On the discontinuous Galerkin approximation of Maxwell's equations

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Outline

- Maxwell's equations
- Discontinuous Galerkin (dG) approximation
- Correctness for dG spectral problem
  - [AE & JLG, SINUM 23; hal-04145808]
- Asymptotic optimality for dG time-harmonic problem
  - [TCF & AE, hal-04216433, hal-04589791]
- Some further insights on spectral correctness

Maxwell's equations

- Functional setting
- Compactness
- Spectral and time-harmonic problems

Maxwell's equations

- Space-time PDEs posed on $D \times J$ with $D \subset \mathbb{R}^3$, $J := (0, T)$
- Find $(\boldsymbol{H}, \boldsymbol{E}) : D \times J \to \mathbb{R}^3 \times \mathbb{R}^3$ s.t.

$$\begin{aligned}
&\partial_t(\mu \boldsymbol{H}) + \nabla{\times}\boldsymbol{E} = \boldsymbol{0} && \text{(Faraday)}\\
&\partial_t(\epsilon \boldsymbol{E}) - \nabla{\times}\boldsymbol{H} = -\boldsymbol{j} && \text{(Ampère)}\\
&\nabla{\cdot}(\mu \boldsymbol{H}) = 0 && \text{(Gauss)}\\
&\nabla{\cdot}(\epsilon \boldsymbol{E}) = \rho && \text{(Gauss)}
\end{aligned}$$

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- Data: ρ (charge density) and $\boldsymbol{j}$ (current) s.t. $\partial_t \rho + \nabla\cdot\boldsymbol{j} = 0$

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- Material properties: ϵ (electric permittivity), μ (magnetic permeability)
- Data: ρ (charge density) and $\boldsymbol{j}$ (current) s.t. $\partial_t \rho + \nabla \cdot \boldsymbol{j} = 0$
- Prescribe ICs $(\boldsymbol{H}_0, \boldsymbol{E}_0) : D \to \mathbb{R}^3 \times \mathbb{R}^3$
- Focus on bounded Lipschitz domain D: enforce BC on $\Gamma := \partial D$
  - Simplest BCs: perfect magnetic or electric conductor

$$\boldsymbol{H} \times \boldsymbol{n}|_\Gamma = \boldsymbol{0} \quad \text{or} \quad \boldsymbol{E} \times \boldsymbol{n}|_\Gamma = \boldsymbol{0}$$

  - Other possible BCs: impedance, transparent (far field), ...

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- Recall

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  - if $\nabla\cdot(\mu \boldsymbol{H}_0) = 0$, then $\nabla\cdot(\mu \boldsymbol{H}) = 0$ at all times
  - if $\nabla\cdot(\epsilon \boldsymbol{E}_0) = \rho_0$, then $\nabla\cdot(\epsilon \boldsymbol{E}) = \rho$ at all times

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- Actually, $\nabla\cdot(\mu \boldsymbol{H}) = 0$ does not always imply $\mu \boldsymbol{H} \in \mathrm{im}(\nabla\times)$ (depends on domain topology), but the converse is true!

- The topology-blind statement of the involution on $\boldsymbol{H}$ is

$$\mu \boldsymbol{H} \in \mathrm{im}(\nabla\times)$$

- Similarly, in the absence of free charges ($\boldsymbol{j} = \boldsymbol{0}$), the topology-blind statement of the involution on $\boldsymbol{E}$ is

$$\epsilon \boldsymbol{E} \in \mathrm{im}(\nabla\times)$$

Functional setting

- Graph spaces for gradient, curl, or divergence

$$H^1(D), \quad \boldsymbol{H}(\mathbf{curl};D) := \{\boldsymbol{h} \in \boldsymbol{L}^2(D) \mid \nabla\times\boldsymbol{h} \in \boldsymbol{L}^2(D)\}, \quad \boldsymbol{H}(\mathrm{div};D)$$

- Hilbert spaces equipped with natural graph norm, e.g.,

$$\|\boldsymbol{h}\|^2_{\boldsymbol{H}(\mathbf{curl};D)} := \|\boldsymbol{h}\|^2_{\boldsymbol{L}^2} + \ell_D^2\|\nabla\times\boldsymbol{h}\|^2_{\boldsymbol{L}^2}$$

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- De Rham sequences (with and without BC)

$$H^1_0(D) \xrightarrow{\nabla_0} \boldsymbol{H}_0(\mathbf{curl};D) \xrightarrow{\nabla_0\times} \boldsymbol{H}_0(\mathrm{div};D) \xrightarrow{\nabla_0\cdot} L^2_0(D)$$

$$L^2(D) \xleftarrow{\nabla\cdot} \boldsymbol{H}(\mathrm{div};D) \xleftarrow{\nabla\times} \boldsymbol{H}(\mathbf{curl};D) \xleftarrow{\nabla} H^1(D)$$

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- All operators have closed range

- Pairs of adjoint operators: $(\nabla_0, -\nabla\cdot)$, $(\nabla_0\times, \nabla\times)$, $(-\nabla_0\cdot, \nabla)$

Functional setting for involutions

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- Rewriting of involutions using Closed Range Theorem (orthogonalities meant in $\boldsymbol{L}^2$) [Hiptmair 02]

$$\mu\boldsymbol{H} \in \mathrm{im}(\nabla\times) = \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp, \quad \epsilon\boldsymbol{E} \in \mathrm{im}(\nabla_0\times) = \boldsymbol{H}(\mathbf{curl} = \mathbf{0}; D)^\perp$$

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- Topology-blind statements!
  - $\boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp} \subset \boldsymbol{H}(\mathrm{div} = 0; D)$ with equality iff Γ is connected
  - $\boldsymbol{H}(\mathbf{curl} = \mathbf{0}; D)^{\perp} \subset \boldsymbol{H}_0(\mathrm{div} = 0; D)$ with equality iff D is simply connected

  See [Dautray, Lions 90; Amrouche, Bernardi, Dauge, Girault, 98]

Compactness

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  - pcw. constant material properties (or multiplier property in $\boldsymbol{H}^s$, $s \in (0, \frac{1}{2}]$)

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$$\boldsymbol{X}^{\mathrm{c}}_{\mu,0} := \{\boldsymbol{h} \in \boldsymbol{H}_0(\mathbf{curl}; D) \mid \mu\boldsymbol{h} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}\}$$
$$\boldsymbol{X}^{\mathrm{c}}_{\epsilon} := \{\boldsymbol{e} \in \boldsymbol{H}(\mathbf{curl}; D) \mid \epsilon\boldsymbol{e} \in \boldsymbol{H}(\mathbf{curl} = \mathbf{0}; D)^{\perp}\}$$

There is $s \in (0, \frac{1}{2}]$ s.t. for all $\boldsymbol{h} \in \boldsymbol{X}^{\mathrm{c}}_{\mu,0}$ and all $\boldsymbol{e} \in \boldsymbol{X}^{\mathrm{c}}_{\epsilon}$,

$$|\boldsymbol{h}|_{\boldsymbol{H}^s(D)} \lesssim \ell_D^{1-s}\|\nabla_0{\times}\boldsymbol{h}\|_{\boldsymbol{L}^2}, \quad |\boldsymbol{e}|_{\boldsymbol{H}^s(D)} \lesssim \ell_D^{1-s}\|\nabla{\times}\boldsymbol{e}\|_{\boldsymbol{L}^2}$$

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- Improved regularity shift for constant properties: There is $s' \in (\frac{1}{2}, 1]$ s.t. for all $\boldsymbol{\eta} \in \boldsymbol{X}^{\mathrm{c}}_0$ and all $\boldsymbol{\varepsilon} \in \boldsymbol{X}^{\mathrm{c}}$,

$$|\boldsymbol{\eta}|_{\boldsymbol{H}^{s'}(D)} \lesssim \ell_D^{1-s'} \|\nabla_0\times\boldsymbol{\eta}\|_{\boldsymbol{L}^2}, \quad |\boldsymbol{\varepsilon}|_{\boldsymbol{H}^{s'}(D)} \lesssim \ell_D^{1-s'} \|\nabla\times\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2}$$

with $\boldsymbol{X}^{\mathrm{c}}_0 := \boldsymbol{H}_0(\mathbf{curl}; D) \cap \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}$, $\boldsymbol{X}^{\mathrm{c}} := \boldsymbol{H}(\mathbf{curl}; D) \cap \boldsymbol{H}(\mathbf{curl} = \mathbf{0}; D)^{\perp}$

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- See [Weber, 80; Birman & Solomyak, 87; Costabel, 90; Amrouche, Bernardi, Dauge, Girault, 98; Jochmann, 99; Bonito, Guermond & Luddens, 13]

Spectral problem

- For dimensional consistency,
  - vacuum properties ϵ_0, μ_0; speed of light: $\mathfrak{c} := (\mu_0 \epsilon_0)^{-\frac{1}{2}}$
  - reference frequency $\omega_D := \mathfrak{c}\ell_D^{-1}$

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- Find nonzero $\lambda \in \mathbb{C}$ and nonzero $(\boldsymbol{H}, \boldsymbol{E}) \in \boldsymbol{X}^{\mathrm{c}}_{\mu,0} \times \boldsymbol{X}^{\mathrm{c}}_{\epsilon}$ s.t.

$$-\nabla\times\boldsymbol{E} = \frac{\omega_D}{\lambda}\mu\boldsymbol{H}, \qquad \nabla_0\times\boldsymbol{H} = \frac{\omega_D}{\lambda}\epsilon\boldsymbol{E}$$

(eigenvalue λ is nondimensional)

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- Eigenfunctions are involution-preserving

$$\boldsymbol{H} \in \boldsymbol{X}^{c}_{\mu,0} \iff \left\{\boldsymbol{H} \in \boldsymbol{H}_0(\mathbf{curl}; D) \wedge \mu\boldsymbol{H} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}\right\}$$

$$\boldsymbol{E} \in \boldsymbol{X}^{c}_{\epsilon} \iff \left\{\boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl}; D) \wedge \epsilon\boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl} = \mathbf{0}; D)^{\perp}\right\}$$

Boundary-value operator for spectral problem

- Introduce $\boldsymbol{L}^2$-orthogonal projections

$$\boldsymbol{\Pi}_0^{\mathrm{c}} : \boldsymbol{L}^2(D) \to \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D), \quad \boldsymbol{\Pi}^{\mathrm{c}} : \boldsymbol{L}^2(D) \to \boldsymbol{H}(\mathbf{curl} = \mathbf{0}; D)$$

Involutions mean that $\boldsymbol{\Pi}_0^{\mathrm{c}}(\mu \boldsymbol{H}) = \mathbf{0}$, $\boldsymbol{\Pi}^{\mathrm{c}}(\epsilon \boldsymbol{E}) = \mathbf{0}$

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- Boundary-value operator $T : L \to L := \boldsymbol{L}^2(D) \times \boldsymbol{L}^2(D)$

- For all $(\boldsymbol{f}, \boldsymbol{g}) \in L$, $T(\boldsymbol{f}, \boldsymbol{g})$ is the unique pair $(\boldsymbol{H}, \boldsymbol{E}) \in \boldsymbol{X}^{\mathrm{c}}_{\mu,0} \times \boldsymbol{X}^{\mathrm{c}}_{\epsilon} \subset L$ solving the well-posed problem

$$-\nabla\times\boldsymbol{E} = \omega_D(\boldsymbol{I} - \mathbf{\Pi}_0^{\mathrm{c}})(\mu \boldsymbol{f}), \quad \nabla_0\times\boldsymbol{H} = \omega_D(\boldsymbol{I} - \mathbf{\Pi}^{\mathrm{c}})(\epsilon \boldsymbol{g})$$

  By construction, $(\boldsymbol{I} - \mathbf{\Pi}_0^{\mathrm{c}})(\mu \boldsymbol{f}) \in \mathrm{im}(\nabla\times)$ and $(\boldsymbol{I} - \mathbf{\Pi}^{\mathrm{c}})(\epsilon \boldsymbol{g}) \in \mathrm{im}(\nabla_0\times)$

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- Since $\boldsymbol{X}_{\mu,0}^{\mathrm{c}} \times \boldsymbol{X}_{\epsilon}^{\mathrm{c}} \hookrightarrow \boldsymbol{H}^s(D) \times \boldsymbol{H}^s(D)$, T is a compact operator
- $\big(\lambda, (\boldsymbol{H}, \boldsymbol{E})\big)$, $\lambda \neq 0$, is a Maxwell eigenpair iff

$$T(\boldsymbol{H}, \boldsymbol{E}) = \lambda(\boldsymbol{H}, \boldsymbol{E})$$

Time-harmonic Maxwell's equations

- To fix ideas, enforce BC on $\boldsymbol{E}$: $\boldsymbol{E} \in \boldsymbol{H}_0(\mathbf{curl}; D)$, $\boldsymbol{H} \in \boldsymbol{H}(\mathbf{curl}; D)$
- Fix frequency $\omega > 0$, time-harmonic behavior: $\partial_t \to \mathrm{i}\omega$

$$\mathrm{i}\omega\mu\boldsymbol{H} + \nabla_0\times\boldsymbol{E} = \mathbf{0}, \qquad \mathrm{i}\omega\epsilon\boldsymbol{E} - \nabla\times\boldsymbol{H} = -\boldsymbol{j}$$

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- Eliminate $\boldsymbol{H} \to$ Second-order formulation: Find $\boldsymbol{E} \in \boldsymbol{H}_0(\mathbf{curl}; D)$ s.t.

$$-\omega^2\epsilon\boldsymbol{E} + \nabla\times(\nu\nabla_0\times\boldsymbol{E}) = \boldsymbol{J}$$

with $\nu := \mu^{-1}$ and $\boldsymbol{J} := -\mathrm{i}\omega\boldsymbol{j}$

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- The involution is

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- To fix ideas, enforce BC on $\boldsymbol{E}$: $\boldsymbol{E} \in \boldsymbol{H}_0(\mathbf{curl}; D)$, $\boldsymbol{H} \in \boldsymbol{H}(\mathbf{curl}; D)$
- Fix frequency $\omega > 0$, time-harmonic behavior: $\partial_t \to \mathrm{i}\omega$

$$\mathrm{i}\omega\mu\boldsymbol{H} + \nabla_0{\times}\boldsymbol{E} = \boldsymbol{0}, \qquad \mathrm{i}\omega\epsilon\boldsymbol{E} - \nabla{\times}\boldsymbol{H} = -\boldsymbol{j}$$

- Eliminate $\boldsymbol{H}$ $\longrightarrow$ Second-order formulation: Find $\boldsymbol{E} \in \boldsymbol{H}_0(\mathbf{curl}; D)$ s.t.

$$-\omega^2\epsilon\boldsymbol{E} + \nabla{\times}(\nu\nabla_0{\times}\boldsymbol{E}) = \boldsymbol{J}$$

  with $\nu := \mu^{-1}$ and $\boldsymbol{J} := -\mathrm{i}\omega\boldsymbol{j}$
- The involution is

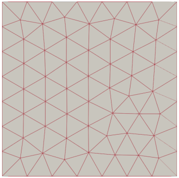
$$\omega^2\epsilon\boldsymbol{E} + \boldsymbol{J} \in \mathrm{im}(\nabla\times) = \boldsymbol{H}_0(\mathbf{curl} = \boldsymbol{0}; D)^{\perp}$$

- Coercive problem with compact perturbation $\longrightarrow$ Fredholm's alternative
  - well-posed problem if ω is not a resonant frequency

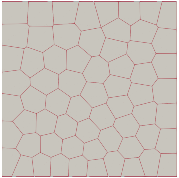
Discontinuous Galerkin approximation

Mesh and broken polynomial spaces

- $\mathcal{T}_h$: shape-regular mesh covering D exactly



simplicial mesh



polygonal mesh

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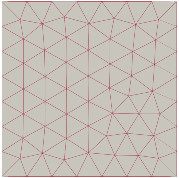

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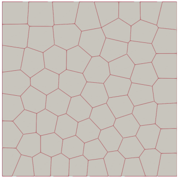

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$$\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) := \{\boldsymbol{v}_h \in \boldsymbol{L}^2(D) \mid \boldsymbol{v}_h|_K \in \mathbb{P}_{k,d}(K;\mathbb{R}^d),\ \forall K \in \mathcal{T}_h\}$$

- Nonconforming approximation space: $\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \not\subset \boldsymbol{H}(\mathbf{curl}; D)$
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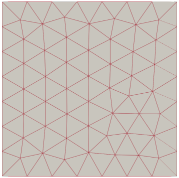

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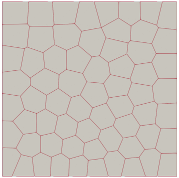

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- dG textbooks: [Hesthaven & Warburton 08; Di Pietro & AE, 12]

Jumps and stabilization

- Mesh interface $F \in \mathcal{F}_h^{\circ}$ s.t. $F = \partial K_l \cap \partial K_r$
  - oriented by unit normal $\boldsymbol{n}_F$ pointing from K_l to K_r
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$$[\![\boldsymbol{v}_h]\!]_F^{\mathrm{c}} := (\boldsymbol{v}_h|_{K_l}|_F - \boldsymbol{v}_h|_{K_r}|_F) \times \boldsymbol{n}_F$$

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- Stabilization bilinear forms

$$s_h^{\mathrm{H}}(\boldsymbol{H}_h, \boldsymbol{h}_h) := \sum_{F \in \mathcal{F}_h} ([\![\boldsymbol{H}_h]\!]_F^{\mathrm{c}}, [\![\boldsymbol{h}_h]\!]_F^{\mathrm{c}})_{\boldsymbol{L}^2(F)} \qquad s_h^{\mathrm{E}}(\boldsymbol{E}_h, \boldsymbol{e}_h) := \sum_{F \in \mathcal{F}_h^\circ} ([\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}, [\![\boldsymbol{e}_h]\!]_F^{\mathrm{c}})_{\boldsymbol{L}^2(F)}$$

  Jump seminorms: $|\boldsymbol{h}_h|_h^{\mathrm{H}} := s_h^{\mathrm{H}}(\boldsymbol{h}_h, \boldsymbol{h}_h)^{\frac{1}{2}}$, $|\boldsymbol{e}_h|_h^{\mathrm{E}} := s_h^{\mathrm{E}}(\boldsymbol{e}_h, \boldsymbol{e}_h)^{\frac{1}{2}}$

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- Literature
  - Discrete gradient for diffusion problems introduced in [Bassi et al., 97] and analyzed in [Brezzi et al., 00]
  - Weak consistency and compactness properties [Burman & AE, 08; Buffa & Ortner, 09; Di Pietro & AE, 09]
  - dG methods with discrete curl for Maxwell's equations [Perugia, Schötzau & Monk, 02; Houston et al., 05]

Example of weak consistency property

- Consistency defect: For all $\boldsymbol{h}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$ and all $\boldsymbol{e} \in \boldsymbol{H}(\mathbf{curl}; D)$,

$$\delta(\boldsymbol{h}_h, \boldsymbol{e}) := (\boldsymbol{h}_h, \nabla\times\boldsymbol{e})_{\boldsymbol{L}^2} - (\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{h}_h), \boldsymbol{e})_{\boldsymbol{L}^2}$$

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- Sketch of proof. Using $\boldsymbol{L}^2$-orthogonal projection $\boldsymbol{\Pi}_h^{\mathrm{b}}$ onto $\boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$,

$$\delta(\boldsymbol{h}_h, \boldsymbol{\varepsilon}) = \sum_{F\in\mathcal{F}_h} ([\![\boldsymbol{h}_h]\!]_F^{\mathrm{c}}, \{\!\{\boldsymbol{\varepsilon} - \boldsymbol{\Pi}_h^{\mathrm{b}}(\boldsymbol{\varepsilon})\}\!\}_F)_{\boldsymbol{L}^2(F)}$$

  Use approximation properties of $\boldsymbol{\Pi}_h^{\mathrm{b}}$ and $|\boldsymbol{\varepsilon}|_{\boldsymbol{H}^{s'}(D)} \lesssim \ell_D^{1-s'} \|\nabla\times\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2}$

Spectral correctness

Discrete spectral problem

- Recall spectral problem: Find nonzero $\lambda \in \mathbb{C}$ and nonzero $(\boldsymbol{H}, \boldsymbol{E}) \in \boldsymbol{X}^{\mathrm{c}}_{\mu,0} \times \boldsymbol{X}^{\mathrm{c}}_{\epsilon}$ s.t.

$$-(\nabla\times\boldsymbol{E}, \boldsymbol{h})_{\boldsymbol{L}^2} + (\nabla_0\times\boldsymbol{H}, \boldsymbol{e})_{\boldsymbol{L}^2} = \frac{\omega_D}{\lambda}\big((\mu\boldsymbol{H}, \epsilon\boldsymbol{E}), (\boldsymbol{h}, \boldsymbol{e})\big)_L$$

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- Discrete bilinear form (stabilization weights: $\kappa_{\mathrm{H}} := (\mu_0/\epsilon_0)^{\frac{1}{2}}$, $\kappa_{\mathrm{E}} := (\epsilon_0/\mu_0)^{\frac{1}{2}}$)

$$\begin{aligned} a_h\big((\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)\big) := & -(\boldsymbol{C}^{k,\ell}_h(\boldsymbol{E}_h), \boldsymbol{h}_h)_{\boldsymbol{L}^2} + (\boldsymbol{C}^{k,\ell}_{h,0}(\boldsymbol{H}_h), \boldsymbol{e}_h)_{\boldsymbol{L}^2} \\ & + \kappa_{\mathrm{H}} s^{\mathrm{H}}_h(\boldsymbol{H}_h, \boldsymbol{h}_h) + \kappa_{\mathrm{E}} s^{\mathrm{E}}_h(\boldsymbol{E}_h, \boldsymbol{e}_h) \end{aligned}$$

Discrete involutions

- Curl-free subspaces of broken polynomial spaces

$$\boldsymbol{P}_{k0}^{\mathrm{c}}(\mathbf{curl}=\mathbf{0};\mathcal{T}_h) := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \cap \boldsymbol{H}_0(\mathbf{curl}=\mathbf{0};D)$$
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- Lemma. The discrete involutions satisfied by any eigenpair $(\boldsymbol{H}_h, \boldsymbol{E}_h)$ are

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- Curl-free subspaces need to be “sufficiently rich” to enjoy suitable approximation properties

- On simplicial meshes, these subspaces are composed of Nédélec (edge) finite elements, and several effective interpolation operators exist!

Main result

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- Spectral correctness also using CIP-stabilized FEM on split meshes (Alfeld or Clough–Tocher) [AE & JLG, 24, hal-04478683]

Asymptotic optimality, time-harmonic problem

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- Explicit-frequency analysis in [Melenk & Sauter, 10; TCF & Nicaise, 20; Lafontaine, Spence & Wunsch, 22]

Helmholtz problem

- Consider first Helmholtz problem (positive material properties ϑ, ν)

$$-\omega^2 \vartheta u - \nabla\cdot(\nu \nabla u) = f, \quad u|_\Gamma = 0$$

- Asymptotic optimality of H^1-conforming FEM approximation proved using duality argument [Schatz, 74]

$$(1 - c(h)) |\!|\!| u - u_h |\!|\!| \leq \inf_{v_h \in P_k^{\mathrm{b}}(\mathcal{T}_h)\cap H_0^1(D)} |\!|\!| u - v_h |\!|\!|, \quad \lim_{h\to 0} c(h) = 0$$

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- Explicit-frequency analysis in [Melenk & Sauter, 10; TCF & Nicaise, 20; Lafontaine, Spence & Wunsch, 22]

- dG approximation of Helmholtz problem analyzed in [TCF, 23]
  - bound consistency defect of discrete gradient
  - deal with nonconforming setting and stabilization

Maxwell's problem with conforming approximation

- (Recall) Given $\boldsymbol{J} \in \boldsymbol{L}^2(D)$, find $\boldsymbol{E} \in \boldsymbol{H}_0(\mathbf{curl}; D)$ s.t.

$$-\omega^2(\epsilon\boldsymbol{E}, \boldsymbol{e})_{\boldsymbol{L}^2} + (\nu\nabla_0\times\boldsymbol{E}, \nabla_0\times\boldsymbol{e})_{\boldsymbol{L}^2} = (\boldsymbol{J}, \boldsymbol{e})_{\boldsymbol{L}^2} \quad \forall\boldsymbol{e} \in \boldsymbol{H}_0(\mathbf{curl}; D)$$

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$$(1 - c(h))|||\boldsymbol{E} - \boldsymbol{E}_h||| \le \inf_{\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)\cap\boldsymbol{H}_0(\mathbf{curl};D)} |||\boldsymbol{E} - \boldsymbol{v}_h|||, \quad \lim_{h\to 0} c(h) = 0$$

with energy norm $|||\boldsymbol{v}|||^2 := \omega^2\|\epsilon^{\frac{1}{2}}\boldsymbol{v}\|^2_{L^2(D)} + \|\nu^{\frac{1}{2}}\nabla_0\times\boldsymbol{v}\|^2_{\boldsymbol{L}^2}$

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  - impedance BCs in [Melenk & Sauter, 23], explicit-frequency analysis, smooth and connected boundary
  - perfect conductor BCs in [TCF & AE, 24], general domain and material properties, frequency-dependence not made explicit

- Discrete problem: Find $\boldsymbol{E}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$ s.t.

$$b_h(\boldsymbol{E}_h, \boldsymbol{e}_h) = (\boldsymbol{J}, \boldsymbol{e}_h)_{\boldsymbol{L}^2} \quad \forall \boldsymbol{e}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$$

with discrete bilinear form

$$b_h(\boldsymbol{E}_h, \boldsymbol{e}_h) := -\omega^2(\epsilon \boldsymbol{E}_h, \boldsymbol{e}_h)_{\boldsymbol{L}^2} + (\nu \boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{E}_h), \boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{e}_h))_{\boldsymbol{L}^2} + s_h(\boldsymbol{E}_h, \boldsymbol{e}_h)$$

and symmetric, positive-semidefinite stabilization bilinear form s_h

Example: Interior penalty dG

- Interior penalty dG bilinear form

$$b_h^{\mathrm{IPDG}}(\boldsymbol{E}_h, \boldsymbol{e}_h) := -\omega^2(\epsilon \boldsymbol{E}_h, \boldsymbol{e}_h)_{\boldsymbol{L}^2} + (\nu \nabla_h \times \boldsymbol{E}_h, \nabla_h \times \boldsymbol{e}_h)_{\boldsymbol{L}^2} + \eta_* s_h^{\mathrm{IPDG}}(\boldsymbol{E}_h, \boldsymbol{e}_h) + \sum_{F \in \mathcal{F}_h} \left\{ (\{\!\{\nu \nabla_h \times \boldsymbol{E}_h\}\!\}_F, [\![\boldsymbol{e}_h]\!]_F^{\mathrm{c}})_{\boldsymbol{L}^2(F)} + ([\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}, \{\!\{\nu \nabla_h \times \boldsymbol{e}_h\}\!\}_F)_{\boldsymbol{L}^2(F)} \right\}$$

with stabilization bilinear form ($\nu_F := \max_{K \supset F} \nu|_K$)

$$s_h^{\mathrm{IPDG}}(\boldsymbol{E}_h, \boldsymbol{e}_h) := \sum_{F \in \mathcal{F}_h} \frac{\nu_F}{h_F} ([\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}, [\![\boldsymbol{e}_h]\!]_F^{\mathrm{c}})_{\boldsymbol{L}^2(F)}$$

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$$+ \sum_{F \in \mathcal{F}_h} \left\{ (\{\!\{\boldsymbol{\nu} \nabla_h \times \boldsymbol{E}_h\}\!\}_F, [\![\boldsymbol{e}_h]\!]_F^{\mathrm{c}})_{\boldsymbol{L}^2(F)} + ([\![\boldsymbol{E}_h]\!]_F^{\mathrm{c}}, \{\!\{\boldsymbol{\nu} \nabla_h \times \boldsymbol{e}_h\}\!\}_F)_{\boldsymbol{L}^2(F)} \right\}$$

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- b_h can be rewritten using discrete curl operators upon setting

$$s_h(\boldsymbol{E}_h, \boldsymbol{e}_h) := \eta_* s_h^{\mathrm{IPDG}}(\boldsymbol{E}_h, \boldsymbol{e}_h) - (\nu \boldsymbol{L}_{h,0}^{\ell}(\boldsymbol{E}_h), \boldsymbol{L}_{h,0}^{\ell}(\boldsymbol{e}_h))_{\boldsymbol{L}^2}$$

and positivity of s_h requires taking $\eta_* > 0$ large enough

Main result on dG approximation

- Error $\boldsymbol{e} := \boldsymbol{E} - \boldsymbol{E}_h$ lives in $\boldsymbol{V}_\sharp := \boldsymbol{H}_0(\mathbf{curl}; D) + \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$
  - natural extension of $[\![\cdot]\!]_F^{\mathrm{c}}$ and $\boldsymbol{C}_{h,0}^{k,\ell}$ to $\boldsymbol{V}_\sharp$
  - assume s_h can be extended to $\boldsymbol{V}_\sharp \to s_\sharp$
  - equip $\boldsymbol{V}_\sharp$ with extended energy norm

$$|\!|\!|\boldsymbol{v}|\!|\!|_{\sharp\mathrm{s}}^2 := \omega^2 \|\epsilon^{\frac{1}{2}} \boldsymbol{v}\|_{\boldsymbol{L}^2}^2 + \|\nu^{\frac{1}{2}} \boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{v})\|_{\boldsymbol{L}^2}^2 + s_\sharp(\boldsymbol{v}, \boldsymbol{v})$$

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- Theorem [TCF & AE, 24] Assume simplicial meshes, $k \geq 1$, and some minimal assumption on stabilization. Then, with $\lim_{h\to 0} c(h) = 0$,

$$(1 - c(h))\, |\!|\!|\boldsymbol{e}|\!|\!|_{\sharp\mathrm{s}}^2 \leq (1 + c(h)) \inf_{\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)} |\!|\!|\boldsymbol{E} - \boldsymbol{v}_h|\!|\!|_{\sharp\mathrm{s}}^2 + \underbrace{2\rho^{-1} |\!|\!|\boldsymbol{e}|\!|\!|_{\sharp\mathrm{s}} \min_{\boldsymbol{\Phi}_h^{\mathrm{c}} \in \boldsymbol{P}_\ell^{\mathrm{c}}(\mathcal{T}_h)} |\!|\!|\nu \nabla_0 \times \boldsymbol{E} - \boldsymbol{\Phi}_h^{\mathrm{c}}|\!|\!|_{\mathrm{ap}*}}_{\text{nonconformity}\times\text{consistency defect}}$$

Consistency defect can be tamed by increasing ℓ for smooth solutions

Further insight on spectral correctness

- Roadmap
- Poincaré–Steklov inequalities and inf-sup stability
- Duality argument

Convergence in operator norm (1/3)

- Recall $\boldsymbol{L}^2$-orthogonal projections

$$\boldsymbol{\Pi}_0^{\mathrm{c}} : \boldsymbol{L}^2(D) \to \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D), \quad \boldsymbol{\Pi}^{\mathrm{c}} : \boldsymbol{L}^2(D) \to \boldsymbol{H}(\mathbf{curl} = \mathbf{0}; D)$$

- For all $(\boldsymbol{f}, \boldsymbol{g}) \in L := \boldsymbol{L}^2(D) \times \boldsymbol{L}^2(D)$, $T(\boldsymbol{f}, \boldsymbol{g})$ is the unique pair $(\boldsymbol{H}, \boldsymbol{E}) \in \boldsymbol{X}_{\mu,0}^{\mathrm{c}} \times \boldsymbol{X}_{\epsilon}^{\mathrm{c}}$ solving the well-posed problem

$$-\nabla \times \boldsymbol{E} = \omega_D (\boldsymbol{I} - \boldsymbol{\Pi}_0^{\mathrm{c}})(\mu \boldsymbol{f}), \quad \nabla_0 \times \boldsymbol{H} = \omega_D (\boldsymbol{I} - \boldsymbol{\Pi}^{\mathrm{c}})(\epsilon \boldsymbol{g})$$

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- $\big(\lambda, (\boldsymbol{H}, \boldsymbol{E})\big)$, $\lambda \neq 0$, is a Maxwell eigenpair iff

$$T(\boldsymbol{H}, \boldsymbol{E}) = \lambda(\boldsymbol{H}, \boldsymbol{E})$$

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- What is the discrete counterpart?

Convergence in operator norm (2/3)

- Introduce discrete $\boldsymbol{L}^2$-orthogonal projections

$$\boldsymbol{\Pi}_{h0}^{\mathrm{c}} : \boldsymbol{L}^2(D) \longrightarrow \boldsymbol{P}_{k,0}^{\mathrm{c}}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h), \quad \boldsymbol{\Pi}_{h}^{\mathrm{c}} : \boldsymbol{L}^2(D) \longrightarrow \boldsymbol{P}_{k}^{\mathrm{c}}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$$

and set $\boldsymbol{X}_{\mu,h0}^{\mathrm{c}} := \{\boldsymbol{h}_h \in \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \mid \boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\mu \boldsymbol{h}_h) = \mathbf{0}\}$, $\boldsymbol{X}_{\epsilon,h}^{\mathrm{c}} := \ldots$

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- For all $(\boldsymbol{f}, \boldsymbol{g}) \in L$, $T_h(\boldsymbol{f}, \boldsymbol{g})$ is the unique pair $(\boldsymbol{H}_h, \boldsymbol{E}_h) \in \boldsymbol{X}^{\mathrm{c}}_{\mu,h0} \times \boldsymbol{X}^{\mathrm{c}}_{\epsilon,h}$ solving the well-posed problem (proof to come!)

$$a_h\big((\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)\big) = \omega_D\big(((\boldsymbol{I} - \boldsymbol{\Pi}^{\mathrm{c}}_{h0})(\mu \boldsymbol{f}), (\boldsymbol{I} - \boldsymbol{\Pi}^{\mathrm{c}}_{h})(\epsilon \boldsymbol{g})), (\boldsymbol{h}_h, \boldsymbol{e}_h)\big)_L$$

for all $(\boldsymbol{h}_h, \boldsymbol{e}_h) \in L_h := \boldsymbol{P}^{\mathrm{b}}_k(\mathcal{T}_h) \times \boldsymbol{P}^{\mathrm{b}}_k(\mathcal{T}_h)$, with discrete bilinear form

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- $\big(\lambda_h, (\boldsymbol{H}_h, \boldsymbol{E}_h)\big)$, $\lambda_h \neq 0$, is a discrete Maxwell eigenpair iff

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$$\lim_{h\to 0} \|T - T_h\|_{\mathcal{L}(L;L)} = 0$$

- Two key arguments to prove this result
  - stability by deflated inf-sup condition using discrete Poincaré–Steklov inequalities
  - duality argument

Discrete Poincaré–Steklov inequalities

- Weak PS inequalities

$$\|\boldsymbol{h}\|_{\boldsymbol{L}^2(D)} = \ell_D \|\nabla_0 \times \boldsymbol{h}\|_{(X^{\mathrm{c}})'}, \qquad \forall \boldsymbol{h} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}$$
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- Discrete setting? The difficulty is that

$$\boldsymbol{P}^c_{k,0}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp \not\subset \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp \quad \ldots$$

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- Lemma [AE & JLG, 23] Discrete PS inequalities hold with dual norms augmented by jump seminorms

$$\|\boldsymbol{h}_h\|_{\boldsymbol{L}^2(D)} \lesssim \ell_D \|\nabla_0\times\boldsymbol{h}_h\|_{(\boldsymbol{X}^{\mathrm{c}})'} + h^{\frac{1}{2}} |\boldsymbol{h}_h|_h^{\mathrm{H}}, \qquad \forall \boldsymbol{h}_h \in \boldsymbol{X}^{\mathrm{c}}_{\mu,h0}$$
$$\|\boldsymbol{e}_h\|_{\boldsymbol{L}^2(D)} \lesssim \ell_D \|\nabla\times\boldsymbol{e}_h\|_{(\boldsymbol{X}_0^{\mathrm{c}})'} + h^{\frac{1}{2}} |\boldsymbol{e}_h|_h^{\mathrm{E}}, \qquad \forall \boldsymbol{e}_h \in \boldsymbol{X}^{\mathrm{c}}_{\epsilon,h}$$

(Hidden constant in $\lesssim$ depends on contrast factors μ/μ_0, ϵ/ϵ_0)

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with $\boldsymbol{H}_0(\mathbf{curl}; D)$-conforming averaging operator from [AE & JLG, 17]

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  with $\boldsymbol{H}_0(\mathbf{curl}; D)$-conforming averaging operator from [AE & JLG, 17]

- Since $\boldsymbol{\xi} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}$, weak PS inequality gives

$$\|\boldsymbol{\xi}\|_{\boldsymbol{L}^2} \leq \ell_D \|\nabla_0 \times \boldsymbol{\xi}\|_{(\boldsymbol{X}^{\mathrm{c}})'} = \ell_D \|\nabla_0 \times \boldsymbol{h}_h^{\mathrm{c}}\|_{(\boldsymbol{X}^{\mathrm{c}})'}$$

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- Let $\boldsymbol{h}_h \in \boldsymbol{X}^{\mathrm{c}}_{\mu,h0}$; set

$$\boldsymbol{h}_h^{\mathrm{c}} := \mathcal{I}_{h0}^{\mathrm{c,av}}(\boldsymbol{h}_h), \qquad \boldsymbol{\xi} := \boldsymbol{h}_h^{\mathrm{c}} - \boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{h}_h^{\mathrm{c}})$$

with $\boldsymbol{H}_0(\mathbf{curl}; D)$-conforming averaging operator from [AE & JLG, 17]

- Since $\boldsymbol{\xi} \in \boldsymbol{H}_0(\mathbf{curl} = \mathbf{0}; D)^{\perp}$, weak PS inequality gives

$$\|\boldsymbol{\xi}\|_{\boldsymbol{L}^2} \le \ell_D \|\nabla_0 \times \boldsymbol{\xi}\|_{(\boldsymbol{X}^{\mathrm{c}})'} = \ell_D \|\nabla_0 \times \boldsymbol{h}_h^{\mathrm{c}}\|_{(\boldsymbol{X}^{\mathrm{c}})'}$$

- Triangle inequality and approximation properties of $\mathcal{I}_{h0}^{\mathrm{c,av}}$ give

$$\begin{aligned} \|\boldsymbol{\xi}\|_{\boldsymbol{L}^2} &\le \ell_D \|\nabla_0 \times (\boldsymbol{h}_h - \boldsymbol{h}_h^{\mathrm{c}})\|_{(\boldsymbol{X}^{\mathrm{c}})'} + \ell_D \|\nabla_0 \times \boldsymbol{h}_h\|_{(\boldsymbol{X}^{\mathrm{c}})'} \\ &\le \|\boldsymbol{h}_h - \boldsymbol{h}_h^{\mathrm{c}}\|_{\boldsymbol{L}^2} + \ell_D \|\nabla_0 \times \boldsymbol{h}_h\|_{(\boldsymbol{X}^{\mathrm{c}})'} \\ &\lesssim h^{\frac{1}{2}} |\boldsymbol{h}_h|_h^{\mathrm{H}} + \ell_D \|\nabla_0 \times \boldsymbol{h}_h\|_{(\boldsymbol{X}^{\mathrm{c}})'} \end{aligned}$$

Sketch of proof (2/2)

- Commuting approximation operators for Nédélec and Raviart–Thomas FEM; see [AE & JLG, 21 (vol. I)] and [Schöberl 01; Christiansen, Winther 06]

$$\mathcal{J}^{\mathrm{c}}_{h0} : \boldsymbol{L}^2(D) \to \boldsymbol{P}^{\mathrm{c}}_{k0}(\mathcal{T}_h), \quad \mathcal{J}^{\mathrm{d}}_{h0} : \boldsymbol{L}^2(D) \to \boldsymbol{P}^{\mathrm{d}}_{k0}(\mathcal{T}_h)$$

- $\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{h}^{\mathrm{c}}_h)) \in \boldsymbol{P}^{\mathrm{c}}_{k0}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$ by commuting property

$$\nabla_0{\times}(\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{h}^{\mathrm{c}}_h))) = \mathcal{J}^{\mathrm{d}}_{h0}(\nabla_0{\times}(\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{h}^{\mathrm{c}}_h))) = \mathcal{J}^{\mathrm{d}}_{h0}(\mathbf{0}) = \mathbf{0}$$

Sketch of proof (2/2)

- Commuting approximation operators for Nédélec and Raviart–Thomas FEM; see [AE & JLG, 21 (vol. I)] and [Schöberl 01; Christiansen, Winther 06]

$$\mathcal{J}_{h0}^{\mathrm{c}} : \boldsymbol{L}^2(D) \to \boldsymbol{P}_{k0}^{\mathrm{c}}(\mathcal{T}_h), \quad \mathcal{J}_{h0}^{\mathrm{d}} : \boldsymbol{L}^2(D) \to \boldsymbol{P}_{k0}^{\mathrm{d}}(\mathcal{T}_h)$$

- $\mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{h}_h^{\mathrm{c}})) \in \boldsymbol{P}_{k0}^{\mathrm{c}}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$ by commuting property

$$\nabla_0 \times (\mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{h}_h^{\mathrm{c}}))) = \mathcal{J}_{h0}^{\mathrm{d}}(\nabla_0 \times (\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{h}_h^{\mathrm{c}}))) = \mathcal{J}_{h0}^{\mathrm{d}}(\mathbf{0}) = \mathbf{0}$$

- So $\boldsymbol{h}_h^{\mathrm{c}} - \mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{\xi}) = \mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{h}_h^{\mathrm{c}} - \boldsymbol{\xi}) = \mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{\Pi}_0^{\mathrm{c}}(\boldsymbol{h}_h^{\mathrm{c}})) \in \boldsymbol{P}_{k0}^{\mathrm{c}}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$

Sketch of proof (2/2)

- Commuting approximation operators for Nédélec and Raviart–Thomas FEM; see [AE & JLG, 21 (vol. I)] and [Schöberl 01; Christiansen, Winther 06]

$$\mathcal{J}^{\mathrm{c}}_{h0} : \boldsymbol{L}^2(D) \to \boldsymbol{P}^{\mathrm{c}}_{k0}(\mathcal{T}_h), \quad \mathcal{J}^{\mathrm{d}}_{h0} : \boldsymbol{L}^2(D) \to \boldsymbol{P}^{\mathrm{d}}_{k0}(\mathcal{T}_h)$$

- $\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{h}^{\mathrm{c}}_h)) \in \boldsymbol{P}^{\mathrm{c}}_{k0}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$ by commuting property

$$\nabla_0{\times}(\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{h}^{\mathrm{c}}_h))) = \mathcal{J}^{\mathrm{d}}_{h0}(\nabla_0{\times}(\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{h}^{\mathrm{c}}_h))) = \mathcal{J}^{\mathrm{d}}_{h0}(\mathbf{0}) = \mathbf{0}$$

- So $\boldsymbol{h}^{\mathrm{c}}_h - \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\xi}) = \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{h}^{\mathrm{c}}_h - \boldsymbol{\xi}) = \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{h}^{\mathrm{c}}_h)) \in \boldsymbol{P}^{\mathrm{c}}_{k0}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$
- Since $\mu\boldsymbol{h}_h \in \boldsymbol{P}^{\mathrm{c}}_{k0}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^{\perp}$ by assumption, this gives

$$\begin{aligned}\|\mu^{\frac{1}{2}}\boldsymbol{h}_h\|^2_{\boldsymbol{L}^2} &= (\mu\boldsymbol{h}_h, \boldsymbol{h}_h - \boldsymbol{h}^{\mathrm{c}}_h)_{\boldsymbol{L}^2} + (\mu\boldsymbol{h}_h, \boldsymbol{h}^{\mathrm{c}}_h - \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\xi}))_{\boldsymbol{L}^2} + (\mu\boldsymbol{h}_h, \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\xi}))_{\boldsymbol{L}^2} \\ &= (\mu\boldsymbol{h}_h, \boldsymbol{h}_h - \boldsymbol{h}^{\mathrm{c}}_h)_{\boldsymbol{L}^2} + (\mu\boldsymbol{h}_h, \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\xi}))_{\boldsymbol{L}^2}\end{aligned}$$

Sketch of proof (2/2)

- Commuting approximation operators for Nédélec and Raviart–Thomas FEM; see [AE & JLG, 21 (vol. I)] and [Schöberl 01; Christiansen, Winther 06]

$$\mathcal{J}^{\mathrm{c}}_{h0} : \boldsymbol{L}^2(D) \to \boldsymbol{P}^{\mathrm{c}}_{k0}(\mathcal{T}_h), \quad \mathcal{J}^{\mathrm{d}}_{h0} : \boldsymbol{L}^2(D) \to \boldsymbol{P}^{\mathrm{d}}_{k0}(\mathcal{T}_h)$$

- $\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{h}^{\mathrm{c}}_h)) \in \boldsymbol{P}^{\mathrm{c}}_{k0}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$ by commuting property

$$\nabla_0 \times (\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{h}^{\mathrm{c}}_h))) = \mathcal{J}^{\mathrm{d}}_{h0}(\nabla_0 \times (\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{h}^{\mathrm{c}}_h))) = \mathcal{J}^{\mathrm{d}}_{h0}(\mathbf{0}) = \mathbf{0}$$

- So $\boldsymbol{h}^{\mathrm{c}}_h - \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\xi}) = \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{h}^{\mathrm{c}}_h - \boldsymbol{\xi}) = \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\Pi}^{\mathrm{c}}_0(\boldsymbol{h}^{\mathrm{c}}_h)) \in \boldsymbol{P}^{\mathrm{c}}_{k0}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$
- Since $\mu \boldsymbol{h}_h \in \boldsymbol{P}^{\mathrm{c}}_{k0}(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^{\perp}$ by assumption, this gives

$$\begin{aligned} \|\mu^{\frac{1}{2}} \boldsymbol{h}_h\|^2_{\boldsymbol{L}^2} &= (\mu \boldsymbol{h}_h, \boldsymbol{h}_h - \boldsymbol{h}^{\mathrm{c}}_h)_{\boldsymbol{L}^2} + (\mu \boldsymbol{h}_h, \boldsymbol{h}^{\mathrm{c}}_h - \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\xi}))_{\boldsymbol{L}^2} + (\mu \boldsymbol{h}_h, \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\xi}))_{\boldsymbol{L}^2} \\ &= (\mu \boldsymbol{h}_h, \boldsymbol{h}_h - \boldsymbol{h}^{\mathrm{c}}_h)_{\boldsymbol{L}^2} + (\mu \boldsymbol{h}_h, \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\xi}))_{\boldsymbol{L}^2} \end{aligned}$$

- Since $\mathcal{J}^{\mathrm{c}}_{h0}$ is $\boldsymbol{L}^2$-stable, we conclude that

$$\|\boldsymbol{h}_h\|_{\boldsymbol{L}^2} \lesssim \|\boldsymbol{h}_h - \boldsymbol{h}^{\mathrm{c}}_h\|_{\boldsymbol{L}^2} + \|\boldsymbol{\xi}\|_{\boldsymbol{L}^2} \lesssim h^{\frac{1}{2}} |\boldsymbol{h}_h|^{\mathrm{H}}_h + \ell_D \|\nabla_0 \times \boldsymbol{h}_h\|_{(\boldsymbol{X}^{\mathrm{c}})'}$$

Inf-sup stability

- Mesh-dependent norm on $L_h := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \times \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$,

$$\|(\boldsymbol{h}_h, \boldsymbol{e}_h)\|_{\flat,h} := \omega_D^{\frac{1}{2}} \|(\mu^{\frac{1}{2}} \boldsymbol{h}_h, \epsilon^{\frac{1}{2}} \boldsymbol{e}_h)\|_L + \kappa_{\mathrm{H}}^{\frac{1}{2}} \left\{ \|h^{\frac{1}{2}} \boldsymbol{C}_{h0}(\boldsymbol{h}_h)\|_{\boldsymbol{L}^2} + |\boldsymbol{h}_h|_h^{\mathrm{H}} \right\} + \kappa_{\mathrm{E}}^{\frac{1}{2}} \left\{ \|h^{\frac{1}{2}} \boldsymbol{C}_h(\boldsymbol{e}_h)\|_{\boldsymbol{L}^2} + |\boldsymbol{e}_h|_h^{\mathrm{E}} \right\}$$

(Notice $h^{\frac{1}{2}}$-weighted curls as expected in Friedrichs systems [AE & JLG, 06])

Inf-sup stability

- Mesh-dependent norm on $L_h := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \times \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$,

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(Notice $h^{\frac{1}{2}}$-weighted curls as expected in Friedrichs systems [AE & JLG, 06])

- Deflated inf-sup condition: For all $(\boldsymbol{H}_h, \boldsymbol{E}_h) \in \boldsymbol{X}^{\mathrm{c}}_{\mu,h0} \times \boldsymbol{X}^{\mathrm{c}}_{\epsilon,h}$,

$$\omega_D^{\frac{1}{2}} \|(\boldsymbol{H}_h, \boldsymbol{E}_h)\|_{\flat,h} \lesssim \sup_{(\boldsymbol{h}_h, \boldsymbol{e}_h) \in L_h} \frac{|a_h\big((\boldsymbol{H}_h, \boldsymbol{E}_h), (\boldsymbol{h}_h, \boldsymbol{e}_h)\big)|}{\|(\mu^{\frac{1}{2}}\boldsymbol{h}_h, \epsilon^{\frac{1}{2}}\boldsymbol{e}_h)\|_L}$$

(different norms, different spaces)

(proof uses techniques for Friedrichs systems and discrete PS inequalities)

Inf-sup stability

- Mesh-dependent norm on $L_h := \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h) \times \boldsymbol{P}_k^{\mathrm{b}}(\mathcal{T}_h)$,

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(different norms, different spaces)

(proof uses techniques for Friedrichs systems and discrete PS inequalities)

- Corollary. Discrete BVP problem defining $T_h : L \to L_h$ is well-posed

Duality argument (1/2)

- Let $(\boldsymbol{f}, \boldsymbol{g}) \in L$
- Let $(\boldsymbol{H}, \boldsymbol{E}) := T(\boldsymbol{f}, \boldsymbol{g}) \in \boldsymbol{X}^{\mathrm{c}}_{\boldsymbol{\mu},0} \times \boldsymbol{X}^{\mathrm{c}}_{\boldsymbol{\epsilon}}$
- Let $(\boldsymbol{H}_h, \boldsymbol{E}_h) := T_h(\boldsymbol{f}, \boldsymbol{g}) \in \boldsymbol{X}^{\mathrm{c}}_{\boldsymbol{\mu},h0} \times \boldsymbol{X}^{\mathrm{c}}_{\boldsymbol{\epsilon},h}$

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- The goal is to prove that $\lim_{h\to 0} \|(\mu^{\frac{1}{2}}\boldsymbol{\delta h}, \epsilon^{\frac{1}{2}}\boldsymbol{\delta e})\|_L = 0$ with the errors

$$\boldsymbol{\delta h} := \boldsymbol{H} - \boldsymbol{H}_h, \qquad \boldsymbol{\delta e} := \boldsymbol{E} - \boldsymbol{E}_h$$

Duality argument (1/2)

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- Let $(\boldsymbol{H},\boldsymbol{E}) := T(\boldsymbol{f},\boldsymbol{g}) \in \boldsymbol{X}^{\mathrm{c}}_{\mu,0} \times \boldsymbol{X}^{\mathrm{c}}_{\epsilon}$
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$$\boldsymbol{\delta h} := \boldsymbol{H} - \boldsymbol{H}_h, \qquad \boldsymbol{\delta e} := \boldsymbol{E} - \boldsymbol{E}_h$$

- Dual problem: Find $(\boldsymbol{\eta},\boldsymbol{\varepsilon}) \in \boldsymbol{X}^{\mathrm{c}}_0 \times \boldsymbol{X}^{\mathrm{c}}$ s.t. (involution with constant properties!!)

$$-\nabla_0\times\boldsymbol{\eta} = \omega_D(\boldsymbol{I} - \boldsymbol{\Pi}^{\mathrm{c}})(\epsilon\boldsymbol{\delta e}), \qquad \nabla\times\boldsymbol{\varepsilon} = \omega_D(\boldsymbol{I} - \boldsymbol{\Pi}^{\mathrm{c}}_0)(\mu\boldsymbol{\delta h})$$

Duality argument (1/2)

- Let $(\boldsymbol{f}, \boldsymbol{g}) \in L$
- Let $(\boldsymbol{H}, \boldsymbol{E}) := T(\boldsymbol{f}, \boldsymbol{g}) \in \boldsymbol{X}^{\mathrm{c}}_{\mu,0} \times \boldsymbol{X}^{\mathrm{c}}_{\epsilon}$
- Let $(\boldsymbol{H}_h, \boldsymbol{E}_h) := T_h(\boldsymbol{f}, \boldsymbol{g}) \in \boldsymbol{X}^{\mathrm{c}}_{\mu,h0} \times \boldsymbol{X}^{\mathrm{c}}_{\epsilon,h}$

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- Improved regularity shift, $s' \in (\frac{1}{2}, 1]$

$$|\boldsymbol{\eta}|_{\boldsymbol{H}^{s'}} \lesssim \ell_D^{1-s'}\|\nabla_0\times\boldsymbol{\eta}\|_{\boldsymbol{L}^2}, \qquad |\boldsymbol{\varepsilon}|_{\boldsymbol{H}^{s'}} \lesssim \ell_D^{1-s'}\|\nabla\times\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2}$$

(Notice that $\ell_D\big(\mu_0^{\frac{1}{2}}\|\nabla_0\times\boldsymbol{\eta}\|_{\boldsymbol{L}^2} + \epsilon_0^{\frac{1}{2}}\|\nabla\times\boldsymbol{\varepsilon}\|_{\boldsymbol{L}^2}\big) \lesssim \|(\mu^{\frac{1}{2}}\boldsymbol{\delta h}, \epsilon^{\frac{1}{2}}\boldsymbol{\delta e})\|_L$)

Duality argument (2/2)

- Error representation

$$\omega_D \|(\mu^{\frac{1}{2}} \boldsymbol{\delta h}, \epsilon^{\frac{1}{2}} \boldsymbol{\delta e})\|_L^2 = \theta_{\text{app}} + \theta_{\text{gal}} + \theta_{\text{crl}} + \theta_{\text{div}}$$

Duality argument (2/2)

- Error representation

$$\omega_D \|(\mu^{\frac{1}{2}} \boldsymbol{\delta h}, \epsilon^{\frac{1}{2}} \boldsymbol{\delta e})\|_L^2 = \theta_{\text{app}} + \theta_{\text{gal}} + \theta_{\text{crl}} + \theta_{\text{div}}$$

  - **Approximation error:** $\theta_{\text{app}} := a_h((\boldsymbol{\delta h}, \boldsymbol{\delta e}), ((\boldsymbol{I} - \boldsymbol{\Pi}_h^{\text{b}})(\boldsymbol{\eta}), (\boldsymbol{I} - \boldsymbol{\Pi}_h^{\text{b}})(\boldsymbol{\varepsilon})))$

Duality argument (2/2)

- Error representation

$$\omega_D \|(\mu^{\frac{1}{2}}\boldsymbol{\delta h}, \epsilon^{\frac{1}{2}}\boldsymbol{\delta e})\|_L^2 = \theta_{\mathrm{app}} + \theta_{\mathrm{gal}} + \theta_{\mathrm{crl}} + \theta_{\mathrm{div}}$$

  - **Approximation error:** $\theta_{\mathrm{app}} := a_h((\boldsymbol{\delta h}, \boldsymbol{\delta e}), ((\boldsymbol{I} - \boldsymbol{\Pi}_h^{\mathrm{b}})(\boldsymbol{\eta}), (\boldsymbol{I} - \boldsymbol{\Pi}_h^{\mathrm{b}})(\boldsymbol{\varepsilon})))$
  - **Galerkin orthogonality error** caused by inconsistency on rhs: $\theta_{\mathrm{gal}} := \omega_D\left\{((\boldsymbol{\Pi}_0^{\mathrm{c}} - \boldsymbol{\Pi}_{h0}^{\mathrm{c}})(\mu \boldsymbol{f}), \boldsymbol{\eta}_h)_{\boldsymbol{L}^2} + ((\boldsymbol{\Pi}^{\mathrm{c}} - \boldsymbol{\Pi}_h^{\mathrm{c}})(\epsilon \boldsymbol{g}), \boldsymbol{\varepsilon}_h)_{\boldsymbol{L}^2}\right\}$

Duality argument (2/2)

- Error representation

$$\omega_D \|(\mu^{\frac{1}{2}}\boldsymbol{\delta h}, \epsilon^{\frac{1}{2}}\boldsymbol{\delta e})\|_L^2 = \theta_{\text{app}} + \theta_{\text{gal}} + \theta_{\text{crl}} + \theta_{\text{div}}$$

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  - **Galerkin orthogonality error** caused by inconsistency on rhs: $\theta_{\text{gal}} := \omega_D \left\{ ((\boldsymbol{\Pi}_0^{\text{c}} - \boldsymbol{\Pi}_{h0}^{\text{c}})(\mu \boldsymbol{f}), \boldsymbol{\eta}_h)_{\boldsymbol{L}^2} + ((\boldsymbol{\Pi}^{\text{c}} - \boldsymbol{\Pi}_h^{\text{c}})(\epsilon \boldsymbol{g}), \boldsymbol{\varepsilon}_h)_{\boldsymbol{L}^2} \right\}$
  - **Curl commuting error:** ($\boldsymbol{\eta}, \boldsymbol{\varepsilon}$ are not polynomials!) $\theta_{\text{crl}} := \left\{ (\boldsymbol{h}_h, \nabla \times \boldsymbol{\varepsilon})_{\boldsymbol{L}^2} - (\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{h}_h), \boldsymbol{\varepsilon})_{\boldsymbol{L}^2} \right\} - \left\{ (\boldsymbol{e}_h, \nabla_0 \times \boldsymbol{\eta})_{\boldsymbol{L}^2} - (\boldsymbol{C}_h^{k,\ell}(\boldsymbol{e}_h), \boldsymbol{\eta})_{\boldsymbol{L}^2} \right\}$

Duality argument (2/2)

- Error representation

$$\omega_D \|(\mu^{\frac{1}{2}}\boldsymbol{\delta h}, \epsilon^{\frac{1}{2}}\boldsymbol{\delta e})\|_L^2 = \theta_{\mathrm{app}} + \theta_{\mathrm{gal}} + \theta_{\mathrm{crl}} + \theta_{\mathrm{div}}$$

  - **Approximation error:** $\theta_{\mathrm{app}} := a_h((\boldsymbol{\delta h}, \boldsymbol{\delta e}), ((\boldsymbol{I} - \boldsymbol{\Pi}_h^{\mathrm{b}})(\boldsymbol{\eta}), (\boldsymbol{I} - \boldsymbol{\Pi}_h^{\mathrm{b}})(\boldsymbol{\varepsilon})))$
  - **Galerkin orthogonality error** caused by inconsistency on rhs: $\theta_{\mathrm{gal}} := \omega_D \big\{ ((\boldsymbol{\Pi}_0^{\mathrm{c}} - \boldsymbol{\Pi}_{h0}^{\mathrm{c}})(\mu \boldsymbol{f}), \boldsymbol{\eta}_h)_{\boldsymbol{L}^2} + ((\boldsymbol{\Pi}^{\mathrm{c}} - \boldsymbol{\Pi}_h^{\mathrm{c}})(\epsilon \boldsymbol{g}), \boldsymbol{\varepsilon}_h)_{\boldsymbol{L}^2} \big\}$
  - **Curl commuting error:** ($\boldsymbol{\eta}, \boldsymbol{\varepsilon}$ are not polynomials!) $\theta_{\mathrm{crl}} := \big\{ (\boldsymbol{h}_h, \nabla \times \boldsymbol{\varepsilon})_{\boldsymbol{L}^2} - (\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{h}_h), \boldsymbol{\varepsilon})_{\boldsymbol{L}^2} \big\} - \big\{ (\boldsymbol{e}_h, \nabla_0 \times \boldsymbol{\eta})_{\boldsymbol{L}^2} - (\boldsymbol{C}_h^{k,\ell}(\boldsymbol{e}_h), \boldsymbol{\eta})_{\boldsymbol{L}^2} \big\}$
  - **Divergence conformity error:** ($\boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\mu \boldsymbol{\delta h}) = \boldsymbol{0} \not\Rightarrow \boldsymbol{\Pi}_0^{\mathrm{c}}(\mu \boldsymbol{\delta h}) = \boldsymbol{0}$) $\theta_{\mathrm{div}} := \omega_D \big\{ (\boldsymbol{\delta h}, \boldsymbol{\Pi}_0^{\mathrm{c}}(\mu \boldsymbol{\delta h}))_{\boldsymbol{L}^2} + (\boldsymbol{\delta e}, \boldsymbol{\Pi}^{\mathrm{c}}(\epsilon \boldsymbol{\delta e}))_{\boldsymbol{L}^2} \big\}$

Duality argument (2/2)

- Error representation

$$\omega_D \|(\mu^{\frac{1}{2}} \boldsymbol{\delta h}, \epsilon^{\frac{1}{2}} \boldsymbol{\delta e})\|_L^2 = \theta_{\text{app}} + \theta_{\text{gal}} + \theta_{\text{crl}} + \theta_{\text{div}}$$

  - **Approximation error:** $\theta_{\text{app}} := a_h((\boldsymbol{\delta h}, \boldsymbol{\delta e}), ((\boldsymbol{I} - \boldsymbol{\Pi}_h^{\text{b}})(\boldsymbol{\eta}), (\boldsymbol{I} - \boldsymbol{\Pi}_h^{\text{b}})(\boldsymbol{\varepsilon})))$
  - **Galerkin orthogonality error** caused by inconsistency on rhs: $\theta_{\text{gal}} := \omega_D \{ ((\boldsymbol{\Pi}_0^{\text{c}} - \boldsymbol{\Pi}_{h0}^{\text{c}})(\mu \boldsymbol{f}), \boldsymbol{\eta}_h)_{\boldsymbol{L}^2} + ((\boldsymbol{\Pi}^{\text{c}} - \boldsymbol{\Pi}_h^{\text{c}})(\epsilon \boldsymbol{g}), \boldsymbol{\varepsilon}_h)_{\boldsymbol{L}^2} \}$
  - **Curl commuting error:** ($\boldsymbol{\eta}, \boldsymbol{\varepsilon}$ are not polynomials!) $\theta_{\text{crl}} := \{ (\boldsymbol{h}_h, \nabla\times\boldsymbol{\varepsilon})_{\boldsymbol{L}^2} - (\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{h}_h), \boldsymbol{\varepsilon})_{\boldsymbol{L}^2} \} - \{ (\boldsymbol{e}_h, \nabla_0\times\boldsymbol{\eta})_{\boldsymbol{L}^2} - (\boldsymbol{C}_h^{k,\ell}(\boldsymbol{e}_h), \boldsymbol{\eta})_{\boldsymbol{L}^2} \}$
  - **Divergence conformity error:** ($\boldsymbol{\Pi}_{h0}^{\text{c}}(\mu \boldsymbol{\delta h}) = \boldsymbol{0} \not\Rightarrow \boldsymbol{\Pi}_0^{\text{c}}(\mu \boldsymbol{\delta h}) = \boldsymbol{0}$) $\theta_{\text{div}} := \omega_D \{ (\boldsymbol{\delta h}, \boldsymbol{\Pi}_0^{\text{c}}(\mu \boldsymbol{\delta h}))_{\boldsymbol{L}^2} + (\boldsymbol{\delta e}, \boldsymbol{\Pi}^{\text{c}}(\epsilon \boldsymbol{\delta e}))_{\boldsymbol{L}^2} \}$

- All terms bounded using improved regularity shift on dual solution and a priori estimate from deflated inf-sup condition

Duality argument (2/2)

- Error representation

$$\omega_D \|(\mu^{\frac{1}{2}}\boldsymbol{\delta h}, \epsilon^{\frac{1}{2}}\boldsymbol{\delta e})\|_L^2 = \theta_{\mathrm{app}} + \theta_{\mathrm{gal}} + \theta_{\mathrm{crl}} + \theta_{\mathrm{div}}$$

  - **Approximation error:** $\theta_{\mathrm{app}} := a_h((\boldsymbol{\delta h}, \boldsymbol{\delta e}), ((\boldsymbol{I} - \boldsymbol{\Pi}_h^{\mathrm{b}})(\boldsymbol{\eta}), (\boldsymbol{I} - \boldsymbol{\Pi}_h^{\mathrm{b}})(\boldsymbol{\varepsilon})))$
  - **Galerkin orthogonality error** caused by inconsistency on rhs: $\theta_{\mathrm{gal}} := \omega_D\{((\boldsymbol{\Pi}_0^{\mathrm{c}} - \boldsymbol{\Pi}_{h0}^{\mathrm{c}})(\mu\boldsymbol{f}), \boldsymbol{\eta}_h)_{\boldsymbol{L}^2} + ((\boldsymbol{\Pi}^{\mathrm{c}} - \boldsymbol{\Pi}_h^{\mathrm{c}})(\epsilon\boldsymbol{g}), \boldsymbol{\varepsilon}_h)_{\boldsymbol{L}^2}\}$
  - **Curl commuting error:** ($\boldsymbol{\eta}, \boldsymbol{\varepsilon}$ are not polynomials!) $\theta_{\mathrm{crl}} := \{(\boldsymbol{h}_h, \nabla\times\boldsymbol{\varepsilon})_{\boldsymbol{L}^2} - (\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{h}_h), \boldsymbol{\varepsilon})_{\boldsymbol{L}^2}\} - \{(\boldsymbol{e}_h, \nabla_0\times\boldsymbol{\eta})_{\boldsymbol{L}^2} - (\boldsymbol{C}_h^{k,\ell}(\boldsymbol{e}_h), \boldsymbol{\eta})_{\boldsymbol{L}^2}\}$
  - **Divergence conformity error:** ($\boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\mu\boldsymbol{\delta h}) = \boldsymbol{0} \not\Rightarrow \boldsymbol{\Pi}_0^{\mathrm{c}}(\mu\boldsymbol{\delta h}) = \boldsymbol{0}$) $\theta_{\mathrm{div}} := \omega_D\{(\boldsymbol{\delta h}, \boldsymbol{\Pi}_0^{\mathrm{c}}(\mu\boldsymbol{\delta h}))_{\boldsymbol{L}^2} + (\boldsymbol{\delta e}, \boldsymbol{\Pi}^{\mathrm{c}}(\epsilon\boldsymbol{\delta e}))_{\boldsymbol{L}^2}\}$

- All terms bounded using improved regularity shift on dual solution and a priori estimate from deflated inf-sup condition

- Altogether, $\|T - T_h\|_{\mathcal{L}(L;L)} \lesssim (h/\ell_D)^{\sigma}$ with $\sigma := \min(s, s' - \frac{1}{2})$, i.e.,

$$\|(\mu^{\frac{1}{2}}\boldsymbol{\delta h}, \epsilon^{\frac{1}{2}}\boldsymbol{\delta e})\|_L \lesssim (h/\ell_D)^{\sigma}\|(\mu^{\frac{1}{2}}\boldsymbol{f}, \epsilon^{\frac{1}{2}}\boldsymbol{g})\|_L$$

Duality argument (2/2)

- Error representation

$$\omega_D \|(\mu^{\frac{1}{2}}\boldsymbol{\delta h}, \epsilon^{\frac{1}{2}}\boldsymbol{\delta e})\|_L^2 = \theta_{\mathrm{app}} + \theta_{\mathrm{gal}} + \theta_{\mathrm{crl}} + \theta_{\mathrm{div}}$$

  - **Approximation error:** $\theta_{\mathrm{app}} := a_h((\boldsymbol{\delta h}, \boldsymbol{\delta e}), ((\boldsymbol{I} - \boldsymbol{\Pi}_h^{\mathrm{b}})(\boldsymbol{\eta}), (\boldsymbol{I} - \boldsymbol{\Pi}_h^{\mathrm{b}})(\boldsymbol{\varepsilon})))$
  - **Galerkin orthogonality error** caused by inconsistency on rhs: $\theta_{\mathrm{gal}} := \omega_D\left\{((\boldsymbol{\Pi}_0^{\mathrm{c}} - \boldsymbol{\Pi}_{h0}^{\mathrm{c}})(\mu\boldsymbol{f}), \boldsymbol{\eta}_h)_{\boldsymbol{L}^2} + ((\boldsymbol{\Pi}^{\mathrm{c}} - \boldsymbol{\Pi}_h^{\mathrm{c}})(\epsilon\boldsymbol{g}), \boldsymbol{\varepsilon}_h)_{\boldsymbol{L}^2}\right\}$
  - **Curl commuting error:** ($\boldsymbol{\eta}, \boldsymbol{\varepsilon}$ are not polynomials!) $\theta_{\mathrm{crl}} := \left\{(\boldsymbol{h}_h, \nabla\times\boldsymbol{\varepsilon})_{\boldsymbol{L}^2} - (\boldsymbol{C}_{h,0}^{k,\ell}(\boldsymbol{h}_h), \boldsymbol{\varepsilon})_{\boldsymbol{L}^2}\right\} - \left\{(\boldsymbol{e}_h, \nabla_0\times\boldsymbol{\eta})_{\boldsymbol{L}^2} - (\boldsymbol{C}_h^{k,\ell}(\boldsymbol{e}_h), \boldsymbol{\eta})_{\boldsymbol{L}^2}\right\}$
  - **Divergence conformity error:** $(\boldsymbol{\Pi}_{h0}^{\mathrm{c}}(\mu\boldsymbol{\delta h}) = \boldsymbol{0} \not\Rightarrow \boldsymbol{\Pi}_0^{\mathrm{c}}(\mu\boldsymbol{\delta h}) = \boldsymbol{0})$ $\theta_{\mathrm{div}} := \omega_D\left\{(\boldsymbol{\delta h}, \boldsymbol{\Pi}_0^{\mathrm{c}}(\mu\boldsymbol{\delta h}))_{\boldsymbol{L}^2} + (\boldsymbol{\delta e}, \boldsymbol{\Pi}^{\mathrm{c}}(\epsilon\boldsymbol{\delta e}))_{\boldsymbol{L}^2}\right\}$

- All terms bounded using improved regularity shift on dual solution and a priori estimate from deflated inf-sup condition

- Altogether, $\|T - T_h\|_{\mathcal{L}(L;L)} \lesssim (h/\ell_D)^\sigma$ with $\sigma := \min(s, s' - \frac{1}{2})$, i.e.,

$$\|(\mu^{\frac{1}{2}}\boldsymbol{\delta h}, \epsilon^{\frac{1}{2}}\boldsymbol{\delta e})\|_L \lesssim (h/\ell_D)^\sigma \|(\mu^{\frac{1}{2}}\boldsymbol{f}, \epsilon^{\frac{1}{2}}\boldsymbol{g})\|_L$$

!! Thank you for your attention !!