Data Driven Robust Optimization Exam

19/03/2018

The exam is made of two independant parts. If necessary, you can admit the results of previous questions. All documents authorized, all electronical device forbidden.

Some usefull recalls.

1. An SOCP constraint take the form $a^T x + b + \|c^T x + d\| \leq 0$.
2. If $(g_i)_{i \in [\![1,d]\!]}$ are concave functions with $\cap_{i=1}^d ri(\text{dom}(g_i) \neq \emptyset$ we have

$$\Big(\sum_{i=1}^d g_i(\cdot)\Big)_\star(v) = \sup_{(v^i)_{i \in [\![1,d]\!]}} \Big\{ \sum_{i=1}^d (g_i)_\star(v^i) \;\Big|\; \sum_{i=1}^d v^i = v \Big\}$$

3. $\big\{w^T w \leq xy, x \geq 0, y \geq 0\big\}$ is equivalent to $\left\| \begin{pmatrix} 2w \\ x - y \end{pmatrix} \right\| \leq x + y$.
4. The value at risk of level ε is defined by

$$VaR^{\mathbb{P}}_\varepsilon(\boldsymbol{X}) := \inf \big\{ t \mid \mathbb{P}(\boldsymbol{X} \leq t) \geq 1 - \varepsilon \big\}$$

5. for $\varepsilon \in (0, 0.5]$,

$$\forall \tilde{u} \sim (\mu, \Sigma), \quad \mathbb{P}(\tilde{u}^T v \leq \alpha) \geq 1 - \varepsilon \quad \iff \quad \mu^T v \leq \alpha - \sqrt{\frac{1-\varepsilon}{\varepsilon}} \sqrt{v^T \Sigma v},$$

where $\tilde{u} \sim (\mu, \Sigma)$ means that $\mathbb{E}\big[\tilde{u}\big] = \mu$ and $var(\tilde{u}) = \Sigma$.

A simple example

1. Robust quadratic constraints

We are interested in the following quadratic constraint $f(u, x) := -\sum_{i=1}^d \frac{1}{2} x_i u^T Q_i u \leq 0$, were all matrices Q_i are positive definite, where $u \in \mathbb{R}^{n_u}$ and $x \in \mathbb{R}^{n_x}_+$.

(a) (1 point) Let $f_i(u) = -\frac{1}{2} u^T Q_i u$. Compute $(f_i)_\star(v) := \inf_{u \in \mathbb{R}^{n_u}} v^T u - f_i(u)$

Solution: f_i is strictly concave. By differentiation $Qu^\sharp = -v$ thus $(f_i)_\star(v) = -\frac{1}{2} v^T Q^{-1} v$.

(b) (1 point) Show that

$$f_\star(v, x) := \inf_u v^T u - f(u, x) = \sup_{(v^i)_{i \in [\![1,d]\!]}} \Big\{ -\frac{1}{2} \sum_{i=1}^d \frac{(v^i)^T Q_i^{-1} v^i}{x_i} \;\Big|\; \sum_{i=1}^d v^i = v \Big\}$$

Solution: We have, for any $x_i > 0$, $(x_i f_i)_\star(v^i) = x_i (f_i)_\star(\frac{v^i}{x_i})$.

2. Application

We are interested in the following problem

$$\begin{aligned} \min_{x\in\mathbb{R}^2_+} \quad & c^T x && \text{(1a)} \\ s.t. \quad & \tilde{z} = x_1 \tilde{u}^T Q_1 \tilde{u} + x_2 \tilde{u}^T Q_2 \tilde{u} && \text{(1b)} \\ & \mathbb{P}(\tilde{z} \geq 0) \geq 0.9 && \text{(1c)} \\ & Ax \leq b && \text{(1d)} \end{aligned}$$

where $Q_i \in M_4(\mathbb{R})$ are positive definite matrices, and $\tilde{u}$ is a random variable that can take values in $\{a_1, a_2, a_3\}$. We have a sample of 100 realisations of $\tilde{u}$, given in the following table.

a_1	a_2	a_3
30	20	50

(a) (2 points) We set $f(\tilde{u}, x) = -\frac{1}{2}\sum_{i=1}^2 x_i \tilde{u}^T Q_i \tilde{u}$. Show that $f_\star(v, x) \geq s$ is equivalent to

$$\begin{cases} \alpha_1 + \alpha_2 \leq 2s \\ \alpha_i x_i \geq (v^i)^T Q_i^{-1} v^i \quad i = 1, 2 \\ v^1 + v^2 = v \end{cases}$$

Solution: $f_\star(v, x) = \sup_{v^1+v^2=v} -\frac{1}{2}\sum_{i=1}^2 \frac{(v^i)^T Q_i^{-1} v^i}{x_i}$ thus $f_\star(v, x) \geq s$ is equivalent to

$$\begin{cases} \displaystyle\sum_{i=1}^2 \frac{(v^i)^T Q_i^{-1} v^i}{x_i} \leq 2s \\ v^1 + v^2 = v \end{cases}$$

or

$$\begin{cases} \alpha_1 + \alpha_2 \leq 2s \\ \alpha_i x_i \geq (v^i)^T Q_i^{-1} v^i \quad i = 1, 2 \\ v^1 + v^2 = v \end{cases}$$

(b) (1 point) Show that, in this problem, $f_\star(v, x) \geq s$ can be written as SOCP constraints.

Solution:

$$\begin{cases} \alpha_1 + \alpha_2 \leq 2s \\ \left\| \begin{pmatrix} 2Q_i^{-1/2} v^i \\ \alpha_i - x_i \end{pmatrix} \right\| \leq x_i + \alpha_i \quad i = 1, 2 \\ v^1 + v^2 = v \end{cases}$$

(c) (4 points) Leveraging the χ^2 test, explicit (giving numerical values to all possible parameters - see table at the end) a SOCP problem whose solution is a feasible solution for Problem 1 with 80% confidence (in the sampling). Precise the size of each variables, and the number of SOCP constraints and linear constraints.

Solution:

$$
\begin{array}{lll}
\min\limits_{x} & c^T x & \\
s.t. & Ax \leq b, x \geq 0 & \\
& \alpha_1 + \alpha_2 \leq 2s & \\
& \left\| \begin{pmatrix} 2Q_i^{-1/2} v^i \\ \alpha_i - x_i \end{pmatrix} \right\| \leq x_i + \alpha_i & i = 1,2 \\
& v^1 + v^2 = v & \\
& t - s \leq 0 & \\
& \beta + 10\Big[\eta + \lambda \frac{3.21}{100} + 2\lambda - 2(0.3\sigma_1 + 0.2\sigma_2 + 0.5\sigma_3)\Big] \leq t & \\
& \left\| \begin{pmatrix} 2\sigma_j \\ \eta - w_j \end{pmatrix} \right\| \leq 2\lambda + \eta - w_j & j = 1,2,3 \\
& 0 \leq w_j \leq \lambda + \eta & j = 1,2,3 \\
& a_j^T v - w_j \leq \beta & j = 1,2,3
\end{array}
$$

(d) (1 point) How many sample are needed to ensure the same guarantee through a sampling approach ? What would actually happen ?

Solution: $2/(0.2 \times 0.1) - 1 = 99$ samples. With confidence at least $(1 - 0.2)^{0.99}$ all three realizations would be taken, and the constraint would be realized almost-surely.

A new data-driven approach

We will now assume that $0 < \varepsilon \leq 0.5$.

3. Estimated variance and covariance

We are interested in the following optimization problem

$$
\begin{array}{ll}
\min\limits_{x \in \mathbb{R}^d} & c^T x \\
s.t. & \mathbb{P}(f(\tilde{u}, x) \leq 0) \geq 1 - \varepsilon
\end{array}
$$

where $f(u, x)$ is a function concave in u, and convex in x.

We define the following trust region

$$
\mathcal{P}^{CS}(\Gamma_1, \Gamma_2) = \Big\{ \mathbb{P} \quad | \quad \|\mathbb{E}^{\mathbb{P}}(\tilde{u}) - \hat{\mu}\|_2 \leq \Gamma_1, \quad |||var^{\mathbb{P}}(\tilde{u}) - \hat{\Sigma}||| \leq \Gamma_2 \Big\},
$$

where $var^{\mathbb{P}}$ is the variance operator, $|||A||| := \sup_{\|x\|_2 \leq 1} \|Ax\|_2$ is the operator norm, and $\hat{\mu}$ (resp. $\hat{\Sigma}$) is an estimator of the expectation of $\tilde{u}$ (resp. of the covariance matrix of $\tilde{u}$). We assume that Γ_1 and Γ_2 have be choosen such that $\mathbb{P}_S^*\Big(\mathbb{P}^* \in \mathcal{P}^{CS}(\Gamma_1, \Gamma_2)\Big) \geq 1 - \alpha$.

(a) (2 points) We call $R(\mu, \Sigma)$ the set of probabilities such that $\mathbb{P} \in R(\mu, \Sigma)$ if and only if $\mathbb{E}^{\mathbb{P}}\big[\tilde{u}\big] = \mu$ and $var^{\mathbb{P}}(\tilde{u}) = \Sigma$. Show that

$$
\sup_{\mathbb{P} \in R(\mu, \Sigma)} VaR_{\varepsilon}^{\mathbb{P}}(v^T \tilde{u}) = \mu^T v + \sqrt{\frac{1 - \varepsilon}{\varepsilon}} \sqrt{v^T \Sigma v}.
$$

Solution:

$$\mu^T v + \sqrt{\frac{1-\varepsilon}{\varepsilon}}\sqrt{v^T \Sigma v} \leq \alpha$$

iff

$$\forall \mathbb{P} \in R(\mu, \Sigma), \quad \mathbb{P}(\tilde{u}^T v \leq \alpha) \geq 1 - \varepsilon$$

iff

$$\forall \mathbb{P} \in R(\mu, \Sigma), VaR_\varepsilon^{\mathbb{P}}(v^T \tilde{u}) \leq \alpha$$

(b) (1 point) Show that $\sup_{|||A||| \leq 1} w^T A w = w^T w$.

Solution: By Cauchy-Schwartz we have $w^T A w \leq \|w\| \|Aw\| \leq \|w\|^2 |||A||| \leq \|w\|^2$. And the inequality is attained for $A = I$.

(c) (2 points) Show that

$$\sup_{\mathbb{P} \in \mathcal{P}^{CS}(\Gamma_1, \Gamma_2)} VaR_\varepsilon^{\mathbb{P}}(v^T \tilde{u}) = \hat{\mu}^T v + \Gamma_1 \|v\|_2 + \sqrt{\frac{1-\varepsilon}{\varepsilon}}\sqrt{v^T(\hat{\Sigma} + \Gamma_2 I)v}$$

Solution:

$$\begin{aligned}
\sup_{\mathbb{P} \in \mathcal{P}^{CS}(\Gamma_1, \Gamma_2)} VaR_\varepsilon^{\mathbb{P}}(v^T \tilde{u}) &= \sup_{\|\mu - \hat{mu}\|_2 \leq \Gamma_1, |||\Sigma - \hat{\Sigma}||| \leq \Gamma_2} \mu^T v + \sqrt{\frac{1-\varepsilon}{\varepsilon}}\sqrt{v^T \Sigma v} \\
&= \hat{\mu}^T v + \Gamma_1 \|v\| + \sqrt{\frac{1-\varepsilon}{\varepsilon}}\sqrt{\sup_{|||\Sigma - \hat{\Sigma}||| \leq \Gamma_2} v^T \Sigma v}
\end{aligned}$$

Which, coupled with the previous question, yields the result.

(d) (2 points) Show that

$$\mathcal{U}^{CS} := \left\{\hat{\mu} + y + C^T w \quad | \quad \exists y, w \in \mathbb{R}^d s.t. \|y\|_2 \leq \Gamma_1, \quad \|w\|_2 \leq \sqrt{\frac{1-\varepsilon}{\varepsilon}}\right\},$$

with $C^T C = \hat{\Sigma} + \Gamma_2 I$ implies a probabilistic guarantee of level $1 - \varepsilon$ for $f(\tilde{u}, x) \leq 0$ with confidence $1 - \alpha$.

Solution: We have

$$\begin{aligned}
\delta^\star(v | \mathcal{U}^{CS}) &= \sup_{\|y\|_2 \leq \Gamma_1, \|w\|_2 \leq \sqrt{\frac{1-\varepsilon}{\varepsilon}}} v^T(\hat{\mu} + y + C^T w) \\
&= v^T \hat{\mu} + \Gamma_1 \|y\| + \sqrt{\frac{1-\varepsilon}{\varepsilon}} \sup_{\|w\| \leq 1} \|Cw\| \\
&= \sup_{\mathbb{P} \in \mathcal{P}^{CS}(\Gamma_1, \Gamma_2)} VaR_\varepsilon^{\mathbb{P}}(v^T \tilde{u})
\end{aligned}$$

(e) (3 points) Give a data driven robust formulation, leveraging $\mathcal{P}^{CS}$ that guarantee $\mathbb{P}^*(f(\tilde{u}, x) \leq 0) \geq 1 - \varepsilon$ with confidence $1 - \alpha$. This formulation should be expressed as a set of linear and SOCP constraints and a linear inequality over the partial concave conjugate of f.

Solution: Applying the generic DDRO method, with confidence region $\mathcal{P}^{CS}$, we obtain

$$\begin{cases} f_\star(v,x) \geq s \\ t - s \leq 0 \\ \hat{\mu}^T v + \Gamma_1 \|v\|_2 + \sqrt{\frac{1-\varepsilon}{\varepsilon}} \|C^T v\| \leq t \end{cases}$$

where $C^T C = \hat{\Sigma} + \Gamma_2 I$

4. Estimated Variance and Covariance - extensions
   (a) (3 points) Assume now that we know that $\tilde{u} \in U$ almost-surely, where $U := \{u \in \mathbb{R}^{n_u} \mid Du \leq e\}$ is a non-empty polytope. Improve the data-driven SOCP formulation.

Solution: We have

$$\begin{aligned} \delta^\star_P(v) &= \max_{u: Du \leq e} v^T u \\ &= \min_{\lambda \geq 0} \max_u v^T u + \lambda^T (e - Du) \\ &= \min_{\lambda \geq 0} \lambda^T e \\ &\quad s.t.\ D^T \lambda = v \end{aligned}$$

Thus the following set of constraints imply a probabilistic guarantee.

$$\begin{cases} f_\star(v,x) \geq s \\ t - s \leq 0 \\ \hat{\mu}^T v + \Gamma_1 \|v - w\|_2 + \sqrt{\frac{1-\varepsilon}{\varepsilon}} \|C^T (v - w)\| \leq t_1 \\ \lambda^T e \leq t_2 \\ \lambda \geq 0 \\ D^T \lambda = v \\ t_1 + t_2 \leq t \end{cases}$$

   (b) (1 point) For given v, solve $\max_{u \in \mathcal{U}^{CS}} v^T u$.
   (c) (4 points) Forgetting the support constraint, instead of an SOCP representation we would like to use outer-linear approximation of the robust formulation. Give the pseudo-code of a constraint generation method.

Solution: Let C be such that $C^T C = \hat{\Sigma} + \Gamma_2 I$.

1. Set $v_0 = \hat{\mu}$, $k = 0$.
2. Solve

$$\begin{aligned} \min_{x,v} \quad & c^T x \\ s.t. \quad & f_\star(v,x) \geq s \\ & t - s \leq 0 \\ & v_k^T u \leq t \qquad\qquad \forall \kappa = 0 \dots k \end{aligned}$$

which define v_k.

3. set $u_{k+1} = \hat{\mu} + \Gamma_1 \frac{v_k}{\|v_k\|} + \sqrt{\frac{1-\varepsilon}{\varepsilon}} \frac{Cv}{\|Cv\|}$

4. If $v_k^T u_{k+1} \leq t$ STOP, else set $k = k + 1$, go to 2.

CHI-SQUARED PERCENTAGE POINTS

ν	0.1%	0.5%	1.0%	2.5%	5.0%	10.0%	12.5%	20.0%	25.0%	33.3%	50.0%
1	0.000	0.000	0.000	0.001	0.004	0.016	0.025	0.064	0.102	0.186	0.455
2	0.002	0.010	0.020	0.051	0.103	0.211	0.267	0.446	0.575	0.811	1.386
3	0.024	0.072	0.115	0.216	0.352	0.584	0.692	1.005	1.213	1.568	2.366
4	0.091	0.207	0.297	0.484	0.711	1.064	1.219	1.649	1.923	2.378	3.357
5	0.210	0.412	0.554	0.831	1.145	1.610	1.808	2.343	2.675	3.216	4.351
6	0.381	0.676	0.872	1.237	1.635	2.204	2.441	3.070	3.455	4.074	5.348
7	0.598	0.989	1.239	1.690	2.167	2.833	3.106	3.822	4.255	4.945	6.346
8	0.857	1.344	1.646	2.180	2.733	3.490	3.797	4.594	5.071	5.826	7.344
9	1.152	1.735	2.088	2.700	3.325	4.168	4.507	5.380	5.899	6.716	8.343
10	1.479	2.156	2.558	3.247	3.940	4.865	5.234	6.179	6.737	7.612	9.342

CHI-SQUARED PERCENTAGE POINTS

ν	60.0%	66.7%	75.0%	80.0%	87.5%	90.0%	95.0%	97.5%	99.0%	99.5%	99.9%
1	0.708	0.936	1.323	1.642	2.354	2.706	3.841	5.024	6.635	7.879	10.828
2	1.833	2.197	2.773	3.219	4.159	4.605	5.991	7.378	9.210	10.597	13.816
3	2.946	3.405	4.108	4.642	5.739	6.251	7.815	9.348	11.345	12.838	16.266
4	4.045	4.579	5.385	5.989	7.214	7.779	9.488	11.143	13.277	14.860	18.467
5	5.132	5.730	6.626	7.289	8.625	9.236	11.070	12.833	15.086	16.750	20.515
6	6.211	6.867	7.841	8.558	9.992	10.645	12.592	14.449	16.812	18.548	22.458
7	7.283	7.992	9.037	9.803	11.326	12.017	14.067	16.013	18.475	20.278	24.322
8	8.351	9.107	10.219	11.030	12.636	13.362	15.507	17.535	20.090	21.955	26.125
9	9.414	10.215	11.389	12.242	13.926	14.684	16.919	19.023	21.666	23.589	27.877
10	10.473	11.317	12.549	13.442	15.198	15.987	18.307	20.483	23.209	25.188	29.588