Exercises : Duality

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Exercise 1 (Dual formulation)**.** *Let $g : \mathbb{R}^n \to \mathbb{R}^m$. Show that*

1. $\mathbb{I}_{g(x)=0} = \sup_{\lambda \in \mathbb{R}^m} \lambda^\top g(x)$
2. $\mathbb{I}_{g(x)\leq 0} = \sup_{\lambda \in \mathbb{R}^m_+} \lambda^\top g(x)$
3. $\mathbb{I}_{g(x)\in C} = \sup_{\lambda \in -C^+} \lambda^\top g(x)$ *where C is a closed convex cone, and $C^+ := \{\lambda \in \mathbb{R}^m \mid \lambda^\top c \geq 0, \forall c \in C\}$.*

Exercise 2 (Linear Programming)**.** *Consider the following linear problem (LP)*

$$(P) \quad \underset{x\geq 0}{\text{Min}} \quad c^\top x \qquad s.t. \quad Ax = b$$

1. *Show that the dual of (P) is an LP.*
2. *Show that the dual of the dual of (P) is equivalent to (P).*

Exercise 3 (Quadratically Constrained Quadratic Programming)**.** *Consider the problem*

$$(QCQP) \quad \underset{x\in\mathbb{R}^n}{\text{Min}} \quad \frac{1}{2}x^\top P_0 x + q_0^\top x + r_0 \qquad \frac{1}{2}x^\top P_i x + q_i^\top x + r_i \leq 0 \quad \forall i \in [m]$$

where $P_0 \in S^n_{++}$ and $P_i \in S^n_+$.

1. *Show by duality that there exists, for $\mu \in \mathbb{R}^m_+$, P_μ, q_μ and r_μ, we have $g(\mu) = -\frac{1}{2}q_\mu P_\mu^{-1} + r_\mu$ such that $val(P) \geq g(\mu)$.*
2. *Give an easy condition under which $val(P) = \sup_{\mu\geq 0} g(\mu)$.*

Exercise 4 (Conic Programming)**.** *Let $K \subset \mathbb{R}^n$ be a closed convex pointed cone, and denote $x \preceq_K y$ iff $y \in x + K$. Consider the following program, with $A \in M_{m,n}$ and $b \in \mathbb{R}^m$.*

$$(P) \quad \underset{x\in\mathbb{R}^n}{\text{Min}} \quad c^\top x \qquad s.t. \quad Ax = b, \quad x \preceq_K 0$$

1. *Show that (P) is a convex optimization problem.*
2. *Denote $\mathcal{L}(x, \lambda, \mu) = c^\top x + \lambda^\top(Ax - b) + \mu^\top x$. Show that $\text{val}(P) = \text{Min}_{x\in\mathbb{R}^n} \sup_{\lambda\in\mathbb{R}^m, \mu\in K^+} \mathcal{L}(x, \lambda, \mu)$.*
3. *Give a dual problem to (P).*

Exercise 5 (Duality gap)**.** *Consider the following problem*

$$\underset{x\in\mathbb{R}, y\in\mathbb{R}^+_*}{\text{Min}} \quad e^{-x} \qquad s.t. \quad x^2/y \leq 0$$

1. *Find the optimal solution of this problem.*
2. *Write and solve the (Lagrangian) dual problem. Is there a duality gap ?*

Exercise 6 (Two-way partitionning)**.** *Let $W \in S_n$ be a symmetric matrix, consider the following problem.*

$$(P) \quad \underset{x\in\mathbb{R}^n}{\text{Min}} \quad x^\top W x \qquad s.t. \quad x_i^2 = 1 \qquad \forall i \in [n]$$

1. *Consider a set of n element that you want to partition in 2 subsets, with a cost $c_{i,j}$ if i and j are in the same set, and a cost $-c_{i,j}$ if they are in a different set. Justify that it can be solved by solving (P).*

2. *Is (P) a convex problem ?*

3. *Show that, for any $\lambda \in \mathbb{R}^n$ such that $W + \operatorname{diag}(\lambda) \succeq 0$, we have $\operatorname{val}(P) \geq -\sum \lambda_i$. Deduce a lower bound on $\operatorname{val}(P)$.*

Exercise 7 (Linear SVM : duality)**.** *Consider the following problem (see : `https://www.youtube.com/watch?v=IOetFPgsMUc` for background)*

$$\begin{aligned} \min_{w \in \mathbb{R}^d, b \in \mathbb{R}} & \frac{1}{2}\|w\|^2 \\ s.t. \quad & y_i(w^\top x_i + b) \geq 1 \qquad \forall i \in [n] \\ & \eta_i \geq 0 \qquad \forall i \in [n] \end{aligned}$$

1. *In which case can we guarantee strong duality ?*

2. *Write the dual of this optimization problem and express the optimal primal solution $(w^\sharp, b^\sharp)$ in terms of the optimal dual solution.*