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CONVERGENCE OF EXPLICIT RUNGE–KUTTA DISCONTINUOUS GALERKIN APPROXIMATIONS OF THE FIRST-ORDER FORM OF MAXWELL'S EQUATIONS WITH LOW REGULARITY SOLUTIONS*

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Abstract. We establish a convergence result for the approximation of low-regularity solutions to time-dependent PDE systems that have an involution structure similar to Maxwell's equations and the linear wave equations. The approximation is based on an explicit Runge–Kutta (ERK) time-stepping and the discontinuous Galerkin (dG) method with stabilization (so-called upwind fluxes) in space. The regularity setting only assumes that the exact solution and its first time derivative are in $L^\infty(J; H^s(D))$ with a Sobolev regularity index s in $(0, \frac{1}{2})$ (here, J is the time interval and D the space domain), and that its second time derivative is in $L^\infty(J; L^2(D))$. The two main tools for the convergence analysis are a Ritz projection in space that leverages recent convergence results in operator norm for the dG approximation of the steady form of the PDE, and the L^2-stability under a standard CFL condition of three-stage, third-order and four-stage, fourth-order ERK schemes. These latter results are known in the literature, but we provide here a somewhat simpler argument to prove the L^2-stability.

Key words. curl-curl problem, grad-div problem, duality argument, involution, spectral approximation, finite elements, Maxwell's equations, linear wave equations

MSC codes. 65N25, 35Q61, 65N15, 65N30, 35L05, 35P15

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1. Introduction.

We want to solve time-dependent PDE systems that have an involution structure similar to Maxwell's equations and the linear wave equations. More precisely, we are interested in approximating in time and space problems of the following type,

$$\partial_t u + A(u) = 0, \qquad u(0) = u^0, \tag{1.1}$$

where the linear operator $A : D(A) \subset L \to L$ is unbounded and maximal monotone, and the initial datum u^0 is in L. For the applications we have in mind, L is a (real or complex) Hilbert space with inner product $(\cdot,\cdot)_L$. Recall that A being maximal monotone means that $\Re((A(v), v)_L) \geq 0$ for all $v \in D(A)$, and that, for all $\lambda \in \mathbb{R}_{>0}$ and all $f \in L$, there exists $v \in D(A)$ so that $\lambda v + A(v) = f$. Recall also that maximal monotone operators are closed and densely defined; cf. Brezis [7, section 7.1]. Recall finally that the Hille–Yosida theorem implies that for all $u^0 \in D(A)$, the system (1.1) has a unique solution in $C^1([0, +\infty); L) \cap C^0([0, +\infty); D(A))$.

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The class of problems of interest to us in this paper has an additional property that implies that solutions to (1.1) have some smoothness and can therefore be well approximated. More specifically, denoting by $\mathrm{im}(A)$ the image of A (or range of A), we assume that there exists a subspace H^s dense in L with compact embedding, i.e., $H^s \Subset L$, so that

$$\mathrm{im}(A) \cap D(A) \subset H^s. \tag{1.2}$$

In the applications considered here, H^s is a fractional-order Sobolev space with smoothness index s in $(0, \frac{1}{2})$, and L is the space of square integrable functions/vector fields. The compactness of the injection $H^s \Subset L$ implies that functions in $\mathrm{im}(A) \cap D(A)$ can be well approximated in L. Notice in passing that, as $\mathrm{im}(A)$ is closed, Banach's closed range theorem implies that $\ker(A^\mathsf{T})^\perp = \mathrm{im}(A)$ (here, orthogonality is meant in L). Hence, if $u^0 \in \mathrm{im}(A)$, then $u(t)$ remains in $\mathrm{im}(A)$ owing to the identity

$$(u(t), v)_L = (u^0, v)_L - \int_0^t (A(u(\xi)), v)_L \,\mathrm{d}\xi = 0 \qquad \forall v \in \ker(A^\mathsf{T}). \tag{1.3}$$

Then the Hille–Yosida theorem gives $u(t) \in \mathrm{im}(A) \cap D(A) \subset H^s$, which, in turn, implies that $u(t)$ can be well approximated in L as well. The implication

$$u^0 \in \mathrm{im}(A) \Longrightarrow u(t) \in \mathrm{im}(A) \quad \forall t > 0, \tag{1.4}$$

is called involution in Boillat [5], Dafermos [13], and we henceforth adopt this terminology. In the context of Maxwell's equations, the fact that $u(t) \in \ker(A^\mathsf{T})^\perp = \mathrm{im}(A)$ is called Gauss's law of electromagnetism.

The problem addressed in this paper is about the time and space approximation of (1.1). The objective of this paper is to prove that, when properly stabilized, the discontinuous Galerkin method combined with explicit Runge–Kutta (ERK) schemes is convergent in the L^2-norm. We do so by only relying on the involution property (1.4), the smoothness property (1.2) induced by the involution, and the assumption on the initial data that $u_0 \in D(A^2) \cap \mathrm{im}(A)$, which, when combined, give $u(t) \in H^s$ for all $t > 0$. The two key ingredients of the proof are as follows: (i) The approximation of the steady-state problem $A(v) = f$ with $f \in \ker(A^\mathsf{T})^\perp = \mathrm{im}(A)$ using a (properly stabilized) discontinuous Galerkin (dG) method converges in the operator norm in L^2 with rate h^s, where s is the smoothness index in (1.2). This result has been established in [17] for the dG approximation of the first-order formulations of the Maxwell and linear wave operators with discontinuous coefficients, irrespective of the topology of space domain and for any $s > 0$. The smoothness property (1.2) induced by the involution is essential to prove this result. (ii) There exist ERK methods that are L^2-stable under a standard CFL condition. We focus in this paper on three-stage, third-order and four-stage, fourth-order ERK schemes. The L^2-stability of third-order schemes has been proved in Burman, Ern, and Fernández [8] and Zhang and Shu [30] (see Remark 4.2 below), and the L^2-stability of fourth-order schemes has been proved in Xu et al. [29]; see also Sun and Shu [25, 26]. In this paper, we present slightly different proofs that somewhat simplify the argumentation.

The convergence in operator norm of the space approximation of the steady-state problem is proved in [17] for any positive smoothness index s. This result crucially uses that the dG approximation is properly stabilized. For Maxwell's equations, this entails penalizing the jumps of the tangential components of both the magnetic and the electric fields across mesh interfaces and weakly enforcing the boundary conditions.

We leverage this result in the convergence analysis herein by introducing the Ritz projection associated with the steady-state problem when forming the time-discrete error equations. This way of proceeding in the convergence analysis of time-dependent PDEs is classical; we refer the reader to Wheeler [28] for the original idea in the context of the heat equation, and we refer to [15, Chaps. 77 and 78] for examples in the context of time-dependent Friedrichs's systems.

We emphasize that establishing error estimates when the smoothness index s in (1.2) is allowed to be smaller that $\frac{1}{2}$ is a major roadblock in the literature. Considering situations for which $s \in (0, \frac{1}{2})$ is not an academic exercise as working with heterogeneous materials and nonsmooth boundaries systematically leads to smoothness indices that are less than $\frac{1}{2}$. The restriction $s > \frac{1}{2}$ is invoked in most papers dedicated to the analysis of Maxwell's equations; see, in particular, Ciarlet and Zou [11, Thm. 4.2], Fezoui et al. [18, eq. (16)], and Li [22, Thm. 5.1]. The reader is also referred to Hesthaven and Warburton [19, Thm. 4.2] and Campos Pinto, and Sonnendrücker [9], where $s \geq 2$ is assumed and to Lanteri, Scheid, and Viquerat [21, sect. 4.2], where $s \geq 1$ is assumed. The originality of the present paper is to show that, by using the results from [17], it is possible to prove convergence for any $s > 0$. Interestingly, the present result only holds when the dG approximation is properly stabilized, i.e., when upwinding is used, confirming previous numerical observations reported in the literature; see Remarks 3.6 and 3.10.

The main convergence results established in the paper are Theorem 5.3 for third-order ERK methods and Theorem 5.7 for fourth-order ERK methods. We emphasize that these results hold for *any* three-stage, third-order and any fourth-stage, fourth-order ERK schemes. The generic smoothness of the solution that is used in the proofs is

$$\|u\|_{L^\infty((0,\infty);H^s)} < \infty, \quad \|\partial_t u\|_{L^\infty((0,\infty);H^s)} < \infty, \quad \|\partial_{tt} u\|_{L^\infty((0,\infty);L)} < \infty. \tag{1.5}$$

These bounds are natural in the context of the Hille–Yosida theorem. They are proved by invoking generic regularity assumptions on the initial datum u^0 and by using (1.2)–(1.4). The reader is referred to Lemma 2.1 where this result is proved for Maxwell's equations. A similar result is asserted in Lemma 2.4 for the linear wave equation. Since we assume that the regularity in space can be very low, i.e., $s \in (0, \frac{1}{2})$, we do not exploit the full convergence capacity of the third- and fourth-order ERK methods in the error estimate proved in the paper. The crucial property of these methods we invoke instead is their L^2-stability under the standard CFL condition on the time step, as established in Proposition 4.1 below. This property is not available for second-order, two-stage ERK methods or for the forward Euler method.

The paper is organized as follows. In section 2, we show that Maxwell's equations and the linear wave equation have the generic properties invoked in the main convergence results of the paper. We introduce in section 3 the dG setting and the associated notation. The key properties of the Ritz projection are recalled in Theorem 3.5 in the case of Maxwell's equations and in Theorem 3.9 for the linear wave equations. In section 4, we introduce the fully discrete setting. We first rewrite the space semidiscrete problem in a unified setting and then use ERK schemes for the time discretization. Our main convergence results (Theorems 5.3 and 5.7) are stated and proved in section 5. We revisit the proofs of L^2-stability of the ERK schemes in section 6.

2. Examples. We show in this section that the generic properties mentioned above hold true for Maxwell's equations and the linear wave equation. Although everything that is said in the paper regarding space approximation could be written using the unified formalism of finite element exterior calculus (see Arnold, Falk, and Winther [3]), we prefer to use the vector calculus formalism to be more explicit even though the structures of the grad-div and curl-curl operators are similar.

2.1. Maxwell's equations. We consider Maxwell's equations in $\mathbb{R}^3$. Let $D \subset \mathbb{R}^d$, $d = 3$, be an open, bounded, connected polyhedron with Lipschitz boundary. We seek a pair of vector fields $(\boldsymbol{H}, \boldsymbol{E}) : D \to \mathbb{R}^d \times \mathbb{R}^d$ such that, for all $t > 0$,

$$\partial_t(\mu \boldsymbol{H}) + \nabla \times \boldsymbol{E} = \boldsymbol{0}, \qquad \partial_t(\epsilon \boldsymbol{E}) - \nabla_0 \times \boldsymbol{H} = \boldsymbol{0}, \qquad \boldsymbol{H}|_{\partial D} \times \boldsymbol{n}_D = \boldsymbol{0}, \tag{2.1}$$

where $\boldsymbol{H}$ is the magnetic field, $\boldsymbol{E}$ the electric field, and $\boldsymbol{n}_D$ the unit outward normal at the boundary ∂D of D. Here, both $\nabla\times$ and $\nabla_0\times$ denote the curl operator, but the operator $\nabla\times$ acts on vector fields without boundary conditions, whereas $\nabla_0\times$ only acts on fields with zero tangential trace. The operators $\nabla\times$ and $\nabla_0\times$ are adjoint to each other. Other boundary conditions can be considered as well. The magnetic permeability μ and the electrical permittivity ϵ are assumed to be time independent but can be heterogeneous. We assume that there is a partition of D into a finite number of disjoint Lipschitz polyhedra such that μ and ϵ are piecewise smooth on this partition.

Maxwell's equations fit the framework of the Hille–Yosida theorem by setting

$$u := (\boldsymbol{H}, \boldsymbol{E}), \tag{2.2a}$$

$$L := \boldsymbol{L}^2(D) \times \boldsymbol{L}^2(D), \quad ((\boldsymbol{H}, \boldsymbol{E}), (\boldsymbol{h}, \boldsymbol{e}))_L := (\mu \boldsymbol{H}, \boldsymbol{h})_{\boldsymbol{L}^2(D)} + (\epsilon \boldsymbol{E}, \boldsymbol{e})_{\boldsymbol{L}^2(D)}, \tag{2.2b}$$

$$A(u) := \left(\frac{1}{\mu} \nabla \times \boldsymbol{E}, -\frac{1}{\epsilon} \nabla_0 \times \boldsymbol{H} \right) \quad \text{with} \quad D(A) = \boldsymbol{H}_0(\mathbf{curl}; D) \times \boldsymbol{H}(\mathbf{curl}; D). \tag{2.2c}$$

Moreover, a key property of (2.1) is that $\mu\boldsymbol{H}$ is in the image of the curl operator (i.e., $\mu \boldsymbol{H} \in \operatorname{im} \nabla\times$) and $\epsilon\boldsymbol{E}$ is in the image of the curl operator acting on vector fields with zero tangential trace at the boundary (i.e., $\epsilon \boldsymbol{E} \in \operatorname{im} \nabla_0\times$). Since $\operatorname{im} \nabla\times = (\ker \nabla_0\times)^\perp$ and $\operatorname{im} \nabla_0\times = (\ker \nabla\times)^\perp$ (here, orthogonality is meant in $\boldsymbol{L}^2(D)$), the involution property of the solutions to (2.1) can be reformulated as follows:

$$\begin{aligned} &(\mu \boldsymbol{H}^0, \epsilon \boldsymbol{E}^0) \in (\ker \nabla_0\times)^\perp \times (\ker \nabla\times)^\perp \\ &\quad \Longrightarrow (\mu \boldsymbol{H}(t), \epsilon \boldsymbol{E}(t)) \in (\ker \nabla_0\times)^\perp \times (\ker \nabla\times)^\perp \quad \forall t > 0, \end{aligned} \tag{2.3}$$

which are Gauss's laws of magnetism and electrostatics. In addition to implying Gauss's laws, the involution property (2.3) implies that the solutions to (2.1) have some smoothness. More precisely, setting $\boldsymbol{X}_{\mu 0} \times \boldsymbol{X}_\epsilon := D(A) \cap \operatorname{im}(A)$, i.e.,

$$\boldsymbol{X}_{\mu 0} := \{ \boldsymbol{h} \in \boldsymbol{H}_0(\mathbf{curl}; D) \mid \mu \boldsymbol{h} \in (\ker \nabla_0\times)^\perp \}, \tag{2.4a}$$

$$\boldsymbol{X}_\epsilon := \{ \boldsymbol{e} \in \boldsymbol{H}(\mathbf{curl}; D) \mid \epsilon \boldsymbol{e} \in (\ker \nabla\times)^\perp \}, \tag{2.4b}$$

it is well established in the literature that there exists $s > 0$ so that

$$\boldsymbol{X}_{\mu 0} \times \boldsymbol{X}_\epsilon \subset \boldsymbol{H}^s(D) \times \boldsymbol{H}^s(D). \tag{2.5}$$

This is a well-known consequence of the elliptic regularity theory; see e.g., Bonito, Guermond, and Luddens [6], Jochmann [20]. In general, we have $s \in (0, \frac{1}{2})$, which is the case of interest in this paper. When μ and ϵ are constant or smooth, the smoothness index s is equal to $\frac{1}{2}$ in Lipschitz domains (see Costabel [12, Thm. 2], Birman and Solomyak [4, Thm. 3.1], Weber [27]), the smoothness index s is in the interval $(\frac{1}{2}, 1]$ if D is a Lipschitz polyhedron (see Amrouche et al. [2, Prop. 3.7]), and $s = 1$ if D is convex (see [2, Thm. 2.17]). The reader is also referred to Ciarlet [10, Thm. 16] for situations with heterogeneous properties for which smoothness is established with $s > \frac{1}{2}$.

The regularity of the solution to (2.1) that is invoked in the paper and stated in (1.5) takes the following form in the context of Maxwell's equations.

LEMMA 2.1 (smoothness). *Assume that the initial data satisfy* $(\boldsymbol{H}^0, \boldsymbol{E}^0) \in D(A^2) \cap \mathrm{im}(A)$. *Let* $s > 0$ *be the smoothness index so that* (2.5) *holds true. Then the solution to* (2.1) *is such that, for all* $T > 0$,

$$(\boldsymbol{H}, \boldsymbol{E}) \in L^\infty((0,T); \boldsymbol{H}^s(D) \times \boldsymbol{H}^s(D)), \tag{2.6a}$$

$$(\partial_t \boldsymbol{H}, \partial_t \boldsymbol{E}) \in L^\infty((0,T); \boldsymbol{H}^s(D) \times \boldsymbol{H}^s(D)), \tag{2.6b}$$

$$(\partial_{tt} \boldsymbol{H}, \partial_{tt} \boldsymbol{E}) \in L^\infty((0,T); \boldsymbol{L}^2(D) \times \boldsymbol{L}^2(D)). \tag{2.6c}$$

Proof. This is just an application of the Hille–Yosida theorem, but we sketch the proof for completeness. The proof consists of bootstrapping energy estimates by leveraging the assumed regularity of the initial condition. Let $T > 0$ be a fixed time.

(i) Taking one time derivative of (2.1), testing the PDEs with the pair $(\partial_t \boldsymbol{H}, \partial_t \boldsymbol{E})$, and using that $(\partial_t \boldsymbol{H})|_{\partial D} \times \boldsymbol{n}_D = \boldsymbol{0}$, gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{ \|\mu^{\frac{1}{2}} \partial_t \boldsymbol{H}(t)\|^2_{\boldsymbol{L}^2(D)} + \|\epsilon^{\frac{1}{2}} \partial_t \boldsymbol{E}(t)\|^2_{\boldsymbol{L}^2(D)} \Big\} = 0.$$

Since $(\boldsymbol{H}^0, \boldsymbol{E}^0) \in \boldsymbol{H}_0(\mathbf{curl}; D) \times \boldsymbol{H}(\mathbf{curl}; D)$, this gives that

$$\begin{aligned} &\|\mu^{-\frac{1}{2}} \nabla\times \boldsymbol{E}\|^2_{L^\infty((0,T);\boldsymbol{L}^2(D))} + \|\epsilon^{-\frac{1}{2}} \nabla_0\times \boldsymbol{H}\|^2_{L^\infty((0,T);\boldsymbol{L}^2(D))} \\ &\quad = \|\mu^{-\frac{1}{2}} \nabla\times \boldsymbol{E}^0\|^2_{\boldsymbol{L}^2(D)} + \|\epsilon^{-\frac{1}{2}} \nabla_0\times \boldsymbol{H}^0\|^2_{\boldsymbol{L}^2(D)} < \infty, \end{aligned}$$

where we used (2.1) to replace the time derivatives by the curl operators and the assumption $(\boldsymbol{H}^0, \boldsymbol{E}^0) \in D(A)$, thereby proving that $(\boldsymbol{H}(t), \boldsymbol{E}(t)) \in D(A)$ for all $t \in (0,T)$. Moreover, as $(\boldsymbol{H}(t), \boldsymbol{E}(t)) \in \mathrm{im}(A)$ for all $t \in (0,T)$ owing to (2.3) and the assumption $(\boldsymbol{H}^0, \boldsymbol{E}^0) \in \mathrm{im}(A)$, we conclude that both $\boldsymbol{H}$ and $\boldsymbol{E}$ are in $L^\infty((0,T); \boldsymbol{H}^s(D))$. This establishes (2.6a).

(ii) Taking two time derivatives of (2.1), testing with the pair $(\partial_{tt} \boldsymbol{H}, \partial_{tt} \boldsymbol{E})$, using $(\partial_{tt} \boldsymbol{H})|_{\partial D} \times \boldsymbol{n}_D = \boldsymbol{0}$, and $\mu^{\frac{1}{2}} \partial_{tt} \boldsymbol{H}(0) = -\mu^{-\frac{1}{2}} \nabla\times(\epsilon^{-1} \nabla_0\times \boldsymbol{H}^0)$, $\epsilon^{\frac{1}{2}} \partial_{tt} \boldsymbol{E}(0) = -\epsilon^{-\frac{1}{2}} \nabla_0\times(\mu^{-1} \nabla\times \boldsymbol{E}^0)$, and reasoning as above gives

$$\begin{aligned} &\|\mu^{\frac{1}{2}} \partial_{tt} \boldsymbol{H}\|^2_{L^\infty((0,T);\boldsymbol{L}^2(D))} + \|\epsilon^{\frac{1}{2}} \partial_{tt} \boldsymbol{E}\|^2_{L^\infty((0,T);\boldsymbol{L}^2(D))} \\ &\quad = \|\mu^{-\frac{1}{2}} \nabla\times(\epsilon^{-1} \nabla_0\times \boldsymbol{H}^0)\|^2_{\boldsymbol{L}^2(D)} + \|\epsilon^{-\frac{1}{2}} \nabla_0\times(\mu^{-1} \nabla\times \boldsymbol{E}^0)\|^2_{\boldsymbol{L}^2(D)} < \infty. \end{aligned}$$

As we also have $\nabla\times\partial_t \boldsymbol{E} = -\partial_{tt}(\mu \boldsymbol{H})$ and $\nabla_0\times\partial_t \boldsymbol{H} = \partial_{tt}(\epsilon \boldsymbol{E})$, this estimate implies that $\partial_t \boldsymbol{H}(t) \in \boldsymbol{H}_0(\mathbf{curl}; D)$, $\partial_t \boldsymbol{E}(t) \in \boldsymbol{H}(\mathbf{curl}; D)$ because $(\boldsymbol{H}_0, \boldsymbol{E}_0) \in D(A^2)$, i.e., $(\partial_t \boldsymbol{H}(t), \partial_t \boldsymbol{E}(t)) \in D(A)$ for all $t \in (0,T)$. Finally, observing that $(\partial_t \boldsymbol{H}(t), \partial_t \boldsymbol{E}(t)) \in \mathrm{im}(A)$, we conclude that $(\partial_t \boldsymbol{H}(t), \partial_t \boldsymbol{E}(t)) \in D(A) \cap \mathrm{im}(A)$ for all $t \in (0,T)$. This establishes (2.6b) and (2.6c). ∎

Remark 2.2 (involution). In view of Lemma 2.1, the involution property of the solution to (2.1) can be reformulated as $(\boldsymbol{H}(0), \boldsymbol{E}(0)) \in \boldsymbol{X}_{\mu 0} \times \boldsymbol{X}_\epsilon \Longrightarrow (\boldsymbol{H}(t), \boldsymbol{E}(t)) \in \boldsymbol{X}_{\mu 0} \times \boldsymbol{X}_\epsilon$, for all $t > 0$.

Remark 2.3 (right-hand side). In some applications, Ampère's law, i.e., the second equation in (2.1), comes with a source term $\boldsymbol{j}$ representing a current density and satisfying the charge conservation equation $\partial_t \rho + \nabla\cdot \boldsymbol{j} = 0$, where ρ is the charge density. Then the involution satisfied by the electric field takes the form $\nabla\cdot(\epsilon\boldsymbol{E})=\rho$. The functional setting described in the paper is unchanged if there are no charges or the charge distribution is time independent, i.e., $\partial_t\rho$ vanishes identically and $\boldsymbol{j}$ is divergence-free. The error analysis of ERK schemes then becomes somewhat more intricate as the representation of the semigroup action is now given by Duhamel's formula. This entails quadrature errors for the time integration of $\boldsymbol{j}$ that must be taken into account, but overall the stability and convergence results established in the paper are unchanged. The situation is more delicate when the charge distribution is time dependent. This case requires a modification of the functional setting that is left to future work.

2.2. Linear wave equation. Let us now consider the linear wave equation in $\mathbb{R}^d$ with $d\in\{2,3\}$. Assuming again that $D\subset\mathbb{R}^d$ is an open, bounded, connected polyhedron with Lipschitz boundary, we seek a pair of scalar and vector fields $(p,\boldsymbol{v}):D\to\mathbb{R}\times\mathbb{R}^d$ such that, for all $t>0$,

$$\partial_t(\kappa p)+\nabla_0\cdot\boldsymbol{v}=0,\qquad \partial_t(\rho\boldsymbol{v})+\nabla p=\boldsymbol{0},\qquad \boldsymbol{v}|_{\partial D}\cdot\boldsymbol{n}_D=0, \tag{2.7}$$

where p is the pressure and $\boldsymbol{v}$ the velocity, and $\nabla_0\cdot$ denotes the divergence operator acting on fields with zero normal trace. Other boundary conditions can be considered as well. Moreover, the compressibility κ and the density ρ are time-independent coefficients that can be heterogeneous. We assume that there is a partition of D into a finite number of disjoint Lipschitz polyhedra such that κ and ρ are piecewise smooth on this partition.

The linear wave equation fits the framework of the Hille–Yosida theorem by setting

$$u:=(p,\boldsymbol{v}), \tag{2.8a}$$

$$L:=L^2(D)\times\boldsymbol{L}^2(D),\quad ((p,\boldsymbol{v}),(q,\boldsymbol{w}))_L:=(\kappa p,q)_{L^2(D)}+(\rho\boldsymbol{v},\boldsymbol{w})_{\boldsymbol{L}^2(D)}, \tag{2.8b}$$

$$A(u):=\left(\frac{1}{\kappa}\nabla_0\cdot\boldsymbol{v},\frac{1}{\rho}\nabla p\right)\quad\text{with}\quad D(A)=H^1(D)\times\boldsymbol{H}_0(\mathrm{div};D). \tag{2.8c}$$

Moreover, since $\mathrm{im}(\nabla)=\ker(\nabla_0\cdot)^\perp$ and $\mathrm{im}(\nabla_0\cdot)=\ker(\nabla)^\perp$, the involution property associated with (2.7) is

$$\begin{aligned}(\kappa p^0,\rho\boldsymbol{v}^0)&\in(\ker\nabla)^\perp\times(\ker\nabla_0\cdot)^\perp\\ &\Longrightarrow(\kappa p(t),\rho\boldsymbol{v}(t))\in(\ker\nabla)^\perp\times(\ker\nabla_0\cdot)^\perp\quad\forall t>0.\end{aligned} \tag{2.9}$$

The involution on the pressure reduces to $\int_D\kappa(x)p(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}=0$ for all $t>0$ because D is connected, i.e., $\ker\nabla=\mathrm{span}\{1\}=\mathbb{P}_0$. The involution on the velocity amounts to $\rho\boldsymbol{v}\in(\ker\nabla_0\cdot)^\perp$, which implies in particular that $\nabla\times(\rho\boldsymbol{v})=\boldsymbol{0}$. Let us introduce

$$X_\kappa:=\{q\in H^1(D)\mid\kappa q\in(\ker\nabla)^\perp\}, \tag{2.10a}$$

$$\boldsymbol{X}_{\rho 0}:=\{\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{div};D)\mid\rho\boldsymbol{v}\in(\ker\nabla_0\cdot)^\perp\}. \tag{2.10b}$$

It is well established in the literature that there exists $s>0$ so that

$$\boldsymbol{X}_{\rho 0}\subset\boldsymbol{H}^s(D). \tag{2.11}$$

The smoothness index s is larger than $\frac{1}{2}$ if ρ is constant or smooth. The index s is in general smaller than $\frac{1}{2}$ if the mass distribution ρ is discontinuous. Obviously, we have $X_\kappa\subset H^1(D)$. Hence, combining (2.9) and (2.11), we infer that the solution to (2.7) satisfies $(p(t),\boldsymbol{v}(t))\in H^1(D)\times\boldsymbol{H}^s(D)$ for all $t>0$.

LEMMA 2.4 (smoothness). *Assume that the initial data for the linear wave equation satisfy* $(p^0, \boldsymbol{v}^0) \in D(A^2) \cap \mathrm{im}(A)$. *Let* $s > 0$ *be the smoothness index so that* (2.11) *holds true. Then the solution to* (2.7) *is such that, for all* $T > 0$,

$$(p, \boldsymbol{v}) \in L^\infty((0,T); H^1(D) \times \boldsymbol{H}^s(D)), \tag{2.12a}$$

$$(\partial_t p, \partial_t \boldsymbol{v}) \in L^\infty((0,T); H^1(D) \times \boldsymbol{H}^s(D)), \tag{2.12b}$$

$$(\partial_{tt} p, \partial_{tt} \boldsymbol{v}) \in L^\infty((0,T); L^2(D) \times \boldsymbol{L}^2(D)). \tag{2.12c}$$

Proof. Proceed as in the proof of Lemma 2.1. ∎

Remark 2.5 (topology of D). The involution on the velocity implies $\nabla \times \boldsymbol{v}(t) = \boldsymbol{0}$ for all $t > 0$, since $(\ker \nabla_0 \cdot)^\perp = \nabla H^1(D)$. Notice though that the identity $\nabla \times \boldsymbol{v}(t) = \boldsymbol{0}$ fully describes the involution only if D is simply connected because $(\ker \nabla_0 \cdot)^\perp = \ker \nabla\times$ if and only if D is simply connected. The involution that is independent of the topology of D is $\boldsymbol{v}(t) \in (\ker \nabla_0 \cdot)^\perp$ for all $t > 0$.

3. DG setting.

We describe in this section the dG setting that is used to approximate (2.1) in space. Although all the material that follows could be reformulated into one single abstract framework, we refrain from doing so to facilitate the reading of the paper. We separately introduce the discrete operators that are specific for Maxwell's equations and the linear wave equation. The material herein is standard and is only recalled for completeness. To be dimensionally consistent, we introduce a length scale, ℓ_D, associated with D; it can be, for instance, the diameter of D. We denote the simulation time by $T > 0$.

3.1. Meshes, averages, and jumps.

Let $(\mathcal{T}_h)_{h \in \mathcal{H}}$ be a shape-regular family of affine simplicial meshes such that each mesh covers D exactly and is compatible with the partition of D associated with the discontinuities of the material properties. The meshes are assumed to be quasi-uniform. The symbol K denotes a generic mesh cell, h_K its diameter, and $\boldsymbol{n}_K$ its outward unit normal. We define $\tilde{h}$ as the piecewise constant function on $\mathcal{T}_h$ such that $\tilde{h}|_K = h_K$ for all $K \in \mathcal{T}_h$; we set $h := \|\tilde{h}\|_{L^\infty(D)}$. The set of mesh faces, $\mathcal{F}_h$, is split into the subset of mesh interfaces, say $\mathcal{F}_h^\circ$, and the subset of mesh boundary faces, say $\mathcal{F}_h^\partial$. Each interface is shared by two distinct mesh cells which we denote K_l, K_r. Each boundary face is shared by one mesh cell, K_l, and the boundary, ∂D. For every mesh face $F \in \mathcal{F}_h$, h_F denotes the diameter of F. Every mesh interface $F \in \mathcal{F}_h^\circ$ is oriented by the unit normal, $\boldsymbol{n}_F$, pointing from K_l to K_r. The orientation of $\boldsymbol{n}_F$ is fixed once and for all, and this determines the labeling of the two mesh cells sharing F as K_l and K_r; see also Remark 3.3. Every boundary face $F \in \mathcal{F}_h^\partial$ is oriented by the unit normal $\boldsymbol{n}_F := \boldsymbol{n}_D$.

The approximation is done using polynomials of degree $k \geq 0$. We emphasize that it is legitimate to take $k = 0$ for the theory developed in this paper. The vector space over $\mathbb{R}$ or $\mathbb{C}$ (depending on the context) composed of d-variate polynomials of total degree at most k is denoted $\mathbb{P}_{k,d}$. We set $\boldsymbol{\mathbb{P}}_{k,d} := [\mathbb{P}_{k,d}]^d$ and consider the scalar- and vector-valued broken polynomial spaces

$$P_k^{\mathrm{b}}(\mathcal{T}_h) := \{v_h \in L^2(D) \mid v_{h|K} \in \mathbb{P}_{k,d} \ \forall K \in \mathcal{T}_h\}, \tag{3.1a}$$

$$\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) := \{\boldsymbol{w}_h \in \boldsymbol{L}^2(D) \mid \boldsymbol{w}_{h|K} \in \boldsymbol{\mathbb{P}}_{k,d} \ \forall K \in \mathcal{T}_h\}. \tag{3.1b}$$

We define the L^2- and $\boldsymbol{L}^2$-orthogonal projections onto the broken polynomial spaces $P_k^{\mathrm{b}}(\mathcal{T}_h)$ and $\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$ as

$$\Pi_h^{\mathrm{b}} : L^2(D) \to P_k^{\mathrm{b}}(\mathcal{T}_h), \qquad \boldsymbol{\Pi}_h^{\mathrm{b}} : \boldsymbol{L}^2(D) \to \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h). \tag{3.2}$$

For all $K \in \mathcal{T}_h$, all $F \in \mathcal{F}_h$ with $F \subset \partial K$, all $v_h \in P_k^{\mathrm{b}}(\mathcal{T}_h)$, and all $\boldsymbol{w}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$, we define the local trace operators such that $\gamma_{K,F}^{\mathrm{g}}(v_h)(\boldsymbol{x}) := v_h|_K(\boldsymbol{x})$, $\gamma_{K,F}^{\mathrm{g}}(\boldsymbol{w}_h)(\boldsymbol{x}) := \boldsymbol{w}_h|_K(\boldsymbol{x})$, $\gamma_{K,F}^{\mathrm{c}}(\boldsymbol{w}_h)(\boldsymbol{x}) := \boldsymbol{w}_h|_K(\boldsymbol{x}) \times \boldsymbol{n}_F$, $\gamma_{K,F}^{\mathrm{d}}(\boldsymbol{w}_h)(\boldsymbol{x}) := \boldsymbol{w}_h|_K(\boldsymbol{x}) \cdot \boldsymbol{n}_F$ for all $\boldsymbol{x} \in F$. Here, the superscript $\mathrm{x} \in \{\mathrm{g}, \mathrm{c}, \mathrm{d}\}$ refers to the gradient, curl, and divergence operators, since each trace is naturally associated with the corresponding operator. Then, for all $F = \partial K_l \cap \partial K_r \in \mathcal{F}_h^{\circ}$, we define the jump and average operators such that

(3.3a)
$$[\![v_h]\!]_F^{\mathrm{g}} := \gamma_{K_l,F}^{\mathrm{g}}(v_h) - \gamma_{K_r,F}^{\mathrm{g}}(v_h), \qquad \{\!\{v_h\}\!\}_F^{\mathrm{g}} := \frac{1}{2}\big(\gamma_{K_l,F}^{\mathrm{g}}(v_h) + \gamma_{K_r,F}^{\mathrm{g}}(v_h)\big),$$
(3.3b)
$$[\![\boldsymbol{w}_h]\!]_F^{\mathrm{x}} := \gamma_{K_l,F}^{\mathrm{x}}(\boldsymbol{w}_h) - \gamma_{K_r,F}^{\mathrm{x}}(\boldsymbol{w}_h), \qquad \{\!\{\boldsymbol{w}_h\}\!\}_F^{\mathrm{x}} := \frac{1}{2}\big(\gamma_{K_l,F}^{\mathrm{x}}(\boldsymbol{w}_h) + \gamma_{K_r,F}^{\mathrm{x}}(\boldsymbol{w}_h)\big).$$

To allow for more compact expressions, we also set $[\![\boldsymbol{w}_h]\!]_F^{\mathrm{x}} := \{\!\{\boldsymbol{w}_h\}\!\}_F^{\mathrm{x}} := \gamma_{K_l,F}^{\mathrm{x}}(\boldsymbol{w}_h)$ for all $F = \partial K_l \cap \partial D \in \mathcal{F}_h^{\partial}$.

3.2. Maxwell's equations. In this section, we exemplify the dG space semidiscretization for Maxwell's equations. The magnetic permeability and electric permittivity of vacuum are denoted μ_0, ϵ_0, respectively. To simplify the presentation, we assume throughout the paper that the ratios $\mu_0^{-1}\mu$ and $\epsilon_0^{-1}\epsilon$ are of order unity, so that these ratios can be hidden in the generic constants used in the error analysis. Setting $\mathfrak{c} := (\mu_0\epsilon_0)^{-\frac{1}{2}}$, we introduce the quantity $\omega := \mathfrak{c}\ell_D^{-1}$ which scales as the reciprocal of a time scale.

3.2.1. Discrete bilinear form and involution. We define the jump bilinear forms such that for all $\boldsymbol{H}_h, \boldsymbol{h}_h, \boldsymbol{E}_h, \boldsymbol{e}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$,

$$s_h^{\mathrm{c}}(\boldsymbol{H}_h, \boldsymbol{h}_h) := \sum_{F \in \mathcal{F}_h} ([\![\boldsymbol{H}_h]\!]_F^{\mathrm{c}}, [\![\boldsymbol{h}_h]\!]_F^{\mathrm{c}})_{\boldsymbol{L}^2(F)}, \tag{3.4a}$$
$$s_h^{\mathrm{c},\circ}(\boldsymbol{E}_h, \boldsymbol{e}_h) := \sum_{F \in \mathcal{F}_h^{\circ}} ([\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}, [\![\boldsymbol{e}_h]\!]_F^{\mathrm{c}})_{\boldsymbol{L}^2(F)}, \tag{3.4b}$$

which induce the seminorms $|\boldsymbol{h}_h|_{\mathrm{J}}^{\mathrm{c}} := s_h^{\mathrm{c}}(\boldsymbol{h}_h, \boldsymbol{h}_h)^{\frac{1}{2}}$ and $|\boldsymbol{e}_h|_{\mathrm{J}}^{\mathrm{c},\circ} := s_h^{\mathrm{c},\circ}(\boldsymbol{e}_h, \boldsymbol{e}_h)^{\frac{1}{2}}$. The discrete curl operators

$$\boldsymbol{C}_{h0} : \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \to \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h), \qquad \boldsymbol{C}_h : \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \to \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \tag{3.5}$$

are defined as follows: For all $(\boldsymbol{h}_h, \boldsymbol{e}_h) \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \times \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$, $\boldsymbol{C}_{h0}(\boldsymbol{h}_h)$ and $\boldsymbol{C}_h(\boldsymbol{e}_h)$ are the unique members of $\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$ such that the following identities hold true for all $\boldsymbol{e}'_h, \boldsymbol{h}'_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$,

$$(\boldsymbol{C}_{h0}(\boldsymbol{h}_h), \boldsymbol{e}'_h)_{\boldsymbol{L}^2(D)} := (\nabla_h \times \boldsymbol{h}_h, \boldsymbol{e}'_h)_{\boldsymbol{L}^2(D)} + \sum_{F \in \mathcal{F}_h} ([\![\boldsymbol{h}_h]\!]_F^{\mathrm{c}}, \{\!\{\boldsymbol{e}'_h\}\!\}_F^{\mathrm{g}})_{\boldsymbol{L}^2(F)}, \tag{3.6a}$$
$$(\boldsymbol{C}_h(\boldsymbol{e}_h), \boldsymbol{h}'_h)_{\boldsymbol{L}^2(D)} := (\nabla_h \times \boldsymbol{e}_h, \boldsymbol{h}'_h)_{\boldsymbol{L}^2(D)} + \sum_{F \in \mathcal{F}_h^{\circ}} ([\![\boldsymbol{e}_h]\!]_F^{\mathrm{c}}, \{\!\{\boldsymbol{h}'_h\}\!\}_F^{\mathrm{g}})_{\boldsymbol{L}^2(F)}, \tag{3.6b}$$

where $\nabla_h \times$ denotes the broken curl operator (evaluated cellwise). The following integration by parts formula holds true:

$$(\boldsymbol{C}_{h0}(\boldsymbol{h}_h), \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)} = (\boldsymbol{h}_h, \boldsymbol{C}_h(\boldsymbol{e}_h))_{\boldsymbol{L}^2(D)} \quad \forall (\boldsymbol{h}_h, \boldsymbol{e}_h) \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \times \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h). \tag{3.7a}$$

Setting $L_h := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \times \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$, we define the bilinear form $a_h : L_h \times L_h \to \mathbb{R}$ such that

$$\begin{aligned}(3.8)\qquad a_h\big((\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)\big) := &-(\boldsymbol{C}_h(\boldsymbol{E}_h), \boldsymbol{h}_h)_{\boldsymbol{L}^2(D)} + (\boldsymbol{C}_{h0}(\boldsymbol{H}_h), \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)} \\ &+ \kappa_{\mathrm{H}} s_h^{\mathrm{c}}(\boldsymbol{H}_h, \boldsymbol{h}_h) + \kappa_{\mathrm{E}} s_h^{\mathrm{c},\circ}(\boldsymbol{E}_h, \boldsymbol{e}_h)\end{aligned}$$

with $\kappa_{\mathrm{H}} := \mu_0 \omega \ell_D = (\mu_0/\epsilon_0)^{\frac{1}{2}}$ and $\kappa_{\mathrm{E}} := \epsilon_0 \omega \ell_D = (\epsilon_0/\mu_0)^{\frac{1}{2}}$ (notice that $\kappa_{\mathrm{H}}\kappa_{\mathrm{E}} = 1$). The Riesz representative of the bilinear form a_h is denoted $A_h : L_h \to L_h$. This operator is such that, for all $(\boldsymbol{H}_h, \boldsymbol{E}_h)$ in L_h, the following identity holds true, for all $(\boldsymbol{h}_h, \boldsymbol{e}_h) \in L_h$:

$$(3.9)\qquad \big(A_h(\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)\big)_{\boldsymbol{L}^2(D)\times\boldsymbol{L}^2(D)} := a_h\big((\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)\big).$$

The semidiscrete version of Maxwell's equations (2.1) consists of seeking the pair $(\boldsymbol{H}_h, \boldsymbol{E}_h) \in C^1([0,T]; L_h)$ so that

$$(3.10)\qquad \partial_t(\epsilon \boldsymbol{E}_h, \mu \boldsymbol{H}_h) + A_h(\boldsymbol{H}_h, \boldsymbol{E}_h) = \boldsymbol{0}, \qquad (\boldsymbol{H}_h(0), \boldsymbol{E}_h(0)) = (\boldsymbol{H}_h^0, \boldsymbol{E}_h^0)$$

with initial data $(\boldsymbol{H}_h^0, \boldsymbol{E}_h^0)$ approximating $(\boldsymbol{H}^0, \boldsymbol{E}^0)$. Assuming that $(\boldsymbol{E}_h^0, \boldsymbol{H}_h^0) \in \mathrm{im}(A_h)$, the pair $(\epsilon \boldsymbol{E}_h, \mu \boldsymbol{H}_h)$ remains in the image of the operator A_h as t grows, i.e.,

$$(3.11)\qquad (\epsilon \boldsymbol{E}_h(t), \mu \boldsymbol{H}_h(t)) \in \mathrm{im}(A_h) \quad \forall t > 0.$$

What makes this simple observation nontrivial is that $\mathrm{im}(A_h)$ is a genuine subset of L_h. Recall that $\mathrm{im}(A_h) = \ker(A_h^{\mathsf{T}})^{\perp}$. Let us now characterize $\ker(A_h^{\mathsf{T}})$. Let $\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)$ be the $\boldsymbol{H}(\mathbf{curl}; D)$-conforming finite element space composed of the piecewise Nédélec polynomials of the first family and of degree $k \geq 0$, and let $\boldsymbol{P}_{k0}^{\mathrm{c}}(\mathcal{T}_h)$ be its subspace composed of the discrete fields with zero tangential trace on ∂D. We set

$$(3.12a)\qquad \boldsymbol{P}_k^{\mathrm{c}}(\mathbf{curl} = \boldsymbol{0}; \mathcal{T}_h) := \{\boldsymbol{e}_h \in \boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h) \mid \nabla \times \boldsymbol{e}_h = \boldsymbol{0}\},$$
$$(3.12b)\qquad \boldsymbol{P}_{k0}^{\mathrm{c}}(\mathbf{curl} = \boldsymbol{0}; \mathcal{T}_h) := \{\boldsymbol{h}_h \in \boldsymbol{P}_{k0}^{\mathrm{c}}(\mathcal{T}_h) \mid \nabla_0 \times \boldsymbol{h}_h = \boldsymbol{0}\}.$$

Notice that $\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)$ is not a subspace of $\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$ (actually $\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h) \subsetneq \boldsymbol{P}_{k+1}^{\mathrm{b}}(\mathcal{T}_h)$), but we are going to make use of the following well-known results (see, e.g., [14, Chap. 15 and 18]):

$$(3.13a)\qquad \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \cap \ker \nabla\times = \boldsymbol{P}_k^{\mathrm{c}}(\mathbf{curl} = \boldsymbol{0}; \mathcal{T}_h),$$
$$(3.13b)\qquad \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \cap \ker \nabla_0\times = \boldsymbol{P}_{k0}^{\mathrm{c}}(\mathbf{curl} = \boldsymbol{0}; \mathcal{T}_h).$$

The following result is established in [17, Lem. 3.1].

Lemma 3.1 (discrete involution). *The following holds true:*

$$(3.14)\qquad \ker(A_h^{\mathsf{T}}) = \boldsymbol{P}_{k0}^{\mathrm{c}}(\mathbf{curl} = \boldsymbol{0}; \mathcal{T}_h) \times \boldsymbol{P}_k^{\mathrm{c}}(\mathbf{curl} = \boldsymbol{0}; \mathcal{T}_h).$$

Lemma 3.1 gives $\mathrm{im}(A_h) = \boldsymbol{P}_{k0}^{\mathrm{c}}(\mathbf{curl} = \boldsymbol{0}; \mathcal{T}_h)^{\perp} \times \boldsymbol{P}_k^{\mathrm{c}}(\mathbf{curl} = \boldsymbol{0}; \mathcal{T}_h)^{\perp}$. Accordingly, we set

$$(3.15a)\qquad \boldsymbol{X}_{\mu 0,h} := \{\boldsymbol{h}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \mid \mu \boldsymbol{h}_h \in \boldsymbol{P}_{k0}^{\mathrm{c}}(\mathbf{curl} = \boldsymbol{0}; \mathcal{T}_h)^{\perp}\},$$
$$(3.15b)\qquad \boldsymbol{X}_{\epsilon,h} := \{\boldsymbol{e}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \mid \epsilon \boldsymbol{e}_h \in \boldsymbol{P}_k^{\mathrm{c}}(\mathbf{curl} = \boldsymbol{0}; \mathcal{T}_h)^{\perp}\}.$$

With this notation, we observe that the solution to (3.10) satisfies the involution

$$(3.16)\quad (\boldsymbol{H}_h(0), \boldsymbol{E}_h(0)) \in \boldsymbol{X}_{\mu 0,h} \times \boldsymbol{X}_{\epsilon,h} \Longrightarrow (\boldsymbol{H}_h(t), \boldsymbol{E}_h(t)) \in \boldsymbol{X}_{\mu 0,h} \times \boldsymbol{X}_{\epsilon,h} \quad \forall t > 0.$$

Remark 3.2 (quadrangles and hexaedrons). The identities in (3.13) do not hold for quadrangles and hexaedrons with the definition of $\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$ given in (3.2). In this case, the dG space $\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$ has to be slightly enriched as explained in Perrier [23, section 5].

Remark 3.3 (face orientation). Notice that the stabilization bilinear forms defined in (3.4) and the consistency bilinear forms appearing on the right-hand side of (3.6) are quadratic in terms of the jumps $[\![\boldsymbol{h}_h]\!]_F^{\mathrm{c}}$, $[\![\boldsymbol{e}_h]\!]_F^{\mathrm{c}}$ and averages $\{\!\{\boldsymbol{h}_h\}\!\}_F^{\mathrm{c}}$, $\{\!\{\boldsymbol{e}_h\}\!\}_F^{\mathrm{c}}$; therefore, they do not depend on the choice that is made to orient the mesh interfaces.

3.2.2. Deflated inf-sup condition and Ritz projection. We equip L with the norm $\|(\boldsymbol{f},\boldsymbol{g})\|_L := \big(\|\mu^{\frac12}\boldsymbol{f}\|^2_{\boldsymbol{L}^2(D)} + \|\epsilon^{\frac12}\boldsymbol{g}\|^2_{\boldsymbol{L}^2(D)}\big)^{\frac12}$, and we equip L_h with the mesh-dependent norm

(3.17)

$$\|(\boldsymbol{h}_h,\boldsymbol{e}_h)\|_{\flat,h} := \omega^{\frac12}\|(\boldsymbol{h}_h,\boldsymbol{e}_h)\|_L \\ + \kappa_{\mathrm{H}}^{\frac12}\big\{\|\tilde{h}^{\frac12}\boldsymbol{C}_{h0}(\boldsymbol{h}_h)\|_{\boldsymbol{L}^2(D)} + |\boldsymbol{h}_h|_{\mathrm{J}}^{\mathrm{c}}\big\} + \kappa_{\mathrm{E}}^{\frac12}\big\{\|\tilde{h}^{\frac12}\boldsymbol{C}_{h}(\boldsymbol{e}_h)\|_{\boldsymbol{L}^2(D)} + |\boldsymbol{e}_h|_{\mathrm{J}}^{\mathrm{c},\circ}\big\}.$$

The following two results established in [16, 17] play a key role in the paper.

LEMMA 3.4 (deflated inf-sup condition). *There exists $c>0$ so that, for all $h\in\mathcal{H}$ and for every pair $(\boldsymbol{H}_h,\boldsymbol{E}_h)$ in $\boldsymbol{X}_{\mu0,h}\times\boldsymbol{X}_{\epsilon,h}$, the following holds true:*

$$\omega^{\frac12}\|(\boldsymbol{H}_h,\boldsymbol{E}_h)\|_{\flat,h} \le c \sup_{(\boldsymbol{h}_h,\boldsymbol{e}_h)\in L_h} \frac{|a_h\big((\boldsymbol{H}_h,\boldsymbol{E}_h),(\boldsymbol{h}_h,\boldsymbol{e}_h)\big)|}{\|(\boldsymbol{h}_h,\boldsymbol{e}_h)\|_L}. \tag{3.18}$$

THEOREM 3.5 (Ritz projection). *There is c, so that for all $h\in\mathcal{H}$ and all $(\boldsymbol{H},\boldsymbol{E})\in \boldsymbol{X}_{\mu0}\times\boldsymbol{X}_{\epsilon}$, there exists a unique pair $(\boldsymbol{\Pi}_{0,h}(\boldsymbol{H}),\boldsymbol{\Pi}_h(\boldsymbol{E}))\in\boldsymbol{X}_{\mu0,h}\times\boldsymbol{X}_{\epsilon,h}$ so that*

(3.19a)

$$a_h\big((\boldsymbol{\Pi}_{0,h}(\boldsymbol{H}),\boldsymbol{\Pi}_h(\boldsymbol{E})),(\boldsymbol{h}_h,\boldsymbol{e}_h)\big) = a\big((\boldsymbol{H},\boldsymbol{E}),(\boldsymbol{h}_h,\boldsymbol{e}_h)\big) \quad \forall(\boldsymbol{h}_h,\boldsymbol{e}_h)\in L_h,$$

(3.19b)

$$\|(\boldsymbol{H}-\boldsymbol{\Pi}_{0,h}(\boldsymbol{H}),\boldsymbol{E}-\boldsymbol{\Pi}_h(\boldsymbol{E}))\|_L \le c\ell_D^{1-s}h^s\big(\mu_0^{\frac12}\|\nabla\times\boldsymbol{H}\|_{\boldsymbol{L}^2(D)} + \epsilon_0^{\frac12}\|\nabla\times\boldsymbol{E}|_{\boldsymbol{L}^2(D)}\big),$$

with $a\big((\boldsymbol{H},\boldsymbol{E}),(\boldsymbol{h}_h,\boldsymbol{e}_h)\big) := -(\nabla\times\boldsymbol{E},\boldsymbol{h}_h)_{\boldsymbol{L}^2(D)} + (\nabla_0\times\boldsymbol{H},\boldsymbol{e}_h)_{\boldsymbol{L}^2(D)}$.

Remark 3.6 (stabilization versus centered flux). Notice that it is essential that the stabilization coefficients κ_{H}, κ_{E} be nonzero for equality to hold in (3.14). When the dG numerical flux is centered, i.e., $\kappa_{\mathrm{H}}=0$ and $\kappa_{\mathrm{E}}=0$, we only have $\mathrm{im}(A_h)\subsetneq \boldsymbol{P}_{k0}^{\mathrm{c}}(\mathbf{curl}=\mathbf{0};\mathcal{T}_h)^\perp\times\boldsymbol{P}_k^{\mathrm{c}}(\mathbf{curl}=\mathbf{0};\mathcal{T}_h)^\perp$. This nevertheless implies that a discrete version of Gauss's laws is satisfied (as observed, e.g., in Campos Pinto and Sonnendrücker [9, section 5.4]), but this is not enough for the approximation to be convergent under the stringent smoothness assumptions considered in this paper. More precisely, it is essential that $\kappa_{\mathrm{H}}>0$ and $\kappa_{\mathrm{E}}>0$ to establish the deflated inf-sup condition (3.18), which, in turn, gives the estimate (3.19b). The importance of stabilization is now well recognized in the literature; see, e.g., Alvarez et al. [1], Hesthaven and Warburton [19]. In conclusion, the combination of (3.14) and (3.18) can be viewed as the discrete counterparts of (1.2), i.e., $\mathrm{im}(A)\cap D(A)\subset H^s$ only when $\kappa_{\mathrm{H}}>0$ and $\kappa_{\mathrm{E}}>0$.

3.3. Linear wave equations. In this section, we exemplify the dG space semi-discretization for the linear wave equations. Let κ_0,ρ_0 be reference scales for κ and ρ, respectively. We assume throughout the paper that the ratios $\kappa_0^{-1}\kappa$ and $\rho_0^{-1}\rho$ are of order unity. Setting $\mathfrak{c} := (\kappa_0\rho_0)^{-\frac12}$, we introduce the quantity $\omega := \mathfrak{c}\ell_D^{-1}$ which scales as the reciprocal of a time scale.

3.3.1. Discrete bilinear form and involution. We define the jump bilinear forms such that, for all $p_h, q_h \in P_k^{\mathrm{b}}(\mathcal{T}_h)$ and all $\boldsymbol{v}_h, \boldsymbol{w}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$,

$$s_h^{\mathrm{g},\circ}(p_h, q_h) := \sum_{F\in\mathcal{F}_h^\circ} ([\![p_h]\!]_F^{\mathrm{g}}, [\![q_h]\!]_F^{\mathrm{g}})_{L^2(F)}, \tag{3.20a}$$

$$s_h^{\mathrm{d}}(\boldsymbol{v}_h, \boldsymbol{w}_h) := \sum_{F\in\mathcal{F}_h} ([\![\boldsymbol{v}_h]\!]_F^{\mathrm{d}}, [\![\boldsymbol{w}_h]\!]_F^{\mathrm{d}})_{L^2(F)}, \tag{3.20b}$$

which induce the seminorms $|q_h|_h^{\mathrm{g},\circ} := s_h^{\mathrm{g},\circ}(q_h, q_h)^{\frac{1}{2}}$ and $|\boldsymbol{v}_h|_h^{\mathrm{d}} := s_h^{\mathrm{d}}(\boldsymbol{v}_h, \boldsymbol{v}_h)^{\frac{1}{2}}$. The discrete gradient and divergence operators

$$\boldsymbol{G}_h : P_k^{\mathrm{b}}(\mathcal{T}_h) \to \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h), \qquad D_{h0} : \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \to P_k^{\mathrm{b}}(\mathcal{T}_h) \tag{3.21}$$

are defined as follows: For all $(q_h, \boldsymbol{v}_h) \in P_k^{\mathrm{b}}(\mathcal{T}_h) \times \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$, $\boldsymbol{G}_h(q_h)$ and $D_{h0}(\boldsymbol{v}_h)$ are the unique members of $\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$ and $P_k^{\mathrm{b}}(\mathcal{T}_h)$ such that the following identities hold true for all $q_h' \in P_k^{\mathrm{b}}(\mathcal{T}_h)$ and $\boldsymbol{v}_h' \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$:

$$(\boldsymbol{G}_h(p_h), \boldsymbol{v}_h')_{\boldsymbol{L}^2(D)} := (\nabla_h p_h, \boldsymbol{v}_h')_{\boldsymbol{L}^2(D)} - \sum_{F\in\mathcal{F}_h^\circ} ([\![q_h]\!]_F^{\mathrm{g}}, \{\!\{\boldsymbol{v}_h'\}\!\}_F^{\mathrm{d}})_{L^2(F)}, \tag{3.22a}$$

$$(D_{h0}(\boldsymbol{v}_h), q_h')_{L^2(D)} := (\nabla_h\cdot \boldsymbol{v}_h, q_h')_{L^2(D)} - \sum_{F\in\mathcal{F}_h} ([\![\boldsymbol{v}_h]\!]_F^{\mathrm{d}}, \{\!\{q_h'\}\!\}_F^{\mathrm{g}})_{L^2(F)}, \tag{3.22b}$$

where ∇_h and $\nabla_h\cdot$ denote the broken gradient and divergence operators (evaluated cellwise). The following integration by parts formula holds true:

$$(\boldsymbol{G}_h(q_h), \boldsymbol{v}_h)_{\boldsymbol{L}^2(D)} = -(q_h, D_{h0}(\boldsymbol{v}_h))_{L^2(D)} \quad \forall (q_h, \boldsymbol{v}_h) \in P_k^{\mathrm{b}}(\mathcal{T}_h) \times \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h). \tag{3.23}$$

Setting $L_h := P_k^{\mathrm{b}}(\mathcal{T}_h) \times \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$, we define the bilinear form $a_h : L_h \times L_h \to \mathbb{R}$ such that

$$\begin{aligned} a_h\big((p_h, \boldsymbol{u}_h), (q_h, \boldsymbol{v}_h)\big) := (D_{h0}(\boldsymbol{u}_h), q_h)_{L^2(D)} + (\boldsymbol{G}_h(p_h), \boldsymbol{v}_h)_{\boldsymbol{L}^2(D)} \\ + \kappa_{\mathrm{P}} s_h^{\mathrm{g},\circ}(p_h, q_h) + \kappa_{\mathrm{V}} s_h^{\mathrm{d}}(\boldsymbol{u}_h, \boldsymbol{v}_h) \end{aligned} \tag{3.24}$$

with $\kappa_{\mathrm{P}} := \kappa_0 \omega \ell_D = (\kappa_0/\rho_0)^{\frac{1}{2}}$ and $\kappa_{\mathrm{V}} := \rho_0 \omega \ell_D = (\rho_0/\kappa_0)^{\frac{1}{2}}$ (notice that $\kappa_{\mathrm{P}}\kappa_{\mathrm{V}} = 1$). The Riesz representative of the bilinear form a_h is the operator $A_h : L_h \to L_h$ such that the following identity holds true for all pairs $(p_h, \boldsymbol{u}_h), (q_h, \boldsymbol{w}_h) \in L_h$:

$$\big(A_h(p_h, \boldsymbol{u}_h), (q_h, \boldsymbol{v}_h)\big)_{L^2(D)\times\boldsymbol{L}^2(D)} := a_h\big((p_h, \boldsymbol{u}_h), (q_h, \boldsymbol{v}_h)\big). \tag{3.25}$$

The semidiscrete version of the linear wave equations (2.7) consists of seeking the pair $(p_h, \boldsymbol{u}_h) \in C^1([0, T]; L_h)$ so that

$$\partial_t(\kappa p_h, \rho \boldsymbol{u}_h) + A_h(p_h, \boldsymbol{u}_h) = 0, \qquad (p_h(0), \boldsymbol{u}_h(0)) = (p_h^0, \boldsymbol{u}_h^0) \tag{3.26}$$

with initial data $(p_h^0, \boldsymbol{u}_h^0)$ approximating $(p^0, \boldsymbol{u}^0)$. Assuming that $(p_h^0, \boldsymbol{u}_h^0) \in \mathrm{im}(A_h)$, the pair $(\kappa p_h, \rho \boldsymbol{u}_h)$ remains in the image of the operator A_h as t grows, i.e.,

$$(\kappa p_h(t), \rho \boldsymbol{u}_h(t)) \in \mathrm{im}(A_h) \quad \forall t > 0. \tag{3.27}$$

We now characterize $\ker(A_h^{\mathsf{T}}) = \mathrm{im}(A_h)^\perp$. We first observe that $\{q_h \in P^{\mathrm{b}}(\mathcal{T}_h) \mid \nabla q_h = \boldsymbol{0}\} = \mathbb{P}_0$. Let $\boldsymbol{P}_{k0}^{\mathrm{d}}(\mathcal{T}_h)$ be the $\boldsymbol{H}_0(\mathrm{div}; D)$-conforming space composed of the piecewise Raviart–Thomas finite elements of degree $k \geq 0$ with zero normal trace on ∂D. We set

$$\boldsymbol{P}_{k0}^{\mathrm{d}}(\mathrm{div} = 0; \mathcal{T}_h) := \{\boldsymbol{v}_h \in \boldsymbol{P}_{k0}^{\mathrm{d}}(\mathcal{T}_h) \mid \nabla_0\cdot \boldsymbol{v}_h = 0\}. \tag{3.28}$$

Notice that $\boldsymbol{P}^{\mathrm{d}}_{k0}(\mathcal{T}_h)$ is not a subspace of $\boldsymbol{P}^{\mathrm{b}}_{k}(\mathcal{T}_h)$ (actually $\boldsymbol{P}^{\mathrm{d}}_{k0}(\mathcal{T}_h) \subsetneq \boldsymbol{P}^{\mathrm{b}}_{k+1}(\mathcal{T}_h)$), but the following well-known result holds (see, e.g., [14, Lem. 14.9 and Thm. 18.10]):

$$\boldsymbol{P}^{\mathrm{b}}_{k}(\mathcal{T}_h) \cap \ker \nabla_0\cdot = \boldsymbol{P}^{\mathrm{d}}_{k0}(\mathrm{div}=0;\mathcal{T}_h). \tag{3.29}$$

Lemma 3.7 (discrete involution). *The following holds true:*

$$\ker(A_h^{\mathsf{T}}) = \mathbb{P}_0 \times \boldsymbol{P}^{\mathrm{d}}_{k0}(\mathrm{div}=0;\mathcal{T}_h). \tag{3.30}$$

Lemma 3.7 gives $\mathrm{im}(A_h) = \mathbb{P}_0^{\perp} \times \boldsymbol{P}^{\mathrm{d}}_{k0}(\mathrm{div}=0;\mathcal{T}_h)^{\perp}$. Accordingly, we set

$$X_{\kappa,h} := \left\{ q_h \in P^{\mathrm{b}}_{k}(\mathcal{T}_h) \mid \int_D \kappa q_h \,\mathrm{d}x = 0 \right\}, \tag{3.31a}$$

$$\boldsymbol{X}_{\rho 0,h} := \{ \boldsymbol{v}_h \in \boldsymbol{P}^{\mathrm{b}}_{k}(\mathcal{T}_h) \mid \rho \boldsymbol{v}_h \in \boldsymbol{P}^{\mathrm{d}}_{k0}(\mathrm{div}=0;\mathcal{T}_h)^{\perp} \}. \tag{3.31b}$$

With this notation, we observe that the solution to (3.26) satisfies the involution

$$(p_h(0),\boldsymbol{u}_h(0)) \in X_{\kappa,h} \times \boldsymbol{X}_{\rho 0,h} \Longrightarrow (p_h(t),\boldsymbol{u}_h(t)) \in X_{\kappa,h} \times \boldsymbol{X}_{\rho 0,h} \quad \forall t>0. \tag{3.32}$$

3.3.2. Deflated inf-sup condition and Ritz projection. We equip L with the norm $\|(f,\boldsymbol{g})\|_L := \big(\|\kappa^{\frac12} f\|^2_{L^2(D)} + \|\rho^{\frac12}\boldsymbol{v}\|^2_{\boldsymbol{L}^2(D)}\big)^{\frac12}$ and we equip L_h with the mesh-dependent norm

(3.33)

$$\|(p_h,\boldsymbol{u}_h)\|_{\flat,h} := \omega^{\frac12}\|(p_h,\boldsymbol{u}_h)\|_L + \kappa_{\mathrm{P}}^{\frac12}\big\{\|\tilde{h}^{\frac12}\boldsymbol{G}_h(p_h)\|_{\boldsymbol{L}^2(D)} + |p_h|^{\mathrm{g},\circ}_{\mathrm{J}}\big\} + \kappa_{\mathrm{V}}^{\frac12}\big\{\|\tilde{h}^{\frac12} D_{h0}(\boldsymbol{u}_h)\|_{L^2(D)} + |\boldsymbol{u}_h|^{\mathrm{d}}_{\mathrm{J}}\big\}.$$

The following two results established in [16, 17] play a key role in the paper.

Lemma 3.8 (deflated inf-sup condition). *There exists $c>0$ so that, for all $h \in \mathcal{H}$ and for every pair $(p_h,\boldsymbol{u}_h)$ in $X_{\kappa,h}\times \boldsymbol{X}_{\rho 0,h}$, the following holds true:*

$$\omega^{\frac12}\|(p_h,\boldsymbol{u}_h)\|_{\flat,h} \le c \sup_{(q_h,\boldsymbol{v}_h)\in L_h} \frac{|a_h\big((p_h,\boldsymbol{u}_h),(q_h,\boldsymbol{v}_h)\big)|}{\|(q_h,\boldsymbol{v}_h)\|_L}. \tag{3.34}$$

Theorem 3.9 (Ritz projection). *There is c so that, for all $h\in\mathcal{H}$ and all $(p,\boldsymbol{u}) \in X_\kappa \times \boldsymbol{X}_{\rho 0}$, there exists a unique pair $(\Pi_h(p),\boldsymbol{\Pi}_{0,h}(\boldsymbol{u})) \in X_{\kappa,h}\times\boldsymbol{X}_{\rho 0,h}$ so that*

$$a_h\big((\Pi_h(p),\boldsymbol{\Pi}_{0,h}(\boldsymbol{u})),(q_h,\boldsymbol{v}_h)\big) = a\big((p,\boldsymbol{u}),(q_h,\boldsymbol{v}_h)\big) \quad \forall (q_h,\boldsymbol{v}_h)\in L_h, \tag{3.35a}$$

$$\|(p-\Pi_h(p),\boldsymbol{u}-\boldsymbol{\Pi}_{0,h}(\boldsymbol{u}))\|_L \le c\ell_D^{1-s}h^s(\kappa_0^{\frac12}\|\nabla p\|_{\boldsymbol{L}^2(D)} + \rho_0^{\frac12}\|\nabla\cdot\boldsymbol{u}\|_{\boldsymbol{L}^2(D)}), \tag{3.35b}$$

with $a\big((p,\boldsymbol{u}),(q_h,\boldsymbol{v}_h)\big) := (\nabla_0\cdot\boldsymbol{u},q_h)_{L^2(D)} + (\nabla p,\boldsymbol{v}_h)_{\boldsymbol{L}^2(D)}$.

Remark 3.10 (smoothnesss index). The deflated inf-sup condition and the error estimate for the Ritz projection are proved in [16] under the assumption $s>\frac12$, but the duality argument from [17] can be easily adapted to prove the deflated inf-sup condition and the error estimate for the Ritz projection in the present setting where the coefficients κ and ρ are discontinuous and the smoothness index s is in $(0,\frac12)$. Moreover, as in Remark 3.6, we emphasize that the combination of (3.30) and (3.34) can be viewed as the discrete counterparts of $\mathrm{im}(A)\cap D(A) \subset H^s$ only when $\kappa_{\mathrm{H}}>0$ and $\kappa_{\mathrm{E}}>0$. The importance of stabilization is now well recognized in the literature; see, e.g., Stanglmeier et al. [24].

4. Fully discrete setting. In this section, we first rewrite the space semidiscrete problems derived in the previous section in a unified setting and then use ERK schemes for the time discretization.

4.1. Unified setting for the space semidiscrete problem. Since the proposed method can be used for solving a large class of problems having the same structure as Maxwell's equations and the linear wave equations, we now adopt the abstract notation introduced in section 1. This will facilitate the stability and convergence analysis reported below. Recall that we want to solve the time-dependent problem (1.1), where the linear operator $A : D(A) \subset L \to L$ is unbounded and maximal monotone, and L is a Hilbert space equipped with the inner product $(\cdot,\cdot)_L$. To allow for a bit more generality, we consider here complex linear spaces. Recall also that we assume that there exists $s \in (0, \frac{1}{2})$ and a subspace H^s dense in L with compact embedding, i.e., $H^s \Subset L$, so that

$$X := \mathrm{im}(A) \cap D(A) \subset H^s. \tag{4.1}$$

We denote $|v|_X := \|A(v)\|_L$. We set $\|v\|_X := \|v\|_L + \ell|v|_X$ where ℓ is a length scale that makes the above definition dimensionally consistent. A simple choice is to set $\ell := \ell_D$. We denote by $|v|_{H^s}$ any seminorm on H^s such that $\|v\|_L + \ell^s|v|_{H^s}$ is a norm equivalent to $\|v\|_{H^s}$. Recalling Lemmas 2.1 and 2.4, the above property means that under natural assumptions on the initial data u^0, the solution to (1.1) satisfies the following smoothness property:

$$\|u\|_{L^\infty((0,\infty);X)} < \infty, \quad \|\partial_t u\|_{L^\infty((0,\infty);X)} < \infty, \quad \|\partial_{tt} u\|_{L^\infty((0,\infty);L)} < \infty. \tag{4.2}$$

We now approximate "in space" the problem (1.1) by introducing a sequence $(L_h)_{h\in\mathcal{H}}$ of finite-dimensional subspaces of L equipped with the inner product $(\cdot,\cdot)_L$. We then consider the ODE system

$$\partial_t u_h + A_h(u_h) = 0, \quad u_h(0) = u_h^0, \tag{4.3}$$

where $A_h : L_h \to L_h$ is a linear operator approximating A. Let $\Pi_h^{\mathrm{b}} : L \to L_h$ denote the L-orthogonal projection. We formalize the approximation properties of Π_h^{b} and A_h by assuming that there exists a linear operator $\Pi_h : X \to L_h$ (which we call the Ritz projection) and a constant $c > 0$ such that, for all $u \in X$ and all $h \in \mathcal{H}$,

$$(A_h \circ \Pi_h)(u) = (\Pi_h^{\mathrm{b}} \circ A)(u), \tag{4.4a}$$

$$\|\Pi_h(u) - u\|_L \le c\ell^{1-s}h^s|u|_X, \tag{4.4b}$$

$$\|\Pi_h^{\mathrm{b}}(u) - u\|_L \le ch^s|u|_{H^s}. \tag{4.4c}$$

Recall that we only assume $s \in (0, \frac{1}{2})$. The estimate (4.4b) holds true for the dG approximation of Maxwell's equations (see Theorem 3.5) and the linear wave equation (see Theorem 3.9). The notion of involution invoked in Lemmas 3.1 and 3.7 and in the deflated inf-sup conditions (3.4) and (3.8) is essential to establish these error estimates.

We finish by saying a few words on the dissipation properties of A_h. Just like the operator A satisfies $\Re((A(v), v)_L) \ge 0$ for all $v \in D(A)$, we assume that $A_h + A_h^{\mathsf{H}}$ is nonnegative (where A_h^{H} denotes the Hermitian transpose of A_h). More precisely, we set

$$s_h(v_h, w_h) := \left(\frac{1}{2}(A_h + A_h^{\mathsf{H}})(v_h), w_h\right)_L \qquad \forall v_h, w_h \in L_h, \tag{4.5}$$

and assume that the Hermitian sesquilinear form $s(\cdot,\cdot)$ is nonnegative, i.e.,

$$s_h(v_h, v_h) = \Re\big((A_h(v_h), v_h)_L\big) \geq 0 \qquad \forall v_h \in L_h. \tag{4.6}$$

We henceforth refer to this property by saying that A_h is dissipative. We denote

$$|v_h|_S := \big(s_h(v_h, v_h)\big)^{\frac{1}{2}}, \quad (v_h, w_h)_S := s_h(v_h, w_h) \qquad \forall v_h, w_h \in L_h. \tag{4.7}$$

The above definitions imply that

$$\Re\big((A_h(v_h), v_h)_L\big) = |v_h|_S^2 \qquad \forall v_h \in L_h. \tag{4.8}$$

Notice that (4.5) can be viewed as an integration by parts formula since

$$(A_h(v_h), w_h)_L = -(v_h, A_h(w_h))_L + 2s_h(v_h, w_h) \qquad \forall v_h, w_h \in L_h. \tag{4.9}$$

One important question regarding ERK schemes is their L^2-stability under the standard CFL condition $\tau C(A_h) \leq \sigma$, where $C(A_h)$ is a constant depending on the stability properties of the discrete operator A_h. To be able to estimate $C(A_h)$, we assume that there exists a constant c_L so that the following holds true for all $h \in \mathcal{H}$:

$$\|A_h(v_h)\|_L \leq c_L h^{-1} \|v_h\|_L \qquad \forall v_h \in L_h. \tag{4.10}$$

In the dG context, this inequality is established by invoking discrete inverse and trace inequalities, and using that the underlying operator A is a first-order differential operator. We also notice that (4.10) readily implies that $|v_h|_S^2 \leq c_L h^{-1} \|v_h\|_L^2$.

4.2. ERK schemes. For the time discretization of the ODE system (4.3), we consider three-stage, third-order and four-stage, fourth-order ERK schemes, which we call ERK(3,3) and ERK(4,4). The time step is taken constant for simplicity, but the present resuts still hold true with variable time steps for the ERK(3,3) method. Given $u_h^{n,0} := u_h^n$, where u_h^n is the approximation of u at time t^n, and given a time step τ, the computation of the update u_h^{n+1} by the ERK(3,3) scheme is done in three stages as follows:

$$u_h^{n,1} = u_h^{n,0} - \tau A_h u_h^{n,0}, \tag{4.11a}$$

$$u_h^{n,2} = u_h^{n,1} - \frac{1}{2}\tau A_h(u_h^{n,1} - u_h^{n,0}), \tag{4.11b}$$

$$u_h^{n+1} := u_h^{n,3} = u_h^{n,2} - \frac{1}{3}\tau A_h(u_h^{n,2} - u_h^{n,1}). \tag{4.11c}$$

The computation of the update u_h^{n+1} by the ERK(4,4) scheme is done in four stages as follows:

$$u_h^{n,1} = u_h^{n,0} - \tau A_h u_h^{n,0}, \tag{4.12a}$$

$$u_h^{n,2} = u_h^{n,1} - \frac{1}{2}\tau A_h(u_h^{n,1} - u_h^{n,0}), \tag{4.12b}$$

$$u_h^{n,3} = u_h^{n,2} - \frac{1}{3}\tau A_h(u_h^{n,2} - u_h^{n,1}), \tag{4.12c}$$

$$u_h^{n+1} := u_h^{n,4} = u_h^{n,3} - \frac{1}{4}\tau A_h(u_h^{n,3} - u_h^{n,2}). \tag{4.12d}$$

Let $P_l(z) := \sum_{m\in\{0:l\}} \frac{1}{m!} z^m$ be the Taylor polynomial approximating the exponential function at order $l \in \mathbb{N}$ at 0. A straightforward computation shows that (4.11) amounts to $u_h^{n+1} = P_3(-\tau A_h)(u_h^n)$ and that (4.12) amounts to $u_h^{n+1} = P_4(-\tau A_h)(u_h^n)$.

Equations (4.11) and (4.12) each provide one possible realization of ERK(3,3) and ERK(4,4) schemes. More generally, ERK schemes are specified by means of their Butcher array. However, in the absence of external forcing and for linear evolution PDEs, as in (4.3), every ERK(3,3) scheme and every ERK(4,4) scheme eventually reduces, after elimination of the intermediate stages, to the identities $u_h^{n+1} = P_3(-\tau A_h)(u_h^n)$ and $u_h^{n+1} = P_4(-\tau A_h)(u_h^n)$, respectively. Therefore, to establish L^2-stability, it is sufficient to consider the schemes (4.11) and (4.12) as they simplify some of the forthcoming algebra.

PROPOSITION 4.1 (L^2-stability). *Assume that A_h satisfies* (4.10) *and is dissipative (see* (4.6)*). There exists $\sigma_0 > 0$ so that, for all $h \in \mathcal{H}$ and all τ satisfying the following standard (hyperbolic) CFL condition,*

$$c_L \tau h^{-1} \le \sigma_0, \tag{4.13}$$

we have

$$\|P_3(-\tau A_h)\|_{\mathcal{L}(L_h)} \le 1, \qquad \|P_4(-\tau A_h)^2\|_{\mathcal{L}(L_h)} \le 1. \tag{4.14}$$

Consequently, if the sequence $(u_h^n)_{n\in\mathbb{N}}$ is generated by the ERK(3,3) *scheme, we have $\|u_h^n\|_L \le \|u_h^0\|_L$, and if the sequence $(u_h^n)_{n\in\mathbb{N}}$ is generated by the ERK*(4,4) *scheme, we have $\|u_h^n\|_L \le \max\big(\|u_h^0\|_L, \|u_h^1\|_L\big)$.*

Remark 4.2 (literature). The L^2-stability of ERK(3,3) schemes under the standard hyperbolic CFL condition (4.13) is established in Burman, Ern, and Fernández [8] for Friedrichs systems approximated with stabilized continuous/discontinuous finite elements, and it is established in Zhang and Shu [30] for scalar conservation laws approximated with stabilized discontinuous finite elements and one specific ERK(3,3) scheme. In particular, the L^2-stability of ERK(3,3) schemes in the form $\|u_h^{n+1}\|_L \le \|u_h^n\|_L$ readily follows from Burman, Ern, and Fernández [8, Lem. 4.2], and the estimate in Step (iii) of the proof of Lemma 4.3 therein with $\alpha_h^n = \beta_h^n = \gamma_h^n = 0$ and $\Lambda_h^n = 0$. Concerning the L^2-stability of the ERK(4,4) schemes under the CFL condition (4.13), we refer the reader to Xu et al. [29]; see also Sun and Shu [25, 26]. For completeness, we provide somewhat simpler proofs of these statements in section 6.

5. Convergence. Our objective in this section is to prove convergence under the following minimal regularity assumptions,

$$\|\partial_{tt} u\|_{L^\infty((0,\infty);L)} < \infty, \quad \|\partial_t u\|_{L^\infty((0,\infty);X)} < \infty, \quad \|u\|_{L^\infty((0,\infty);X)} < \infty, \tag{5.1}$$

where $s > 0$. We recall that these assumptions are met owing to the involution property (1.4) provided the initial datum u_0 satisfies elementary structural properties; see, e.g., Lemma 2.1. In what follows, the symbol C denotes a generic positive constant whose value can change at each occurrence provided it is independent of the mesh and time-step sizes.

5.1. Preliminaries. Let $u_h^0 \in L_h$ be a reasonable approximation of u^0 which we recall is assumed to belong to the space X defined in (4.1). For instance, setting $u_h^0 := \Pi_h^{\mathrm{b}}(u_0)$, we have

$$\|u^0 - u_h^0\|_L \le C h^s |u^0|_{H^s}. \tag{5.2}$$

To establish the consistency of the approximating scheme, we are going to use the following key identity satisfied by the exact solution, which is a direct consequence of (4.4a) and (1.1):

$$\Pi_h^{\mathrm{b}}\big(\partial_t u(t)\big) = -A_h\big(\Pi_h\big(u(t)\big)\big). \tag{5.3}$$

5.2. ERK(3,3). We start by introducing the discrete errors

$$e_h^n := e_h^{n,0} := u_h^{n,0} - \Pi_h\big(u(t^n)\big), \tag{5.4a}$$

$$e_h^{n,1} := u_h^{n,1} - \Pi_h\big(u(t^n) + \tau\partial_t u(t^n)\big), \tag{5.4b}$$

$$e_h^{n,2} := u_h^{n,2} - \Pi_h\big(u(t^n) + \tau\partial_t u(t^n)\big), \tag{5.4c}$$

$$e_h^{n+1} := e_h^{n,3} := u_h^{n,3} - \Pi_h\big(u(t^{n+1})\big). \tag{5.4d}$$

Remark 5.1 ($e_h^{n,2}$). If full regularity were assumed or if an $L^\infty(L^2)$-bound on the third-order time derivative were available so that we would have $\partial_{tt}u \in L^\infty(X)$, it would then be more convenient to set $e_h^{n,2} := u_h^{n,2} - \Pi_h\big(u(t^n) + \tau\partial_t u(t^n) + \frac{1}{2}\tau^2\partial_{tt}u(t^n)\big)$, but comparing $u_h^{n,2}$ to $u(t^n) + \tau\partial_t u(t^n)$ is sufficient for our purpose since we just want to establish convergence by invoking as little regularity as possible.

LEMMA 5.2 (error equation). *The error equation for the ERK*(3,3) *scheme is*

$$e_h^{n+1} = P_3(-\tau A_h)(e_h^n) + \tau \sum_{m\in\{0:2\}} Q_{3,m}(-\tau A_h)(R_h^{n,m}), \tag{5.5}$$

where $Q_{3,0}(z) := 1 + \frac{1}{2}z + \frac{1}{6}z^2$, $Q_{3,1}(z) := 1 + \frac{1}{3}z$, $Q_{3,2}(z) := 1$, *and*

$$R_h^{n,0} := (\Pi_h^{\mathrm{b}} - \Pi_h)\big(\partial_t u(t^n)\big), \tag{5.6a}$$

$$R_h^{n,1} := \tau\Pi_h^{\mathrm{b}}\big(\partial_{tt}u(t^n)\big), \tag{5.6b}$$

$$R_h^{n,2} := \tau^{-1}\Pi_h\big(u(t^{n+1}) - u(t^n) - \tau\partial_t u(t^n)\big). \tag{5.6c}$$

Proof. Using the identity (5.3) satisfied by the exact solution, we write the following identities (tautologies):

$$\Pi_h\big(u(t^n) + \tau\partial_t u(t^n)\big) = \Pi_h\big(u(t^n)\big) - \tau A_h\big(\Pi_h\big(u(t^n)\big)\big) - \tau R_h^{n,0},$$

$$\Pi_h\big(u(t^n) + \tau\partial_t u(t^n)\big) = \Pi_h\big(u(t^n) + \tau\partial_t u(t^n)\big) - \tau A_h\big(\Pi_h\big(\tau\partial_t u(t^n)\big)\big) - \tau R_h^{n,1},$$

$$\Pi_h\big(u(t^{n+1})\big) = \Pi_h\big(u(t^n) + \tau\partial_t u(t^n)\big) - \tau R_h^{n,2}.$$

Subtracting these relations from (4.11), we infer that the following discrete error equations hold true:

$$e_h^{n,1} = e_h^{n,0} - \tau A_h e_h^{n,0} + \tau R_h^{n,0},$$

$$e_h^{n,2} = e_h^{n,1} - \frac{1}{2}\tau A_h(e_h^{n,1} - e_h^{n,0}) + \tau R_h^{n,1},$$

$$e_h^{n,3} = e_h^{n,2} - \frac{1}{3}\tau A_h(e_h^{n,2} - e_h^{n,1}) + \tau R_h^{n,2}.$$

We arrive at the assertion by eliminating the intermediate stages and using $e_h^n := e_h^{n,0}$, $e_h^{n+1} := e_h^{n,3}$. ∎

THEOREM 5.3 (convergence). *Assume that the initial datum* u_0 *is such that the solution to* (1.1) *satisfies the generic regularity assumption* (5.1) *and that the discrete initial condition is chosen so that* (5.2) *holds true. Let* $(u_h^n)_{n\in\mathbb{N}}$ *be the time-discrete sequence produced by the ERK*(3,3) *scheme. There exists* σ_0 *and a constant* C *so that for all* $h \in \mathcal{H}$, *all* τ *satisfying the standard CFL condition* (4.13), *and all* $m \geq 1$, *the following error estimate holds:*

$$\|u(t^m) - u_h^m\|_L \leq C\Big\{\ell^{1-s}h^s|u|_{L^\infty((0,t^m);X)} + t^m\Big\{\tau\|\partial_{tt}u\|_{L^\infty((0,t^m);L)} + \ell^{1-s}h^s|\partial_t u|_{L^\infty((0,t^m);X)}\Big\}\Big\}. \tag{5.7}$$

Proof. Starting from (5.5) and invoking the triangle inequality gives

$$\|e_h^{n+1}\|_L \leq \|P_3(-\tau A_h)(e_h^n)\|_L + \tau \sum_{m\in\{0:2\}} \|Q_{3,m}(-\tau A_h)(R_h^{n,m})\|_L.$$

We bound the first term on the right-hand side by invoking (4.14) from Proposition 4.1, and the second one by using the CFL condition (4.13), and the inverse inequality (4.10). This gives

$$\|e_h^{n+1}\|_L \leq \|e_h^n\|_L + C\tau \sum_{m\in\{0:2\}} \|R_h^{n,m}\|_L.$$

It remains to bound the three residuals on the right-hand side. We have

$$\|R_h^{n,0}\|_L \leq \|(I-\Pi_h)(\partial_t u(t^n))\|_L \leq C\ell^{1-s}h^s|\partial_t u(t^n)|_X$$

and

$$\|R_h^{n,1}\|_L \leq \tau\|\partial_{tt}u(t^n)\|_L.$$

Moreover, since $u(t^{n+1}) - u(t^n) - \tau\partial_t u(t^n) = \int_{t^n}^{t^{n+1}} (t^{n+1}-\xi)\partial_t u(\xi)\,\mathrm{d}\xi$, we infer that

$$\|R_h^{n,2}\|_L \leq \tau\|\partial_{tt}u\|_{L^\infty((t^n,t^{n+1});L)}.$$

Putting the above bounds together, summing over $n \in \{1{:}m\}$ for all $m \geq 1$, and observing that $\sum_{n\in\{1:m\}} \tau = t^m$ implies that

$$\|e_h^m\|_L \leq \|e_h^0\|_L + Ct^m\Big\{\tau\|\partial_{tt}u\|_{L^\infty((0,t^m);L)} + \ell^{1-s}h^s|\partial_t u|_{L^\infty((0,t^m);X)}\Big\}.$$

Recalling the assumption (5.2) on the initial condition and invoking the triangle inequality together with the approximation property (4.4b) of Π_h proves the claim. ∎

Remark 5.4 (convergence order). The convergence order from Theorem 5.3 is $O(\tau + h^s)$. Notice that we do not exploit here the higher-order truncation error in time of ERK(3,3). No superfluous or unrealistic smoothness is invoked. The pivotal property of this scheme is its L^2-stability as established in Proposition 4.1. It is also critical to invoke the Ritz projection and its properties listed in (4.4a)–(4.4b).

5.3. ERK(4,4). We start by introducing the discrete errors

$$e_h^n := e_h^{n,0} := u_h^{n,0} - \Pi_h\big(u(t^n)\big), \tag{5.8a}$$

$$e_h^{n,1} := u_h^{n,1} - \Pi_h\big(u(t^n) + \tau\partial_t u(t^n)\big), \tag{5.8b}$$

$$e_h^{n,2} := u_h^{n,2} - \Pi_h\big(u(t^n) + \tau\partial_t u(t^n)\big), \tag{5.8c}$$

$$e_h^{n,3} := u_h^{n,3} - \Pi_h\big(u(t^n) + \tau\partial_t u(t^n)\big), \tag{5.8d}$$

$$e_h^{n+1} := e_h^{n,4} := u_h^{n,4} - \Pi_h\big(u(t^{n+1})\big). \tag{5.8e}$$

Remark 5.5 ($e_h^{n,2}$, $e_h^{n,3}$). If full regularity were assumed or if an $L^\infty(L^2)$-bound on the third- and fourth-order time derivatives were available so that we would have $\partial_{tt}u \in L^\infty(X)$ and $\partial_{ttt}u \in L^\infty(X)$, it would be more convenient to set $e_h^{n,2}$ as in Remark 5.1 and $e_h^{n,3} := u_h^{n,2} - \Pi_h\big(u(t^n) + \tau\partial_t u(t^n) + \frac{1}{2}\tau^2\partial_{tt}u(t^n) + \frac{1}{6}\tau^3\partial_{ttt}u(t^n)\big)$.

LEMMA 5.6 (error equation). *The error equation for the ERK(4,4) scheme is*

$$e_h^{n+1} = P_4(-\tau A_h)(e_h^n) + \tau \sum_{m\in\{0:3\}} Q_{4,m}(-\tau A_h)(R_h^{n,m}), \tag{5.9}$$

where $Q_{4,0}(s) := 1 + \frac{1}{2}s + \frac{1}{6}s^2 + \frac{1}{24}s^3$, $Q_{4,1}(s) := 1 + \frac{1}{3}s + \frac{1}{12}s^2$, $Q_{4,2}(s) := 1 + \frac{1}{4}s$, $Q_{4,3}(s) = 1$, $R_h^{n,0}$ *and* $R_h^{n,1}$ *are defined by* (5.6a) *and* (5.6b), *respectively,* $R_h^{n,2} = 0$, *and* $R_h^{n,3} := \tau^{-1}\Pi_h\big(u(t^{n+1}) - u(t^n) - \tau\partial_t u(t^n)\big)$.

Proof. Using the same identities (tautologies) as for ERK(3,3) and subtracting these relations from (4.12), we infer that the following discrete error equations hold true:

$$\begin{aligned}
e_h^{n,1} &= e_h^{n,0} - \tau A_h e_h^{n,0} + \tau R_h^{n,0},\\
e_h^{n,2} &= e_h^{n,1} - \frac{1}{2}\tau A_h(e_h^{n,1} - e_h^{n,0}) + \tau R_h^{n,1},\\
e_h^{n,3} &= e_h^{n,2} - \frac{1}{3}\tau A_h(e_h^{n,2} - e_h^{n,1}) + \tau R_h^{n,2},\\
e_h^{n,4} &= e_h^{n,2} - \frac{1}{4}\tau A_h(e_h^{n,3} - e_h^{n,2}) + \tau R_h^{n,3}.
\end{aligned}$$

Eliminating the intermediate stages leads to the assertion. □

THEOREM 5.7 (convergence). *Assume that the exact solution satisfies the regularity assumption* (5.1) *and that the discrete initial condition is chosen so that* (5.2) *holds true. Let* $(u_h^n)_{n\in\mathbb{N}}$ *be the time-discrete sequence produced by the ERK*(4,4) *scheme. There exists* σ_0 *and a constant* C *so that for all* $h\in\mathcal{H}$, *all* τ *satisfying the standard CFL condition* (4.13), *and all* $m\geq 1$, *the following error estimate holds:*

$$\begin{aligned}
\|u(t^m) - u_h^m\|_L \leq C\Big\{&\ell^{1-s}h^s|u|_{L^\infty((0,t^m);X)}\\
&+ t^m\Big\{\tau\|\partial_{tt}u\|_{L^\infty((0,t^m);L)} + \ell^{1-s}h^s|\partial_t u|_{L^\infty((0,t^m);X)}\Big\}\Big\}.
\end{aligned} \tag{5.10}$$

Proof. Considering the error equation (5.9) on two successive time steps, we infer that

$$\begin{aligned}
e_h^{n+2} = P_4(-\tau A_h)^2(e_h^n) &+ \tau \sum_{m\in\{0:3\}} P_4(-\tau A_h)\big(Q_{4,m}(-\tau A_h)(R_h^{n,m})\big)\\
&+ \tau \sum_{m\in\{0:3\}} Q_{4,m}(-\tau A_h)(R_h^{n+1,m}).
\end{aligned}$$

We now invoke the triangle inequality, the bound (4.14) from Proposition 4.1, the CFL condition (4.13), and the inverse inequality (4.10) to infer that

$$\|e_h^{n+2}\|_L \leq \|e_h^n\|_L + C\tau \sum_{m\in\{0:3\}} \big\{\|R_h^{n,m}\|_L + \|R_h^{n+1,m}\|_L\big\}.$$

All the residuals can be bounded by reasoning as in the proof of Theorem 5.3. Summing over $n\in\{1{:}m\}$ for all $m\geq 2$, we infer that

$$\|e_h^m\|_L \leq \|e_h^\mu\|_L + Ct^m\Big\{\tau\|\partial_{tt}u\|_{L^\infty((t^\mu,t^m);L)} + \ell^{1-s}h^s|\partial_t u|_{L^\infty((t^\mu,t^m);X)}\Big\}$$

with $\mu := m \bmod 2$. Altogether, this gives

$$\|e_h^m\|_L \leq \max\big\{\|e_h^0\|_L, \|e_h^1\|_L\big\} + Ct^m\Big\{\tau\|\partial_{tt}u\|_{L^\infty((0,t^m);L)} + \ell^{1-s}h^s|\partial_t u|_{L^\infty((0,t^m);X)}\Big\}.$$

To bound $\|e_h^1\|_L$, we use (5.9) for $n=0$ and invoke the same arguments as above. Finally, recalling the assumption (5.2) on the initial condition and invoking the triangle inequality together with the approximation property (4.4b) of Π_h proves the claim. ∎

6. L^2-stability of ERK(3,3) and ERK(4,4). In this section, we recall the L^2-stability argument for ERK schemes presented in Xu et al. [29] and simplify some of the argumentation.

Our starting point is the integration by parts formula (4.9) which allows us to compute quantities of the form $\Re\big((A_h^k(v_h), A_h^l(v_h))_L\big)$ for all $k>l\geq 0$.

LEMMA 6.1 (induction formula). *Let $k>l\geq 0$ be two natural numbers. Set $n:=\lfloor\frac{k-l}{2}\rfloor$. Then the following identity holds true for all v_h in L_h:*

$$\Re\big((A_h^k(v_h), A_h^l(v_h))_L\big) = \sum_{m\in\{1:n\}}(-1)^{m+1}2\Re\big((A_h^{k-m}(v_h), A_h^{l+m-1}(v_h))_S\big) + (-1)^n\begin{cases}\|A_h^{l+n}(v_h)\|_L^2 & \text{if } k-l=2n,\\ |A_h^{l+n}(v_h)|_S^2 & \text{if } k-l=2n+1.\end{cases} \tag{6.1}$$

Proof. By induction using the integration by parts formula (4.9). ∎

LEMMA 6.2 (quadratic identity). *Let $s\geq 1$ and consider $(s+1)$ real numbers $(a_k)_{k\in\{0:s\}}$ defining the polynomial $\mathcal{P}(z):=\sum_{k\in\{0:s\}}a_kz^k$. The following holds true for all $v_h\in L_h$:*

$$\|\mathcal{P}(-\tau A_h)(v_h)\|_L^2 = \|v_h\|_L^2 + \sum_{k\in\{1:s\}}a_k^L\tau^{2k}\|A_h^k(v_h)\|_L^2 + \sum_{k,l\in\{0:s-1\}}a_{kl}^S\tau^{k+l+1}\Re\big((A_h^k(v_h), A_h^l(v_h))_S\big) \tag{6.2}$$

with the coefficients $(a_k^L)_{k\in\{1:s\}}$ and $(a_{kl}^S)_{k,l\in\{0:s-1\}}$ such that $a_{kl}^S=a_{lk}^S$ and

$$a_k^L = a_k^2 + 2\sum_{m=1}^{\min(k,s-k)}(-1)^m a_{k+m}a_{k-m} \qquad \forall k\in\{1:s\}, \tag{6.3a}$$

$$a_{kl}^S = 2\sum_{m=1}^{\min(l+1,s-k)}(-1)^{k+l+m}a_{k+m}a_{l-m+1} \qquad \forall k\in\{0:s-1\}\ \ \forall l\in\{0:k\}. \tag{6.3b}$$

Proof. A direct calculation shows that

$$\begin{aligned}
&\|\mathcal{P}(-\tau A_h)(v_h)\|_L^2\\
&= \|v_h\|_L^2 + \sum_{k\in\{1:s\}}a_k^2\tau^{2k}\|A_h^k(v_h)\|_L^2\\
&\quad + \sum_{k\in\{0:s\}}\sum_{l\in\{0:k-1\}}2a_ka_l(-1)^{k+l}\tau^{k+l}\Re\big((A_h^k(v_h), A_h^l(v_h)\big)_L\\
&= \|v_h\|_L^2 + \sum_{k\in\{1:s\}}a_k^2\tau^{2k}\|A_h^k(v_h)\|_L^2
\end{aligned}$$

$$+\sum_{k\in\{1:s\}}\sum_{l\in\{0:k-1\}} 2a_k a_l(-1)^{k+l}\tau^{k+l}\Bigg\{\sum_{m\in\{1:n\}}(-1)^{m+1}2\Re\big((A_h^{k-m}(v_h),A_h^{l+m-1}(v_h))_S\big)$$
$$+(-1)^n\left\{\begin{matrix}\|A_h^{l+n}(v_h)\|_L^2 & \text{if } k-l=2n\\ |A_h^{l+n}(v_h)|_S^2 & \text{if } k-l=2n+1\end{matrix}\right\}\Bigg\},$$

where we used (6.1) in the last step. The assertion follows from straightforward manipulations on the indices and by using the definitions of a_k^L and a_{kl}^S. $\square$

The arrays a^L and a^S for ERK(2,2), ERK(3,3), ERK(4,4) are shown below:

$$\text{(6.4a)}\qquad \text{ERK(2,2)}\qquad a^L=\begin{pmatrix}0\\ \frac14\end{pmatrix},\qquad a^S=\begin{pmatrix}-2 & 1\\ 1 & -1\end{pmatrix},$$

$$\text{(6.4b)}\qquad \text{ERK(3,3)}\qquad a^L=\begin{pmatrix}0\\ -\frac{1}{12}\\ \frac{1}{36}\end{pmatrix},\qquad a^S=\begin{pmatrix}-2 & 1 & -\frac13\\ 1 & -\frac23 & \frac13\\ -\frac13 & \frac13 & -\frac16\end{pmatrix},$$

$$\text{(6.4c)}\qquad \text{ERK(4,4)}\qquad a^L=\begin{pmatrix}0\\ 0\\ -\frac{1}{72}\\ \frac{1}{576}\end{pmatrix},\qquad a^S=\begin{pmatrix}-2 & 1 & -\frac13 & \frac{1}{12}\\ 1 & -\frac23 & \frac14 & -\frac{1}{12}\\ -\frac13 & \frac14 & -\frac{1}{12} & \frac{1}{24}\\ \frac{1}{12} & -\frac{1}{12} & \frac{1}{24} & -\frac{1}{72}\end{pmatrix}.$$

LEMMA 6.3 (L^2-stability). *Assume that there exists an integer $r\geq 1$ so that the following two conditions hold:* (i) *the $(r-1)\times(r-1)$ matrix with entries $(a_{kl}^S)_{k,l\in\{0:r-1\}}$ is negative definite;* (ii) *$a_r^L<0$ and $a_l^L=0$ for all $l\in\{1:r-1\}$. Assume also that the discrete operator A_h satisfies the estimate* (4.10) *and is dissipative. There exists $\sigma_0>0$ so that, if the standard CFL condition* (4.13) *holds true, then $\|\mathcal{P}(-\tau A_h)\|_{\mathcal{L}(L_h)}\leq 1$.*

Proof. Let $v_h\in L_h$. For all $k\in\{r+1:s\}$, invoking the CFL condition (4.13) yields

$$\tau^k\|A_h^k(v_h)\|_L\leq (c_L\tau h^{-1})^{k-r}\tau^r\|A_h^r(v_h)\|_L\leq \sigma_0^{k-r}\tau^r\|A_h^r(v_h)\|_L.$$

For all $k\in\{r:s-1\}$ and all $l\in\{0:r-1\}$, we have

$$\begin{aligned}\tau^{k+l+1}|(A_h^k(v_h),A_h^l(v_h))_S| &\leq \tau^{k+l+1}|A_h^k(v_h)|_S|A_h^l(v_h)|_S\\ &\leq \tau^{k+l+1}(c_Lh^{-1})^{\frac12}(c_Lh^{-1})^{k-r}\|A_h^r(v_h)\|_L|A_h^l(v_h)|_S\\ &\leq \tau^{r+l+\frac12}(c_L\tau h^{-1})^{\frac12}(c_L\tau h^{-1})^{k-r}\|A_h^r(v_h)\|_L|A_h^l(v_h)|_S\\ &\leq \frac12\sigma_0^{2(k-r)}\tau^{2r}\|A_h^r(v_h)\|_L^2+\frac12\sigma_0\tau^{2l+1}|A_h^l(v_h)|_S^2.\end{aligned}$$

Since $(A_h^k(v_h),A_h^l(v_h))_S$ and $(A_h^l(v_h),A_h^k(v_h))_S$ have the same module, the same bound holds for all $l\in\{r:s-1\}$ and all $k\in\{0:r-1\}$. Finally, for all $k\in\{r:s-1\}$ and all $l\in\{r:s-1\}$, we have

$$\begin{aligned}\tau^{k+l+1}|(A_h^k(v_h),A_h^l(v_h))_S| &\leq \tau^{2r}(c_L\tau h^{-1})(c_L\tau h^{-1})^{k+l-2r}\|A_h^r(v_h)\|_L^2\\ &\leq \tau^{2r}\sigma_0^{k+l-2r+1}\|A_h^r(v_h)\|_L^2.\end{aligned}$$

Inserting the above bounds into (6.2) and since $a_l^L=0$ for all $l\in\{1:r-1\}$, we infer that

$$\begin{aligned}(6.5)\quad \|\mathcal{P}(-\tau A_h)(v_h)\|_L^2 &\le \|v_h\|_L^2 + \tilde{a}_r^L\tau^{2r}\|A_h^r(v_h)\|_L^2 \\ &+ \sum_{k,l=0}^{r-1} a_{kl}^S \tau^{k+l+1}\Re\big((A_h^k(v_h), A_h^l(v_h))_S\big) + \sum_{l=0}^{r-1}\left(\sum_{k=r}^{s-1}|a_{kl}^S|\right)\sigma_0\tau^{2l+1}|A_h^l(v_h)|_S^2\end{aligned}$$

with

$$\tilde{a}_r^L := a_r^L + \sum_{k=r+1}^{s}\sigma_0^{2(k-r)}|a_k^L| + \sum_{k=r}^{s-1}\sum_{l=0}^{r-1}\sigma_0^{2(k-r)}|a_{kl}^S| + \sum_{k=r}^{s-1}\sum_{l=r}^{s-1}\sigma_0^{k+l-2r+1}|a_{kl}^S|.$$

Since $a_r^L < 0$, the above expression shows that it is possible to choose $\sigma_0 > 0$ small enough so that $\tilde{a}_r^L < 0$. Furthermore, since the submatrix $(a_{kl}^S)_{k,l\in\{0:r-1\}}$ is negative definite, it is possible to choose $\sigma_0 > 0$ small enough so that

$$\sum_{k,l=0}^{r-1} a_{kl}^S \tau^{k+l+1}\Re\big((A_h^k(v_h), A_h^l(v_h))_S\big) + \sum_{l=0}^{r-1}\left(\sum_{k=r}^{s-1}|a_{kl}^S|\right)\sigma_0\tau^{2l+1}|A_h^l(v_h)|_S^2 \le 0.$$

This completes the proof. ∎

Corollary 6.4 (L^2-stability of ERK(3,3)). *The statement of Proposition* 4.1 *holds true for ERK*(3,3).

Proof. We consider the polynomial $\mathcal{P}(z) := P_3(z)$. An inspection of the matrix a^S in (6.4b) shows that the assumption of Lemma 6.3 holds true for $r = 2$. Indeed, we see that $a_1^L = 0$ and $a_2^L = -\frac{1}{2} < 0$, and the 2×2 block

$$\begin{pmatrix}-2 & 1\\ 1 & -\frac{2}{3}\end{pmatrix}$$

is negative definite. ∎

Corollary 6.5 (L^2-stability of ERK(4,4)). *The statement of Proposition* 4.1 *holds true for ERK*(4,4).

Proof. We consider first the polynomial $\mathcal{P}(z) := P_4(z)$. From (6.4c), we see that $a_1^L = 0$, $a_2^L = 0$, and $a_3^L = -\frac{1}{72} < 0$, but the 3×3 block $a^S(0{:}2, 0{:}2)$ is not negative definite. But for the scheme ERK(8,4) composed of two steps of the method ERK(4,4), a direct calculation using the polynomial $\mathcal{P}(z) = P_4(z)^2$ shows that we have $a_1^L = 0$, $a_2^L = 0$, and $a_3^L = -\frac{1}{36} < 0$, and that the 4×4 block $a^S(0{:}3, 0{:}3)$ is proportional to

$$\begin{pmatrix}-144 & 144 & -96 & 48\\ 144 & -192 & 144 & -78\\ -96 & 144 & -114 & 65\\ 48 & -78 & 65 & -37\end{pmatrix}.$$

This matrix is not negative definite, but the 3×3 block $a^S(0{:}2, 0{:}2)$ is, which is sufficient to apply Lemma 6.3 with $r = 3$. ∎

Remark 6.6 (additional dissipation). A slightly stronger result than the assertion in Lemma 6.3 is that, under the same assumptions, there exists $\gamma_0 > 0$ so that, if the standard CFL condition (4.13) holds true, then for all $v_h \in L_h$,

$$\|\mathcal{P}(-\tau A_h)(v_h)\|_L^2 + \gamma_0\tau \sum_{k\in\{0:s-1\}} \tau^{2k}|A_h^k(v_h)|_S^2 \le \|v_h\|_L^2.$$

This follows from (6.5) by invoking that the submatrix $(a^S_{kl})_{k,l\in\{0:r-1\}}$ is negative definite (so that its eigenvalues are bounded from above away from zero) and that, for all $k \in \{r : s-1\}$, $\gamma_0\tau^{2k+1}|A^k_h(v_h)|^2_S \le \gamma_0\sigma_0^{2(k-r)+1}\tau^{2r}\|A^r_h(v_h)\|^2_L$, which can be bounded by $-\tilde{a}^L_r\tau^{2r}\|A^r_h(v_h)\|^2_L$ by choosing $\gamma_0 > 0$ small enough.

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