendix A. Helmholtz decompositions. In this appendix, we recall some useful results on Helmholtz decompositions; these results are mostly drawn from [2], [26], and [32]. We also give a short proof of Lemma 2.2. The reader is also referred to [4, Chap. 3], [37, section 4.1], and [38, Chap. 3].

A.1. Topology of D. Recall that D is an open, bounded, Lipschitz polyhedron of $\mathbb{R}^3$. We denote by Γ_0 the boundary of the only unbounded connected component of $\mathbb{R}^3\backslash\overline{D}$. If ∂D is not connected, i.e., $\partial D \neq \Gamma_0$, we denote by $\{\Gamma_i\}_{i\in\{1:I\}}$ the connected components of ∂D that are different from Γ_0 (see, e.g., [32, p. 37], [2, p. 835], [26, p. 217]). If D is not simply connected, we assume that there exist J cuts ($(d-1)$-dimensional smooth manifolds) $\{\Sigma_j\}_{j\in\{1:J\}}$ that make the open set $D^\Sigma := D\backslash\bigcup_{j\in\{1:J\}}\Sigma_j$ simply connected. Additional regularity assumptions on these cuts as stated in [2, Hyp. 3.3, p. 836] are assumed to hold true.

A.2. Helmholtz decompositions. We consider the subspaces

$$\boldsymbol{H}^\Gamma(\mathrm{div}=0;D) := \{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div}=0;D) \mid \int_{\Gamma_i}\boldsymbol{v}{\cdot}\boldsymbol{n}\,\mathrm{d}s = 0\ \forall i\in\{1:I\}\}, \tag{A.1a}$$

$$\boldsymbol{H}_0^\Sigma(\mathrm{div}=0;D) := \{\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{div}=0;D) \mid \int_{\Sigma_j}\boldsymbol{v}{\cdot}\boldsymbol{n}\,\mathrm{d}s = 0\ \forall j\in\{1:J\}\}. \tag{A.1b}$$

The following $\boldsymbol{L}^2$-orthogonal decompositions hold true:

$$\boldsymbol{L}^2(D) = \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D) \overset{\perp}{\oplus} \boldsymbol{H}^\Gamma(\mathrm{div}=0;D), \tag{A.2a}$$

$$\boldsymbol{L}^2(D) = \boldsymbol{H}(\mathbf{curl}=\mathbf{0};D) \overset{\perp}{\oplus} \boldsymbol{H}_0^\Sigma(\mathrm{div}=0;D). \tag{A.2b}$$

In other words, we have

$$\boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)^\perp = \boldsymbol{H}^\Gamma(\mathrm{div}=0;D), \tag{A.3a}$$

$$\boldsymbol{H}(\mathbf{curl}=\mathbf{0};D)^\perp = \boldsymbol{H}_0^\Sigma(\mathrm{div}=0;D). \tag{A.3b}$$

For all $q \in L^2(D)$ such that $q|_{D^\Sigma} \in H^1(D^\Sigma)$, we denote by $\nabla_\Sigma q$ the broken gradient of q such that $(\nabla_\Sigma q)(\boldsymbol{x}) = (\nabla q|_{D^\Sigma})(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in D$. For all $i \in \mathbb{N}$, let c_i denote any real number. We define

$$H^1_\Gamma(D) := \{q \in H^1(D) \mid q|_{\Gamma_0} = 0, q|_{\Gamma_i} = c_i \ \forall i \in \{1{:}I\}\}, \tag{A.4a}$$

$$H^1_\Sigma(D) := \{q \in L^2(D) \mid q|_{D^\Sigma} \in H^1(D^\Sigma), [\![q]\!]|_{\Sigma_j} = c_j \ \forall j \in \{1{:}J\}\}. \tag{A.4b}$$

Then, the following $\boldsymbol{L}^2$-orthogonal decompositions also hold true:

$$\boldsymbol{L}^2(D) = \nabla H^1_\Gamma(D) \overset{\perp}{\oplus} \nabla\times \boldsymbol{H}(\mathbf{curl}; D), \tag{A.5a}$$

$$\boldsymbol{L}^2(D) = \nabla_\Sigma H^1_\Sigma(D) \overset{\perp}{\oplus} \nabla_0\times \boldsymbol{H}_0(\mathbf{curl}; D). \tag{A.5b}$$

A.3. Proof of Lemma 2.2. We only prove that the operators $\nabla\times : \boldsymbol{X}^{\mathrm{c}}_\epsilon \to \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$ and $\nabla_0\times : \boldsymbol{X}^{\mathrm{c}}_{\mu,0} \to \boldsymbol{H}(\mathbf{curl} = \mathbf{0}; D)^\perp$ are isomorphisms, since the proof for the other two operators is similar. (Recall that the spaces $\boldsymbol{X}^{\mathrm{c}}_\epsilon$ and $\boldsymbol{X}^{\mathrm{c}}_{\mu,0}$ are defined in (2.9).)

(1) Integration by parts readily shows that $\nabla\times\boldsymbol{e} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$ for all $\boldsymbol{e} \in \boldsymbol{X}^{\mathrm{c}}_\epsilon$ and $\nabla_0\times\boldsymbol{h} \in \boldsymbol{H}(\mathbf{curl} = \mathbf{0}; D)^\perp$ for all $\boldsymbol{h} \in \boldsymbol{X}^{\mathrm{c}}_{\mu,0}$.

(2) *Injectivity.* Let $\boldsymbol{e} \in \boldsymbol{X}^{\mathrm{c}}_\epsilon$ be such that $\nabla\times\boldsymbol{e} = \mathbf{0}$. Then, $\boldsymbol{e} \in \boldsymbol{H}(\mathbf{curl} = \mathbf{0}; D)$, and by definition of $\boldsymbol{X}^{\mathrm{c}}_\epsilon$, we infer that $(\epsilon\boldsymbol{e}, \boldsymbol{e})_{\boldsymbol{L}^2(D)} = 0$. This proves that $\boldsymbol{e} = \mathbf{0}$. Hence, the operator $\nabla\times : \boldsymbol{X}^{\mathrm{c}}_\epsilon \to \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$ is injective. The proof that the operator $\nabla_0\times : \boldsymbol{X}^{\mathrm{c}}_{\mu,0} \to \boldsymbol{H}(\mathbf{curl} = \mathbf{0}; D)^\perp$ is injective is similar.

(3) *Surjectivity.* Let $\boldsymbol{\theta} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$. Since every $q \in H^1_\Gamma(D)$ satisfies $\nabla q \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)$, we have $\boldsymbol{\theta} \in (\nabla H^1_\Gamma(D))^\perp$. Owing to (A.5a), we infer that there is $\boldsymbol{e}' \in \boldsymbol{H}(\mathbf{curl}; D)$ such that $\nabla\times\boldsymbol{e}' = \boldsymbol{\theta}$. Let $p \in H^1_\Sigma(D)$ be the unique function solving $(\epsilon\nabla_\Sigma p, \nabla_\Sigma q)_{\boldsymbol{L}^2(D)} = (\epsilon\boldsymbol{e}', \nabla_\Sigma q)_{\boldsymbol{L}^2(D)}$ for all $q \in H^1_\Sigma(D)$. Set $\boldsymbol{e} := \boldsymbol{e}' - \nabla_\Sigma p$. We have $\nabla\times\boldsymbol{e} = \nabla\times\boldsymbol{e}' = \boldsymbol{\theta}$, and $\epsilon\boldsymbol{e} \in (\nabla_\Sigma H^1_\Sigma(D))^\perp$. Owing to (A.5b), we infer that $\epsilon\boldsymbol{e}$ is in the range of the $\nabla_0\times$ operator. Integration by parts readily implies that $\epsilon\boldsymbol{e} \in \boldsymbol{H}(\mathbf{curl} = \mathbf{0}; D)^\perp$. In summary, we have shown that $\boldsymbol{e} \in \boldsymbol{X}^{\mathrm{c}}_\epsilon$ and that $\nabla\times\boldsymbol{e} = \boldsymbol{\theta}$. This proves the surjectivity of the operator $\nabla\times : \boldsymbol{X}^{\mathrm{c}}_\epsilon \to \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$. The proof that the operator $\nabla_0\times : \boldsymbol{X}^{\mathrm{c}}_{\mu,0} \to \boldsymbol{H}(\mathbf{curl} = \mathbf{0}; D)^\perp$ is surjective is similar.

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