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# $\boldsymbol{H}(\mathbf{curl})$-RECONSTRUCTION OF PIECEWISE POLYNOMIAL FIELDS WITH APPLICATION TO *hp*-A POSTERIORI NONCONFORMING ERROR ANALYSIS FOR MAXWELL'S EQUATIONS*

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**Abstract.** We devise and analyze a novel $\boldsymbol{H}(\mathbf{curl})$-reconstruction operator for piecewise polynomial fields on shape-regular simplicial meshes. The (nonpolynomial) reconstruction is devised over the mesh vertex patches using the partition of unity induced by hat basis functions in combination with local Helmholtz decompositions. Our main focus is on homogeneous tangential boundary conditions. We prove that the difference between the reconstructed $\boldsymbol{H}_0(\mathbf{curl})$-field and the original, piecewise polynomial field, measured in the broken curl norm and in the $\boldsymbol{L}^2$-norm, can be bounded in terms of suitable jump norms of the original field. The bounds are always $h$-optimal, and $p$-suboptimal by $\frac{1}{2}$-order for the broken curl norm and by $\frac{3}{2}$-order for the $\boldsymbol{L}^2$-norm. An auxiliary result of independent interest is a novel broken curl, divergence-preserving Poincaré inequality on vertex patches. Moreover, the $\boldsymbol{L}^2$-norm estimate can be improved to $\frac{1}{2}$-order suboptimality under a (reasonable) assumption on the uniform elliptic regularity pickup for a Poisson problem with Neumann conditions over the vertex patches. We also discuss extensions of the $\boldsymbol{H}_0(\mathbf{curl})$-reconstruction operator to the prescription of mixed boundary conditions, to agglomerated polytopal meshes, and to convex domains. Finally, we showcase an important application of the $\boldsymbol{H}(\mathbf{curl})$-reconstruction operator to the $hp$-a posteriori nonconforming error analysis of Maxwell's equations. We focus on the (symmetric) interior penalty discontinuous Galerkin approximation of some simplified forms of Maxwell's equations.

**Key words.** $H$(curl)-reconstruction, $hp$-analysis, broken Poincaré inequality, a posteriori error analysis, nonconforming approximation, discontinuous Galerkin, Maxwell's equations

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## 1. Introduction.

Given a piecewise polynomial field on a shape-regular simplicial mesh covering exactly a Lipschitz polyhedral domain, we show in the paper the existence of a (nonpolynomial) $\boldsymbol{H}(\mathbf{curl})$-conforming field whose distance to the original field in the broken curl norm and in the $\boldsymbol{L}^2$-norm is bounded in terms of suitable jumps (and boundary values) of the original piecewise polynomial field, namely its tangential jumps (and boundary values) and the normal jumps (and boundary values) of its curl. All the estimates are local and $h$-optimal, and they are $p$-suboptimal by only $\frac{1}{2}$-order in the broken curl norm and by $\frac{3}{2}$-order in the $\boldsymbol{L}^2$-norm. Here, $h$ denotes the mesh size and $p$ the polynomial degree. We emphasize that, as an auxiliary result to establish the $\boldsymbol{L}^2$-norm estimate, we derive a novel broken curl, divergence-preserving Poincaré inequality on the mesh vertex patches that is of independent interest. We also observe that the $\boldsymbol{L}^2$-estimate can be improved to $\frac{1}{2}$-order $p$-suboptimality under a (reasonable) assumption on the uniform elliptic regularity pickup for a Poisson problem with Neumann conditions over the mesh vertex patches.

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We present an explicit construction of the (nonpolynomial) $\boldsymbol{H}(\mathbf{curl})$-conforming field over the mesh vertex patches using the partition of unity induced by hat basis functions in combination with local Helmholtz decompositions. The crucial advantage of resorting to a local construction over the mesh vertex patches is that the associated subdomains have a trivial topology, thereby circumventing any dependence of the estimates on the topology of the original domain. For simplicity, we devise the construction so that the (nonpolynomial) $\boldsymbol{H}(\mathbf{curl})$-conforming field satisfies homogeneous tangential boundary conditions, i.e., sits in $\boldsymbol{H}_0(\mathbf{curl}; D)$, but we also discuss the extension to more general mixed boundary conditions. Another relevant extension is the case of agglomerated polytopal meshes, where the same $hp$-estimates as on simplicial meshes can be established by considering the underlying simplicial submesh under the assumption that this latter mesh has locally the same size as the original polytopal mesh. Finally, we also discuss a variant where the $\boldsymbol{H}(\mathbf{curl})$-conforming field is defined globally without considering the mesh vertex patches. On convex domains, this leads to a $p$-optimal $\boldsymbol{L}^2$-estimate without any further assumption, but the estimate is only global and the constant depends on the topology of the global domain.

Reconstructing $\boldsymbol{H}(\mathbf{curl})$-conforming fields from piecewise polynomial fields on a shape-regular simplicial mesh has already been quite explored in the literature. A first possibility is to devise a polynomial $\boldsymbol{H}(\mathbf{curl})$-reconstruction using Nédélec (edge) finite elements defined by prescribing the degrees of freedom as the averages of those of the original, piecewise polynomial field. Such averaging operators have been considered in [8], and a systematic analysis can be found in [22] with and without boundary prescription, leading to (local) $h$-optimal estimates in the broken curl and $\boldsymbol{L}^2$-norms (see also [6, 37] for the seminal work in the $H^1$-context). However, defining a reconstruction by prescription of degrees of freedom can hardly lead to $p$-optimal estimates because of the repeated use of discrete trace (and inverse) inequalities needed in the analysis (see [7] for further insight in the $H^1$-context). A second possibility is to solve local, discrete minimization problems using again Nédélec and Raviart–Thomas finite elements. Following the seminal work in [5, 26, 27] in the $H^1/\boldsymbol{H}(\mathrm{div})$-context, recent advances leading, in particular, to full $p$-optimality in the $\boldsymbol{H}(\mathbf{curl})$-context were accomplished in [13, 15]. [13] poses local problems on edge patches, but does not achieve $\boldsymbol{H}(\mathbf{curl})$-conformity. Instead, [15] poses local problems on vertex patches and achieves $\boldsymbol{H}(\mathbf{curl})$-conformity. However, in both works, the original piecewise polynomial field must satisfy a certain orthogonality property, typically associated with Galerkin's orthogonality when solving a curl-curl problem. We also refer the reader to [11] for a simplified construction of [15] and to [16] for the use of the technique from [15] to devise a stable, local, commuting operator in $\boldsymbol{H}(\mathbf{curl})$ with $hp$-optimal approximation properties. The above discussion shows that a gap remains in the literature when it comes to devising an $\boldsymbol{H}(\mathbf{curl})$-conforming reconstruction from an arbitrary piecewise polynomial field and ensuring $hp$-approximation properties.

The main application we have in mind for the present $\boldsymbol{H}(\mathbf{curl})$-reconstruction operator is the $hp$-a posteriori nonconforming error analysis of some simplified forms of Maxwell's equations involving the curl-curl operator and a zero-order term. In the case of elliptic problems, the idea of invoking an $H^1$-reconstruction operator based on a global Helmholtz decomposition to estimate the so-called nonconforming approximation error can be traced back to [18] in two dimensions and to [10] in three dimensions with Dirichlet boundary conditions (see also [4, 1], where the Helmholtz decomposition is also invoked). A further advance has been accomplished recently in [21] in the context of a discretization by the hybrid high-order method introduced in [20] for elliptic problems including mixed Dirichlet–Neumann boundary conditions.

Specifically, the $H^1$-reconstruction operator considered in [21] is devised locally on mesh vertex patches which all have a simple topology.

We consider the (symmetric) interior penalty discontinuous Galerkin (dG) method for space discretization. This approximation technique is classical for elliptic problems (see, e.g., [3, 19] and the references therein) and has been developed and analyzed in the context of Maxwell's equations in [38, 30]. Here, we perform the *hp*-a posteriori error analysis hinging on the present $\boldsymbol{H}_0(\mathbf{curl})$-reconstruction operator, focusing on homogeneous Dirichlet boundary conditions. This analysis offers, to our knowledge, two groundbreaking features: it is the first time that a full *hp*-analysis is achieved, and it is the first time that the analysis hinges on local (vertex patch) Helmholtz decompositions to estimate the nonconforming error. We emphasize that while the theoretical derivation of the error estimate relies on the $\boldsymbol{H}_0(\mathbf{curl})$-reconstruction operator, this reconstruction does not need to be computed in practice. Previous a posteriori error estimates bounded the nonconforming error using an averaging operator, as in [31, 32, 12], and were therefore only $h$-optimal. We notice, in passing, that [12] derived an *hp*-a posteriori bound on the conforming error and addressed the frequency-dependence of the a posteriori estimates in the indefinite regime by using a duality argument à la Schatz (see [39]).

The paper is organized as follows. In section 2, we briefly present the continuous and discrete settings. In section 3, we state our main results on the $\boldsymbol{H}_0(\mathbf{curl})$-reconstruction operator and comment on various possible extensions. The main result of this section is Theorem 3.7. In section 4, we establish some preparatory results. In particular, we highlight the local broken curl, divergence-preserving Poincaré inequality from Lemma 4.3. In section 5, we prove our main result. Finally, in section 6, we consider the approximation of Maxwell's equations by the (symmetric) interior penalty dG method and develop an *hp*-version a posteriori error analysis based on the $\boldsymbol{H}_0(\mathbf{curl})$-reconstruction operator devised in section 3. The main result of this section is Theorem 6.8.

## 2. Setting.

In this section, we briefly describe the continuous and discrete settings.

### 2.1. Continuous setting.

Let $D$ be an open, bounded, connected Lipschitz set (domain) in $\mathbb{R}^d$ with boundary $\partial D$ and outward unit normal $\boldsymbol{n}_D$. We assume that $D$ is a polyhedron so that it can be covered exactly by a simplicial mesh. To fix the ideas, we assume that $d=3$, but all of what follows readily extends to $d=2$. We do not make any simplifying assumption on the topology of $D$, i.e., $D$ may be multiply connected and its boundary be composed of several connected components.

We use boldface font for vectors, vector fields, and functional spaces composed of such fields. We use standard notation for Lebesgue and Sobolev spaces, and fractional-order Sobolev spaces are equipped with the standard Sobolev–Slobodeckii seminorm based on the double integral (see, e.g., [23, Definition 2.14]). To alleviate the notation, the inner product and associated norm in the spaces $L^2(D)$ and $\boldsymbol{L}^2(D)$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively, whereas we add a subscript $\omega$ when we consider the Lebesgue spaces over a subdomain $\omega \subset D$. We consider the Hilbert Sobolev spaces

$$\boldsymbol{H}(\mathbf{curl};\omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\omega) \mid \nabla\times\boldsymbol{v} \in \boldsymbol{L}^2(\omega)\}, \tag{2.1a}$$

$$\boldsymbol{H}_0(\mathbf{curl};\omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\omega) \mid \gamma^{\mathrm{c}}_{\partial\omega}(\boldsymbol{v}) = \boldsymbol{0}\}, \tag{2.1b}$$

$$\boldsymbol{H}(\mathrm{div};\omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\omega) \mid \nabla\cdot\boldsymbol{v} \in L^2(\omega)\}, \tag{2.1c}$$

$$\boldsymbol{H}_0(\mathrm{div};\omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\omega) \mid \gamma^{\mathrm{d}}_{\partial\omega}(\boldsymbol{v}) = 0\}, \tag{2.1d}$$

where $\gamma^{\mathrm{c}}_{\partial\omega} : \boldsymbol{H}(\mathbf{curl};\omega) \to \boldsymbol{H}^{-\frac{1}{2}}(\partial\omega)$ (resp., $\gamma^{\mathrm{d}}_{\partial\omega} : \boldsymbol{H}(\mathrm{div};\omega) \to H^{-\frac{1}{2}}(\partial\omega)$) is the extension by density of the tangential (resp., normal) trace operator such that $\gamma^{\mathrm{c}}_{\partial\omega}(\boldsymbol{v}) = \boldsymbol{v}|_{\partial\omega}\times\boldsymbol{n}_\omega$ (resp., $\gamma^{\mathrm{d}}_{\partial\omega}(\boldsymbol{v}) = \boldsymbol{v}|_{\partial\omega}\cdot\boldsymbol{n}_\omega$) for smooth fields. The subscript $_0$ is used for the nabla operator to indicate that the operator acts on functions or fields respecting suitable homogeneous Dirichlet conditions. Thus, $(\nabla, -\nabla_0\cdot)$, $(\nabla\times, \nabla_0\times)$, and $(\nabla\cdot, -\nabla_0)$ are pairs of adjoint operators; for instance, $(\nabla_0\times\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v}, \nabla\times\boldsymbol{w})$ for all $(\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{H}_0(\mathbf{curl}; D) \times \boldsymbol{H}(\mathbf{curl}; D)$.

**2.2. Mesh and polynomial spaces.** Let $\mathcal{T}_h$ be an affine simplicial mesh covering $D$ exactly. A generic mesh cell is denoted by $K$, its diameter by $h_K$, and its outward unit normal by $\boldsymbol{n}_K$. We write $\mathcal{F}_h$ for the set of mesh faces, $\mathcal{F}_h^{\mathrm{int}}$ for the subset of mesh interfaces (shared by two distinct mesh cells, $K_l$, $K_r$), and $\mathcal{F}_h^{\mathrm{bd}}$ for the subset of mesh boundary faces (shared by one mesh cell, $K_l$, and the boundary, $\partial D$). Every mesh interface $F \in \mathcal{F}_h^{\mathrm{int}}$ is oriented by the unit normal, $\boldsymbol{n}_F$, pointing from $K_l$ to $K_r$ (the orientation is arbitrary, but fixed). Every boundary face $F \in \mathcal{F}_h^{\mathrm{bd}}$ is oriented by the unit normal $\boldsymbol{n}_F := \boldsymbol{n}_D|_F$. The diameter of a generic face $F \in \mathcal{F}_h$ is denoted by $h_F$.

The set of mesh vertices is denoted by $\mathcal{V}_h$ and is decomposed into the subset of interior vertices, $\mathcal{V}_h^{\mathrm{int}}$, and the subset of boundary vertices, $\mathcal{V}_h^{\mathrm{bd}}$. For all $\mathbf{a} \in \mathcal{V}_h$, $\mathcal{T}_{\mathbf{a}}$ denotes the collection of the mesh cells which share $\mathbf{a}$ and $D_{\mathbf{a}}$ the corresponding open subdomain (often called vertex patch). The set of mesh faces belonging to $\overline{D_{\mathbf{a}}}$, say $\overline{\mathcal{F}}_{\boldsymbol{a}}$, is partitioned as $\overline{\mathcal{F}}_{\boldsymbol{a}} = \mathcal{F}_{\mathbf{a}} \cup \mathcal{F}_{\setminus\boldsymbol{a}}$, where $\mathcal{F}_{\mathbf{a}}$ is the collection of the mesh faces which share $\mathbf{a}$ and $\mathcal{F}_{\setminus\boldsymbol{a}}$ is the collection of all the mesh faces lying on $\partial D_{\mathbf{a}}$ and not containing $\mathbf{a}$. We notice that the two sets $\mathcal{F}_{\mathbf{a}}$ and $\mathcal{F}_{\setminus\boldsymbol{a}}$ are disjoint and that the set $\mathcal{F}_{\mathbf{a}}$ contains mesh boundary faces if $\mathbf{a}$ lies on the boundary. Furthermore, for all $\mathbf{a} \in \mathcal{V}_h$, we set $h_{\mathbf{a}} := \mathrm{diam}(D_{\mathbf{a}})$ and let $\psi_{\mathbf{a}}$ be the hat basis function equal to 1 at $\mathbf{a}$ and having support in $D_{\mathbf{a}}$. Recall that the hat basis functions satisfy the following partition-of-unity property:

$$\sum_{\mathbf{a}\in\mathcal{V}_h} \psi_{\mathbf{a}} = 1. \tag{2.2}$$

For all $K \in \mathcal{T}_h$, $\mathcal{F}_K$ (resp., $\mathcal{V}_K$) is the collection of the mesh faces composing $\partial K$ (resp., mesh vertices in $\partial K$). For all $F \in \mathcal{F}_K$ and every piecewise smooth field $\boldsymbol{v}$ on $\mathcal{T}_h$, we define the local trace operators such that $\gamma^{\mathrm{g}}_{K,F}(\boldsymbol{v})(\boldsymbol{x}) := \boldsymbol{v}|_K(\boldsymbol{x})$, $\gamma^{\mathrm{c}}_{K,F}(\boldsymbol{v})(\boldsymbol{x}) := \boldsymbol{v}|_K(\boldsymbol{x})\times\boldsymbol{n}_F$, $\gamma^{\mathrm{d}}_{K,F}(\boldsymbol{v})(\boldsymbol{x}) := \boldsymbol{v}_h|_K(\boldsymbol{x}) \cdot \boldsymbol{n}_F$ for a.e. $\boldsymbol{x} \in F$. Then, for all $F \in \mathcal{F}_h^{\mathrm{int}}$ and $\mathrm{x} \in \{\mathrm{g}, \mathrm{c}, \mathrm{d}\}$, we define the jump and average operators such that

$$[\![\boldsymbol{v}]\!]^{\mathrm{x}}_F := \gamma^{\mathrm{x}}_{K_l,F}(\boldsymbol{v}) - \gamma^{\mathrm{x}}_{K_r,F}(\boldsymbol{v}), \quad \{\!\{\boldsymbol{v}\}\!\}^{\mathrm{x}}_F := \frac{1}{2}\big(\gamma^{\mathrm{x}}_{K_l,F}(\boldsymbol{v}) + \gamma^{\mathrm{x}}_{K_r,F}(\boldsymbol{v})\big). \tag{2.3}$$

We also set $[\![\boldsymbol{v}]\!]^{\mathrm{x}}_F := \{\!\{\boldsymbol{v}\}\!\}^{\mathrm{x}}_F := \gamma^{\mathrm{x}}_{K_l,F}(\boldsymbol{v})$ for all $F \in \mathcal{F}_h^{\mathrm{bd}}$. For all $K \in \mathcal{T}_h$, we define the jump across the boundary of $K$ as $[\![\bullet]\!]^{\mathrm{x}}_{\partial K}|_F := \iota_{K,F}[\![\bullet]\!]^{\mathrm{x}}_F$ for all $F \in \mathcal{F}_K$, with $\iota_{K,F} := \boldsymbol{n}_K\cdot\boldsymbol{n}_F = \pm 1$. Let $\nabla_h$ denote the broken gradient operator acting elementwise.

Let $p \geq 1$ be the polynomial degree (see Remark 3.8 below for the case $p = 0$). Let $\mathbb{P}_{p,d}$ be the space composed of $d$-variate polynomials of total degree at most $p$ and set $\boldsymbol{\mathbb{P}}_{p,d} := [\mathbb{P}_{p,d}]^d$. We consider the broken polynomial space

$$\boldsymbol{P}^{\mathrm{b}}_p(\mathcal{T}_h) := \{\boldsymbol{v}_h \in \boldsymbol{L}^2(D) \mid \boldsymbol{v}_h|_K \in \boldsymbol{\mathbb{P}}_{p,d}\, \forall K \in \mathcal{T}_h\}, \tag{2.4}$$

and we denote the $\boldsymbol{L}^2$-orthogonal projection onto $\boldsymbol{P}^{\mathrm{b}}_p(\mathcal{T}_h)$ as

$$\boldsymbol{\Pi}^{\mathrm{b}}_h : \boldsymbol{L}^2(D) \to \boldsymbol{P}^{\mathrm{b}}_p(\mathcal{T}_h). \tag{2.5}$$

We use the notation $P^{\mathrm{b}}_p(\mathcal{T}_{\mathbf{a}})$ for the broken polynomial space based on the local mesh $\mathcal{T}_{\mathbf{a}}$ for a mesh vertex $\mathbf{a} \in \mathcal{V}_h$.

**2.3. *hp*-approximation tools.** Let $\kappa_{\mathcal{T}_h}$ denote the shape-regularity parameter of the mesh $\mathcal{T}_h$ and $C(\kappa_{\mathcal{T}_h})$ denote a generic constant solely depending on $\kappa_{\mathcal{T}_h}$ and on $d$ (but independent of the mesh size and the polynomial degree), and whose value can change at each occurrence. Sometimes, we also abbreviate as $A \lesssim B$ the inequality $A \leq C(\kappa_{\mathcal{T}_h})B$ for positive numbers $A$ and $B$.

LEMMA 2.1 (discrete trace inequality). *There exists $C(\kappa_{\mathcal{T}_h})$ so that, for all $v \in \mathbb{P}_{p,d}(K)$, all $K \in \mathcal{T}_h$, and all $p \geq 1$,*

$$\|v\|_{\partial K} \leq C(\kappa_{\mathcal{T}_h}) \left( \frac{p^2}{h_K} \right)^{\frac{1}{2}} \|v\|_K. \tag{2.6}$$

*Proof.* A proof can be found in [42] (with explicit constant in terms of $d$). ∎

LEMMA 2.2 (local $L^2$-orthogonal projection). *Let $\delta \in (0, \frac{1}{2}]$. There exists $C(\kappa_{\mathcal{T}_h})$ so that, for all $v \in H^{\frac{1}{2}+\delta}(K)$, all $K \in \mathcal{T}_h$, and all $p \geq 1$,*

$$\|v - \Pi_K^p(v)\|_{\partial K} \leq C(\kappa_{\mathcal{T}_h}) \left( \frac{h_K}{p} \right)^{\delta} |v|_{H^{\frac{1}{2}+\delta}(K)}, \tag{2.7}$$

*where $\Pi_K^p$ denotes the $L^2$-orthogonal projection onto $\mathbb{P}_{p,d}$.*

*Proof.* A proof can be found in [36, Corollary 1.2.]. ∎

LEMMA 2.3 (local modified *hp*-Karkulik–Melenk operator). *There exists $C(\kappa_{\mathcal{T}_h})$ such that, for all $\mathbf{a} \in \mathcal{V}_h$, there is a local interpolation operator $\mathcal{I}_{\mathbf{a}}^g : H^1(D_{\mathbf{a}}) \to P_p^{\mathrm{b}}(\mathcal{T}_{\mathbf{a}}) \cap H^1(D_{\mathbf{a}})$ such that for all $v \in H^1(D_{\mathbf{a}})$, all $K \in \mathcal{T}_{\mathbf{a}}$, and all $p \geq 1$,*

$$\left( \frac{p}{h_K} \right) \|v - \mathcal{I}_{\mathbf{a}}^g(v)\|_K + \left( \frac{p}{h_K} \right)^{\frac{1}{2}} \|v - \mathcal{I}_{\mathbf{a}}^g(v)\|_{\partial K} + \|\nabla \mathcal{I}_{\mathbf{a}}^g(v)\|_K \leq C(\kappa_{\mathcal{T}_h}) \|\nabla v\|_{D_{\mathbf{a}}}. \tag{2.8}$$

*Proof.* See [35, 33] (see also [21, Corollary 2.5] for using the $H^1$-seminorm on the right-hand side). ∎

**2.4. Jump lifting operator.** For every field $\boldsymbol{v}_h \in \boldsymbol{P}_{p+1}^{\mathrm{b}}(\mathcal{T}_h)$, $\nabla_h \times \boldsymbol{v}_h$ denotes the broken curl of $\boldsymbol{v}_h$ (evaluated cellwise). The reason we consider the space $\boldsymbol{P}_{p+1}^{\mathrm{b}}(\mathcal{T}_h)$ is that, in what follows, we shall consider the product of a hat basis function times a field in $\boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h)$. We define the jump lifting operator $\boldsymbol{L}_h^p : \boldsymbol{P}_{p+1}^{\mathrm{b}}(\mathcal{T}_h) \to \boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h)$ such that

$$(\boldsymbol{L}_h^p(\boldsymbol{v}_h), \boldsymbol{\phi}_h) := \sum_{F \in \mathcal{F}_h} ([\![\boldsymbol{v}_h]\!]_F^{\mathrm{c}}, \{\!\{\boldsymbol{\phi}_h\}\!\}_F^{\mathrm{g}})_{\boldsymbol{L}^2(F)} \quad \forall \boldsymbol{\phi}_h \in \boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h). \tag{2.9}$$

LEMMA 2.4 (bound on lifting operator). *There exists $C(\kappa_{\mathcal{T}_h})$ so that for all $\boldsymbol{v}_h \in \boldsymbol{P}_{p+1}^{\mathrm{b}}(\mathcal{T}_h)$, all $K \in \mathcal{T}_h$, and all $p \geq 1$,*

$$\|\boldsymbol{L}_h^p(\boldsymbol{v}_h)\|_K \leq C(\kappa_{\mathcal{T}_h}) \left\{ \sum_{F \in \mathcal{F}_K} \left( \frac{p^2}{h_K} \right) \|[\![\boldsymbol{v}_h]\!]_F^{\mathrm{c}}\|_F^2 \right\}^{\frac{1}{2}}. \tag{2.10}$$

*Proof.* Apply the Cauchy–Schwarz inequality and the discrete trace inequality (2.6). ∎

**3. Main results on $\boldsymbol{H}(\mathbf{curl})$-reconstruction.** This section presents, and comments on extensions of, our main result on $\boldsymbol{H}_0(\mathbf{curl})$-reconstruction. Specifically, we construct an operator

$$\boldsymbol{R}^{\mathrm{c}} : \boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h) \to \boldsymbol{H}_0(\mathbf{curl}; D), \tag{3.1}$$

such that, for all $\boldsymbol{E}_h \in \boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h)$, the error $\boldsymbol{E}_h - \boldsymbol{R}^{\mathrm{c}}(\boldsymbol{E}_h)$ is bounded in the broken curl norm and in the $\boldsymbol{L}^2$-norm by suitable jumps of $\boldsymbol{E}_h$ and $\nabla_h \times \boldsymbol{E}_h$ across the mesh interfaces and by suitable traces on the mesh boundary faces.

The devising of $\boldsymbol{R}^{\mathrm{c}}(\boldsymbol{E}_h)$ combines local (nonpolynomial) constructions on the vertex patches associated with each mesh vertex $\mathbf{a} \in \mathcal{V}_h$. For all $\mathbf{a} \in \mathcal{V}_h$, we define the following functional spaces:

$$\boldsymbol{X}_0^{\mathrm{c}}(D_{\mathbf{a}}) := \boldsymbol{H}_0(\mathbf{curl}; D_{\mathbf{a}}) \cap \boldsymbol{H}(\mathrm{div}=0; D_{\mathbf{a}}), \tag{3.2a}$$

$$\boldsymbol{X}^{\mathrm{c}}(D_{\mathbf{a}}) := \boldsymbol{H}(\mathbf{curl}; D_{\mathbf{a}}) \cap \boldsymbol{H}_0(\mathrm{div}=0; D_{\mathbf{a}}), \tag{3.2b}$$

where $\boldsymbol{H}(\mathrm{div}=0; D_{\mathbf{a}})$ (resp., $\boldsymbol{H}_0(\mathrm{div}=0; D_{\mathbf{a}})$) is the subspace of $\boldsymbol{H}(\mathrm{div}; D_{\mathbf{a}})$ (resp., $\boldsymbol{H}_0(\mathrm{div}; D_{\mathbf{a}})$) composed of divergence-free fields. We also set

$$X_0^{\mathrm{g}}(D_{\mathbf{a}}) := H_0^1(D_{\mathbf{a}}), \tag{3.3a}$$

$$X^{\mathrm{g}}(D_{\mathbf{a}}) := H^1(D_{\mathbf{a}})/\mathbb{R}. \tag{3.3b}$$

Definition 3.1 (patchwise and global $\boldsymbol{H}_0(\mathbf{curl})$-reconstruction). *Let $\boldsymbol{E}_h \in \boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h)$. For all $\mathbf{a} \in \mathcal{V}_h$, let $\boldsymbol{U}_{\mathbf{a}} \in \boldsymbol{X}_0^{\mathrm{c}}(D_{\mathbf{a}})$ solve the well-posed problem*

(3.4)
$$(\nabla_0 \times \boldsymbol{U}_{\mathbf{a}}, \nabla_0 \times \boldsymbol{W}_{\mathbf{a}})_{D_{\mathbf{a}}} = (\nabla_h \times (\psi_{\mathbf{a}} \boldsymbol{E}_h) + \boldsymbol{L}_h^p(\psi_{\mathbf{a}} \boldsymbol{E}_h), \nabla_0 \times \boldsymbol{W}_{\mathbf{a}})_{D_{\mathbf{a}}} \quad \forall \boldsymbol{W}_{\mathbf{a}} \in \boldsymbol{X}_0^{\mathrm{c}}(D_{\mathbf{a}}),$$

*and let $\theta_{\mathbf{a}} \in X_0^{\mathrm{g}}(D_{\mathbf{a}})$ solve the following well-posed problem:*

$$(\nabla \theta_{\mathbf{a}}, \nabla v_{\mathbf{a}})_{D_{\mathbf{a}}} = (\psi_{\mathbf{a}} \boldsymbol{E}_h, \nabla v_{\mathbf{a}})_{D_{\mathbf{a}}} \quad \forall v_{\mathbf{a}} \in X_0^{\mathrm{g}}(D_{\mathbf{a}}). \tag{3.5}$$

*Set*

$$\boldsymbol{E}_{\mathbf{a}} := \boldsymbol{U}_{\mathbf{a}} + \nabla \theta_{\mathbf{a}}, \tag{3.6}$$

*and extending $\boldsymbol{E}_{\mathbf{a}}$ by zero to $D$, set*

$$\boldsymbol{R}^{\mathrm{c}}(\boldsymbol{E}_h) := \sum_{\mathbf{a} \in \mathcal{V}_h} \boldsymbol{E}_{\mathbf{a}}. \tag{3.7}$$

*Remark* 3.2 (local problems). As the vertex patch $D_{\mathbf{a}}$ is a simply connected Lipschitz domain with connected boundary, the local problems (3.4) and (3.5) are well-posed. Moreover, an equivalent definition is

$$\begin{aligned}
\boldsymbol{U}_{\mathbf{a}} &:= \arg \min_{\boldsymbol{\rho}_{\mathbf{a}} \in \boldsymbol{X}_0^{\mathrm{c}}(D_{\mathbf{a}})} \|\nabla_0 \times \boldsymbol{\rho}_{\mathbf{a}} - \{\nabla_h \times (\psi_{\mathbf{a}} \boldsymbol{E}_h) + \boldsymbol{L}_h^p(\psi_{\mathbf{a}} \boldsymbol{E}_h)\}\|_{D_{\mathbf{a}}}, \\
\theta_{\mathbf{a}} &:= \arg \min_{\xi_{\mathbf{a}} \in X_0^{\mathrm{g}}(D_{\mathbf{a}})} \|\nabla \xi_{\mathbf{a}} - \psi_{\mathbf{a}} \boldsymbol{E}_h\|_{D_{\mathbf{a}}}.
\end{aligned}$$

A simple but crucial observation is that $\boldsymbol{E}_{\mathbf{a}} \in \boldsymbol{H}_0(\mathbf{curl}; D_{\mathbf{a}})$ so that

$$\boldsymbol{R}^{\mathrm{c}}(\boldsymbol{E}_h) \in \boldsymbol{H}_0(\mathbf{curl}; D). \tag{3.8}$$

We now set

$$\boldsymbol{\delta}_{\mathbf{a}} := \boldsymbol{E}_{\mathbf{a}} - \psi_{\mathbf{a}} \boldsymbol{E}_h \quad \forall \mathbf{a} \in \mathcal{V}_h \tag{3.9}$$

and estimate $\boldsymbol{\delta}_{\mathbf{a}}$ in the broken curl norm and in the $\boldsymbol{L}^2$-norm.

LEMMA 3.3 (local bound in broken curl norm). *There exists $C(\kappa_{\mathcal{T}_h})$ so that, for all $\boldsymbol{E}_h \in \boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h)$ and all $\mathbf{a} \in \mathcal{V}_h$,*

$$\|\nabla_h \times \boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}} \leq C(\kappa_{\mathcal{T}_h}) \left\{ \sum_{F \in \mathcal{F}_{\mathbf{a}}} \left(\frac{h_F}{p}\right) \|[\![\nabla_h \times \boldsymbol{E}_h]\!]_F^{\mathrm{d}}\|_F^2 + \left(\frac{p^2}{h_F}\right) \|[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2 \right\}^{\frac{1}{2}}. \tag{3.10}$$

*Proof.* The proof can be found in section 5.1. ◻

LEMMA 3.4 (local $\boldsymbol{L}^2$-bound using broken curl, divergence-preserving Poincaré inequality). *There exists $C(\kappa_{\mathcal{T}_h})$ so that, for all $\boldsymbol{E}_h \in \boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h)$ and all $\mathbf{a} \in \mathcal{V}_h$,*

$$\|\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}} \leq C(\kappa_{\mathcal{T}_h}) p \left\{ \sum_{F \in \mathcal{F}_{\mathbf{a}}} \left(\frac{h_F}{p}\right)^3 \|[\![\nabla_h \times \boldsymbol{E}_h]\!]_F^{\mathrm{d}}\|_F^2 + h_F \|[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2 \right\}^{\frac{1}{2}}. \tag{3.11}$$

*Proof.* The proof can be found in section 5.2. ◻

The $\boldsymbol{L}^2$-estimate from Lemma 3.4 can be improved by invoking the elliptic regularity pickup of the Poisson problem with homogeneous Neumann boundary conditions on the vertex patches. The improvement consists in removing the broken curl contribution from (3.11) and also the global scaling factor $p$ in front of the opening brace. We need, however, a uniform estimate of the underlying constants over all the vertex patches in the mesh. It is natural to expect that these constants can be bounded in terms of the mesh shape-regularity. As we did not find a precise result in the literature supporting this claim, we formulate it as an assumption. We hope that this will stimulate further research in this direction to prove (or disprove) the assumption.

*Assumption* 3.5 (elliptic regularity pickup for Neumann problem). There exist $C(\kappa_{\mathcal{T}_h})$ and $\delta := \delta(\kappa_{\mathcal{T}_h}) > 0$ so that, for all $\mathbf{a} \in \mathcal{V}_h$ and all $\boldsymbol{g} \in \boldsymbol{X}^{\mathrm{g}}(D_{\mathbf{a}}) := [X^{\mathrm{g}}(D_{\mathbf{a}})]^d$, the unique solution $\zeta_{\boldsymbol{g}} \in X^{\mathrm{g}}(D_{\mathbf{a}})$ of the Neumann problem $(\nabla \zeta_{\boldsymbol{g}}, \nabla \rho)_{D_{\mathbf{a}}} = (\boldsymbol{g}, \nabla \rho)_{D_{\mathbf{a}}}$ for all $\rho \in X^{\mathrm{g}}(D_{\mathbf{a}})$ (i.e., $\Delta \zeta_{\boldsymbol{g}} = \nabla \cdot \boldsymbol{g}$ in $D_{\mathbf{a}}$ and $\boldsymbol{n}_{D_{\mathbf{a}}} \cdot \nabla \zeta_{\boldsymbol{g}} = \boldsymbol{n}_{D_{\mathbf{a}}} \cdot \boldsymbol{g}$ on $\partial D_{\mathbf{a}}$) satisfies the estimate

$$\|\zeta_{\boldsymbol{g}}\|_{D_{\mathbf{a}}} + h_{\mathbf{a}}^{\frac{3}{2}+\delta} |\nabla \zeta_{\boldsymbol{g}}|_{H^{\frac{1}{2}+\delta}(D_{\mathbf{a}})} \leq C(\kappa_{\mathcal{T}_h}) h_{\mathbf{a}}^2 \|\nabla \boldsymbol{g}\|_{D_{\mathbf{a}}}. \tag{3.12}$$

LEMMA 3.6 (local $\boldsymbol{L}^2$-bound using Sobolev embedding on vertex patches). *Under Assumption* 3.5, *there exists $C(\kappa_{\mathcal{T}_h})$ so that, for all $\boldsymbol{E}_h \in \boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h)$ and all $\mathbf{a} \in \mathcal{V}_h$,*

$$\|\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}} \leq C(\kappa_{\mathcal{T}_h}) \left\{ \sum_{F \in \mathcal{F}_{\mathbf{a}}} h_F \|[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2 \right\}^{\frac{1}{2}}. \tag{3.13}$$

*Proof.* The proof can be found in section 5.3. ◻

We are now ready to state global estimates on $\boldsymbol{R}^{\mathrm{c}}(\boldsymbol{E}_h) - \boldsymbol{E}_h$.

THEOREM 3.7 (bound on $\boldsymbol{R}^{\mathrm{c}}(\boldsymbol{E}_h) - \boldsymbol{E}_h$). *There exists $C(\kappa_{\mathcal{T}_h})$ so that, for all $\boldsymbol{E}_h \in \boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h)$,*

$$\begin{aligned} \|\nabla_h \times (\boldsymbol{R}^{\mathrm{c}}(\boldsymbol{E}_h) - \boldsymbol{E}_h)\| \leq C(\kappa_{\mathcal{T}_h}) \Bigg\{ & \sum_{K \in \mathcal{T}_h} \left(\frac{h_K}{p}\right) \|[\![\nabla_h \times \boldsymbol{E}_h]\!]_{\partial K}^{\mathrm{d}}\|_{\partial K}^2 \\ & + \left(\frac{p^2}{h_K}\right) \|[\![\boldsymbol{E}_h]\!]_{\partial K}^{\mathrm{c}}\|_{\partial K}^2 \Bigg\}^{\frac{1}{2}}, \end{aligned} \tag{3.14a}$$

(3.14b)
$$\|\boldsymbol{R}^{\mathrm{c}}(\boldsymbol{E}_h)-\boldsymbol{E}_h\| \leq C(\kappa_{\mathcal{T}_h})p\left\{\sum_{K\in\mathcal{T}_h}\left(\frac{h_K}{p}\right)^3 \|[\![\nabla_h\times\boldsymbol{E}_h]\!]^{\mathrm{d}}_{\partial K}\|^2_{\partial K}+h_K\|[\![\boldsymbol{E}_h]\!]^{\mathrm{c}}_{\partial K}\|^2_{\partial K}\right\}^{\frac{1}{2}}.$$

*Moreover, under Assumption* 3.5, *we have*

$$\|\boldsymbol{R}^{\mathrm{c}}(\boldsymbol{E}_h)-\boldsymbol{E}_h\| \leq C(\kappa_{\mathcal{T}_h})\left\{\sum_{K\in\mathcal{T}_h} h_K\|[\![\boldsymbol{E}_h]\!]^{\mathrm{c}}_{\partial K}\|^2_{\partial K}\right\}^{\frac{1}{2}}. \tag{3.14c}$$

*Proof.* The bounds (3.14) readily follow from the local estimates established in Lemmas 3.3, 3.4, and 3.6 once we observe that, for all $K\in\mathcal{T}_h$, $(\boldsymbol{R}^{\mathrm{c}}(\boldsymbol{E}_h)-\boldsymbol{E}_h)|_K = \sum_{\mathbf{a}\in\mathcal{V}_K}\boldsymbol{\delta}_{\mathbf{a}}|_K$ owing to the partition-of-unity property (2.2). Invoking the triangle inequality and the mesh shape-regularity concludes the proof. ∎

*Remark* 3.8 ($p=0$). The operator $\boldsymbol{R}^{\mathrm{c}}$ can also be employed on $\boldsymbol{P}^{\mathrm{b}}_0(\mathcal{T}_h)\subset\boldsymbol{P}^{\mathrm{b}}_1(\mathcal{T}_h)$.

*Remark* 3.9 ($p$-suboptimality). The $\frac{1}{2}$-order suboptimality in (3.10) stems from the bound (2.10) on the jump lifting operator (which itself follows from the discrete trace inequality (2.6)). The $\frac{3}{2}$-order suboptimality in (3.11) inherits the $\frac{1}{2}$-order suboptimality from (3.10) to which a one-order suboptimality originating from the broken curl, divergence-preserving Poincaré inequality is added. As this latter inequality does not involve discrete functions, no gain in the polynomial degree can be achieved. Finally, the $\frac{1}{2}$-order suboptimality in (3.13) stems from a Sobolev embedding invoked in a duality argument, which circumvents the need to invoke the broken curl estimate (3.10).

*Remark* 3.10 (agglomerated polytopal meshes). Let $\mathcal{T}_h$ be a simplicial background mesh covering $D$ exactly and let $\tilde{\mathcal{T}}_H$ be an agglomerated polytopal mesh built from $\mathcal{T}_h$ in such a way that every $\tilde{K}\in\tilde{\mathcal{T}}_H$ is the union of a uniformly bounded number of simplicial cells $K\in\mathcal{T}_{\tilde{K}}\subset\mathcal{T}_h$; see Figure 1 for an illustration. The above assumption on the mesh implies that there exists a constant $C_{\mathrm{sh}}\geq 1$ such that

$$h_K\leq h_{\tilde{K}}\leq C_{\mathrm{sh}}h_K \quad \forall K\in\mathcal{T}_{\tilde{K}}. \tag{3.15}$$

Since $\boldsymbol{P}^{\mathrm{b}}_p(\tilde{\mathcal{T}}_H)\subset\boldsymbol{P}^{\mathrm{b}}_p(\mathcal{T}_h)$, an $\boldsymbol{H}_0(\mathbf{curl})$-reconstruction of any field $\boldsymbol{E}_h\in\boldsymbol{P}^{\mathrm{b}}_p(\tilde{\mathcal{T}}_H)$ can be devised by considering the $\boldsymbol{H}_0(\mathbf{curl})$-reconstruction operator defined on $\boldsymbol{P}^{\mathrm{b}}_p(\mathcal{T}_h)$. Since $\boldsymbol{E}_h\in\boldsymbol{P}^{\mathrm{b}}_p(\tilde{\mathcal{T}}_H)$ and its broken curl only jump across the interfaces on $\mathcal{T}_h$ that

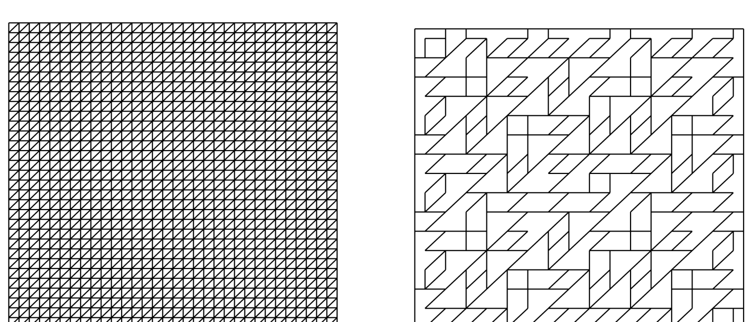

FIG. 1. *Triangular mesh $\mathcal{T}_h$ consisting of* 2,048 *elements (left); agglomerated polygonal mesh $\tilde{\mathcal{T}}_H$ consisting of* 514 *elements (right).*

are subsets of interfaces of $\tilde{\mathcal{T}}_H$, the error $\boldsymbol{R}^{\mathrm{c}}(\boldsymbol{E}_h) - \boldsymbol{E}_h$ can still be bounded as in (3.14), both in the broken curl norm and in the $\boldsymbol{L}^2$-norm, where the summations on the right-hand side run over the mesh cells in $\tilde{\mathcal{T}}_H$ and the local mesh size $h_{\tilde{K}}$ can be used owing to (3.15). Notice that the condition (3.15) does not accommodate polytopal meshes with tiny faces. This case is left to future work. (We mention [9], which derives a posteriori estimates for elliptic problems on polytopal meshes with tiny faces, but this work assumes that the domain is simply connected and does not rely on a partition of unity based on a simplicial background mesh.)

*Remark* 3.11 (mixed boundary conditions). The $\boldsymbol{H}_0(\mathbf{curl})$-reconstruction from Definition 3.1 can be adapted with minor modifications to handle mixed boundary conditions. Let $\Gamma_{\mathrm{D}} \subset \partial D$ and $\Gamma_{\mathrm{N}} := \partial D \backslash \Gamma_{\mathrm{D}}$, with $|\Gamma_{\mathrm{N}}| \neq 0$, be Lipschitz subsets with connected interior that can be covered exactly by mesh faces. Let $\boldsymbol{H}_{\Gamma_{\mathrm{D}}}(\mathbf{curl}; D)$ be the subspace of $\boldsymbol{H}(\mathbf{curl}; D)$ prescribing a homogeneous Dirichlet boundary condition only on $\Gamma_{\mathrm{D}}$. To devise a global $\boldsymbol{H}_{\Gamma_{\mathrm{D}}}(\mathbf{curl})$-reconstruction, the local problems (3.4) and (3.5) defined on each vertex patch are now posed on the function spaces $\boldsymbol{X}^{\mathrm{c}}_{\Gamma_{\mathrm{D}}}(D_{\mathbf{a}}) := \boldsymbol{H}_{\partial D_{\mathbf{a}} \backslash \Gamma_{\mathrm{N}}}(\mathbf{curl}; D_{\mathbf{a}}) \cap \boldsymbol{H}_{\partial D_{\mathbf{a}} \cap \Gamma_{\mathrm{N}}}(\mathrm{div} = 0; D_{\mathbf{a}})$ (where the subscripts indicate where the tangential and the normal boundary condition is prescribed, respectively) and $X^{\mathrm{g}}_{\Gamma_{\mathrm{D}}}(D_{\mathbf{a}}) := H^1_{\partial D_{\mathbf{a}} \backslash \Gamma_{\mathrm{N}}}(D_{\mathbf{a}})$ for all $\mathbf{a} \in \mathcal{V}_h$. Then one sets again $\boldsymbol{E}_{\mathbf{a}} := \boldsymbol{U}_{\mathbf{a}} + \nabla \theta_{\mathbf{a}}$, and extending $\boldsymbol{E}_{\mathbf{a}}$ by zero to $D$, one sets $\boldsymbol{R}^{\mathrm{c}}(\boldsymbol{E}_h) := \sum_{\mathbf{a} \in \mathcal{V}_h} \boldsymbol{E}_{\mathbf{a}}$, so that $\boldsymbol{R}^{\mathrm{c}}(\boldsymbol{E}_h) \in \boldsymbol{H}_{\Gamma_{\mathrm{D}}}(\mathbf{curl}; D)$. The local broken curl estimate now relies on the local Helmholtz decomposition with mixed boundary conditions discussed in Remark 4.2 below. The estimate reads as follows:

$$\|\nabla_h \times \boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}} \leq C(\kappa_{\mathcal{T}_h}) \left\{ \sum_{F \in \mathcal{F}_{\mathbf{a}} \backslash \mathcal{F}_h^{\mathrm{N}}} \left( \frac{h_F}{p} \right) \|[\![\nabla_h \times \boldsymbol{E}_h]\!]^{\mathrm{d}}_F\|_F^2 + \left( \frac{p^2}{h_F} \right) \|[\![\boldsymbol{E}_h]\!]^{\mathrm{c}}_F\|_F^2 \right\}^{\frac{1}{2}},$$

where $\mathcal{F}_h^{\mathrm{N}}$ denotes the subset of mesh boundary faces lying on $\Gamma_{\mathrm{N}}$. The local $\boldsymbol{L}^2$-error estimate relies on the broken Poincaré with mixed boundary conditions discussed in Remark 4.5 below (which requires one more conjecture, unless pure Neumann conditions are enforced). The estimate reads as follows:

$$\|\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}} \leq C(\kappa_{\mathcal{T}_h}) p \left\{ \sum_{F \in \mathcal{F}_{\mathbf{a}} \backslash \mathcal{F}_h^{\mathrm{N}}} \left( \frac{h_F}{p} \right)^3 \|[\![\nabla_h \times \boldsymbol{E}_h]\!]^{\mathrm{d}}_F\|_F^2 + h_F \|[\![\boldsymbol{E}_h]\!]^{\mathrm{c}}_F\|_F^2 \right\}^{\frac{1}{2}}.$$

Finally, an extension of the bound from Lemma 3.6 to mixed boundary conditions is not expected to hold because the elliptic regularity pickup stated in Assumption 3.5 is not expected to hold.

*Remark* 3.12 (globally defined $\boldsymbol{H}_0(\mathbf{curl})$-reconstruction and convex domains). One can also consider a globally defined $\boldsymbol{H}_0(\mathbf{curl})$-reconstruction operator, say $\tilde{\boldsymbol{R}}^{\mathrm{c}} : \boldsymbol{P}^{\mathrm{b}}_p(\mathcal{T}_h) \to \boldsymbol{H}_0(\mathbf{curl}; D)$, without posing auxiliary problems on vertex patches exploiting the partition of unity. Assume first that $D$ has trivial topology. Then, one considers $\boldsymbol{U}_D \in \boldsymbol{X}^{\mathrm{c}}_0(D) := \boldsymbol{H}_0(\mathbf{curl}; D) \cap \boldsymbol{H}(\mathrm{div} = 0; D)$ solving the well-posed problem

$$(\nabla_0 \times \boldsymbol{U}_D, \nabla_0 \times \boldsymbol{W}_D) = (\nabla_h \times \boldsymbol{E}_h + \boldsymbol{L}^p_h(\boldsymbol{E}_h), \nabla_0 \times \boldsymbol{W}_D) \quad \forall \boldsymbol{W}_D \in \boldsymbol{X}^{\mathrm{c}}_0(D),$$

and $\theta_D \in H^1_0(D)$ solving the well-posed problem

$$(\nabla \theta_D, \nabla v_D) = (\boldsymbol{E}_h, \nabla v_D) \quad \forall v_D \in H^1_0(D),$$

and one sets $\tilde{\boldsymbol{R}}^{\rm c}(\boldsymbol{E}_h) := \boldsymbol{U}_D + \nabla\theta_D$. The error $\tilde{\boldsymbol{R}}^{\rm c}(\boldsymbol{E}_h) - \boldsymbol{E}_h$ can again be bounded by the right-hand sides in (3.14), both in the broken curl norm and in the $\boldsymbol{L}^2$-norm. The proofs, which follow arguments similar to those presented in section 5, are omitted for brevity. One advantage of the globally defined $\boldsymbol{H}_0(\mathbf{curl})$-reconstruction operator is achieved when the domain $D$ is convex. Indeed, in this case, full elliptic regularity pickup can be invoked when deriving the $\boldsymbol{L}^2$-error bound, leading to the $hp$-optimal estimate

$$\|\tilde{\boldsymbol{R}}^{\rm c}(\boldsymbol{E}_h) - \boldsymbol{E}_h\| \le C(\kappa_{\mathcal{T}_h}) \left\{ \sum_{K\in\mathcal{T}_h} \left(\frac{h_K}{p}\right) \|[\![\boldsymbol{E}_h]\!]^{\rm c}_{\partial K}\|^2_{\partial K} \right\}^{\frac{1}{2}}.$$

However, the broken curl norm bound cannot be improved and remains $p$-suboptimal by $\frac{1}{2}$-order. Finally, the above global construction can also be deployed if the domain $D$ has a more general topology upon posing the problem defining $\boldsymbol{U}_D$ in $\boldsymbol{H}_0(\mathbf{curl}; D) \cap \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}$, where $\boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)$ is the subspace of $\boldsymbol{H}_0(\mathbf{curl}; D)$ composed of curl-free fields and orthogonalities are meant in $\boldsymbol{L}^2$. However, the constants in the bounds on $\tilde{\boldsymbol{R}}^{\rm c}(\boldsymbol{E}_h) - \boldsymbol{E}_h$ now depend on the topology of $D$.

**4. Preparatory results.** In this section, we collect three preparatory results. The first result is a (classical) local Helmholtz decomposition to be invoked in the proof of Lemma 3.3. The second one is a local broken curl, divergence-preserving Poincaré inequality to be invoked in the proof of Lemma 3.4. Finally, the third one is a local Sobolev embedding to be invoked in the proof of Lemma 3.6.

LEMMA 4.1 (local Helmholtz decomposition). *For all $\mathbf{a} \in \mathcal{V}_h$ and all $\boldsymbol{v} \in \boldsymbol{L}^2(D_{\mathbf{a}})$, there exist $\boldsymbol{\psi} \in \boldsymbol{X}^{\rm c}_0(D_{\mathbf{a}})$ (see (3.2a)) and $\xi \in X^{\rm g}(D_{\mathbf{a}})$ (see (3.3)), so that*

$$\boldsymbol{v} = \nabla_0 \times \boldsymbol{\psi} + \nabla\xi \quad \text{in } D_{\mathbf{a}}, \tag{4.1a}$$

$$\|\boldsymbol{v}\|^2_{D_{\mathbf{a}}} = \|\nabla_0 \times \boldsymbol{\psi}\|^2_{D_{\mathbf{a}}} + \|\nabla\xi\|^2_{D_{\mathbf{a}}}. \tag{4.1b}$$

*Proof.* For completeness, we sketch a short proof. We solve the Neumann problem $(\nabla\xi, \nabla w)_{D_{\mathbf{a}}} = (\boldsymbol{v}, \nabla w)_{D_{\mathbf{a}}}$ in $X^{\rm g}(D_{\mathbf{a}})$ and observe that $\nabla\xi - \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div} = 0; D_{\mathbf{a}})$. Using [29, Theorem 3], we infer that there exists $\boldsymbol{\zeta} \in \boldsymbol{H}_0(\mathbf{curl}; D_{\mathbf{a}})$ such that $\nabla_0 \times \boldsymbol{\zeta} = \boldsymbol{v} - \nabla\xi$. Then, we solve the Dirichlet problem $(\nabla\gamma, \nabla w)_{D_{\mathbf{a}}} = (\boldsymbol{\zeta}, \nabla w)_{D_{\mathbf{a}}}$ in $X^{\rm g}_0(D_{\mathbf{a}})$ and set $\boldsymbol{\psi} = \boldsymbol{\zeta} - \nabla\gamma$. The pair $(\boldsymbol{\psi}, \xi)$ satisfies all the required properties. ◻

*Remark* 4.2 (mixed boundary conditions). For mixed boundary conditions, Lemma 4.1 is adapted as follows: For all $\mathbf{a} \in \mathcal{V}_h$ and all $\boldsymbol{v} \in \boldsymbol{L}^2(D_{\mathbf{a}})$, there exist $\boldsymbol{\psi} \in \boldsymbol{X}^{\rm c}_{\Gamma_{\rm D}}(D_{\mathbf{a}})$ and $\xi \in H^1_{\partial D_{\mathbf{a}} \cap \Gamma_{\rm N}}(D_{\mathbf{a}})$, so that (4.1) still holds true. To prove this result, one solves Poisson problems with mixed boundary conditions.

LEMMA 4.3 (local broken curl, divergence-preserving Poincaré inequality). *There exists a constant $C(\kappa_{\mathcal{T}_h})$ so that, for all $\mathbf{a} \in \mathcal{V}_h$, all $\boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{T}_{\mathbf{a}}) := \{\boldsymbol{w} \in \boldsymbol{L}^2(D_{\mathbf{a}}) \mid \nabla_h \boldsymbol{w} \in \boldsymbol{L}^2(D_{\mathbf{a}})\}$, and all $\boldsymbol{v}^{\rm c}_{\mathbf{a}} \in \boldsymbol{H}_0(\mathbf{curl}; D_{\mathbf{a}})$ with $\boldsymbol{v}^{\rm c}_{\mathbf{a}} - \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div} = 0; D_{\mathbf{a}})$, the following holds:*

$$\|\boldsymbol{v}^{\rm c}_{\mathbf{a}} - \boldsymbol{v}\|_{D_{\mathbf{a}}} \le C(\kappa_{\mathcal{T}_h}) h_{\mathbf{a}} \left\{ \|\nabla_h \times (\boldsymbol{v}^{\rm c}_{\mathbf{a}} - \boldsymbol{v})\|^2_{D_{\mathbf{a}}} + \sum_{F \in \overline{\mathcal{F}}_{\mathbf{a}}} h_F^{-1} \|[\![\boldsymbol{v}]\!]^{\rm c}_F\|^2_F \right\}^{\frac{1}{2}}, \tag{4.2}$$

*recalling that the jump operator at $F$ denotes the actual trace whenever $F \in \mathcal{F}^{\rm bd}_h$.*

*Proof.* Let $\mathbf{a} \in \mathcal{V}_h$, let $\boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{T}_{\mathbf{a}})$, and let $\boldsymbol{v}^{\rm c}_{\mathbf{a}} \in \boldsymbol{H}_0(\mathbf{curl}; D_{\mathbf{a}})$ be such that $\boldsymbol{v}^{\rm c}_{\mathbf{a}} - \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div} = 0; D_{\mathbf{a}})$. Recall that $X^{\rm g}(D_{\mathbf{a}}) := H^1(D_{\mathbf{a}})/\mathbb{R}$.

(1) Since $D_{\mathbf{a}}$ is a simply connected Lipschitz domain, we infer from [2, Lemma 3.5] that there exists $\boldsymbol{\phi} \in \boldsymbol{X}^{\rm g}(D_{\mathbf{a}}) := [X^{\rm g}(D_{\mathbf{a}})]^d$ such that $\nabla \times \boldsymbol{\phi} = \boldsymbol{v}^{\rm c}_{\mathbf{a}} - \boldsymbol{v}$ in $D_{\mathbf{a}}$ (notice that $\nabla \cdot \boldsymbol{\phi} = 0$ in $D_{\mathbf{a}}$ as well, but this additional property is not needed here). However,

the stability constant on $\|\nabla\boldsymbol{\phi}\|_{D_{\mathbf{a}}}$ is not known explicitly in terms of the geometry of $D_{\mathbf{a}}$. For interior vertices, the vertex patch is a star-shaped domain with respect to a ball whose radius is comparable to the size of the patch, with constant depending only on the mesh regularity. This result is established in [14, Proposition 8.2] in any dimension $d\geq 2$ by reduction to the case $d=2$ where the result is known from [34]. Invoking [28, Corollary 28] then gives $\boldsymbol{\phi}\in \boldsymbol{X}^{\mathrm{g}}(D_{\mathbf{a}})$ such that $\nabla\times\boldsymbol{\phi}=\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}}-\boldsymbol{v}$ in $D_{\mathbf{a}}$ and a stability constant $C(\kappa_{\mathcal{T}_h})$ only depending on the mesh shape-regularity so that the following bound holds:

$$\|\nabla\boldsymbol{\phi}\|_{D_{\mathbf{a}}}\leq C(\kappa_{\mathcal{T}_h})\|\nabla\times\boldsymbol{\phi}\|_{D_{\mathbf{a}}}=C(\kappa_{\mathcal{T}_h})\|\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}}-\boldsymbol{v}\|_{D_{\mathbf{a}}}. \tag{4.3}$$

For boundary vertices, the star-shapedness of the patch can be affected by the domain boundary so that the radius of the ball can deteriorate, e.g., near re-entrant corners. The way forward is to view each boundary vertex patch as a chain of star-shaped domains, and invoke [28, Theorem 35] to obtain again the bound (4.3). Following [27, Appendix B], we use the notion of shellability and enumerate the simplices in the patch as $\{K_i\}_{i=1}^{n}$ so that two successive simplices in the enumeration share a face, say $F_i=\partial K_i\cap\partial K_{i+1}$. We then define the subdomains $D_{\mathbf{a}}^i:=K_i\cup K_{i+1}$ for all $i\in\{1{:}n-1\}$. As a consequence of mesh shape-regularity, each $D_{\mathbf{a}}^i$ is star-shaped with respect to a $d$-dimensional ball centered at the barycenter of the face $F_i$, with radius comparable to $h_{F_i}$. Moreover, we have the overlap relation $K_{i+1}=D_{\mathbf{a}}^i\cap D_{\mathbf{a}}^{i+1}$. All the conditions stated at the beginning of [28, section 5] are thus fulfilled to consider the boundary vertex patch as a chain of star-shaped domains.

(2) Integrating by parts the curl operator in each cell $K\in\mathcal{T}_{\mathbf{a}}$, we infer that

$$\begin{aligned}\|\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}}-\boldsymbol{v}\|_{D_{\mathbf{a}}}^2&=(\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}}-\boldsymbol{v},\nabla\times\boldsymbol{\phi})_{D_{\mathbf{a}}}\\&=(\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}},\nabla\times\boldsymbol{\phi})_{D_{\mathbf{a}}}-(\boldsymbol{v},\nabla\times\boldsymbol{\phi})_{D_{\mathbf{a}}}\\&=-(\nabla_h\times(\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}}-\boldsymbol{v}),\boldsymbol{\phi})_{D_{\mathbf{a}}}-\sum_{K\in\mathcal{T}_{\mathbf{a}}}(\boldsymbol{v},\boldsymbol{n}_K\times\boldsymbol{\phi})_{\partial K}=:T_1+T_2,\end{aligned}$$

and it remains to bound $T_1$ and $T_2$.

(2a) Bound on $T_1$. Using the Cauchy–Schwarz inequality, recalling that $\boldsymbol{\phi}$ has zero mean-value over $D_{\mathbf{a}}$, and invoking the Poincaré inequality on the vertex patch (see [41, Theorem 3.2] and also [22, Lemma 5.7]), we infer that

$$|T_1|\lesssim h_{\mathbf{a}}\|\nabla_h\times(\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}}-\boldsymbol{v})\|_{D_{\mathbf{a}}}\|\nabla\boldsymbol{\phi}\|_{D_{\mathbf{a}}}\lesssim h_{\mathbf{a}}\|\nabla_h\times(\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}}-\boldsymbol{v})\|_{D_{\mathbf{a}}}\|\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}}-\boldsymbol{v}\|_{D_{\mathbf{a}}},$$

where the second bound follows from (4.3).

(2b) Bound on $T_2$. Reorganizing the summation over the mesh faces in $\overline{\mathcal{F}}_{\boldsymbol{a}}$ as those in $\overline{\mathcal{F}}_{\boldsymbol{a}}\cap\mathcal{F}_h^{\mathrm{int}}$ where $\boldsymbol{\phi}$ has continuous trace and those in $\overline{\mathcal{F}}_{\boldsymbol{a}}\cap\mathcal{F}_h^{\mathrm{bd}}$ where the jump operator denotes the actual trace, we obtain

$$T_2=\sum_{F\in\overline{\mathcal{F}}_{\boldsymbol{a}}}([\![\boldsymbol{v}]\!]_F^{\mathrm{c}},\boldsymbol{n}_F\times\boldsymbol{\phi})_F.$$

Invoking the same arguments as above together with the mesh shape-regularity and a multiplicative trace inequality (see, e.g., [23, Lemma 12.15]), we infer that

$$\begin{aligned}|T_2|&\leq\left\{\sum_{F\in\overline{\mathcal{F}}_{\boldsymbol{a}}}h_F^{-1}\|[\![\boldsymbol{v}]\!]_F^{\mathrm{c}}\|_F^2\right\}^{\frac12}\left\{\sum_{F\in\overline{\mathcal{F}}_{\boldsymbol{a}}}h_F\|\boldsymbol{\phi}\|_F^2\right\}^{\frac12}\\&\lesssim h_{\mathbf{a}}\left\{\sum_{F\in\overline{\mathcal{F}}_{\boldsymbol{a}}}h_F^{-1}\|[\![\boldsymbol{v}]\!]_F^{\mathrm{c}}\|_F^2\right\}^{\frac12}\|\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}}-\boldsymbol{v}\|_{D_{\mathbf{a}}}.\end{aligned}$$

Putting the bounds on $T_1$ and $T_2$ together proves (4.2). ∎

*Remark* 4.4 (extension). A simple inspection of the proof shows that Lemma 4.3 holds true for any $\boldsymbol{v} \in \boldsymbol{L}^2(D_{\mathbf{a}})$ with $\nabla_h \times \boldsymbol{v} \in \boldsymbol{L}^2(D_{\mathbf{a}})$ and $(\boldsymbol{n}_K \times \boldsymbol{v}|_K)|_F \in \boldsymbol{L}^2(F)$ for all $K \in \mathcal{T}_{\mathbf{a}}$ and all $F \in \mathcal{F}_K$.

*Remark* 4.5 (mixed boundary conditions). For mixed boundary conditions, the statement of Lemma 4.3 is adapted as follows: There exists a constant $C(\kappa_{\mathcal{T}_h})$ so that, for all $\mathbf{a} \in \mathcal{V}_h$, all $\boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{T}_{\mathbf{a}})$, and all $\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}} \in \boldsymbol{H}_{\partial D_{\mathbf{a}} \setminus \Gamma_{\mathrm{N}}}(\mathbf{curl}; D_{\mathbf{a}})$ with $\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}} - \boldsymbol{v} \in \boldsymbol{H}_{\partial D_{\mathbf{a}} \cap \Gamma_{\mathrm{N}}}(\mathrm{div} = 0; D_{\mathbf{a}})$, the following holds:

$$\|\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}} - \boldsymbol{v}\|_{D_{\mathbf{a}}} \leq C(\kappa_{\mathcal{T}_h}) h_{\mathbf{a}} \left\{ \|\nabla_h \times (\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}} - \boldsymbol{v})\|_{D_{\mathbf{a}}}^2 + \sum_{F \in \overline{\mathcal{F}}_{\boldsymbol{a}} \setminus \mathcal{F}_h^{\mathrm{N}}} h_F^{-1} \|[\![\boldsymbol{v}]\!]_F^{\mathrm{c}}\|_F^2 \right\}^{\frac{1}{2}}.$$

The proof of this inequality is, however, not straightforward. First, [2, Lemma 3.5] can be replaced by [29, Theorem 3] on regular decompositions with mixed boundary conditions, but the stability constant is not given explicitly in terms of the geometry of $D_{\mathbf{a}}$. To have a stability constant only depending on the mesh shape-regularity, one needs to conjecture that the result of [28] can be extended to mixed boundary conditions. We notice that the extension holds for pure Neumann conditions [28, Theorem 32].

*Remark* 4.6 (local broken-divergence, curl-preserving Poincaré inequality). One can adapt the proof of Lemma 4.3 to establish the following local broken-divergence, curl-preserving Poincaré inequality: There exists a constant $C(\kappa_{\mathcal{T}_h})$ so that, for all $\mathbf{a} \in \mathcal{V}_h$, all $\boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{T}_{\mathbf{a}})$, and all $\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}} \in \boldsymbol{H}_0(\mathrm{div}; D_{\mathbf{a}})$ with $\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}} - \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl} = \mathbf{0}; D_{\mathbf{a}})$, the following holds:

$$\|\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}} - \boldsymbol{v}\|_{D_{\mathbf{a}}} \leq C(\kappa_{\mathcal{T}_h}) h_{\mathbf{a}} \left\{ \|\nabla_h \cdot (\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}} - \boldsymbol{v})\|_{D_{\mathbf{a}}}^2 + \sum_{F \in \overline{\mathcal{F}}_{\boldsymbol{a}}} h_F^{-1} \|[\![\boldsymbol{v}]\!]_F^{\mathrm{d}}\|_F^2 \right\}^{\frac{1}{2}}.$$

LEMMA 4.7 (local Sobolev embedding). *Under Assumption* 3.5 *with* $\delta := \delta(\kappa_{\mathcal{T}_h}) \in (0, \frac{1}{2}]$, *there exists a constant* $C(\kappa_{\mathcal{T}_h})$ *(depending on* $\delta$*) such that, for all* $\mathbf{a} \in \mathcal{V}_h$ *and all* $\boldsymbol{v} \in \boldsymbol{X}^{\mathrm{c}}(D_{\mathbf{a}})$ *(see* (3.2b)*),*

$$|\boldsymbol{v}|_{\boldsymbol{H}^{\frac{1}{2}+\delta}(K)} \leq C(\kappa_{\mathcal{T}_h}) h_{\mathbf{a}}^{\frac{1}{2}-\delta} \|\nabla \times \boldsymbol{v}\|_{D_{\mathbf{a}}} \quad \forall K \in \mathcal{T}_{\mathbf{a}}. \tag{4.4}$$

*Proof.* (1) Let $\mathbf{a} \in \mathcal{V}_h$ and $\boldsymbol{v} \in \boldsymbol{X}^{\mathrm{c}}(D_{\mathbf{a}})$. Invoking the same arguments as in step (1) of the proof of Lemma 4.3, we infer that there exists $\boldsymbol{\phi} \in \boldsymbol{X}^{\mathrm{g}}(D_{\mathbf{a}})$ such that $\nabla \times \boldsymbol{\phi} = \nabla \times \boldsymbol{v}$ in $D_{\mathbf{a}}$, and

$$\|\nabla \boldsymbol{\phi}\|_{D_{\mathbf{a}}} \leq C(\kappa_{\mathcal{T}_h}) \|\nabla \times \boldsymbol{\phi}\|_{D_{\mathbf{a}}} = C(\kappa_{\mathcal{T}_h}) \|\nabla \times \boldsymbol{v}\|_{D_{\mathbf{a}}}. \tag{4.5}$$

Since $\nabla \times (\boldsymbol{v} - \boldsymbol{\phi}) = \mathbf{0}$, there exists a function $p \in X^{\mathrm{g}}(D_{\mathbf{a}})$ such that $\nabla p = \boldsymbol{\phi} - \boldsymbol{v}$. Altogether, this leads to the regular decomposition

$$\boldsymbol{v} = -\nabla p + \boldsymbol{\phi}, \quad p \in X^{\mathrm{g}}(D_{\mathbf{a}}), \quad \boldsymbol{\phi} \in \boldsymbol{X}^{\mathrm{g}}(D_{\mathbf{a}}).$$

(2) Since $\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div} = 0; D)$ by assumption, we infer that $(\nabla p, \nabla \rho)_{D_{\mathbf{a}}} = (\boldsymbol{\phi}, \nabla \rho)_{D_{\mathbf{a}}}$ for all $\rho \in X^{\mathrm{g}}(D_{\mathbf{a}})$. Therefore, we can apply Assumption 3.5 to $\boldsymbol{g} := \boldsymbol{\phi}$, and observing that $\zeta_{\boldsymbol{g}} = p$, we infer that

$$h_{\mathbf{a}}^{\frac{3}{2}+\delta} |\nabla p|_{\boldsymbol{H}^{\frac{1}{2}+\delta}(D_{\mathbf{a}})} \lesssim h_{\mathbf{a}}^2 \|\nabla \boldsymbol{\phi}\|_{D_{\mathbf{a}}} \lesssim h_{\mathbf{a}}^2 \|\nabla \times \boldsymbol{v}\|_{D_{\mathbf{a}}},$$

where the second bound follows from (4.5). Since $|\nabla p|_{\boldsymbol{H}^{\frac{1}{2}+\delta}(K)} \leq |\nabla p|_{\boldsymbol{H}^{\frac{1}{2}+\delta}(D_{\mathbf{a}})}$ for all $K \in \mathcal{T}_{\mathbf{a}}$, we infer that

$$|\nabla p|_{\boldsymbol{H}^{\frac{1}{2}+\delta}(K)} \lesssim h_{\mathbf{a}}^{\frac{1}{2}-\delta} \|\nabla \times \boldsymbol{v}\|_{D_{\mathbf{a}}}. \tag{4.6}$$

(3) For all $K \in \mathcal{T}_{\mathbf{a}}$, invoking the geometric mapping to the reference simplex $\widehat{K}$ and the embedding $|\widehat{\boldsymbol{\phi}}|_{\boldsymbol{H}^{\frac{1}{2}+\delta}(\widehat{K})} \leq \widehat{C}\|\nabla\widehat{\boldsymbol{\phi}}\|_{\widehat{K}}$, we infer that

$$|\boldsymbol{\phi}|_{\boldsymbol{H}^{\frac{1}{2}+\delta}(K)} \lesssim h_K^{\frac{1}{2}-\delta}\|\nabla\boldsymbol{\phi}\|_K \leq h_K^{\frac{1}{2}-\delta}\|\nabla\boldsymbol{\phi}\|_{D_{\mathbf{a}}} \lesssim h_{\mathbf{a}}^{\frac{1}{2}-\delta}\|\nabla\times\boldsymbol{v}\|_{D_{\mathbf{a}}},$$

where the last bound follows again from (4.5) and $h_K \leq h_{\mathbf{a}}$. Combining this bound with (4.6) and invoking the triangle inequality concludes the proof. ∎

**5. Proof of the main results from section 3.** This section contains the proofs of Lemmas 3.3, 3.4, and 3.6.

**5.1. Proof of Lemma 3.3.** Let $\mathbf{a} \in \mathcal{V}_h$. Our goal is bound the error $\boldsymbol{\delta}_{\mathbf{a}} := \boldsymbol{E}_{\mathbf{a}} - \psi_{\mathbf{a}}\boldsymbol{E}_h$ defined in (3.9) in the broken curl norm. Specifically, our goal is to prove that

$$\|\nabla_h\times\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}} \leq C\left\{\sum_{F\in\mathcal{F}_{\mathbf{a}}}\left(\frac{h_F}{p}\right)\|[\![\nabla\times\boldsymbol{E}_h]\!]_F^{\mathrm{d}}\|_F^2 + \sum_{F\in\mathcal{F}_{\mathbf{a}}}\left(\frac{p^2}{h_F}\right)\|[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2\right\}^{\frac{1}{2}}.$$

To this purpose, we apply the Helmholtz decomposition (4.1a) to $\boldsymbol{v} := \nabla_h\times\boldsymbol{\delta}_{\mathbf{a}}$ in the vertex patch $D_{\mathbf{a}}$. This gives $\boldsymbol{\psi} \in \boldsymbol{X}_0^{\mathrm{c}}(D_{\mathbf{a}})$ and $\xi \in X^{\mathrm{g}}(D_{\mathbf{a}})$ such that $\nabla_h\times\boldsymbol{\delta}_{\mathbf{a}} = \nabla_0\times\boldsymbol{\psi} + \nabla\xi$ in $D_{\mathbf{a}}$. Taking the $\boldsymbol{L}^2(D_{\mathbf{a}})$-inner product with $\nabla_h\times\boldsymbol{\delta}_{\mathbf{a}}$, we infer that

$$\|\nabla_h\times\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}}^2 = (\nabla_h\times\boldsymbol{\delta}_{\mathbf{a}}, \nabla_0\times\boldsymbol{\psi})_{D_{\mathbf{a}}} + (\nabla_h\times\boldsymbol{\delta}_{\mathbf{a}}, \nabla\xi)_{D_{\mathbf{a}}} =: \mathrm{I} + \mathrm{II}, \tag{5.1}$$

and it remains to bound I and II.

(1) Bound on I. Since $\nabla_h\times\boldsymbol{\delta}_{\mathbf{a}} = \nabla_0\times\boldsymbol{E}_{\mathbf{a}} - \nabla_h\times(\psi_{\mathbf{a}}\boldsymbol{E}_h) = \nabla_0\times\boldsymbol{U}_{\mathbf{a}} - \nabla_h\times(\psi_{\mathbf{a}}\boldsymbol{E}_h)$ owing to the definition (3.6) of $\boldsymbol{E}_{\mathbf{a}}$, we have

$$\mathrm{I} = (\nabla_0\times\boldsymbol{U}_{\mathbf{a}} - \nabla_h\times(\psi_{\mathbf{a}}\boldsymbol{E}_h), \nabla_0\times\boldsymbol{\psi})_{D_{\mathbf{a}}}.$$

Recalling the definition (3.4) of $\boldsymbol{U}_{\mathbf{a}}$ and since $\boldsymbol{\psi} \in \boldsymbol{X}_0^{\mathrm{c}}(D_{\mathbf{a}})$, this gives

$$\mathrm{I} = (\boldsymbol{L}_h^p(\psi_{\mathbf{a}}\boldsymbol{E}_h), \nabla_0\times\boldsymbol{\psi})_{D_{\mathbf{a}}}.$$

Invoking the Cauchy–Schwarz inequality and the fact that $\psi_{\mathbf{a}}$ is continuous and bounded poitwise by one, we infer that

$$\begin{aligned}\mathrm{I} \leq \|\boldsymbol{L}_h^p(\psi_{\mathbf{a}}\boldsymbol{E}_h)\|_{D_{\mathbf{a}}}\|\nabla_0\times\boldsymbol{\psi}\|_{D_{\mathbf{a}}} &\lesssim \left\{\sum_{F\in\mathcal{F}_{\mathbf{a}}}\left(\frac{p^2}{h_F}\right)\|[\![\psi_{\mathbf{a}}\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2\right\}^{\frac{1}{2}}\|\nabla_0\times\boldsymbol{\psi}\|_{D_{\mathbf{a}}} \\ &\lesssim \left\{\sum_{F\in\mathcal{F}_{\mathbf{a}}}\left(\frac{p^2}{h_F}\right)\|[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2\right\}^{\frac{1}{2}}\|\nabla_h\times\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}},\end{aligned} \tag{5.2}$$

where we used the stability bound (4.1b) in the last estimate.

(2) Bound on II. Since $(\nabla_0\times\boldsymbol{E}_{\mathbf{a}}, \nabla\xi)_{D_{\mathbf{a}}} = 0$, we have

$$\mathrm{II} = -(\nabla_h\times(\psi_{\mathbf{a}}\boldsymbol{E}_h), \nabla\xi)_{D_{\mathbf{a}}}.$$

Let $\mathcal{I}_{\mathbf{a}}^{\mathrm{g}}$ be the modified $hp$-Karkulik–Melenk interpolation operator on the vertex patch $D_{\mathbf{a}}$ from Lemma 2.3. We have

$$\mathrm{II} = -(\nabla_h\times(\psi_{\mathbf{a}}\boldsymbol{E}_h), \nabla(\xi - \mathcal{I}_{\mathbf{a}}^{\mathrm{g}}(\xi)))_{D_{\mathbf{a}}} - (\nabla_h\times(\psi_{\mathbf{a}}\boldsymbol{E}_h), \nabla\mathcal{I}_{\mathbf{a}}^{\mathrm{g}}(\xi))_{D_{\mathbf{a}}} =: \mathrm{II}_1 + \mathrm{II}_2,$$

and we bound the two terms on the right-hand side.

(2a) Bound on $\mathrm{II}_1$. We integrate by parts the gradient operator in all the cells composing $\mathcal{T}_{\mathbf{a}}$. This gives

$$\mathrm{II}_1 = -\sum_{K\in\mathcal{T}_{\mathbf{a}}} (\xi - \mathcal{I}^{\mathrm{g}}_{\mathbf{a}}(\xi), \boldsymbol{n}_K\cdot(\nabla_h\times(\psi_{\mathbf{a}}\boldsymbol{E}_h)))_{\partial K}.$$

All the mesh faces in $\bigcup_{K\in\mathcal{T}_{\mathbf{a}}}\mathcal{F}_K$ are either in $\mathcal{F}_{\mathbf{a}}$ or in $\mathcal{F}_{\setminus\boldsymbol{a}}$. In the first case, we observe that $\xi-\mathcal{I}^{\mathrm{g}}_{\mathbf{a}}(\xi)$ is single-valued across every interface $F\in\mathcal{F}_{\mathbf{a}}\cap\mathcal{F}_h^{\mathrm{int}}$ and that we have set conventionally $[\![\bullet]\!]^{\mathrm{d}}_F := \bullet|_F\cdot\boldsymbol{n}_D$ on every boundary face $F\in\mathcal{F}_{\mathbf{a}}\cap\mathcal{F}_h^{\mathrm{bd}}$. In the second case where $F\in\mathcal{F}_{\setminus\boldsymbol{a}}$, we have $\boldsymbol{n}_F\cdot\nabla_h\times(\psi_{\mathbf{a}}\boldsymbol{E}_h) = \boldsymbol{n}_F\cdot(\psi_{\mathbf{a}}\nabla_h\times\boldsymbol{E}_h)+\boldsymbol{E}_h\cdot(\boldsymbol{n}_F\times\nabla\psi_{\mathbf{a}}) = 0$ since $\psi_{\mathbf{a}}=0$ and $(\boldsymbol{n}_F\times\nabla\psi_{\mathbf{a}})_F = \mathbf{0}$ on $F$. Altogether, this gives

$$\mathrm{II}_1 = -\sum_{F\in\mathcal{F}_{\mathbf{a}}} (\xi - \mathcal{I}^{\mathrm{g}}_{\mathbf{a}}(\xi), [\![\nabla_h\times(\psi_{\mathbf{a}}\boldsymbol{E}_h)]\!]^{\mathrm{d}}_F)_F.$$

We have $\nabla_h\times(\psi_{\mathbf{a}}\boldsymbol{E}_h) = \psi_{\mathbf{a}}\nabla_h\times\boldsymbol{E}_h + (\nabla\psi_{\mathbf{a}})\times\boldsymbol{E}_h$ so that

$$[\![\nabla_h\times\psi_{\mathbf{a}}\boldsymbol{E}_h]\!]^{\mathrm{d}}_F = \psi_{\mathbf{a}}[\![\nabla_h\times\boldsymbol{E}_h]\!]^{\mathrm{d}}_F + \nabla\psi_{\mathbf{a}}\cdot[\![\boldsymbol{E}_h]\!]^{\mathrm{c}}_F,$$

since $\psi_{\mathbf{a}}$ is continuous and $(\boldsymbol{n}_F\times\nabla\psi_{\mathbf{a}})_F$ is single-valued across every interface $F\in\mathcal{F}_{\mathbf{a}}\cap\mathcal{F}_h^{\mathrm{int}}$. This gives

$$\mathrm{II}_1 = -\sum_{F\in\mathcal{F}_{\mathbf{a}}}(\xi-\mathcal{I}^{\mathrm{g}}_{\mathbf{a}}(\xi), \psi_{\mathbf{a}}[\![\nabla_h\times\boldsymbol{E}_h]\!]^{\mathrm{d}}_F)_F - \sum_{F\in\mathcal{F}_{\mathbf{a}}}(\xi-\mathcal{I}^{\mathrm{g}}_{\mathbf{a}}(\xi), \nabla\psi_{\mathbf{a}}\cdot[\![\boldsymbol{E}_h]\!]^{\mathrm{c}}_F)_F.$$

Let $\mathrm{II}_{11}$ and $\mathrm{II}_{12}$ denote the two summations on the right-hand side. Using the Cauchy–Schwarz inequality and $\|\psi_{\mathbf{a}}\|_{L^\infty(F)}=1$, we infer that

$$|\mathrm{II}_{11}| \le \left\{\sum_{F\in\mathcal{F}_{\mathbf{a}}}\left(\frac{h_F}{p}\right)\|[\![\nabla_h\times\boldsymbol{E}_h]\!]^{\mathrm{d}}_F\|_F^2\right\}^{\frac12}\left\{\sum_{F\in\mathcal{F}_{\mathbf{a}}}\left(\frac{p}{h_F}\right)\|\xi-\mathcal{I}^{\mathrm{g}}_{\mathbf{a}}(\xi)\|_F^2\right\}^{\frac12}.$$

We now invoke the approximation result (2.8) on $\mathcal{I}^{\mathrm{g}}_{\mathbf{a}}$. For every $F\in\mathcal{F}_{\mathbf{a}}$, we can pick a mesh cell $K\in\mathcal{T}_{\mathbf{a}}$ of which $F$ is a face and obtain

$$\left(\frac{p}{h_F}\right)^{\frac12}\|\xi-\mathcal{I}^{\mathrm{g}}_{\mathbf{a}}(\xi)\|_F \lesssim \|\nabla\xi\|_K \le \|\nabla\xi\|_{D_{\mathbf{a}}} \le \|\nabla_h\times\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}},$$

where we used (4.1b) in the last bound. This gives

$$|\mathrm{II}_{11}| \lesssim \left\{\sum_{F\in\mathcal{F}_{\mathbf{a}}}\left(\frac{h_F}{p}\right)\|[\![\nabla_h\times\boldsymbol{E}_h]\!]^{\mathrm{d}}_F\|_F^2\right\}^{\frac12}\|\nabla_h\times\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}}.$$

Moreover, using the Cauchy–Schwarz inequality and $\|\nabla\psi_{\mathbf{a}}\|_{L^\infty(F)}\lesssim h_F^{-1}$ gives

$$|\mathrm{II}_{12}| \lesssim \left\{\sum_{F\in\mathcal{F}_{\mathbf{a}}}\left(\frac{h_F}{p}\right)h_F^{-2}\|[\![\boldsymbol{E}_h]\!]^{\mathrm{c}}_F\|_F^2\right\}^{\frac12}\left\{\sum_{F\in\mathcal{F}_{\mathbf{a}}}\left(\frac{p}{h_F}\right)\|\xi-\mathcal{I}^{\mathrm{g}}_{\mathbf{a}}(\xi)\|_F^2\right\}^{\frac12}.$$

Invoking the same arguments as above to bound the factor involving $\xi$, we obtain

$$|\mathrm{II}_{12}| \lesssim \left\{\sum_{F\in\mathcal{F}_{\mathbf{a}}}p^{-3}\left(\frac{p^2}{h_F}\right)\|[\![\boldsymbol{E}_h]\!]^{\mathrm{c}}_F\|_F^2\right\}^{\frac12}\|\nabla_h\times\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}}.$$

Putting everything together, we conclude that

$$|\mathrm{II}_1| \lesssim \left\{ \sum_{F\in\mathcal{F}_{\mathbf{a}}} \left(\frac{h_F}{p}\right) \|[\![\nabla_h\times\boldsymbol{E}_h]\!]_F^{\mathrm{d}}\|_F^2 + p^{-3}\left(\frac{p^2}{h_F}\right)\|[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2 \right\}^{\frac{1}{2}} \|\nabla_h\times\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}}.$$

(2b) Bound on $\mathrm{II}_2$. This time we integrate by parts the curl operator in all the cells composing $\mathcal{T}_{\mathbf{a}}$. Using similar arguments as above, we obtain

$$\mathrm{II}_2 = \sum_{K\in\mathcal{T}_{\mathbf{a}}} (\psi_{\mathbf{a}}\boldsymbol{E}_h, \boldsymbol{n}_K\times\nabla\mathcal{I}_{\mathbf{a}}^{\mathrm{g}}(\xi))_{\partial K} = \sum_{F\in\mathcal{F}_{\mathbf{a}}} (\psi_{\mathbf{a}}[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}, \nabla\mathcal{I}_{\mathbf{a}}^{\mathrm{g}}(\xi))_F.$$

Invoking the Cauchy–Schwarz inequality, the discrete trace inequality (2.6), and $\|\psi_{\mathbf{a}}\|_{L^\infty(F)} = 1$, we infer that

$$\begin{aligned}
|\mathrm{II}_2| &\le \left\{ \sum_{F\in\mathcal{F}_{\mathbf{a}}} \left(\frac{p^2}{h_K}\right)\|[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2 \right\}^{\frac{1}{2}} \left\{ \sum_{F\in\mathcal{F}_{\mathbf{a}}} \left(\frac{h_K}{p^2}\right)\|\nabla\mathcal{I}_{\mathbf{a}}^{\mathrm{g}}(\xi)\|_F^2 \right\}^{\frac{1}{2}} \\
&\lesssim \left\{ \sum_{F\in\mathcal{F}_{\mathbf{a}}} \left(\frac{p^2}{h_K}\right)\|[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2 \right\}^{\frac{1}{2}} \left\{ \sum_{K\in\mathcal{T}_{\mathbf{a}}} \|\nabla\mathcal{I}_{\mathbf{a}}^{\mathrm{g}}(\xi)\|_K^2 \right\}^{\frac{1}{2}} \\
&\lesssim \left\{ \sum_{F\in\mathcal{F}_{\mathbf{a}}} \left(\frac{p^2}{h_K}\right)\|[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2 \right\}^{\frac{1}{2}} \|\nabla\xi\|_{D_{\mathbf{a}}},
\end{aligned}$$

where the last bound follows from the $H^1$-stability of $\mathcal{I}_{\mathbf{a}}^{\mathrm{g}}$ (see (2.8)). Proceeding as above to bound $\|\nabla\xi\|_{D_{\mathbf{a}}}$, we conclude that

$$|\mathrm{II}_2| \lesssim \left\{ \sum_{F\in\mathcal{F}_{\mathbf{a}}} \left(\frac{p^2}{h_K}\right)\|[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2 \right\}^{\frac{1}{2}} \|\nabla_h\times\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}}.$$

(3) Putting the bounds on $\mathrm{II}_1$ and $\mathrm{II}_2$ together and since $p\ge 1$, we conclude the proof. (Notice that we just drop the factor $p^{-3}$ in the bound on $\mathrm{II}_1$.)

**5.2. Proof of Lemma 3.4.** We want to prove that

$$\|\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}} \lesssim p \left\{ \sum_{F\in\mathcal{F}_{\mathbf{a}}} \left(\frac{h_F}{p}\right)^3 \|[\![\nabla_h\times\boldsymbol{E}_h]\!]_F^{\mathrm{d}}\|_F^2 + h_F\|[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2 \right\}^{\frac{1}{2}}.$$

We apply the local broken curl, divergence-preserving Poincaré inequality to $\boldsymbol{v} := \psi_{\mathbf{a}}\boldsymbol{E}_h \in \boldsymbol{P}_{p+1}^{\mathrm{b}}(\mathcal{T}_{\mathbf{a}}) \subset \boldsymbol{H}^1(\mathcal{T}_{\mathbf{a}})$ and select $\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}} := \boldsymbol{E}_{\mathbf{a}}$, which indeed satisfies $\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}} \in \boldsymbol{H}_0(\mathbf{curl}; D_{\mathbf{a}})$ and $\boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}} - \boldsymbol{v} = \boldsymbol{U}_{\mathbf{a}} + \nabla\theta_{\mathbf{a}} - \psi_{\mathbf{a}}\boldsymbol{E}_h \in \boldsymbol{H}(\mathrm{div}=0; D_{\mathbf{a}})$ since $\boldsymbol{U}_{\mathbf{a}} \in \boldsymbol{H}(\mathrm{div}=0; D_{\mathbf{a}})$ by construction and $\nabla\theta_{\mathbf{a}} - \psi_{\mathbf{a}}\boldsymbol{E}_h \in \boldsymbol{H}(\mathrm{div}=0; D_{\mathbf{a}})$ by definition of $\theta_{\mathbf{a}}$. Observing that $\boldsymbol{\delta}_{\mathbf{a}} = \boldsymbol{E}_{\mathbf{a}} - \psi_{\mathbf{a}}\boldsymbol{E}_h = \boldsymbol{v}_{\mathbf{a}}^{\mathrm{c}} - \boldsymbol{v}$, we infer that

$$\begin{aligned}
\|\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}} &\lesssim h_{\mathbf{a}} \left\{ \|\nabla_h\times\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}} + \sum_{F\in\overline{\mathcal{F}}_{\mathbf{a}}} h_F^{-1}\|[\![\psi_{\mathbf{a}}\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2 \right\}^{\frac{1}{2}} \\
&\le h_{\mathbf{a}} \left\{ \|\nabla_h\times\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}} + \sum_{F\in\mathcal{F}_{\mathbf{a}}} h_F^{-1}\|[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2 \right\}^{\frac{1}{2}},
\end{aligned}$$

where we used that $\psi_{\mathbf{a}}$ is a continuous function over $D_{\mathbf{a}}$, vanishing on $\partial D_{\mathbf{a}}$ and bounded by one. Invoking (3.10) to bound the first term on the right-hand side involving the broken curl and using the mesh shape-regularity gives

$$\|\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}} \lesssim \left\{ \sum_{F\in\mathcal{F}_{\mathbf{a}}} \left(\frac{h_F^3}{p}\right) \|[\![\nabla\times\boldsymbol{E}_h]\!]_F^{\mathrm{d}}\|_F^2 + p^2 h_F \|[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2 \right\}^{\frac{1}{2}}.$$

Factoring out $p^2$ completes the proof.

*Remark* 5.1 (definition of $\boldsymbol{\theta}_{\mathbf{a}}$). The above proof highlights the relevance of the local problem (3.5) defining the curl-free part of $\boldsymbol{E}_{\mathbf{a}}$, i.e., $\nabla\theta_{\mathbf{a}}$.

**5.3. Proof of Lemma 3.6.** We want to prove that, provided Assumption 3.5 holds true,

$$\|\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}} \lesssim \left\{ \sum_{F\in\mathcal{F}_{\mathbf{a}}} h_F \|[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2 \right\}^{\frac{1}{2}}.$$

We use the Aubin–Nitsche duality argument to bound $\boldsymbol{\delta}_{\mathbf{a}}$ in the $\boldsymbol{L}^2$-norm on the vertex patch $D_{\mathbf{a}}$. Specifically, we consider the following dual problem: Find $\boldsymbol{Z}_{\mathbf{a}} \in \boldsymbol{X}_0^{\mathrm{c}}(D_{\mathbf{a}})$ such that

$$\nabla\times(\nabla_0\times\boldsymbol{Z}_{\mathbf{a}}) = \boldsymbol{\delta}_{\mathbf{a}} \quad \text{in } D_{\mathbf{a}}, \tag{5.3a}$$

$$\boldsymbol{Z}_{\mathbf{a}}\times\boldsymbol{n}_{D_{\mathbf{a}}} = \boldsymbol{0} \quad \text{on } \partial D_{\mathbf{a}}. \tag{5.3b}$$

This problem is well-posed since $D_{\mathbf{a}}$ has trivial topology and $\boldsymbol{\delta}_{\mathbf{a}}$ is divergence-free. Indeed, $\nabla\cdot\boldsymbol{\delta}_{\mathbf{a}} = \nabla\cdot(\boldsymbol{E}_{\mathbf{a}} - (\psi_{\mathbf{a}}\boldsymbol{E}_h)) = \nabla\cdot\boldsymbol{U}_{\mathbf{a}} + \nabla\cdot(\nabla\theta_{\mathbf{a}} - \psi_{\mathbf{a}}\boldsymbol{E}_h) = 0$ since $\boldsymbol{U}_{\mathbf{a}}$ is divergence-free by construction and the definition (3.5) of $\theta_{\mathbf{a}}$ implies that $\nabla\theta_{\mathbf{a}} - \psi_{\mathbf{a}}\boldsymbol{E}_h$ is divergence-free as well. Recalling the regularity pickup from [17, section 5.2], and [24, Lemma 44.2], we infer that there is $\delta'_{\mathbf{a}} \in (0,\frac{1}{2}]$ (here, $\delta'_{\mathbf{a}}$ can depend on $\mathbf{a}\in\mathcal{V}_h$) so that $\nabla_0\times\boldsymbol{Z}_{\mathbf{a}} \in \boldsymbol{H}^{\frac{1}{2}+\delta'_{\mathbf{a}}}(D_{\mathbf{a}})$. This implies in particular that $\nabla_0\times\boldsymbol{Z}_{\mathbf{a}}$ has meaningful (tangential) traces in $\boldsymbol{L}^2(\partial K)$ for all $K\in\mathcal{T}_{\mathbf{a}}$. Taking the $\boldsymbol{L}^2(D_{\mathbf{a}})$-inner product with $\boldsymbol{\delta}_{\mathbf{a}} = \boldsymbol{U}_{\mathbf{a}} + \nabla\theta_{\mathbf{a}} - \psi_{\mathbf{a}}\boldsymbol{E}_h$ in (5.3a), we obtain

$$\begin{aligned}\|\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}}^2 &= (\nabla\times(\nabla_0\times\boldsymbol{Z}_{\mathbf{a}}),\boldsymbol{\delta}_{\mathbf{a}})_{D_{\mathbf{a}}}\\ &= (\nabla\times(\nabla_0\times\boldsymbol{Z}_{\mathbf{a}}),\boldsymbol{U}_{\mathbf{a}})_{D_{\mathbf{a}}} + (\nabla\times(\nabla_0\times\boldsymbol{Z}_{\mathbf{a}}),\nabla\theta_{\mathbf{a}})_{D_{\mathbf{a}}} - (\nabla\times(\nabla_0\times\boldsymbol{Z}_{\mathbf{a}}),\psi_{\mathbf{a}}\boldsymbol{E}_h)_{D_{\mathbf{a}}}.\end{aligned}$$

Integrating by parts the curl operator, we infer that

$$\begin{aligned}\|\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}}^2 =& (\nabla_0\times\boldsymbol{Z}_{\mathbf{a}},\nabla_0\times\boldsymbol{U}_{\mathbf{a}})_{D_{\mathbf{a}}} - (\nabla_0\times\boldsymbol{Z}_{\mathbf{a}},\nabla_h\times(\psi_{\mathbf{a}}\boldsymbol{E}_h))_{D_{\mathbf{a}}}\\ &- \sum_{K\in\mathcal{T}_{\mathbf{a}}} (\nabla_0\times\boldsymbol{Z}_{\mathbf{a}},(\psi_{\mathbf{a}}\boldsymbol{E}_h)\times\boldsymbol{n}_K)_{\partial K}.\end{aligned}$$

Rearranging the last term on the right-hand side leads to

$$\|\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}}^2 = (\nabla_0\times\boldsymbol{Z}_{\mathbf{a}},\nabla_h\times(\boldsymbol{U}_{\mathbf{a}} - (\psi_{\mathbf{a}}\boldsymbol{E}_h)))_{D_{\mathbf{a}}} - \sum_{F\in\mathcal{F}_{\mathbf{a}}} (\nabla_0\times\boldsymbol{Z}_{\mathbf{a}},\psi_{\mathbf{a}}[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}})_F.$$

Next, invoking the relation (3.4) with $\boldsymbol{W}_{\mathbf{a}} := \boldsymbol{Z}_{\mathbf{a}} \in \boldsymbol{X}_0^{\mathrm{c}}(D_{\mathbf{a}})$ and the definition (2.9) of the lifting operator, we infer that

$$
\begin{aligned}
\|\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}}^2 &= (\nabla_0\times\boldsymbol{Z}_{\mathbf{a}}, \boldsymbol{L}_h^p(\psi_{\mathbf{a}}\boldsymbol{E}_h))_{D_{\mathbf{a}}} - \sum_{F\in\mathcal{F}_{\mathbf{a}}}(\nabla_0\times\boldsymbol{Z}_{\mathbf{a}}, \psi_{\mathbf{a}}[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}})_F \\
&= (\boldsymbol{\Pi}_{\mathbf{a}}^{\mathrm{b}}(\nabla_0\times\boldsymbol{Z}_{\mathbf{a}}), \boldsymbol{L}_h^p(\psi_{\mathbf{a}}\boldsymbol{E}_h))_{D_{\mathbf{a}}} - \sum_{F\in\mathcal{F}_{\mathbf{a}}}(\nabla_0\times\boldsymbol{Z}_{\mathbf{a}}, \psi_{\mathbf{a}}[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}})_F \\
&= -\sum_{F\in\mathcal{F}_{\mathbf{a}}}(\{\!\{\nabla_0\times\boldsymbol{Z}_{\mathbf{a}} - \boldsymbol{\Pi}_{\mathbf{a}}^{\mathrm{b}}(\nabla_0\times\boldsymbol{Z}_{\mathbf{a}})\}\!\}_F^{\mathrm{g}}, \psi_{\mathbf{a}}[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}})_F,
\end{aligned}
\tag{5.4}
$$

where $\boldsymbol{\Pi}_{\mathbf{a}}^{\mathrm{b}}$ denotes the $\boldsymbol{L}^2$-orthogonal projection onto $\boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_{\mathbf{a}})$. Using the Cauchy–Schwarz inequality, $\|\psi_{\mathbf{a}}\|_{L^\infty(F)} = 1$, bounding the mean-values by the traces in the (one or) two cell(s) sharing $F\in\mathcal{F}_{\mathbf{a}}$, and invoking the mesh shape-regularity, we infer that

$$
\begin{aligned}
\|\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}}^2 &\le \left\{\sum_{F\in\mathcal{F}_{\mathbf{a}}}\left(\frac{p}{h_F}\right)^{2\delta}\|\{\!\{\nabla_0\times\boldsymbol{Z}_{\mathbf{a}} - \boldsymbol{\Pi}_{\mathbf{a}}^{\mathrm{b}}(\nabla_0\times\boldsymbol{Z}_{\mathbf{a}})\}\!\}_F^{\mathrm{g}}\|_F^2\right\}^{\frac12}\left\{\sum_{F\in\mathcal{F}_{\mathbf{a}}}\left(\frac{h_F}{p}\right)^{2\delta}\|[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2\right\}^{\frac12} \\
&\lesssim \left\{\sum_{K\in\mathcal{T}_{\mathbf{a}}}\left(\frac{p}{h_K}\right)^{2\delta}\|\nabla_0\times\boldsymbol{Z}_{\mathbf{a}} - \boldsymbol{\Pi}_K^{\mathrm{b}}(\nabla_0\times\boldsymbol{Z}_{\mathbf{a}})\|_{\partial K}^2\right\}^{\frac12}\left\{\sum_{F\in\mathcal{F}_{\mathbf{a}}}\left(\frac{h_F}{p}\right)^{2\delta}\|[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2\right\}^{\frac12},
\end{aligned}
$$

where $\delta\in(0,\frac12]$ is given by Assumption 3.5. Using the approximation result (2.7) on the $L^2$-projection followed by Lemma 4.7 (notice that indeed $\nabla_0\times\boldsymbol{Z}_{\mathbf{a}}\in\boldsymbol{X}^{\mathrm{c}}(D_{\mathbf{a}})$), we infer that, for all $K\in\mathcal{T}_{\mathbf{a}}$,

$$
\left(\frac{p}{h_K}\right)^{\delta}\|\nabla_0\times\boldsymbol{Z}_{\mathbf{a}} - \boldsymbol{\Pi}_K^{\mathrm{b}}(\nabla_0\times\boldsymbol{Z}_{\mathbf{a}})\|_{\partial K} \lesssim |\nabla_0\times\boldsymbol{Z}_{\mathbf{a}}|_{\boldsymbol{H}^{\frac12+\delta}(K)} \lesssim h_{\mathbf{a}}^{\frac12-\delta}\|\boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}}.
$$

Combining the above two bounds and since $h_{\mathbf{a}}\lesssim h_F$ for all $F\in\mathcal{F}_{\mathbf{a}}$ and $p\ge1$, we infer that the assertion holds true. (Notice that we just dropped the factor $p^{-2\delta}$ scaling the tangential jumps of $\boldsymbol{E}_h$ since $\delta$ can be arbitrarily small.)

## 6. Application to *hp*-a posteriori error analysis.

In this section, we show how the $\boldsymbol{H}_0(\mathbf{curl})$-reconstruction operator constructed above can be used in residual-based *hp*-a posteriori error analysis. We focus on the symmetric interior penalty dG approximation of some simplified forms of Maxwell's equations involving the curl-curl operator and a zero-order term. Other nonconforming approximation methods can be considered as well. For simplicity, we assume that Assumption 3.5 holds true.

### 6.1. Model problem.

The model problem consists in finding $\boldsymbol{E}\in\boldsymbol{H}_0(\mathbf{curl};D)$ such that

$$
\omega^2\boldsymbol{\epsilon}\boldsymbol{E} + \nabla\times(\boldsymbol{\nu}\nabla_0\times\boldsymbol{E}) = \boldsymbol{J} \quad \text{in } D, \tag{6.1}
$$

where $\boldsymbol{J}\in\boldsymbol{H}(\mathrm{div};D)$ is the source term. Both material properties $\boldsymbol{\epsilon}$ and $\boldsymbol{\nu}$ can vary in $D$ and are assumed to take positive-definite, symmetric, piecewise constant matrix values with eigenvalues uniformly bounded from above and below away from zero. Notice that, for simplicity, we are enforcing homogeneous Dirichlet boundary conditions (also called perfect electric conductor boundary conditions). The parameter $\omega$ scales as a frequency (reciprocal of a time scale) and can be either a real number (leading to the so-called definite Maxwell problem) or a pure imaginary number (leading to the so-called indefinite problem). Whenever $\omega=0$, the additional property $\nabla\cdot\boldsymbol{J}=0$ is prescribed. For the time being, we assume that $\omega$ is a nonzero real number and refer the reader to Remarks 6.2 and 6.10 for further insight into the other cases.

For a material property $\boldsymbol{\alpha} \in \{\boldsymbol{\epsilon}, \boldsymbol{\nu}\}$, we introduce the inner product and associated norm weighted by $\boldsymbol{\alpha}$, leading to the notation $(\cdot,\cdot)_{\boldsymbol{\alpha}}$ and $\|\cdot\|_{\boldsymbol{\alpha}}$. The weak formulation of (6.1) classically amounts to finding $\boldsymbol{E} \in \boldsymbol{H}_0(\mathbf{curl}; D)$ such that

$$b(\boldsymbol{E}, \boldsymbol{w}) = (\boldsymbol{J}, \boldsymbol{w}) \quad \forall \boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}; D), \tag{6.2}$$

with the bilinear form defined on $\boldsymbol{H}_0(\mathbf{curl}; D) \times \boldsymbol{H}_0(\mathbf{curl}; D)$ such that

$$b(\boldsymbol{v}, \boldsymbol{w}) := \omega^2 (\boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{\epsilon}} + (\nabla_0 \times \boldsymbol{v}, \nabla_0 \times \boldsymbol{w})_{\boldsymbol{\nu}}. \tag{6.3}$$

**6.2. Symmetric interior penalty dG discretization.** We employ the symmetric interior penalty dG method to discretize the weak formulation (6.2) (see [38, 30]). We assume that the mesh $\mathcal{T}_h$ is compatible with the domain partition on which $\boldsymbol{\epsilon}$ and $\boldsymbol{\nu}$ are piecewise constant. For $\boldsymbol{\alpha} \in \{\boldsymbol{\epsilon}, \boldsymbol{\nu}\}$, we set $\boldsymbol{\alpha}_K := \boldsymbol{\alpha}|_K$ for all $K \in \mathcal{T}_h$ and denote by $\alpha_K^\flat$ and $\alpha_K^\sharp$ the smallest and largest eigenvalues of $\boldsymbol{\alpha}_K$, respectively. It is convenient to set

$$\alpha_F^\sharp := \max(\alpha_{K_l}^\sharp, \alpha_{K_r}^\sharp)\, \forall F := \partial K_l \cap \partial K_r \in \mathcal{F}_h^{\mathrm{int}}, \quad \alpha_F^\sharp := \alpha_{K_l}^\sharp\, \forall F := \partial K_l \cap \partial\Omega \in \mathcal{F}_h^{\mathrm{bd}}. \tag{6.4}$$

For all $K \in \mathcal{T}_h$, we also define $\alpha_{\partial K}^\sharp$ so that $\alpha_{\partial K}^\sharp|_F := \alpha_F^\sharp$ for all $F \in \mathcal{F}_K$. To avoid distracting technicalities, we assume that the material properties are only mildly heterogeneous and anisotropic. Thus, we hide anisotropy ratios such as $\alpha_K^\sharp/\alpha_K^\flat$ in the generic constants used in the error analysis. We do the same for contrast factors such as $\alpha_K^\sharp/\alpha_{K'}^\sharp$, where $K'$ lies in some neighborhood of $K$. We refer the reader to Remarks 6.1 and 6.4 for some further insight.

The discrete problem reads as follows: Find $\boldsymbol{E}_h \in \boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h)$ (recall that $p \geq 1$) such that

$$b_h(\boldsymbol{E}_h, \boldsymbol{w}_h) = (\boldsymbol{J}, \boldsymbol{w}_h) \qquad \forall \boldsymbol{w}_h \in \boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h), \tag{6.5}$$

with the discrete bilinear form such that, for all $\boldsymbol{v}_h, \boldsymbol{w}_h \in \boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h)$,

$$\begin{aligned} b_h(\boldsymbol{v}_h, \boldsymbol{w}_h) &:= \omega^2 (\boldsymbol{v}_h, \boldsymbol{w}_h)_{\boldsymbol{\epsilon}} + (\nabla_h \times \boldsymbol{v}_h, \nabla_h \times \boldsymbol{w}_h)_{\boldsymbol{\nu}} + \eta_* s_h(\boldsymbol{v}_h, \boldsymbol{w}_h) \\ &\quad + \sum_{F \in \mathcal{F}_h} \left\{ (\{\!\{\boldsymbol{\nu} \nabla_h \times \boldsymbol{v}_h\}\!\}_F^{\mathrm{g}}, [\![\boldsymbol{w}_h]\!]_F^{\mathrm{c}})_F + ([\![\boldsymbol{v}_h]\!]_F^{\mathrm{c}}, \{\!\{\boldsymbol{\nu} \nabla_h \times \boldsymbol{w}_h\}\!\}_F^{\mathrm{g}})_F \right\}. \end{aligned} \tag{6.6}$$

The stabilization bilinear form $s_h$ is defined as

$$s_h(\boldsymbol{v}_h, \boldsymbol{w}_h) := \sum_{F \in \mathcal{F}_h} \nu_F^\sharp \left( \frac{p^2}{h_F} \right) ([\![\boldsymbol{v}_h]\!]_F^{\mathrm{c}}, [\![\boldsymbol{w}_h]\!]_F^{\mathrm{c}})_F, \tag{6.7}$$

and the user-dependent parameter $\eta_* > 0$ is taken large enough (see below). Using the definition (2.9) of the lifting operator $\boldsymbol{L}_h^p$, we infer that

$$\begin{aligned} b_h(\boldsymbol{v}_h, \boldsymbol{w}_h) &:= \omega^2 (\boldsymbol{v}_h, \boldsymbol{w}_h)_{\boldsymbol{\epsilon}} + (\nabla_h \times \boldsymbol{v}_h, \nabla_h \times \boldsymbol{w}_h)_{\boldsymbol{\nu}} + \eta_* s_h(\boldsymbol{v}_h, \boldsymbol{w}_h) \\ &\quad + (\boldsymbol{L}_h^p(\boldsymbol{w}_h), \nabla_h \times \boldsymbol{v}_h)_{\boldsymbol{\nu}} + (\boldsymbol{L}_h^p(\boldsymbol{v}_h), \nabla_h \times \boldsymbol{w}_h)_{\boldsymbol{\nu}}. \end{aligned} \tag{6.8}$$

We equip the discrete space $\boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h)$ with the norm

$$|\!|\!| \boldsymbol{v}_h |\!|\!|_{\mathrm{dG}}^2 := \omega^2 \|\boldsymbol{v}_h\|_{\boldsymbol{\epsilon}}^2 + \|\nabla_h \times \boldsymbol{v}_h\|_{\boldsymbol{\nu}}^2 + s_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h). \tag{6.9}$$

Recalling (2.10), we infer that

$$\|\boldsymbol{L}_h^p(\boldsymbol{v}_h)\|_{\boldsymbol{\nu}}^2 \leq C(\kappa_{\mathcal{T}_h}) s_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h), \tag{6.10}$$

whence we infer, provided we take $\eta_* \geq \frac{1}{2} + 2C(\kappa_{\mathcal{T}_h})$ with $C(\kappa_{\mathcal{T}_h})$ from (6.10), that the following coercivity property holds:

$$b_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq \frac{1}{2} |||\boldsymbol{v}_h|||_{\mathrm{dG}}^2 \quad \forall \boldsymbol{v}_h \in \boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h). \tag{6.11}$$

*Remark* 6.1 (weighted averages). Whenever the jumps of $\boldsymbol{\nu}$ across the mesh interfaces are large, one can consider weighted ($\boldsymbol{\nu}$-dependent) averages to evaluate the last two terms on the right-hand side of (6.6). In the case of strong anisotropy, the weights depend on the normal-normal component of $\boldsymbol{\nu}$ on both sides of the interface. We refer the reader, e.g., to [25] for an example in the context of a scalar diffusion problems. Accordingly, the definition (2.9) of the lifting operator $\boldsymbol{L}_h^p$ can be modified by using a weighted average on the right-hand side.

*Remark* 6.2 (indefinite case). In the indefinite case, the coercivity property (6.11) is replaced by a Gårding inequality. Assuming that $\omega$ is not a resonant frequency and invoking a duality argument à la Schatz then leads to a discrete inf-sup condition if the mesh size is small enough. We refer the reader to [12, section 5.4] for more details.

**6.3. Estimator and error measure.** The a posteriori error estimator is written as the sum over the mesh cells of local error indicators $\eta_K$ for all $K \in \mathcal{T}_h$. The local error indicator consists of three pieces. The first two respectively measure the residuals of the divergence constraint and of Maxwell's equations:

(6.12a)

$$\eta_{K,\mathrm{div}}^2 := \frac{1}{\epsilon_K^\sharp} \left\{ \frac{1}{\omega^2} \left( \frac{h_K}{p} \right)^2 \|\nabla \cdot (\boldsymbol{J} - \omega^2 \boldsymbol{\epsilon} \boldsymbol{E}_h)\|_K^2 + \omega^2 \left( \frac{h_K}{p} \right) \|[\![\boldsymbol{\epsilon} \boldsymbol{E}_h]\!]_{\partial K}^{\mathrm{d}}\|_{\partial K \setminus \partial\Omega}^2 \right\}$$

and

$$\begin{aligned} \eta_{K,\mathrm{curl}}^2 := \frac{1}{\nu_K^\sharp} \Bigg\{ & \left( \frac{h_K}{p} \right)^2 \|\boldsymbol{J} - \omega^2 \boldsymbol{\epsilon} \boldsymbol{E}_h - \nabla \times (\boldsymbol{\nu} \nabla \times \boldsymbol{E}_h)\|_K^2 \\ & + \left( \frac{h_K}{p} \right) \|[\![\boldsymbol{\nu} \nabla_h \times \boldsymbol{E}_h]\!]_{\partial K}^{\mathrm{c}}\|_{\partial K \setminus \partial\Omega}^2 + \nu_{\partial K}^\sharp \left( \frac{p^2}{h_K} \right) \|[\![\boldsymbol{E}_h]\!]_{\partial K}^{\mathrm{c}}\|_{\partial K}^2 \Bigg\}. \end{aligned} \tag{6.12b}$$

The last part of the estimator is related to the nonconformity of the discrete field $\boldsymbol{E}_h$ and reads

(6.12c)

$$\eta_{K,\mathrm{nc}}^2 := \nu_{\partial K}^\sharp \left( \frac{h_K}{p} \right) \|[\![\nabla_h \times \boldsymbol{E}_h]\!]_{\partial K}^{\mathrm{d}}\|_{\partial K}^2 + \left\{ \omega^2 \epsilon_{\partial K}^\sharp h_K + \nu_{\partial K}^\sharp \left( \frac{p^2}{h_K} \right) \right\} \|[\![\boldsymbol{E}_h]\!]_{\partial K}^{\mathrm{c}}\|_{\partial K}^2.$$

Notice that this latter estimator is suitable under Assumption 3.5. It is convenient to introduce the following shorthand notation:

$$\eta_K^2 := \eta_{K,\mathrm{div}}^2 + \eta_{K,\mathrm{curl}}^2 + \eta_{K,\mathrm{nc}}^2, \qquad \eta_\bullet^2 := \sum_{K \in \mathcal{T}_h} \eta_{\bullet,K}^2, \qquad \eta^2 := \sum_{K \in \mathcal{T}_h} \eta_K^2, \tag{6.13}$$

with $\bullet \in \{\mathrm{div}, \mathrm{curl}, \mathrm{nc}\}$.

The approximation error is defined as

$$\boldsymbol{e} := \boldsymbol{E} - \boldsymbol{E}_h \in \boldsymbol{V}_\sharp := \boldsymbol{H}_0(\mathbf{curl}; D) + \boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h). \tag{6.14}$$

Although the sum defining $\boldsymbol{V}_\sharp$ is not direct, any field $\boldsymbol{v}_h \in \boldsymbol{H}_0(\mathbf{curl}; D) \cap \boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h)$ satisfies $[\![\boldsymbol{v}_h]\!]_F^{\mathrm{c}} = \mathbf{0}$ for all $F \in \mathcal{F}_h$. It is therefore legitimate to extend the jump operator to $\boldsymbol{V}_\sharp$ by setting $[\![\boldsymbol{v}]\!]_F^{\mathrm{c}} := [\![\boldsymbol{v}_h]\!]_F^{\mathrm{c}}$ for all $\boldsymbol{v} = \tilde{\boldsymbol{v}} + \boldsymbol{v}_h \in \boldsymbol{V}_\sharp$. Similarly, the lifting operator $\boldsymbol{L}_h^p$ can be extended to $\boldsymbol{V}_\sharp$ by setting $\boldsymbol{L}_h^p(\boldsymbol{v}) := \boldsymbol{L}_h^p(\boldsymbol{v}_h)$. The discrete bilinear form $b_h$ is then extended to $\boldsymbol{V}_\sharp \times \boldsymbol{V}_\sharp$ by using the expression (6.8), i.e., we set, for all $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}_\sharp$,

$$\begin{aligned} b_\sharp(\boldsymbol{v}, \boldsymbol{w}) := \omega^2(\boldsymbol{v}, \boldsymbol{w})_{\boldsymbol{\epsilon}} + (\nabla_h \times \boldsymbol{v}, \nabla_h \times \boldsymbol{w})_{\boldsymbol{\nu}} + \eta_* s_h(\boldsymbol{v}, \boldsymbol{w}) \\ + (\boldsymbol{L}_h^p(\boldsymbol{w}), \nabla_h \times \boldsymbol{v})_{\boldsymbol{\nu}} + (\boldsymbol{L}_h^p(\boldsymbol{v}), \nabla_h \times \boldsymbol{w})_{\boldsymbol{\nu}}, \end{aligned} \tag{6.15}$$

keeping in mind that $\nabla_h \times \boldsymbol{v} := \nabla_0 \times \boldsymbol{v}$ whenever $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; D)$.

For any subset $\mathcal{T} \subset \mathcal{T}_h$, we define the error measure

$$|\!|\!| \boldsymbol{e} |\!|\!|_{\sharp,\mathcal{T}}^2 := \sum_{K \in \mathcal{T}} \left\{ \omega^2 \|\boldsymbol{e}\|_{\boldsymbol{\epsilon},K}^2 + \|\nabla \times \boldsymbol{e}\|_{\boldsymbol{\nu},K}^2 + \nu_{\partial K}^\sharp \left( \frac{p^2}{h_K} \right) \|[\![\boldsymbol{e}]\!]_{\partial K}^{\mathrm{c}}\|_{\partial K}^2 \right\} \tag{6.16}$$

and omit the subscript $\mathcal{T}$ whenever $\mathcal{T} = \mathcal{T}_h$. When restricted to $\boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h)$, $|\!|\!| \bullet |\!|\!|_\sharp$ is equivalent to the discrete dG norm $|\!|\!| \bullet |\!|\!|_{\mathrm{dG}}$ defined in (6.9) up to a rewriting of the jump contribution as a sum over the mesh cell boundaries rather than over the mesh faces.

*Remark* 6.3 (stabilization weight). We observe that the error measure defined in (6.16) is independent of the stabilization parameter $\eta_*$ introduced in the dG scheme.

*Remark* 6.4 (heterogeneous materials). Whenever one wishes to account for the heterogeneity of the material properties, some weights in the error indicators need to be adapted. For the error indicators defined in (6.12), one needs to consider a contrast factor involving three layers of vertex-based neighbors around $K$, as detailed in [12]. For the nonconformity indicator defined in (6.12c), one layer of neighbors is sufficient. We also notice that it is possible to modify Definition 3.1 so as to reconstruct on each vertex patch an $\boldsymbol{H}_0(\mathbf{curl})$-conforming field by using weighted $L^2$-inner products. However, the estimates satisfied by this modified reconstruction in the broken curl and $\boldsymbol{L}^2$-norms will depend anyway on the contrast factors associated with the underlying weights.

**6.4. *hp*-a posteriori error analysis.** We decompose the approximation error into two components as follows:

$$\boldsymbol{e} = (\boldsymbol{E} - \boldsymbol{E}_c) + (\boldsymbol{E}_c - \boldsymbol{E}_h) =: \boldsymbol{e}_c + \boldsymbol{e}_{nc}, \qquad \boldsymbol{E}_c := \boldsymbol{R}^{\mathrm{c}}(\boldsymbol{E}_h) \in \boldsymbol{H}_0(\mathbf{curl}; D). \tag{6.17}$$

We call $\boldsymbol{e}_c$ the conforming error and $\boldsymbol{e}_{nc}$ the nonconforming error. Notice that we use the $\boldsymbol{H}_0(\mathbf{curl})$-reconstruction operator from section 3 to define the two error components. Actually, the precise definition of $\boldsymbol{E}_c$ is irrelevant for bounding $\boldsymbol{e}_c$ (we only use that $\boldsymbol{e}_c \in \boldsymbol{H}_0(\mathbf{curl}; D)$) and is only relevant for bounding $\boldsymbol{e}_{nc}$.

LEMMA 6.5 (residual). *Let $\eta_{\mathrm{dc}}^2 := \eta_{\mathrm{div}}^2 + \eta_{\mathrm{curl}}^2$. The following holds:*

$$|b_\sharp(\boldsymbol{e}, \boldsymbol{e}_c)| \leq C(\kappa_{\mathcal{T}_h}) \eta_{\mathrm{dc}} |\!|\!| \boldsymbol{e}_c |\!|\!|_\sharp. \tag{6.18}$$

*Proof.* The proof essentially follows the arguments from the proof of [12, Lemma 6.2], up to minor adaptations (the proof therein tracks the dependency of the constants on the heterogeneity of the material properties and deals with the indefinite Maxwell problem). ∎

LEMMA 6.6 (bound on $\boldsymbol{e}_c$). *The following holds:*

$$\|\boldsymbol{e}_c\|_\sharp \le C(\kappa_{\mathcal{T}_h})(\eta_{\mathrm{dc}} + \|\boldsymbol{e}_{nc}\|_\sharp). \tag{6.19}$$

*Proof.* Since $\boldsymbol{e}_c \in \boldsymbol{H}_0(\mathbf{curl}; D)$, we have $s_h(\boldsymbol{e}_c, \boldsymbol{e}_c) = 0$ and $\boldsymbol{L}_h^p(\boldsymbol{e}_c) = \mathbf{0}$, so that

$$\begin{aligned}\|\boldsymbol{e}_c\|_\sharp^2 &= \omega^2 \|\boldsymbol{e}_c\|_{\boldsymbol{\epsilon}}^2 + \|\nabla_0 \times \boldsymbol{e}_c\|_{\boldsymbol{\nu}}^2 \\ &= b_\sharp(\boldsymbol{e}_c, \boldsymbol{e}_c) = b_\sharp(\boldsymbol{e}, \boldsymbol{e}_c) - b_\sharp(\boldsymbol{e}_{nc}, \boldsymbol{e}_c) =: T_1 + T_2.\end{aligned}$$

Applying the bound (6.18) from Lemma 6.5, we get

$$|T_1| \lesssim \eta_{\mathrm{dc}} \|\boldsymbol{e}_c\|_\sharp.$$

Moreover, we observe that

$$T_2 = b_\sharp(\boldsymbol{e}_{nc}, \boldsymbol{e}_c) = \omega^2 (\boldsymbol{e}_{nc}, \boldsymbol{e}_c)_{\boldsymbol{\epsilon}} + (\nabla_h \times \boldsymbol{e}_{nc}, \nabla_0 \times \boldsymbol{e}_c)_{\boldsymbol{\nu}} - (\boldsymbol{L}_h^p(\boldsymbol{E}_h), \nabla_0 \times \boldsymbol{e}_c)_{\boldsymbol{\nu}}.$$

Invoking the Cauchy–Schwarz inequality and the bound (2.10) on the discrete lifting operator, we infer that

$$|T_2| \le 2\Big\{\omega^2 \|\boldsymbol{e}_{nc}\|_{\boldsymbol{\epsilon}}^2 + \|\nabla_h \times \boldsymbol{e}_{nc}\|_{\boldsymbol{\nu}}^2 + \|\boldsymbol{L}_h^p(\boldsymbol{E}_h)\|_{\boldsymbol{\nu}}^2\Big\}^{\frac{1}{2}} \|\boldsymbol{e}_c\|_\sharp \lesssim \|\boldsymbol{e}_{nc}\|_\sharp \|\boldsymbol{e}_c\|_\sharp.$$

Combining the above bounds completes the proof. ∎

LEMMA 6.7 (bound on $\boldsymbol{e}_{nc}$). *Under Assumption* 3.5*, the following holds:*

$$\|\boldsymbol{e}_{nc}\|_\sharp \le C(\kappa_{\mathcal{T}_h}) \eta_{\mathrm{nc}}. \tag{6.20}$$

*Proof.* Recall that, for all $K \in \mathcal{T}_h$, $\boldsymbol{e}_{nc}|_K = \sum_{\mathbf{a} \in \mathcal{V}_K} \boldsymbol{\delta}_{\mathbf{a}}|_K$ with $\boldsymbol{\delta}_{\mathbf{a}} := \boldsymbol{E}_{\mathbf{a}} - \psi_{\mathbf{a}} \boldsymbol{E}_h$ (see (3.9)). Reasoning as in the proof of Theorem 3.7, this gives

$$\|\nabla \times \boldsymbol{e}_{nc}\|_{\boldsymbol{\nu},K}^2 \le \nu_K \sum_{\mathbf{a} \in \mathcal{V}_K} \|\nabla_h \times \boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}}^2.$$

Invoking the bound on $\|\nabla_h \times \boldsymbol{\delta}_{\mathbf{a}}\|_{D_{\mathbf{a}}}$ from Lemma 3.3, and hiding the contrast factor on $\nu$ in the generic constants, we infer that

$$\|\nabla \times \boldsymbol{e}_{nc}\|_{\boldsymbol{\nu},K}^2 \lesssim \sum_{\mathbf{a} \in \mathcal{V}_K} \sum_{F \in \mathcal{F}_{\mathbf{a}}} \left\{ \nu_F^\sharp \left(\frac{h_F}{p}\right) \|[\![\nabla_h \times \boldsymbol{E}_h]\!]_F^{\mathrm{d}}\|_F^2 + \nu_F^\sharp \left(\frac{p^2}{h_F}\right) \|[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2 \right\}.$$

Reasoning similarly to bound $\omega^2 \|\boldsymbol{e}_{nc}\|_{\boldsymbol{\epsilon},K}^2$, but this time invoking the bound from Lemma 3.6, we infer that

$$\omega^2 \|\boldsymbol{e}_{nc}\|_{\boldsymbol{\epsilon},K}^2 \lesssim \sum_{\mathbf{a} \in \mathcal{V}_K} \sum_{F \in \mathcal{F}_{\mathbf{a}}} \omega^2 \epsilon_F^\sharp h_F \|[\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}\|_F^2.$$

Combining the above two bounds together, rearranging the summations over the mesh cells, invoking again the mild heterogeneity of the material properties, and observing that $[\![\boldsymbol{e}_{nc}]\!]_{\partial K}^{\mathrm{c}} = -[\![\boldsymbol{E}_h]\!]_{\partial K}^{\mathrm{c}}$ for all $K \in \mathcal{T}_h$ leads to the expected bound on $\|\boldsymbol{e}_{nc}\|_\sharp$. ∎

THEOREM 6.8 (global upper bound (reliability)). *Under Assumption* 3.5*, the following holds:*

$$\|\!|\boldsymbol{e}\|\!|_\sharp \le C(\kappa_{\mathcal{T}_h})\eta. \tag{6.21}$$

*Proof.* Invoke the triangle inequality together with Lemmas 6.6 and 6.7, and recall the definition of $\eta$ from (6.13). ∎

Finally, we derive local efficiency estimates.

PROPOSITION 6.9 (local lower bound (local efficiency)). *For all $K \in \mathcal{T}_h$, we have*

$$\eta_K \le C(\kappa_{\mathcal{T}_h}) p^{\frac{3}{2}} \left\{ \left(1 + \left(\frac{\omega^2 \epsilon_K^\sharp}{\nu_K^\sharp}\right)^{\frac{1}{2}} \left(\frac{h_K}{p}\right)\right) \|\!|\boldsymbol{E} - \boldsymbol{E}_h\|\!|_{\sharp, K^{\mathrm{f}}} + \mathrm{osc}_{K^{\mathrm{f}}} \right\}, \tag{6.22}$$

*with the data oscillation term*

$$\mathrm{osc}_{K^{\mathrm{f}}}^2 := \sum_{K' \in K^{\mathrm{f}}} \min_{\boldsymbol{J}_h \in \boldsymbol{P}_p^{\mathrm{b}}(\mathcal{T}_h)} \left\{ \frac{1}{\nu_{K'}^\sharp} \left(\frac{h_{K'}}{p}\right)^2 \|\boldsymbol{J} - \boldsymbol{J}_h\|_{K'}^2 + \frac{1}{\omega^2 \epsilon_{K'}^\sharp} \left(\frac{h_{K'}}{p}\right)^2 \|\nabla \cdot (\boldsymbol{J} - \boldsymbol{J}_h)\|_{K'}^2 \right\}, \tag{6.23}$$

*and $K^{\mathrm{f}}$ is the collection of the mesh cells sharing a face with $K$.*

*Proof.* Local bounds on all terms composing the error estimator $\eta_K$ can be found in [12, Theorem 6.5] except for the term involving $[\![\nabla_h \times \boldsymbol{E}_h]\!]_F^{\mathrm{d}}$. Invoking the $H^1$- to $L^2$-norm inverse inequality from [40, Theorem 4.76], we infer that

$$\nu_{\partial K}^\sharp \left(\frac{h_K}{p}\right) \|[\![\nabla_h \times \boldsymbol{E}_h]\!]_{\partial K}^{\mathrm{d}}\|_{\partial K}^2 \lesssim \nu_{\partial K}^\sharp \left(\frac{p^3}{h_K}\right) \|[\![\boldsymbol{E}_h]\!]_{\partial K}^{\mathrm{c}}\|_{\partial K}^2 \lesssim p \|\!|\boldsymbol{E} - \boldsymbol{E}_h\|\!|_{\sharp, K^{\mathrm{f}}}.$$

This completes the proof. ∎

*Remark* 6.10 (indefinite case). Whenever $\omega = 0$, the above $hp$-a posteriori error analysis can be applied with the additional simplification of discarding the $\boldsymbol{L}^2$-norm estimate, so that Assumption 3.5 is no longer relevant. (Notice also that the last term on the right-hand side of (6.23) is no longer relevant since $\boldsymbol{J}$ is divergence-free.) In the time-harmonic regime with a nonzero, pure imaginary $\omega$, the techniques in [12] can be combined with the present $\boldsymbol{H}_0(\mathbf{curl})$-conforming reconstruction.

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